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Primary Rimary Finitely Compactly Packed Modules And Dauns-Primary Submodules Over Noncommutative Rings

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Abstract: In this paper we generalize the concept of primary finitely compactly packed modules over noncommutative rings. This concept generalizes the concepts of primary compactly packed modules over noncommutative rings, and primary finitely compactly packed modules over commutative rings. We first find the relation between primary finitely compactly packed modules and primary compactly packed modules over noncommutative rings. We also prove several results on the primary finitely compactly packed modules over noncommutative rings. In addition, we introduce the definition of Dauns-primary submodules over noncommutative rings, and investigate the relation between this concept and the concepts of Dauns-prime submodules, and primary submodules over noncommutative rings.

Key Words: Primary submodules over noncommutative rings, primary finitely compactly packed modules over noncommutative rings , primary compactly packed modules over noncommutative rings, Dauns-primary submodules over noncommutative rings, Dauns-prime submodules over noncommutative rings.

المقاسات الابتدائية المنتهية المحزومة الرص و المقاسات الجزئية الابتدائية من النمط دونس المعرفة على الحلقات غير الابدالية

ملخص: نعمم في هذا البحث مفهوم المقاسات المنتهية المحزومة الرص على الحلقات غير الابدالية . حيث أن هذا المفهوم هو تعميم لمفهومي المقاسات المحزومة الرص على الحلقات غير الابدالية و المقاسات المنتهية المحزومة الرص على الحلقات الابدالية. في البداية نجد العلاقة بين المقاسات المنتهية المحزومة الرص و المقاسات المحزومة الرص على الحلقات الغير ابدالية، ثم نبرهن بعض النتائج المتعلقة بالمقاسات المنتهية المحزومة الرص على الحلقات غير الابدالية. و نطرح في النهاية مفهوم المقاسات الجزئية الابتدائية من النمط دونس المعرفة على الحلقات غير الابدالية، و نجد العلاقة بينها و بين المقاسات الجزئية الأولية من النمط دونس و المقاسات الجزئية الابتدائية المعرفة على الحلقات غير الابدالية .

1. Introduction

Let R be an arbitrary ring. A nonzero submodule N of an R-module M is primary if for every nonzero $\overline{N} \subseteq N$, $rann(N) = rann(\overline{N})$, where submodule $rann(N) = \{r \mid r \in R \;\; r^n N = 0$ for some positive integer $n\}$. This definition of a primary submodule over noncommutative rings was introduced in [1]. In [2], we generalized the definition of primary compactly packed modules over any arbitrary ring. We define a proper submodule submodule N of an R-module M to be primary compactly packed (pcp) if for each family $\{P_a\}_{a \in l}$ of primary submodules of M with $N \subseteq \bigcup_{a \in l} P_a$, $\exists b \in l$ such that $N \subseteq P_b$. A module M is called pcp if every proper submodule of M is pcp. We generalize the concept of pcp modules over noncommutative rings to the concept of primary finitely compactly packed (pfcp) modules over noncommutative rings. Thus we say that a proper submodule N of an R-module M is pfcp if for each family $\{P_a\}_{a \in l}$ of primary submodules of M with $N \subseteq \bigcup_{a \in l} P_a$, $\exists a_1, a_2, ..., a_n \in l$ such that $N \subseteq \bigcup_{i=1}^{n} P_{a_i}$.

A module M is said to be pfcp if every proper submodule of M is pfcp.

In Section 2 of this paper we give some examples of pfcp modules and find the relation between pcp modules and pfcp modules. We also find the conditions that make a pfcp module pcp.

In Section 3 we investigate some properties of pfcp modules. We also find the necessary and sufficient conditions for any R-module M to be pfcp.

Finally, in Section 4, we introduce the definition of Dauns-primary submodules over noncommutative rings, which is a generalization to the concept of Dauns-prime submodules, over noncommutative rings.

We study the relation between Dauns-primary submodules over noncommutative rings and the concept primary submodules over noncommutative rings.

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2. RELATION BETWEEN PRIMARY COMPACTLY PACKED MODULES AND PRIMARY FINITELY COMPACTLY PACKED MODULES

We recall first the following definitions, see[2].

Definitions 2.1

A proper submodule N of an R-module M is primary compactly packed (pcp) if for each family $\{P_a\}_{a\in l}$ of primary submodules of M with $N \subseteq \bigcup_{a\in l} P_a, \exists b \in l$ such that $N \subseteq P_b$. A module M is said to be pcp if every proper submodule of M is pcp.

