ORDER CONVERGENCE IN INFINITE-DIMENSIONAL VECTOR LATTICES IS NOT TOPOLOGICAL

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Abstract. In this note, we show that the order convergence in a vector lattice X is not topological unless $\dim X < \infty$. Furthermore, we show that, in atomic order continuous Banach lattices, the order convergence is topological on order intervals.

1. Introduction

A net $(x_\alpha)_{\alpha \in A}$ in a vector lattice X is *order convergent* to a vector $x \in X$ if there exists a net $(y_\beta)_{\beta \in B}$ in X such that $y_\beta \downarrow 0$ and, for each $\beta \in B$, there is an $\alpha_\beta \in A$ satisfying $|x_\alpha - x| \le y_\beta$ for all $\alpha \ge \alpha_\beta$. In this case, we write $x_\alpha \xrightarrow{o} x$. It should be clear that an order convergent net has an order bounded tail. A net x_α in X is said to be *unbounded order convergent* to a vector x if, for any $u \in X_+$, $|x_\alpha - x| \wedge u \xrightarrow{o} 0$. In this case, we say that the net x_α *uo*-converges to x, and write $x_\alpha \xrightarrow{uo} x$. Clearly, order convergence implies *uo*-convergence, and they coincide for order bounded nets. For a measure space (Ω, Σ, μ) and a sequence f_n in $L_p(\mu)$ $(0 \le p \le \infty)$, we have $f_n \xrightarrow{uo} 0$ iff $f_n \to 0$ almost everywhere; see, e.g., [8, Remark 3.4]. Hence, $f_n \xrightarrow{o} 0$ in $L_p(\mu)$ iff $f_n \to 0$ almost everywhere and f_n is order bounded in $L_p(\mu)$. It is known that almost everywhere convergence is not topological in general, i.e. there may not be a topology such that the convergence with respect to this topology is the same as *a.e.*-convergence; see for example [13]. Thus, the unbounded order convergence is not topological in general.

A net x_α in a normed lattice X is *unbounded norm convergent* (*un-convergent*) to a vector x if, for all $u \in X_+$, $\big\||x_\alpha - x| \wedge u\big\| \to 0$ (cf. [11, 14, 7, 10]). In this case, we write $x_\alpha \xrightarrow{un} x$. Clearly, norm convergence implies *un*-convergence, and they agree for order bounded nets. Unlike order and unbounded order convergences, *un*-convergence is always topological, and the corresponding topology is referred to as the *un-topology* (see [7, Section 7]). The *un*-topology has been recently investigated in detail in [10].

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Recall that a net x_α in a vector lattice X is *relatively uniformly convergent* to a vector x if there is $u \in X_+$ such that for any $\varepsilon > 0$ there is α_ε satisfying $|x_\alpha - x| \leq \varepsilon u$ for all $\alpha \geq \alpha_\varepsilon$. In this case, we write $x_\alpha \xrightarrow{ru} x$. In Archimedean vector lattices relatively uniformly convergence implies order convergence.

An element $a > 0$ in a vector lattice X is called an *atom* whenever, for every $x \in [0, a]$, there is a real $\lambda \geq 0$ such that $x = \lambda a$. It is known that the band B_a generated by an atom a is a projection band and $B_a = span\{a\}$. A vector lattice X is called *atomic* if the band generated by its atoms is X. For any atom a, let P_a be the band projection corresponding to B_a.

A normed lattice $(X, \|\cdot\|)$ is said to be *order continuous* if, for every net x_α in X with $x_\alpha \downarrow 0$, it holds $\|x_\alpha\| \downarrow 0$ (or, equivalently, $x_\alpha \xrightarrow{o} 0$ in X implies $\|x_\alpha\| \to 0$). Clearly, in order continuous normed lattices, *uo*-convergence implies *un*-convergence.

Since the order convergence could be easily not topological, many researchers investigated classes of ordered topological spaces, in which order convergence of nets (or sequences) agrees with the topological convergence. For instance, in [5], DeMarr proved that a locally convex space (X, τ) can be made into an ordered vector space such that the convergence of nets with respect to τ is equivalent to order convergence if and only if X is normable. In [6, Theorem 1], DeMarr showed that any locally convex space (X, τ) can be embedded into an appropriate ordered vector space E such that $x_\alpha \xrightarrow{\tau} 0$ iff $x_\alpha \xrightarrow{uo} 0$ in E for any net x_α in X.

