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First-order Second-degree Equations Related with Painlevé Equations

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Abstract: The first-order second-degree equations satisfying the Fuchs theorem concerning the absence of movable critical points, related with Painlevé equations, and one-parameter families of solutions which solve the first-order second-degree equations are investigated.

Keywords: Painlevé equations; Fuchs theorem

1 Introduction

Painlevé equations, PI-PVI, which are second order first-degree equations $v'' = F(v', v, z)$ where F is rational in v', algebraic in v and locally analytic in z with the Painlevé property, which were first derived at the beginning of the 20^{th} century by Painlevé and his school [1]. A differential equation is said to have Painlevé property if all solutions are single valued around all movable singularities. Movable means that the position of the singularities varies as a function of the initial values. Painlevé equations may be regarded as the nonlinear counterparts of some classical special functions. They also arise as reductions of solutions of soliton equations solvable by the inverse scattering method (IST). Ablowitz, Ramani, and Segur [2] showed that all the ordinary differential equations (ODE) obtained by the exact similarity transforms from a partial differential equation (PDE) solvable by IST have the Painlevé property. The Painlevé property confirms the integrability properties of a PDE. Wiess, Tabor and Carnevale [3] defined a Painlevé property for PDE that does not refer to that for ODE's. This method is commonly used to investigate the integrability of a given PDE [4, 5]. Painlevé equations can also be obtained as the compatibility condition of the isomonodromy deformation problem. Recently, there have been studies of integrable mappings and discrete systems, including the discrete analogous of the Painlevé equations.

The Riccati equation is the only example for the first-order first-degree equation which has the Painlevé property. By Fuchs theorem, the irreducible form of the first order algebraic differential equation of the second-degree with Painlevé property is given as

$$(v')^2 = (A_2v^2 + A_1v + A_0)v' + B_4v^4 + B_3v^3 + B_2v^2 + B_1v + B_0, \tag{1}$$

where $A_j,\ j = 0, 1, 2$ and $B_k,\ j = 0, 1, 2, 3, 4$ are functions of z and set of parameters denoted by α [6]. Higher order ($n \geq 3$) and second order higher-degree ($k \geq 2$) with Painlevé property were subject to the articles [7–9].

Painlevé equations, PI-PVI, possess a rich internal structure. For example, for certain choice of the parameters, PII-PVI admit one parameter families of solutions, rational, algebraic and expressible in terms of the classical transcendental functions: Airy, Bessel, Weber-Hermite, Whitteker, hypergeometric functions respectively. But, all the known one parameter families of solutions appear as the solutions of Riccati

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equations. In this article, we investigate the one parameter families of solutions of PII-PVI which solves the first-order second-degree equation of the form (1). Let $v(z)$ be a solution of one of the Painlevé equations

$$v'' = P_2(v')^2 + P_1 v' + P_0, \tag{2}$$

where P_0, P_1, P_2 depend on v, z and set of parameters α. Differentiating equation (1), and using (2) to replace v'' , and (1) to replace $(v')^2$, one gets

$$\Phi v' + \Psi = 0, \tag{3}$$

where

$$\begin{aligned}\Phi &= (P_1 - 2A_2 v - A_1)(A_2 v^2 + A_1 v + A_0) + P_2(A_2 v^2 + A_1 v + A_0)^2 + 2P_0 - 4B_4 v^3 \\ &\quad - (3B_3 + A_2')v^2 - (2B_2 + A_1')v - (B_1 + A_0') + 2P_2(B_4 v^4 + B_3 v^3 + B_2 v^2 + B_1 v + B_0), \\ \Psi &= (B_4 v^4 + B_3 v^3 + B_2 v^2 + B_1 v + B_0)[P_2(A_2 v^2 + A_1 v + A_0) + 2P_1 - 2A_2 v - A_1] \\ &\quad - P_0(A_2 v^2 + A_1 v + A_0) - (B_4' v^4 + B_3' v^3 + B_2' v^2 + B_1' v + B_0').\end{aligned} \tag{4}$$

One can determine the coefficients $A_j,\ j = 0, 1, 2$ and $B_k,\ j = 0, 1, 2, 3, 4$ of (1) by setting $\Phi = \Psi = 0$. Therefore, the Painlevé equation (2) admit one-parameter family of solutions characterized by equation (1) if and only if $\Phi = \Psi = 0$. For the sake of the completeness, we will examine all possible cases, and hence, we recover some of the well known one-parameter families of solutions as well as the new ones.

2 Painlevé II Equation

Let v be a solution of the PII equation

$$v'' = 2v^3 + zv + \alpha. \tag{5}$$

In this case, equation (3) takes the form of

$$(\phi_3 v^3 + \phi_2 v^2 + \phi_1 v + \phi_0)v' + \psi_5 v^5 + \psi_4 v^4 + \psi_3 v^3 + \psi_2 v^2 + \psi_1 v + \psi_0 = 0, \tag{6}$$

where

$$\begin{aligned}&\phi_3 = 4(B_4 + \tfrac{1}{2}A_2^2 - 1), && \phi_2 = A_2' + 3B_3 + 3A_1 A_2, \\ &\phi_1 = A_1' + 2B_2 + 2A_0 A_2 + A_1^2 - 2z, && \phi_0 = A_0' + B_1 + A_0 A_1 - 2\alpha, \\ &\psi_5 = 2A_2(B_4 + 1), && \psi_4 = B_4' + A_1 B_4 + 2A_2 B_3 + 2A_1 \\ &\psi_3 = B_3' + A_1 B_3 + 2A_2 B_2 + 2A_0 + zA_2, && \psi_2 = B_2' + A_1 B_2 + 2A_2 B_1 + zA_1 + \alpha A_2, \\ &\psi_1 = B_1' + A_1 B_1 + 2A_2 B_0 + zA_0 + \alpha A_1, && \psi_0 = B_0' + A_1 B_0 + \alpha A_0.\end{aligned} \tag{7}$$

Setting $\psi_5 = 0$ yields $A_2 = 0$ or $B_4 = -1$. If $A_2 = 0$, then one can not choose $A_j, j = 0, 1$ and $B_k, k = 0, 1, ..., 4$ so that $\phi_j = 0, j = 0, 1, 2, 3$ and $\psi_k = 0, k = 0, 1, ..., 4$. If $B_4 = -1$, then $\phi_j = 0, j = 0, 1, 2, 3$, and $\psi_k = 0, k = 0, 1, ..., 4$ if and only if $A_2 = \pm 2, A_1 = 0, A_0 = \pm z$, $B_3 = B_1 = 0$, $B_2 = -z$, $B_0 = -\frac{1}{4}z^2$, and $\alpha = \pm\frac{1}{2}$. With these choices of $A_j,\ B_k$, equation (1) gives the well known equation

$$2v' = \pm(2v^2 + z), \tag{8}$$

which gives the one-parameter family of solutions of PII if $\alpha = \pm 1/2$, [10]. There is no first-order second-degree equation related with PII.

3 Painlevé III Equation

Let v solve the third Painlevé equation PIII

$$v'' = \frac{1}{v}(v')^2 - \frac{1}{z}v' + \gamma v^3 + \frac{1}{z}(\alpha v^2 + \beta) + \frac{\delta}{v}. \tag{9}$$

Then equation (3) takes the following form:

$$(\phi_4 v^4 + \phi_3 v^3 + \phi_2 v^2 + \phi_1 v + \phi_0)v' + \psi_6 v^6 + \psi_5 v^5 + \psi_4 v^4 + \psi_3 v^3 + \psi_2 v^2 + \psi_1 v + \psi_0 = 0, \tag{10}$$

where

$$\begin{array}{l}\phi_4 = 2\gamma - 2B_4 - A_2^2, \quad \phi_3 = \frac{2\alpha}{z} - A_2' - B_3 - A_1A_2 - \frac{1}{z}A_2, \quad \phi_2 = -(A_1' + \frac{1}{z}A_1),\\ \phi_1 = \frac{2\beta}{z} + B_1 + A_0A_1 - A_0' - \frac{1}{z}A_0, \quad \phi_0 = A_0^2 + 2B_0 + 2\delta,\\ \psi_6 = -A_2(B_4 + \gamma), \quad \psi_5 = -(B_4' + \frac{2}{z}B_4 + A_2B_3 + \gamma A_1 + \frac{\alpha}{z}A_2),\\ \psi_4 = A_0B_4 - B_3' - \frac{2}{z}B_3 - \gamma A_0 - \frac{\alpha}{z}A_1 - A_2B_2,\\ \psi_3 = A_0B_3 - B_2' - \frac{2}{z}B_2 - A_2B_1 - \frac{\beta}{z}A_2 - \frac{\alpha}{z}A_0,\\ \psi_2 = A_0B_2 - B_1' - \frac{2}{z}B_1 - A_2B_0 - \frac{\beta}{z}A_1 - \delta A_2,\\ \psi_1 = A_0B_1 - B_0' - \frac{2}{z}B_0 - \frac{\beta}{z}A_0 - \delta A_1, \quad \psi_0 = A_0(B_0 - \delta).\end{array} \tag{11}$$