Now we give the following definition

Definitions 2.2. A proper submodule N of an R-module M is primary finitely compactly packed (pfcp) if for each family $\{P_a\}_{a\in l}$ of primary submodules of M with $N \subseteq \bigcup_{a\in l} P_a$, $\exists a_1, a_2, ..., a_n \in l$ such that $N \subseteq \bigcup_{i=1}^{n} P_{a_i}$. A module M is said to be pfcp if every proper submodule of M is pfcp.

Remark 2.3. It is clear from the definitions that every pcp module is pfcp module, however the converse is not true as illustrated in the following first example.

Examples 2.4

1) Let V be a vector space with dimension greater than 2 over the field $F = \mathbf{Z}/2\mathbf{Z}$.

Then every submodule of V is prime, so every submodule of V is primary. Let e_1 and e_2 be distinct vectors of a basis for V. Let $V_1 = e_1F$, $V_2 = e_2F$, $V_3 = (e_1+e_2)F$, and let $L = V_1 + V_2$. Then $L = \{0, e_1, e_2, e_1+e_2\}$. Thus V_1, V_2, and V_3 are primary submodules of V with the property that $L \subseteq \bigcup_{i=1}^{3} V_i$, but $L \not\subset V_i, \forall i \in \{1,2,3\}$. Thus L is pfcp, however L is not pcp.

2) If M is an R-module that contains only a finite number of primary submodules, then M is pfcp module.

Theorem 2.5 Let M be an R-module in which every finite family of primary submodules of M is totally ordered by inclusion, then M is pcp if and only if M is pfcp.

Proof.
($\rightarrow$) Trivial
($\leftarrow$) Let $N \subseteq \bigcup_{a\in l} P_a$, where P_a is primary submodule for each a . Since M is pfcp, there exist $a_1, a_2, ..., a_n$ such that $N \subseteq \bigcup_{i=1}^{n} P_{a_i}$. Since the family $\{P_{a_i}\}_{i=1}^{n}$ of primary submodules of M is totally ordered by inclusion, there exists $b \in \{a_1, a_2, ..., a_n\}$ such that $\bigcup_{i=1}^{n} P_{a_i} = P_b$. Thus M is pcp.

3. IMPORTANT RESULTS ON PRIMARY FINITELY COMPACTLY PACKED MODULES

Theorem 3.1. If M is pfcp module which has at least one maximal submodule, then M satisfies the ACC on primary submodules.

Proof. Let $N_1 \subseteq N_2 \subseteq ...$ be an ascending chain of primary submodules of M and let $L = \bigcup_i N_i$. If $L=M$ and H is a maximal submodule of M, then $H \subset \bigcup_i N_i$. Since M is pfcp, $\exists n_1, n_2, ..., n_k$ such that $H \subseteq \bigcup_{j=1}^{k} N_{n_j}$. Since $N_1 \subseteq N_2 \subseteq ...$ is an ascending chain, $\exists m \in \{1, 2, ..., k\}$ such that

$$\bigcup_{j=1}^{k} N_{n_j} = N_{n_m} = N_r$$

for some $r \in \{1, 2, 3, ...\}$. Since H is maximal, $H = N_r$. Since $N_r \subseteq N_{r+i} \subseteq \bigcup_i N_i, \forall i = 1, 2, ...$ and N_r is maximal,
either $N_r = N_{r+i}, \forall i = 1, 2, ...,$ thus $N_r = \bigcup_i N_i = M$ which is impossible, or $N_{r+i} = \bigcup_i N_i = M$ which is also impossible. Thus L must be a proper submodule of M. Now since M is pfcp, $\exists n_1, n_2, ..., n_s$ such that $L \subseteq \bigcup_{j=1}^{r} N_{n_j}$. Since $N_1 \subseteq N_2 \subseteq ...$ is an ascending chain, $\exists m \in \{1, 2, ..., r\}$ such that $\bigcup_{j=1}^{r} N_{n_j} = N_{n_m} = N_k$ for some $k \in \{1, 2, 3, ...\}$. Hence

$N_1 \subseteq N_2 \subseteq \ldots \subseteq N_k = N_{k+1} = \ldots$. Therefore the ACC is satisfied for primary submodules.

Since every finitely generated module and every multiplication module has a proper maximal submodule , see[3], then we have the following Corollary.