In [15, Theorem 1], the authors characterized ordered normed spaces in which the order convergence of nets coincides with the norm convergence. Also, they characterized ordered normed spaces in which order convergence and norm convergence coincide for sequences; see [15, Theorem 3].

As an extension of the work in [5, 6, 15], Chuchaev investigated ordered locally convex spaces, where the topological convergence agrees with the order convergence of eventually topologically bounded nets; see, for example, Theorem 2.3, Propositions 2.4, 2.5, and 2.6 in [4]. In addition, he studied ordered locally convex spaces, where the topological convergence is equivalent to the order convergence of sequences; see, for example, Propositions 3.2, 3.3, 3.4 and 3.5 in [4].

In this paper, our main result is that if (X, τ) is a topological vector lattice such that $x_\alpha \xrightarrow{\tau} 0$ iff $x_\alpha \xrightarrow{o} 0$ for any net x_α in X, then $\dim X < \infty$; see Theorem 1. It should be noticed that Theorem 1 was proven in the case of ordered normed spaces with minihedral cones (see [15, Theorem 2]), in the case of Banach lattices (see [16]), and in the case of normed lattices (see [9, Theorem 2]). A useful characterization of *uo*-convergence in atomic vector lattices is given in Proposition 1. In addition, we show that the order convergence is topological on order intervals of atomic order continuous Banach lattices; see Corollary 1. A partial converse of Corollary 1 is given in Theorem 4.

Throughout this paper, all vector lattices are assumed to be Archimedean.

2. Order convergence is not topological

In this section, we show that order convergence is topological only in finite-dimensional vector lattices. We begin with the following technical lemma.

Lemma 1. *Let (X,τ) be a topological vector lattice in which $x_\alpha \xrightarrow{\tau} 0$ implies $x_\alpha \xrightarrow{o} 0$ for any net x_α. The following statements hold.*

(i) *There is a strong unit $e \in X$.*

(ii) *For any net (x_α) in X, if $x_\alpha \xrightarrow{\tau} 0$ then $x_\alpha \xrightarrow{\|\cdot\|_e} 0$, where $\|x\|_e := \inf\{\lambda > 0 : |x| \le \lambda e\}$.*

Proof. Let $\mathcal{N}$ be the zero neighborhood base of τ.

(i) Let $\Delta := \{(y,U) : U \in \mathcal{N}$ and $y \in U\}$ be ordered by $(y_1,U_1) \le (y_2,U_2)$ iff $U_1 \supseteq U_2$. Under this order, Δ is directed upward. For each $\alpha \in \Delta$, let $x_\alpha = x_{(y,U)} := y$. Clearly, $x_\alpha \xrightarrow{\tau} 0$. Now the assumption assures that $x_\alpha \xrightarrow{o} 0$. So there are $(y_0,U_0) \in \Delta$ and $e \in X_+$ such that, for all $(y,U) \ge (y_0,U_0)$, we have $x_{(y,U)} = y \in [-e,e]$. In particular, $U_0 \subseteq [-e,e]$. Since U_0 is absorbing, for every $x \in X$, there is $n \in \mathbb{N}$ satisfying $|x| \in nU_0$ and so $|x| \le ne$. Hence, e is a strong unit.

(ii) Since e is a strong unit then $I_e = X$, where I_e is the ideal generated by e. For each $x \in X$, let $\|x\|_e := \inf\{\lambda > 0 : |x| \le \lambda e\}$. Then $(X, \|\cdot\|_e)$ is a normed lattice; see, for example, [3, Theorem 2.55]. Suppose $x_\alpha \xrightarrow{\tau} 0$. Let U_0 be the zero neighborhood as in part (i). Given $\varepsilon > 0$. Then εU_0 is also a zero neighborhood. Hence, there is α_ε such that $x_\alpha \in \varepsilon U_0$ for all $\alpha \ge \alpha_\varepsilon$. This implies $|x_\alpha| \le \varepsilon e$ for all $\alpha \ge \alpha_\varepsilon$. Hence $x_\alpha \xrightarrow{\|\cdot\|_e} 0$.