Setting $\psi_0 = \psi_6 = 0$ gives

$$A_0(B_0 - \delta) = 0, \quad A_2(B_4 + \gamma) = 0. \tag{12}$$

Thus, one should consider the following four cases separately:
Case 1. $A_0 = A_2 = 0$: If $B_4 = \gamma$, $B_3 = \frac{2\alpha}{z}$, $B_2 = \frac{K}{z^2}$, $B_1 = -\frac{2\beta}{z}$, $B_0 = -\delta$, and $A_1 = \frac{2a_1}{z}$ then, $\phi_j = 0,\ j = 0, 1, ..., 4$, and $\psi_k = 0,\ k = 1, 2, ...5$, where K is arbitrary constant and a_1 is a constant and satisfies

$$\gamma(a_1 + 1) = 0, \quad \alpha(a_1 + 1) = 0, \quad \beta(a_1 - 1) = 0, \quad \delta(a_1 - 1) = 0. \tag{13}$$

Therefore, we have the following subcases: **i**. $a_1^2 - 1 \neq 0,\ \alpha = \beta = \gamma = \delta = 0$ **ii**. $a_1 = -1,\ \beta = \delta = 0$ and **iii**. $a_1 = 1,\ \alpha = \gamma = 0$.

When $\alpha = \beta = \gamma = \delta = 0$, the general solution of PIII is $v(z; 0, 0, 0, 0) = c_1z^{c_2}$ where c_1, c_2 are constants.

When $a_1 = -1,\ \beta = \delta = 0$, (1) gives the following first-order second-degree equation for v

$$(zv' + v)^2 = \gamma z^2v^4 + 2\alpha zv^3 + (K + 1)v^2. \tag{14}$$

The transformations

$$v = \frac{w'}{\gamma^{1/2}(w + 1) + \alpha}, \quad zv' = -\gamma^{1/2}zv^2 + wv, \tag{15}$$

give one-to-one correspondence between solutions v of PIII, and solutions $w(z)$ of the following Riccati equation

$$2zw' - w^2 - 2w + K = 0. \tag{16}$$

$\beta = \delta = 0$ case for PIII was also considered in [10, 11].

When $a_1 = 1,\ \alpha = \gamma = 0$, v satisfies the following first-order second-degree equation

$$(zv' - v)^2 = (K + 1)v^2 - 2\beta zv - \delta z^2. \tag{17}$$

Equation (17) was also considered in [9], and the solution of PIII for $\alpha = \gamma = 0$ was first given in [10].
Case 2. $A_0 \neq 0,\ A_2 \neq 0$: Equation (12) gives $B_4 = -\gamma$ and $B_0 = \delta$. To set $\phi_j = 0,\ j = 0, 1, ..., 4$ and $\psi_k = 0,\ k = 1, 2, ..., 5$, one should choose

$$\begin{array}{c}B_3 = \frac{1}{z}[2\alpha - A_2(2a_1 + 1)], \quad B_2 = -[\frac{a_1^2}{z^2} + \frac{1}{2}A_0A_2],\\ B_1 = -\frac{1}{z}[2\beta + 2a_1A_0 - A_0],\end{array} \tag{18}$$

and $A_1 = \frac{2a_1}{z}$, $A_0^2 = -4\delta$, where $a_1 = \frac{2\alpha}{A_2} - 1$ and $\alpha,\ \beta,\ \gamma,\ \delta$ satisfy

$$\beta\gamma^{1/2} + \alpha(-\delta)^{1/2} + 2\gamma^{1/2}(-\delta)^{1/2} = 0. \tag{19}$$

Then, equation (1) becomes

$$zv' + \gamma^{1/2}zv^2 + [\alpha\gamma^{-1/2} + 1]v + (-\delta)^{1/2}z = 0. \tag{20}$$

Equation (20) was given in the literature before, see for example [10].
Case 3. $A_0 \neq 0,\ A_2 = 0$: Equation (12) gives $B_0 = \delta$. One should choose $B_4 = \gamma$, $B_3 = \frac{2\alpha}{z}$,

$B_2 = -a_1^2$, $B_1 = -\frac{1}{z}[2\beta + 2a_1A_0 - A_0]$, $A_0^2 = -4\delta$, $A_1 = \frac{2a_1}{z}$, and $\alpha = 0$ in order to get $\phi_j = 0,\ j = 0, ..., 4$, and $\psi_k = 0,\ k = 1, ..., 5$, where a_1 is a constant such that

$$\gamma(a_1 + 1) = 0, \quad 2\beta + A_0(a_1 - 1) = 0. \tag{21}$$

So, if $\gamma \neq 0$, then $\beta = -2(-\delta)^{1/2}$ and (1) gives the equation (20) with $\alpha = 0$. If $\gamma = 0$ then (1) gives the equation (17) with $K = -1 - \frac{\beta^2}{\delta}$.

Case 4. $A_0 = 0,\ A_2 \neq 0$: Equation (12) gives $B_4 = -\gamma$, and if $B_0 = -\delta$, $B_3 = \frac{1}{z}[2\alpha - A_2(2a_1 + 1)]$, $B_2 = -\frac{a_1^2}{z^2}$, $B_1 = -\frac{2\beta}{z}$, $A_2^2 = 4\gamma$, $A_1 = \frac{2a_1}{z}$, and $\beta = 0$, then $\phi_j = 0,\ j = 0, ..., 4,\ \psi_k = 0,\ k = 1, ..., 5$, where a_1 is a constant and satisfies

$$\delta(a_1 - 1) = 0, \quad 2\alpha - A_2(a_1 + 1) = 0. \tag{22}$$

So, if $\delta \neq 0$, equation (1) gives the equation (20) for $\beta = 0$. If $\delta = 0$, then v solves the equation (14) with $K = \frac{\alpha^2}{\gamma} - 1$.

4 Painlevé IV Equation

Let v be a solution of the PIV equation

$$v'' = \frac{1}{2v}(v')^2 + \frac{3}{2}v^3 + 4zv^2 + 2(z^2 - \alpha)v + \frac{\beta}{v}. \tag{23}$$

Then equation (3) takes the form of

$$(\phi_4 v^4 + \phi_3 v^3 + \phi_2 v^2 + \phi_1 v + \phi_0)v' + \psi_6 v^6 + \psi_5 v^5 + \psi_4 v^4 + \psi_3 v^3 + \psi_2 v^2 + \psi_1 v + \psi_0 = 0, \tag{24}$$

where

$$\begin{aligned}
&\phi_4 = 3(1 - B_4 - \tfrac{1}{2}A_2^2), \quad \phi_3 = 8z - A_2' - 2B_3 - 2A_1A_2,\\
&\phi_2 = 4(z^2 - \alpha) - B_2 - \tfrac{1}{2}A_1^2 - A_0A_2 - A_1', \quad \phi_1 = -A_0', \quad \phi_0 = \tfrac{1}{2}A_0^2 + B_0 + 2\beta,\\
&\psi_6 = -\tfrac{3}{2}A_2(B_4 + 1), \quad \psi_5 = -(B_4' + \tfrac{1}{2}A_1B_4 + \tfrac{3}{2}A_2B_3 + 4zA_2 + \tfrac{3}{2}A_1),\\
&\psi_4 = \tfrac{1}{2}A_0B_4 - B_3' - \tfrac{1}{2}A_1B_3 - \tfrac{3}{2}A_2B_2 - 2(z^2 - \alpha)A_2 - 4zA_1 - \tfrac{3}{2}A_0,\\
&\psi_3 = \tfrac{1}{2}A_0B_3 - B_2' - \tfrac{1}{2}A_1B_2 - \tfrac{3}{2}A_2B_1 - 2(z^2 - \alpha)A_1 - 4zA_0,\\
&\psi_2 = \tfrac{1}{2}A_0B_2 - B_1' - \tfrac{1}{2}A_1B_1 - \tfrac{3}{2}A_2B_0 - 2(z^2 - \alpha)A_0 - \beta A_2,\\
&\psi_1 = \tfrac{1}{2}A_0B_1 - B_0' - \tfrac{1}{2}A_1B_0 - \beta A_1, \quad \psi_0 = \tfrac{1}{2}A_0(B_0 - 2\beta).
\end{aligned} \tag{25}$$