Corollary 3.2. Let *M be a* pfcp *R*-module. If M is a finitely generated or a multiplication *R*-module, then M satisfies the ACC on primary submodules.

Theorem 3.3. If M is an *R*-module with the property that every nonempty family of primary submodules of M is totally ordered by inclusion and if M satisfies the ACC on primary submodules, then M is pfcp.

Proof. Let N be a submodule of M with the property that $N \subseteq \bigcup_{a \in l} P_a$, where P_a is primary submodule of M for each a. Then by the hypothesis $\{P_a\}$ is totally ordered by inclusion and satisfies the ACC on primary submodules. Therefore there exists $b \in l$ such that $\bigcup P_a \subseteq P_b$. Hence $N \subseteq P_b$ for some $b \in l$. Thus M is pcp. Hence M is pfcp.

Theorem 3.4 Let $\Phi : M \to \overline{M}$ be an *R*-module isomorphism. If M is pfcp module, then $\overline{M}$ is pfcp module.

Proof. Let M be primary compactly packed, and suppose that $\overline{N} \subseteq \bigcup_{a \in l} K_a$ where $\overline{N}$ is a proper submodule of $\overline{M}$ and K_a is a primary submodule of $\overline{M}$ for each $a \in l$. Since Φ is an *R*-module isomorphism, then $\Phi^{-1}(\overline{N}) \subseteq \Phi^{-1}(\bigcup_{a \in g} K_a)$. Thus $\Phi^{-1}(\overline{N}) \subseteq \bigcup_{a \in g} (\Phi^{-1}(K_a))$. Since K_a is a primary submodule of $\overline{M}$ for each $a \in l$, by [1], $\Phi^{-1}(K_a)$ is a primary submodule of M for each $a \in l$. But M is pfcp. Thus there exist $a_1, a_2, ..., a_n \in l$ such that $\Phi^{-1}(\overline{N}) \subseteq \Phi^{-1}(\bigcup_{i=1}^{n} K_{a_i})$.

Therefore $\overline{N} \subseteq \bigcup_{i=1}^{n} K_{a_i}$ for some $a_i \in l$, and hence $\overline{N}$ is pfcp.

Thus $\overline{M}$ is pfcp.

4. Dauns-Primary Submodules.

In this section, we introduce the definition of Dauns-primary submodules and investigate the relation between this concept and the concepts of Dauns-prime submodules and primary submodules.

The following definition was introduced by J.Dauns; [4],[5].

Definition 4.1 Let R be any ring and M any R-module. A submodule N of M is called Dauns-prime if, whenever $rRm \subseteq N$ for $m \in M - N$ and $r \in R$, we have $rM \subseteq N$.

Scott Annin in his Ph.D. thesis, proved the following result, see[6].

Proposition 4.2 Let R be any ring and M any R-module. A submodule N of M is Dauns-prime if and only if $M \mid N$ is prime submodule of M, where

$M \mid N = \{r \mid r \in R,\ rM \subseteq N\}$.

Now, we introduce the definition of Dauns-primary submodules.

Definition 4.3 Let R be any ring and M any module. A submodule N of M is called Dauns-primary if, whenever $rRm \subseteq N$ for $m \in M - N$ and $r \in R$, we have $r^n M \subseteq N$, for some positive integer n.

The relationship between the definition of primary submodules and Dauns-primary submodules is given by the following proposition.

Proposition 4.4 Let R be any ring and let N be a submodule of the R-module M. Then N is a Dauns-primary submodule of M if and only if $M \mid N$ is primary submodule of M.

Proof. For the "if" part, assume that $M \mid N$ is primary submodule of M, and suppose $m \in M - N$ with $rRm \subseteq N$ for some $r \in R$. Then $(Rm + N) \mid N$ is a nonzero submodule of $M \mid N$ that is killed by r. Since $M \mid N$ is primary submodule of M, we conclude that $r^n M \subseteq N$, for some positive integer n.

Conversely, Suppose N is Dauns-primary submodule of M and consider a submodule $0 \neq \overline{N} / N \subseteq M \mid N$. Suppose that $r\overline{N} \subseteq N$, for some $r \in R$. Choose an element $t \in \overline{N} - N$. Then $rRt \subseteq N$. Since N is a Dauns-primary submodule of M, we conclude that $r^n M \subseteq N$, for some positive integer n. Thus $M \mid N$ is primary submodule of M.

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