□

Now we are ready to prove our main result, whose proof is motivated by [16].

Theorem 1. *Let (X,τ) be a topological vector lattice. The following statements are equivalent.*

(i) $\dim X < \infty$.

(ii) *$x_\alpha \xrightarrow{\tau} 0$ iff $x_\alpha \xrightarrow{o} 0$ for any net x_α in X.*

Proof. The implication (i) $\Longrightarrow$ (ii) is trivial.
(ii) $\Longrightarrow$ (i). It follows from Lemma 1, that X has a strong unit e, and $(X = I_e, \|\cdot\|_e)$ is a normed lattice. For a net x_α in X, $x_\alpha \xrightarrow{\|\cdot\|_e} 0 \Rightarrow x_\alpha \xrightarrow{ru} 0 \Rightarrow x_\alpha \xrightarrow{o} 0 \Rightarrow x_\alpha \xrightarrow{\tau} 0$. Combining this with Lemma 1(ii), we get $x_\alpha \xrightarrow{\|\cdot\|_e} 0$ iff $x_\alpha \xrightarrow{\tau} 0$.

Let $(\widehat{X}, \|\cdot\|)$ be the norm completion of $(X, \|\cdot\|_e)$. Then $(\widehat{X}, \|\cdot\|)$ is a Banach lattice. Let $\widehat{I_e}$ be the ideal generated by e in $\widehat{X}$. Then it follows from [1, Theorem 3.4], that $(\widehat{I_e}, \widehat{\|\cdot\|_e})$ is an AM-space with a strong unit e, where $\widehat{\|z\|_e} := \inf\{\lambda > 0 : |z| \leq \lambda e\}$. Now [1, Theorem 3.6] implies that $\widehat{I_e}$ is lattice isometric to $C(K)$-space for some compact Hausdorff space K such that the strong unit e is identified with the constant function $\mathbb{1}$ on K. Clearly, $X = I_e$ is a sublattice of $\widehat{I_e}$ and so, we can identify elements of X with continuous functions on K.

Let $t_0 \in K$ and $g = \chi_{t_0}$ be the characteristic function of $\{t_0\}$. Define

$$F := \{f \in X : f \geq 0 \text{ and } f(t_0) = 1\}.$$

Then F is directed downward under the pointwise ordering. For each $\alpha \in F$, let $f_\alpha = \alpha$. Then, Urysohn's extension lemma assures that $f_\alpha \downarrow g$ pointwise. If $g \notin C(K)$ then $f_\alpha \downarrow 0$ in $C(K)$, and so $f_\alpha \downarrow 0$ in X. That is $f_\alpha \xrightarrow{o} 0$ in X, and hence $f_\alpha \xrightarrow{\tau} 0$ or $f_\alpha \xrightarrow{\|\cdot\|_e} 0$, which is a contradiction since $\|f_\alpha\|_e \geq 1$. Thus, $g \in C(K)$, and so $\{t_0\}$ is open in K. So K is discrete and hence finite. Therefore, $\dim X < \infty$. □

Remark 1. *In* [15, p. 162] *an example is given of an infinite-dimensional ordered normed space* (*which is not a normed lattice*), *where the norm convergence coincides with the order convergence.*

In what follows, we show that, in atomic order continuous Banach lattices, the order convergence can be topologized on order intervals. The following result could be known, but since we do not have an appropriate reference, we include its proof for the sake of completeness.

Proposition 1. *Let X be an atomic vector lattice. Then a net x_α uo-converges iff it converges pointwise.*

Proof. Without loss of generality, we can assume that the net x_α is in X_+ and converges to 0. The forward implication is obvious.