Setting $\psi_0 = \psi_6 = 0$ implies

$$A_2(B_4 + 1) = 0, \quad A_0(B_0 - 2\beta) = 0. \tag{26}$$

respectively. Therefore, there are four subcases: **i.** $A_0 = A_2 = 0$; **ii.** $A_0 = 0, A_2 \neq 0$; **iii.** $A_0 \neq 0, A_2 \neq 0$, and **iv.** $A_0 \neq 0, A_2 = 0$. For the first case, there are no choice of A_j and B_k such that $\Phi = \Psi = 0$. In the second and third cases, one should choose $A_2 = 2\epsilon, A_1 = 4\epsilon z, A_0 = 2(-2\beta)^{1/2}, B_4 = -1, B_3 = -4z$, $B_2 = -4[z^2 + \alpha + \epsilon(-2\beta)^{1/2} + \epsilon]$, $B_1 = -4\epsilon(-2\beta)^{1/2}z, B_0 = 2\beta$, and

$$(-2\beta)^{1/2} + 2\epsilon\alpha + 2 = 0, \tag{27}$$

where $\epsilon = \pm 1$. Then v satisfies the following Riccati equation

$$v' = \epsilon(v^2 + 2zv - 2\alpha - 2\epsilon). \tag{28}$$

Equation (28) was also considered in [12, 13].

When $A_0 \neq 0, A_2 = 0$, one has to choose $A_0 = -4$, $A_1 = 0$, $B_4 = 1$, $B_3 = 4z$, $B_2 = 4(z^2 - \alpha)$, $B_1 = 0$, $B_0 = -4$, and $\beta = -2$. Therefore, one can state the following theorem

Theorem 1 *The Painlevé IV equation admits a one-parameter family of solution characterized by*

$$(v'+2)^2-v^4-4zv^3-4(z^2-\alpha)v^2=0, \tag{29}$$

if and only if $\beta=-2$.

(29) was also given in [14]. If $f(z)=-2z+a$ and $g(z)=-2z-a$, where $a^2=4\alpha$, then (29) can be written as

$$(v'+2)^2=v^2(v-f)(v-g). \tag{30}$$

If $a=0$, that is $\alpha=0$, then (29) reduces to the following Riccati equation

$$v'=\epsilon v(v+2z)-2, \qquad \epsilon=\pm 1. \tag{31}$$

Equation (31) is nothing but equation (28) with $\alpha=0$.

If $a\neq 0$, by using the transformation $v=\frac{fw^2-g}{w^2-1}$ [6], (30) can be transformed to the following Riccati equation:

$$2w'=\epsilon(fw^2-g), \quad \epsilon=\pm 1. \tag{32}$$

By introducing $w=-\frac{2\epsilon y'}{fy},\ x=\frac{-2z+a}{\sqrt{2}}$, (32) can be linearized.

5 Painlevé V Equation

If v solves the PV equation

$$v''=\frac{3v-1}{2v(v-1)}(v')^2-\frac{1}{z}v'+\frac{\alpha}{z^2}v(v-1)^2+\frac{\beta(v-1)^2}{z^2v}+\frac{\gamma}{z}v+\frac{\delta v(v+1)}{v-1}, \tag{33}$$

then equation (3) takes the form of

$$(\phi_5v^5+\phi_4v^4+\phi_3v^3+\phi_2v^2+\phi_1v+\phi_0)v'+\psi_7v^7+\psi_6v^6+\psi_5v^5+\psi_4v^4+\psi_3v^3+\psi_2v^2+\psi_1v+\psi_0=0, \tag{34}$$

where

$$\begin{aligned}
&\phi_5=\tfrac{2\alpha}{z^2}-B_4-\tfrac{1}{2}A_2^2, \quad \phi_4=3B_4+\tfrac{3}{2}A_2^2-\tfrac{6\alpha}{z^2}-A_2'-\tfrac{1}{z}A_2,\\
&\phi_3=2B_3+B_2+\tfrac{1}{2}A_1^2+2A_1A_2+A_0A_2+A_2'+\tfrac{1}{z}A_2-A_1'-\tfrac{1}{z}A_1+\tfrac{2}{z^2}[3\alpha+\beta+\gamma z+\delta z^2],\\
&\phi_2=2B_1+B_2+\tfrac{1}{2}A_1^2+2A_0A_1+A_0A_2+A_1'+\tfrac{1}{z}A_1-A_0'-\tfrac{1}{z}A_0-\tfrac{2}{z^2}[\alpha+3\beta+\gamma z-\delta z^2],\\
&\phi_1=3B_0+\tfrac{3}{2}A_0^2+\tfrac{6\beta}{z^2}+A_0'+\tfrac{1}{z}A_0, \quad \phi_0=-(\tfrac{2\beta}{z^2}+B_0+\tfrac{1}{2}A_0^2),\\
&\psi_7=-\tfrac{1}{2}A_2(B_4+\tfrac{2\alpha}{z^2}), \quad \psi_6=B_4(\tfrac{3}{2}A_2+\tfrac{1}{2}A_1-\tfrac{2}{z})-\tfrac{1}{2}A_2B_3+\tfrac{\alpha}{z^2}(3A_2-A_1)-B_4',\\
&\psi_5=B_4(\tfrac{1}{2}A_1+\tfrac{3}{2}A_0+\tfrac{2}{z})+B_3(\tfrac{3}{2}A_2+\tfrac{1}{2}A_1-\tfrac{2}{z})-\tfrac{1}{2}A_2B_2\\
&\quad-\tfrac{A_2}{z^2}(3\alpha+\beta+\gamma z+\delta z^2)+\tfrac{\alpha}{z^2}(3A_1-A_0)+B_4'-B_3',\\
&\psi_4=B_3(\tfrac{1}{2}A_1+\tfrac{3}{2}A_0+\tfrac{2}{z})+B_2(\tfrac{3}{2}A_2+\tfrac{1}{2}A_1-\tfrac{2}{z})-\tfrac{1}{2}A_2B_1-\tfrac{1}{2}A_0B_4\\
&\quad-\tfrac{A_1}{z^2}(3\alpha+\beta+\gamma z+\delta z^2)+\tfrac{A_2}{z^2}(\alpha+3\beta+\gamma z-\delta z^2)+\tfrac{3\alpha}{z^2}A_0+B_3'-B_2',\\
&\psi_3=B_2(\tfrac{1}{2}A_1+\tfrac{3}{2}A_0+\tfrac{2}{z})+B_1(\tfrac{3}{2}A_2+\tfrac{1}{2}A_1-\tfrac{2}{z})-\tfrac{1}{2}A_2B_0-\tfrac{1}{2}A_0B_3\\
&\quad-\tfrac{A_0}{z^2}(3\alpha+\beta+\gamma z+\delta z^2)+\tfrac{A_1}{z^2}(\alpha+3\beta+\gamma z-\delta z^2)-\tfrac{3\beta}{z^2}A_2+B_2'-B_1',\\
&\psi_2=B_1(\tfrac{1}{2}A_1+\tfrac{3}{2}A_0+\tfrac{2}{z})+B_0(\tfrac{3}{2}A_2+\tfrac{1}{2}A_1-\tfrac{2}{z})+\tfrac{A_0}{z^2}(\alpha+3\beta+\gamma z-\delta z^2)\\
&\quad+\tfrac{\beta}{z^2}(A_2-3A_1)+B_1'-B_0',\\
&\psi_1=B_0(\tfrac{3}{2}A_0+\tfrac{1}{2}A_1+\tfrac{2}{z})-\tfrac{1}{2}A_0B_1+\tfrac{\beta}{z^2}(A_1-3A_0)+B_0', \quad \psi_0=\tfrac{-1}{2}A_0(B_0-\tfrac{2\beta}{z^2}).
\end{aligned} \tag{35}$$

Setting $\psi_7=\psi_0=0$ implies

$$A_0(B_0-\frac{2\beta}{z^2})=0, \quad A_2(B_4+\frac{2\alpha}{z^2})=0. \tag{36}$$

Therefore, one should consider the following four distinct cases.

Case 1. $A_0\neq 0,\ A_2\neq 0$: In this case, (36) implies $B_4=-\frac{1}{4}A_2^2$, and $B_0=-\frac{1}{4}A_0^2$. In order to get

$\phi_j = 0, j = 0, 5$, one should choose $A_0 = \frac{-8\beta}{z^2}$, $A_2^2 = \frac{8\alpha}{z^2}$, then $\phi_j, j = 1, 4$ are identically zero, and $\phi_j = 0,\ j = 2, 3$ if

$$2z^2B_1 + z^2B_2 + \frac{1}{2}z^2A_1^2 + 2z^2A_1A_0 + z\frac{d}{dz}(zA_1) - 2\alpha - 6\beta - 2\gamma z + 2\delta z^2 = 0, \tag{37}$$

$$2z^2B_3 + z^2B_2 + \frac{1}{2}z^2A_1 + 2z^2A_1A_2 - z\frac{d}{dz}(zA_1) + 6\alpha + 2\beta + 2\gamma z + 2\delta z^2 + z^2A_0A_2 = 0, \tag{38}$$

respectively. The equations $\psi_j = 0, j = 1, 6$, give $B_1 = -\frac{1}{2}A_0A_1$ and $B_3 = -\frac{1}{2}A_1A_2$. Then, the equations (37) and (38) give $B_2 = -\frac{1}{4}[(A_0 + A_1 + A_2)^2 + A_1^2 + 2A_0A_2 + 8\delta]$, and

$$\frac{d}{dz}(zA_1) = \frac{z}{2}(A_2 - A_0)(A_0 + A_1 + A_2) + 2\gamma. \tag{39}$$

With these choices $\psi_5 = \psi_2 = 0$ identically, and $\psi_4 = \psi_3 = 0$ if

$$(A_0 + A_1 + A_2)^3 + 8\delta\left(A_1 + 2A_2 - \frac{2}{z}\right) - \frac{4\gamma}{z}(A_0 + A_1 + A_2) = 0, \tag{40}$$

$$(A_0 + A_1 + A_2)^3 + 8\delta\left(A_1 + 2A_0 + \frac{2}{z}\right) + \frac{4\gamma}{z}(A_0 + A_1 + A_2) = 0, \tag{41}$$

respectively.

By adding the above equations, one gets

$$(A_0 + A_1 + A_2)[(A_0 + A_1 + A_2)^2 + 8\delta] = 0. \tag{42}$$

If $A_0 + A_1 + A_2 = 0$, then equations (39), and (40) give $\gamma = 0$ and $\delta(A_2 - A_0 - \frac{2}{z}) = 0$. Thus, if

$$\gamma = 0, \qquad \delta[(2\alpha)^{1/2} - (-2\beta)^{1/2} - 1] = 0, \tag{43}$$

then v solves the following Riccati equation:

$$zv' - (2\alpha)^{1/2}v^2 - [(-2\delta)^{1/2}z - (2\alpha)^{1/2} + (-2\beta)^{1/2}]v + (-2\beta)^{1/2} = 0... \tag{44}$$

If $A_0 + A_1 + A_2 \neq 0$, then equations (42), and (40) give

$$(-2\delta)^{1/2}[1 - (-2\beta)^{1/2} - (2\alpha)^{1/2}] = \gamma, \tag{45}$$

and the equations (39) and (41) are identically satisfied. Therefore, if α, β, γ, and δ satisfy the relation (45), then PV has one-parameter family of solution which is given by the following Riccati equation:

$$zv' - (2\alpha)^{1/2}v^2 - [(-2\delta)^{1/2}z - (2\alpha)^{1/2} + (-2\beta)^{1/2}]v + (-2\beta)^{1/2} = 0... \tag{46}$$

Equation (46) is the well known one-parameter family of solutions of PV [15].

Case 2. $A_0 = A_2 = 0$: In order to make $\phi_j = 0,\ j = 1, ..., 5$ and $\psi_k = 0,\ k = 1, ..., 6$ one should choose $B_4 = \frac{2\alpha}{z^2}$, $B_0 = \frac{-2\beta}{z^2}$, and $A_1,\ B_n,\ n = 1, 2, 3$ satisfy the following equations

$$2B_3 + B_2 + \frac{1}{2}A_1^2 - A_1' - \frac{1}{z}A_1 + \frac{6\alpha}{z^2} + \frac{2\beta}{z^2} + \frac{2\gamma}{z} + 2\delta = 0, \tag{47}$$

$$2B_1 + B_2 + \frac{1}{2}A_1^2 + A_1' + \frac{1}{z}A_1 - \frac{2\alpha}{z^2} - \frac{6\beta}{z^2} - \frac{2\gamma}{z} + 2\delta = 0, \tag{48}$$

$$-B_3' + B_3(\frac{1}{2}A_1 - \frac{2}{z}) + \frac{4\alpha}{z^2}A_1 = 0, \tag{49}$$

$$B_3' - B_2' + B_3(\frac{1}{2}A_1 + \frac{2}{z}) + B_2(\frac{1}{2}A_1 - \frac{2}{z}) - A_1(\frac{3\alpha}{z^2} + \frac{\beta}{z^2} + \frac{\gamma}{z} + \delta) = 0, \tag{50}$$

$$B_2' - B_1' + B_2(\frac{1}{2}A_1 + \frac{2}{z}) + B_1(\frac{1}{2}A_1 - \frac{2}{z}) + A_1(\frac{\alpha}{z^2} + \frac{3\beta}{z^2} + \frac{\gamma}{z} - \delta) = 0, \tag{51}$$

$$B_1' + B_1(\frac{1}{2}A_1 + \frac{2}{z}) - \frac{4\beta}{z^2}A_1 = 0. \tag{52}$$

Adding the equations (49-52) gives

$$A_1[B_1 + B_2 + B_3 + \frac{2}{z^2}(\alpha - \beta) - 2\delta] = 0. \tag{53}$$

Thus, two cases $A_1 = 0$ and $A_1 \neq 0$, should be considered separately.

Case 2.a. $A_1 = 0$: Solving equations (49-52), gives $B_j = \frac{K_j}{z^2}, j = 1, 2, 3$, where K_j are constants, then (47) and (48) give

$$\gamma = \delta = 0, \quad 2K_3 + K_2 + 6\alpha + 2\beta = 0, \quad 2K_1 + K_2 - 2\alpha - 6\beta = 0. \tag{54}$$

If one lets $K_2 = 2K$, then equation (54) gives $K_3 = -(K + 3\alpha + \beta)$, and $K_1 = -(K - \alpha - 3\beta)$. Thus, for $\gamma = \delta = 0$, $v(z)$ satisfies

$$z^2(v')^2 = 2\alpha v^4 - (K + 3\alpha + \beta)v^3 + 2Kv^2 - (K - \alpha - 3\beta)v - 2\beta. \tag{55}$$

The transformation

$$zv' = (v - 1)[w + \mu + \frac{1}{2} - \frac{1}{2}(2\mu - 1)v], \qquad v = -\frac{[(w + \mu + \frac{1}{2})^2 + 2\beta]}{2zw' - (w + \mu + \frac{1}{2})^2 - 2\beta}, \tag{56}$$

where $(2\mu - 1)^2 = 8\alpha$, give one-to-one correspondence between solutions v of (55) and solutions w of the following equation

$$2zw' = w^2 + 2w + K - 3\alpha + 3\beta - 1. \tag{57}$$

The relation between PV with $\gamma = \delta = 0$ and equation (57) was considered in [16].

Case 2.b. $A_1 \neq 0$: In this case, $\Phi = \Psi = 0$ implies that $\alpha + \beta = 0$, $\gamma = (-2\delta)^{1/2}$ and v satisfies

$$zv' = (2\alpha)^{1/2}(v - 1)^2 + \gamma zv. \tag{58}$$

Equation (58) is the special case $\alpha + \beta = 0$ of the one-parameter family of PV, see equation (44).