For the converse, let x_α be a pointwise null in X. Given $u \in X_+$, we need to show that $x_\alpha \wedge u \xrightarrow{o} 0$. Let $\Delta = \mathcal{P}_{fin}(\Omega) \times \mathbb{N}$, where Ω is the collection of all atoms in X. The set Δ is directed w.r. to the following ordering: $(A, n) \leq (B, m)$ if $A \subseteq B$ and $n \leq m$. For each $\delta = (F, n) \in \Delta$, put $y_\delta = \frac{1}{n} \sum\limits_{a \in F} P_a u + \sum\limits_{a \in \Omega \setminus F} P_a u$, where P_a denotes the band projection onto *span*$\{a\}$. It is easy to see that $y_\delta \downarrow 0$ and, for any $\delta \in \Delta$, there is α_δ such that we have $0 \leq x_\alpha \wedge u \leq y_\delta$ for any $\alpha \geq \alpha_\delta$. Therefore, $x_\alpha \wedge u \xrightarrow{o} 0$. □

Unlike Theorem 1, the next theorem shows that *uo*-convergence is topological in any atomic vector lattice.

Theorem 2. *The uo-convergence is topological in atomic vector lattices.*

Proof. By Proposition 1, *uo*-convergence in atomic vector lattices is the same as pointwise convergence and therefore is topological. □

Clearly, o-convergence is nothing than eventually order bounded uo-convergence. Replacing "eventually order bounded" by "order bounded", we obtain the following result in atomic vector lattices.

Corollary 1. *Let X be an atomic vector lattice. Then order convergence is topological on every order bounded subset of X.*

Proof. By Theorem 2, uo-convergence is topological in X and hence on any subset of X in the induced topology. Since order o-convergence coincides with uo-convergence on order intervals, we conclude that order convergence is also topological on order bounded subsets of X. □

The following result extends [7, Theorem 5.3].

Theorem 3. *Let X be a Banach lattice. The following statements are equivalent.*

(i) *For any net x_α in X, $x_\alpha \xrightarrow{uo} 0 \iff x_\alpha \xrightarrow{un} 0$.*
(ii) *For any sequence x_n in X, $x_n \xrightarrow{uo} 0 \iff x_n \xrightarrow{un} 0$.*
(iii) *X is order continuous and atomic.*

Proof. (i) $\Longrightarrow$ (ii) is trivial. (ii) $\Longrightarrow$ (iii) is part of [7, Theorem 5.3]. For (iii) $\Longrightarrow$ (i), suppose X is order continuous and atomic. Then, it follows from [10, Corollary 4.14], that $x_\alpha \xrightarrow{un} 0$ iff $P_a x_\alpha \to 0$ for any atom $a \in X$ and, by Proposition 1, this holds iff $x_\alpha \xrightarrow{uo} 0$. □

The following result is a partial converse of Corollary 1.

Theorem 4. *Assume that there is a Hausdorff locally solid topology τ on an order continuous Banach lattice X such that order convergence and τ-convergence coincide on each order interval of X. Then X is atomic.*

Proof. First we show that τ is a Lebesgue topology. Assume $x_\alpha \xrightarrow{o} 0$, then there exist α_0 and $\nu \in X_+$ such that $(x_\alpha)_{\alpha \geq \alpha_0} \subseteq [-\nu, \nu]$. By the hypothesis, $(x_\alpha)_{\alpha \geq \alpha_0} \xrightarrow{\tau} 0$ in $[-\nu, \nu]$, and so $x_\alpha \xrightarrow{\tau} 0$ in X.

Let $x_\alpha \xrightarrow{uo} 0$. Since X is order continuous, then $x_\alpha \xrightarrow{un} 0$. Suppose now $x_\alpha \xrightarrow{un} 0$, and take $u \in X_+$. Then $\big\||x_\alpha| \wedge u\big\| \to 0$. Since the net $|x_\alpha| \wedge u$ is order bounded, then, by [2, Theorem 4.22], $|x_\alpha| \wedge u \xrightarrow{\tau} 0$ in $[-u, u]$, and so $|x_\alpha| \wedge u \xrightarrow{o} 0$. We conclude $x_\alpha \xrightarrow{uo} 0$. Thus $x_\alpha \xrightarrow{uo} 0 \iff x_\alpha \xrightarrow{un} 0$. It follows from Theorem 3 that X is atomic. □

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