Case 3. $A_0 = 0, A_2 \neq 0$: Equation (36) gives $B_4 = -\frac{2\alpha}{z^2}$. Setting $\phi_5 = 0$ and $\psi_6 = 0$ give $A_2^2 = \frac{8\alpha}{z^2}$ $B_3 = -\frac{1}{2}A_1A_2$, and equations $\phi_j = 0, j = 0, ..., 4$ are satisfied if

$$\begin{aligned} B_2 &= \tfrac{1}{z^2}[z^2A_1' + zA_1 - z^2A_1A_2 - \tfrac{1}{2}z^2A_1^2 - 6\alpha - 2\beta - 2\gamma z - 2\delta z^2], \\ B_1 &= -\tfrac{1}{z^2}[z^2A_1' + zA_1 - \tfrac{1}{2}z^2A_1A_2 - 4\alpha - 4\beta - 2\gamma z], \qquad B_0 = -\tfrac{2\beta}{z^2}, \end{aligned} \tag{59}$$

and A_1 satisfies the following equations:

$$\begin{aligned} z\tfrac{d^2}{dz^2}(zA_1) = {} & \tfrac{d}{dz}(zA_1)[\tfrac{5}{2}zA_2 + \tfrac{3}{2}zA_1 - 1] - 10\alpha(2A_1 + A_2) - 2\beta(A_1 + A_2) \\ & -\gamma(3zA_2 + 2zA_1 - 2) - 2\delta z(zA_1 + 2zA_2 - 2) - \tfrac{1}{4}z^2A_1^2(6A_2 + A_1), \end{aligned} \tag{60}$$

$$\begin{aligned} 2z\tfrac{d^2}{dz^2}(zA_1) = {} & \tfrac{d}{dz}(zA_1)[3zA_2 + zA_1 - 2] - 6\alpha(A_1 + A_2) - 4\beta(A_1 + A_2) \\ & -\gamma(3zA_2 + zA_1 - 4) + 2\delta z(zA_1 + 2) + \tfrac{1}{4}z^2A_1^2(A_2 + A_1), \end{aligned} \tag{61}$$

$$z\tfrac{d^2}{dz^2}(zA_1) = \tfrac{d}{dz}(zA_1)[\tfrac{1}{2}zA_2 - \tfrac{1}{2}zA_1 - 1] + 2\alpha A_1 - 2\beta(A_1 + A_2) + \gamma(zA_1 + 2) + \tfrac{1}{4}z^2A_1^2A_2. \tag{62}$$

Solving equation (60) for $\frac{d^2}{dz^2}(zA_1)$ and using in (61) and (62) give the equations for $\frac{d}{dz}(zA_1)$. Eliminating $\frac{d}{dz}(zA_1)$ between these equations, and using $\alpha = \frac{1}{8}z^2A_2^2$ give

$$(A_1 + A_2)[(A_1 + A_2)^2 + 8\delta] = 0. \tag{63}$$

Therefore, if $A_1 + A_2 \neq 0$, then equation (63) implies that $(A_1 + A_2)^2 + 8\delta = 0$. Equations (60), (61), and (62) are satisfied if $(A_1 + A_2)(zA_2 - 2) + 4\gamma = 0$, and $\beta = 0$. Thus, if $\gamma + (-2\delta)^{1/2}[(2\alpha)^{1/2} - 1] = 0$ and $\beta = 0$, v solves the following Riccati equation

$$zv' = v[(2\alpha)^{1/2}v + (-2\delta)^{1/2}z - (2\alpha)^{1/2})]. \tag{64}$$

Equation (64) is a special case, $\beta = 0$, of (46).

If $A_1 + A_2 = 0$ then equation (60), (61), (62) are satisfied if $(A_2 - \frac{2}{z})(\gamma + 2\delta z) = 0$. Therefore, when $\gamma = \delta = 0$, first-order second-degree equation (1) reduces to the following Riccati equation

$$zv' = (2\alpha)^{1/2} v(v-1) + (-2\beta)^{1/2}(v-1). \quad (65)$$

Equation (65) is the particular case, $K = \alpha - \beta$, of equation (55).

If γ and δ are not both zero, then one has $A_2 = \frac{2}{z}$, and hence $\alpha = \frac{1}{2}$. Thus, we have $B_4 = -\frac{1}{z^2}$, $B_3 = \frac{2}{z^2}$, $B_2 = -\frac{1}{z^2}(2\delta z^2 + 2\gamma z + 2\beta + 1)$, $B_1 = \frac{2}{z^2}(\gamma z + 2\beta)$. Then, for $\alpha = 1/2$ and $\gamma,\ \delta$ not both zero, v solves the following first-order second-degree equation

$$[zv' - v(v-1)]^2 + 2(\delta z^2 + \gamma z + \beta)v^2 - 2(\gamma z + 2\beta)v + 2\beta = 0. \quad (66)$$

If one lets $f(z) = bz + a$, and $g(z) = cz + a$, where $a^2 = 2\beta,\ a(b+c) = 2\gamma,\ bc = 2\delta$, then (66) takes the following form

$$[zv' - v(v-1)]^2 = -(fv - a)(gv - a). \quad (67)$$

If $a = 0$, that is, if $\beta = \gamma = 0$, then equation (67) is reduced to the following Riccati equation

$$zv' = v(v-1) + (-2\delta)^{1/2} zv. \quad (68)$$

Equation (68) is the special case, $\beta = \gamma = 0$ and $\alpha = \frac{1}{2}$, of (46).

If $b = c$, that is, if $\gamma^2 = 4\beta\delta$, then equation (67) is reduced to the following Riccati equation

$$zv' = v(v-1) + [(-2\delta)^{1/2} z + (-2\beta)^{1/2}]v - (-2\beta)^{1/2}. \quad (69)$$

Equation (69) is the special case, $\alpha = \frac{1}{2}$, of (46).

If $\beta \neq 0$, and $\gamma^2 - 4\beta\delta \neq 0$, then the solution of equation (66) is given by $v = \frac{a(w^2+1)}{fw^2+g}$, where $w(z)$ solves the following Riccati equation:

$$2zw' = \epsilon(fw^2 + g), \quad \epsilon = \pm 1. \quad (70)$$

(70) can be transformed to the following linear equation by the transformation $w = -\frac{2\epsilon z y'}{fy}$:

$$z^2 f(z) y'' + azy' + \frac{1}{4} g(z) f^2(z) y = 0. \quad (71)$$

If $b \neq 0$, the change of variable $z = -\frac{a}{b}x$ transforms equation (71) to the equation

$$\ddot{y} - \Big(\frac{1}{x-1} - \frac{1}{x}\Big)\dot{y} + \frac{a^2(x-1)(cx-b)}{4bx^2} y = 0. \quad (72)$$

If $b = 0$, that is $\delta = 0$, then the change of variable $z = \frac{1}{ac}x^2$ transforms equation (71) to the Bessel equation

$$x^2\ddot{y} + x\dot{y} + (x^2 + a^2)y = 0. \quad (73)$$

Case 4. $A_0 \neq 0, A_2 = 0$: In this case, equation (36) gives $B_0 = \frac{2\beta}{z^2}$. Setting $\phi_0 = 0$ and $\psi_1 = 0$ implies that $A_0^2 = -\frac{8\beta}{z^2}$, and $B_1 = -\frac{1}{2}A_0 A_1$. The conditions $\phi_j = 0$ for $j = 1, ..., 5$ are satisfied if

$$\begin{array}{l} B_2 = -\frac{1}{z^2}[z^2 A_1' + zA_1 + z^2 A_0 A_1 + \frac{1}{2} z^2 A_1^2 - 2\alpha - 6\beta - 2\gamma z + 2\delta z^2], \\ B_3 = \frac{1}{z^2}[z^2 A_1' + zA_1 + \frac{1}{2} z^2 A_1^2 - 4\alpha - 4\beta - 2\gamma z], \quad B_4 = \frac{2\alpha}{z^2}, \end{array} \quad (74)$$

and $\psi_j = 0,\ j = 2, ..., 6$ if,

$$\begin{array}{r} z\frac{d^2}{dz^2}(zA_1) = -\frac{d}{dz}(zA_1)[\frac{5}{2}zA_0 + \frac{3}{2}zA_1 + 1] + 2\alpha(A_1 + A_0) + 10\beta(2A_1 + A_0) \\ +\gamma(3zA_0 + 2zA_1 + 2) - 2\delta z(zA_1 + 2zA_0 + 2) - \frac{1}{4}z^2 A_1^2(6A_0 + A_1), \end{array} \quad (75)$$

$$\begin{array}{r} 2z\frac{d^2}{dz^2}(zA_1) = -\frac{d}{dz}(zA_1)[3zA_0 + zA_1 + 2] + 4\alpha(A_1 + A_0) + 6\beta(A_1 + A_0) \\ +\gamma(3zA_0 + zA_1 + 4) + 2\delta z(zA_1 - 2) + \frac{1}{4}z^2 A_1^2(A_0 + A_1), \end{array} \quad (76)$$

$$z\frac{d^2}{dz^2}(zA_1) = \frac{d}{dz}(zA_1)[\tfrac{1}{2}zA_1 - \tfrac{1}{2}zA_0 - 1] + 2\alpha(A_0 + A_1) - 2\beta A_1 - \gamma(zA_1 - 2) + \tfrac{1}{4}z^2A_1^2A_0. \quad (77)$$

Solving equation (75) for $\frac{d^2}{dz^2}(zA_1)$ and using in (76) and (77) give the first order differential equations for A_1. Eliminating $\frac{d}{dz}(zA_1)$ between these equations and using $\beta = \frac{-1}{8}z^2A_0^2$ gives

$$(A_1 + A_0)[(A_1 + A_0)^2 + 8\delta] = 0. \quad (78)$$

If $A_1 + A_0 \neq 0$, then one obtains $(A_1 + A_0)^2 + 8\delta = 0$. The equations (76) and (77) are satisfied if $\alpha = 0$ and $(zA_0 + 2)(A_0 + A_1) = 4\gamma$. Thus, when $\gamma = (-2\delta)^{1/2}[1 - (-\beta)^{1/2}]$, v satisfies the following Riccati equation

$$zv' = [(-2\delta)^{1/2}z + (-2\beta)^{1/2}]v - (-2\beta)^{1/2}. \quad (79)$$

This is the special case, $\alpha = 0$, of equation (46).

If $A_1 + A_0 = 0$, equations(75), (76), (77) are satisfied only if $(zA_0 + 2)(\gamma - 2\delta z) = 0$. Therefore, if $zA_0 + 2 \neq 0$, then one should have $\gamma = \delta = 0$, and v satisfies (65).

If $zA_0 + 2 = 0$, then $\beta = -\frac{1}{2}$ and first-order second-degree equation (1) gives

$$[zv' - (v - 1)]^2 = 2\alpha v^4 - 2(\gamma z + 2\alpha)v^3 - 2(\delta z^2 - \gamma z - \alpha)v^2. \quad (80)$$

The Lie-point discrete symmetry $\bar{v} = \frac{1}{v}, \bar{\alpha} = -\beta, \bar{\beta} = -\alpha, \bar{\gamma} = -\gamma, \bar{\delta} = \delta$ of PV [10] transforms solutions $v(z; \alpha, -\frac{1}{2}, \gamma, \delta)$ of (80) to solutions $\bar{v}(z; \frac{1}{2}, \bar{\beta}, \bar{\gamma}, \bar{\delta})$ of equation (66).

6 Painlevé VI Equation

If v solves the sixth Painlevé equation, PVI

$$\begin{aligned} v'' = {} & \tfrac{1}{2}\{\tfrac{1}{v} + \tfrac{1}{v-1} + \tfrac{1}{v-z}\}(v')^2 - \{\tfrac{1}{z} + \tfrac{1}{z-1} + \tfrac{1}{v-z}\}v' \\ & + \tfrac{v(v-1)(v-z)}{z^2(z-1)^2}\{\alpha + \tfrac{\beta z}{v^2} + \tfrac{\gamma(z-1)}{(v-1)^2} + \tfrac{\delta z(z-1)}{v-z)^2}\}, \end{aligned} \quad (81)$$

then the equation (3) takes the form

$$\begin{aligned} & (\phi_6v^6 + \phi_5v^5 + \phi_4v^4 + \phi_3v^3 + \phi_2v^2 + \phi_1v + \phi_0)v' \\ & + \psi_8v^8 + \psi_7v^7 + \psi_6v^6 + \psi_5v^5 + \psi_4v^4 + \psi_3v^3 + \psi_2v^2 + \psi_1v + \psi_0 = 0, \end{aligned} \quad (82)$$

where

$$\begin{aligned}
&\phi_6 = \tfrac{2\alpha}{z^2(z-1)^2} - B_4 - \tfrac{1}{2}A_2^2, \quad \phi_5 = 2(z+1)B_4 + (z+1)A_2^2 - \tfrac{4\alpha(z+1)}{z^2(z-1)^2} - A_2' - \tfrac{(2z-1)}{z(z-1)}A_2,\\
&\phi_4 = \tfrac{z}{(z-1)}A_2 - \tfrac{(2z-1)}{z(z-1)}(A_1 - A_2) + \tfrac{1}{2}A_1^2 + A_0A_2 + (z+1)A_1A_2 - \tfrac{3}{2}zA_2^2 + B_2 + (z+1)B_3\\
&\quad - 3zB_4 + (z+1)A_2' - A_1' + \tfrac{2}{z^2(z-1)^2}[\alpha(z^2+4z+1) + \beta z + (\delta z + \gamma)(z-1)],\\
&\phi_3 = \tfrac{z}{(z-1)}(A_1 - A_2) - \tfrac{(2z-1)}{z(z-1)}(A_0 - A_1) + 2A_0A_1 - 2zA_1A_2\\
&\quad + 2B_1 - 2zB_3 + (z+1)A_1' - A_0' - zA_2' - \tfrac{4}{z(z-1)^2}[(\alpha+\beta)(z+1) + (\gamma+\delta)(z-1)],\\
&\phi_2 = \tfrac{z}{(z-1)}(A_0 - A_1) + \tfrac{(2z-1)}{z(z-1)}A_0 + \tfrac{3}{2}A_0^2 - (z+1)A_0A_1 - zA_0A_2 - \tfrac{1}{2}zA_1^2 + 3B_0 - (z+1)B_1\\
&\quad - zB_2 - zA_1' + (z+1)A_0' + \tfrac{2}{z(z-1)^2}[\alpha z + \beta(z^2+4z+1) + (\gamma z + \delta)(z-1)],\\
&\phi_1 = -[2(z+1)B_0 + (z+1)A_0^2 + \tfrac{4\beta(z+1)}{(z-1)^2} + zA_0' + \tfrac{z}{(z-1)}A_0],\\
&\phi_0 = z[B_0 + \tfrac{1}{2}A_0^2 + \tfrac{2\beta}{(z-1)^2}],\\
&\psi_8 = -\tfrac{1}{2}A_2[B_4 + \tfrac{2\alpha}{z^2(z-1)^2}],\\
&\psi_7 = B_4[(z+1)A_2 + \tfrac{1}{2}A_1 - \tfrac{2(2z-1)}{z(z-1)}] - \tfrac{1}{2}A_2B_3 + \tfrac{\alpha}{z^2(z-1)^2}[2(z+1)A_2 - A_1] - B_4',\\
&\psi_6 = B_4[\tfrac{3}{2}(A_0 - zA_2) + \tfrac{2z}{(z-1)} + \tfrac{2(2z-1)}{z(z-1)}] + B_3[(z+1)A_2 + \tfrac{1}{2}A_1 - \tfrac{2(2z-1)}{z(z-1)}] - \tfrac{1}{2}A_2B_2 + (z+1)B_4'\\
&\quad - B_3' + \tfrac{\alpha}{z^2(z-1)^2}[2(z+1)A_1 - A_0] - \tfrac{A_2}{z^2(z-1)^2}[\alpha(z^2+4z+1) + \beta z + (\delta z + \gamma)(z-1)],\\
&\psi_5 = B_3[\tfrac{3}{2}(A_0 - zA_2) + \tfrac{2z}{(z-1)} + \tfrac{2(2z-1)}{z(z-1)}] + B_2[(z+1)A_2 + \tfrac{1}{2}A_1 - \tfrac{2(2z-1)}{z(z-1)}] - \tfrac{1}{2}A_2B_1\\
&\quad - B_4[\tfrac{1}{2}zA_1 + (z+1)A_0 + \tfrac{2z}{(z-1)}] + \tfrac{2\alpha(z+1)}{z^2(z-1)^2}A_0 - \tfrac{A_1}{z^2(z-1)^2}[\alpha(z^2+4z+1) + \beta z +\\
&\quad (\delta z + \gamma)(z-1)] + \tfrac{2A_2}{z(z-1)^2}[(\alpha+\beta)(z+1) + (\gamma+\delta)(z-1)] + (z+1)B_3' - zB_4' - B_2',\\
&\psi_4 = B_2[\tfrac{3}{2}(A_0 - zA_2) + \tfrac{2z}{(z-1)} + \tfrac{2(2z-1)}{z(z-1)}] + B_1[(z+1)A_2 + \tfrac{1}{2}A_1 - \tfrac{2(2z-1)}{z(z-1)}] - \tfrac{1}{2}A_2B_0\\
&\quad - B_3[\tfrac{1}{2}zA_1 + (z+1)A_0 + \tfrac{2z}{(z-1)}] + \tfrac{1}{2}zA_0B_4 - \tfrac{A_0}{z^2(z-1)^2}[\alpha(z^2+4z+1) + \beta z + (\delta z + \gamma)(z-1)]\\
&\quad + \tfrac{2A_1}{z(z-1)^2}[(\alpha+\beta)(z+1) + (\gamma+\delta)(z-1)]\\
&\quad - \tfrac{A_2}{z(z-1)^2}[\alpha z + \beta(z^2+4z+1) + \gamma z(z-1) + \delta(z-1)] + (z+1)B_2' - zB_3' - B_1',\\
&\psi_3 = B_1[\tfrac{3}{2}(A_0 - zA_2) + \tfrac{2z}{(z-1)} + \tfrac{2(2z-1)}{z(z-1)}] + B_0[(z+1)A_2 + \tfrac{1}{2}A_1 - \tfrac{2(2z-1)}{z(z-1)}] + \tfrac{1}{2}zA_0B_3\\
&\quad - B_2[\tfrac{1}{2}zA_1 + (z+1)A_0 + \tfrac{2z}{(z-1)}] + \tfrac{2\beta(z+1)}{(z-1)^2}A_2 + \tfrac{2A_0}{z(z-1)^2}[(\alpha+\beta)(z+1) + (\gamma+\delta)(z-1)]\\
&\quad - \tfrac{A_1}{z(z-1)^2}[\alpha z + \beta(z^2+4z+1) + (\gamma z + \delta)(z-1)] + (z+1)B_1' - zB_2' - B_0',\\
&\psi_2 = B_0[\tfrac{3}{2}(A_0 - zA_2) + \tfrac{2z}{(z-1)} + \tfrac{2(2z-1)}{z(z-1)}] - B_1[\tfrac{1}{2}zA_1 + (z+1)A_0 + \tfrac{2z}{(z-1)}] + \tfrac{1}{2}zA_0B_2 + (z+1)B_0'\\
&\quad - zB_1' + \tfrac{\beta}{(z-1)^2}[2(z+1)A_1 - zA_2] - \tfrac{A_0}{z(z-1)^2}[\alpha z + \beta(z^2+4z+1) + (\gamma z + \delta)(z-1)],\\
&\psi_1 = \tfrac{\beta}{(z-1)^2}[2(z+1)A_0 - zA_1] + \tfrac{1}{2}zA_0B_1 - B_0[(z+1)A_0 + \tfrac{1}{2}zA_1 + \tfrac{2z}{(z-1)}] - zB_0',\\
&\psi_0 = \tfrac{z}{2}A_0[B_0 - \tfrac{2\beta}{(z-1)^2}].
\end{aligned} \tag{83}$$

Setting $\psi_8 = \psi_0 = 0$ implies

$$A_0\left(B_0 - \frac{2\beta}{(z-1)^2}\right) = 0, \qquad A_2\left(B_4 + \frac{2\alpha}{z^2(z-1)^2}\right) = 0. \tag{84}$$

To solve these equations one may distinguish between the following four cases:

Case 1. $A_0 \neq 0, A_2 \neq 0$: Equation (84) gives $B_4 = \frac{-2\alpha}{z^2(z-1)^2}$, $B_0 = \frac{2\beta}{(z-1)^2}$. If $A_2 = \frac{2a_2}{z(z-1)}$, $A_0 = \frac{2a_0}{(z-1)}$, $B_3 = -\frac{1}{2}A_1A_2$, and $B_1 = -\frac{1}{2}A_1A_0$ then, $\phi_0 = \phi_6 = \psi_1 = \psi_7 = 0$, where $a_2^2 = 2\alpha$ and $a_0^2 = -2\beta$. Then $\phi_1 = \phi_5 = 0$ identically, and $\phi_j = 0, j = 2,3,4$ if $A_1 = \frac{2(\lambda z + \mu)}{z(z-1)}$, and

$$\begin{aligned}
B_2 = &\tfrac{-2}{z^2(z-1)^2}[(\lambda^2 + a_0\lambda - \gamma - \beta)z^2 + (2\mu\lambda + a_0\lambda + a_0\mu - \mu + 2a_0a_2 - a_0 - \alpha - \beta + \gamma - \delta)z\\
&+ \mu^2 + \mu(a_0+1) + a_0 - \beta + \delta],
\end{aligned} \tag{85}$$

where λ and μ are constants such that $\lambda + \mu + a_0 + a_2 = 0$, $\lambda(a_2 - a_0 - 1) = a_2 - \alpha - \beta - \gamma - \delta$, and $\mu(a_2 - a_0 - 1) = a_0 - \alpha - \beta + \gamma + \delta$. To set $\psi_j = 0,\ j = 2, ..., 6$, λ and μ should satisfy $(\lambda + a_2 - 1)[(\lambda + a_0)^2 - 2\gamma] = 0$, and $(\mu + a_2)[(\mu + a_2)^2 - 2\gamma] = 0$. The above five conditions on λ, and μ are satisfied if

$$(2\alpha)^{1/2} - (-2\beta)^{1/2} - (2\gamma)^{1/2} - (1 - 2\delta)^{1/2} = 1. \tag{86}$$

Therefore, if α, β, γ, and δ satisfy the condition (86), then v satisfies the following Riccati equation:

$$z(z-1)v' = (2\alpha)^{1/2}v^2 - [((-2\beta)^{1/2} + (2\gamma)^{1/2})z + (2\alpha)^{1/2} - (2\gamma)^{1/2}]v + (-2\beta)^{1/2}z. \tag{87}$$

This is the well known one parameter family of solutions of PVI [10, 17, 18].

Case 2. $A_0 = A_2 = 0$: In order to set $\phi_j = 0,\ j = 0, ..., 6$, one should choose

$$\begin{aligned}
&B_4 = \tfrac{2\alpha}{z^2(z-1)^2}, \quad B_0 = -\tfrac{2\beta}{(z-1)^2}, \quad A_1 = \tfrac{a_1}{z},\\
&B_2 = -(z+1)B_3 - \tfrac{1}{2}A_1^2 + A_1' + \tfrac{(2z-1)}{z(z-1)}A_1 - \tfrac{2}{z^2(z-1)^2}[\alpha(z^2+z+1) + \beta z + \gamma(z-1) + \delta z(z-1)],\\
&B_1 = zB_3 - \tfrac{1}{2}(z+1)A_1' - \tfrac{(z^2+2z-1)}{2z(z-1)}A_1 + \tfrac{2}{z(z-1)^2}[(\alpha+\beta)(z+1) + (\gamma+\delta)(z-1)],
\end{aligned} \tag{88}$$

where a_1 is a constant. with these choices $\psi_1 = \psi_7 = 0$ identically, and $\psi_6 = 0$, if B_3 satisfies the following equation:

$$\frac{d}{dz}[z^2(z-1)^2B_3] - \frac{1}{2}a_1 z(z-1)^2B_3 - \frac{2\alpha}{z}[a_1(z+1) - 2z] = 0. \tag{89}$$

Therefore, there are three distinct subcases: **i.** $a_1(a_1-2) \neq 0$; **ii.** $a_1 = 0$, and **iii.** $a_1 = 2$. If $a_1(a_1-2) \neq 0$, then one can not choose B_3 and a_1 so that $\psi_j = 0, j = 2, 3, 4, 5$.

If $a_1 = 0$, then equation (89) gives $B_3 = \frac{-4\alpha z + b_3}{z^2(z-1)^2}$, where b_3 is a constant, and $\psi_j = 0, j = 2, 3, 4, 5$ only if $b_3 = 2(\gamma+\beta)$, and $\alpha = \delta = 0$. Therefore, if $\alpha = \delta = 0$, one-parameter family of solutions of PVI are given by

$$z^2(z-1)^2(v')^2 = 2(v-z)^2[(\gamma+\beta)v - \beta]. \tag{90}$$

If $\gamma + \beta = 0$, then equation (90) reduces to the linear equation

$$z(z-1)v' = -(-2\beta)^{1/2}(v-z), \tag{91}$$

which is a special case, $\alpha = \delta = 0$ and $\gamma + \beta = 0$, of the equation (87).

If $\gamma + \beta \neq 0$, then the transformation

$$v = \frac{1}{(\gamma+\beta)}\{2[z(z-1)\frac{u'}{u} + n(z-1) + mz]^2 + \beta\}, \tag{92}$$

where $2m(m-1) = \gamma$ and $2n(n-1) = -\beta$, transform equation (90) to the hypergeometric equation

$$z(z-1)u'' + [2(n+m+1)z - (2n+1)]u' + (n+m)(n+m-1)u = 0. \tag{93}$$

If $a_1 = 2$, then equation (89) gives $B_3 = \frac{b_3 z - 4\alpha}{z^2(z-1)^2}$, where b_3 is a constant, and $\psi_j = 0, j = 2, 3, 4, 5$ only if $b_3 = 2$ and $\alpha = \beta = 0,\ \gamma = -\delta = \frac{1}{2}$. Therefore, with these values of the parameters, PVI has one-parameter family of solutions given by

$$(z-1)^2(zv' - v)^2 = 2zv(v-1)^2. \tag{94}$$

The transformation $v = 2z[(z-1)\frac{u'}{u} + \frac{1}{\sqrt{2}}]^2$ transforms equation (94) into the hypergeometric equation

$$z(z-1)u'' + (1+\sqrt{2})zu' + \frac{1}{2}u = 0. \tag{95}$$

Case 3. $A_0 \neq 0,\ A_2 = 0$: (84), and $\phi_0 = \phi_6 = 0$ give $B_0 = \frac{2\beta}{(z-1)^2}$, $A_0^2 = \frac{-8\beta}{(z-1)^2}$, and $B_4 = \frac{2\alpha}{z^2(z-1)^2}$ respectively. Then, one can obtain $B_1 = -\frac{1}{2}A_0A_1$, and

$$\begin{aligned}
B_2 &= \tfrac{1}{z^2}A_0 - \tfrac{(z+1)}{2z}A_0A_1 - A_1' - \tfrac{1}{z-1}A_1 - \tfrac{1}{2}A_1^2 + \tfrac{2}{z^2(z-1)^2}[\alpha z + \beta(z^2+z+1)\\
&\quad + \gamma z(z-1) + \delta(z-1)],\\
B_3 &= \tfrac{1}{2z}A_0A_1 - \tfrac{1}{2z^2}A_0 + \tfrac{(z+1)}{2z}A_1' + \tfrac{(z^2+2z-1)}{2z^2(z-1)}A_1 - \tfrac{2}{z^2(z-1)^2}[(\alpha+\beta)(z+1) + (\gamma+\delta)(z-1)],
\end{aligned} \tag{96}$$

by solving $\psi_1 = 0, \phi_j = 0,\ j = 2,3$ respectively. Then, $\phi_1 = \phi_2 = \psi_5 = \psi_7 = \psi_8 = 0$ identically and $\phi_4 = 0$ gives

$$z(z-1)A_1' + (z-1)A_1 - A_0 = 0. \tag{97}$$

If one lets $A_0 = \frac{a_0}{(z-1)}$, where $a_0^2 = -8\beta$. Then equation (97) implies that $A_1 = \frac{a_1}{z} - \frac{a_0}{z(z-1)}$, where a_1 is a constant. The equation $\psi_6 = 0$ now gives

$$(a_1-2)[a_0^2+2a_0a_1+4a_1-8(\gamma+\delta-\alpha)] = 0, \quad (a_0+a_1)[a_0^2+2a_0a_1+4a_1-8(\gamma+\delta+\alpha)] = 0. \tag{98}$$

So, there are four distinct cases: **i.** $a_1 = 2,\ a_0 + a_1 \neq 0$, **ii.** $a_1 \neq 2,\ a_0 + a_1 = 0$, **iii.** $a_1 \neq 2,\ a_0 + a_1 \neq 0$, and **iv.** $a_1 = 2,\ a_0 + a_1 = 0$. If $a_1 = 2,\ a_0 + a_1 \neq 0$ and $a_1 \neq 2,\ a_0 + a_1 = 0$, then one can not choose $B_k,\ k = 0, ..., 4$ such that $\psi_j = 0, j = 2,3,4,5$. When $a_1 \neq 2,\ a_0 + a_1 \neq 0$, $\psi_j = 0, j = 2, ..., 5$ only if $\alpha = \delta = 0, \gamma + \beta = 0$, and then v satisfies (91).

If $a_1 = 2,\ a_0 = -2$, then $\beta = -\frac{1}{2},\ B_0 = \frac{-1}{(z-1)^2},\ B_1 = \frac{2}{(z-1)^2}$ and

$$\begin{aligned} B_2 &= \tfrac{1}{z^2(z-1)^2}[2\alpha z + 2\gamma z(z-1) + 2\delta(z-1) - z^2 - z + 1], \\ B_3 &= \tfrac{1}{z^2(z-1)^2}[(1-2\gamma-2\delta)(z-1) - 2\alpha(z+1)]. \end{aligned} \tag{99}$$

Then $\psi_j = 0,\ j = 2, ..., 5$ identically. Hence, we have the following theorem

Theorem 2 *The Painlevé VI equation admits a one-parameter family of solution characterized by*

$$z^2[(z-1)v' - (v-1)]^2 = v^2\{2\alpha v^2 - [(2\alpha+\lambda)z + 2\alpha - \lambda]v + 2\gamma z^2 + (2\alpha+\lambda-4\gamma)z + 2\gamma - \lambda\}, \tag{100}$$

where $\lambda = 2\gamma + 2\delta - 1$, *if and only if* $\beta = -\frac{1}{2}$.

Equation (100) can be linearized as follows: If $\alpha = \lambda = 0$, then equation (100) can be reduced to the following linear equation:

$$z(z-1)v' = -[(2\gamma)^{1/2}(z-1) - z]v - z. \tag{101}$$

This is a special case, $\alpha = 0$, and $\beta = -\frac{1}{2}$, of equation (87). If $\alpha = 0$ and $\lambda \neq 0$, then the solution of equation (100) is given by

$$v = -\frac{1}{\lambda}[(z-1)w^2 - 2\gamma(z-1) - \lambda], \tag{102}$$

where w satisfies the following Riccati equation

$$2z(z-1)w' = -\epsilon[(z-1)w^2 - 2\gamma(z-1) - \lambda], \quad \epsilon = \pm 1. \tag{103}$$

The transformation $w = \frac{2\epsilon(zy'+ny)}{y}$, where $4n^2 = 1 - 2\delta$, transform equation (103) into the hypergeometric equation:

$$z(z-1)y'' + (2n+1)(z-1)y' - \frac{1}{4}\lambda y = 0 \tag{104}$$

If $\alpha \neq 0$, then equation (100) can be written as

$$z^2[(z-1)v' - (v-1)]^2 = v^2(av - f)(av - g), \tag{105}$$

where $f(z) = b(z-1) + a$, $g(z) = c(z-1) + a$, $a^2 = 2\alpha$, $\quad bc = 2\gamma$, $\quad a(b+c) = 2\alpha + \lambda$. If $b = c$, that is, if $2(2\alpha)^{1/2}(2\gamma)^{1/2} = 2\alpha + 2\gamma + 2\delta - 1$, then equation (105) is reduced to following Riccati equation

$$z(z-1)v' = (2\alpha)^{1/2}v^2 - [(2\gamma)^{1/2}(z-1) - z + (2\alpha)^{1/2}]v - z. \tag{106}$$

(106) is the special case, $\beta = -\frac{1}{2}$, of equation (87).

If $(2\alpha)^{1/2}(2\gamma)^{1/2} \neq 2\alpha + 2\gamma + 2\delta - 1$, then the solution of equation (100) is given by

$$v = \frac{fw^2 - g}{a(w^2-1)}, \tag{107}$$

where w solves the following Riccati equation

$$2z(z-1)w' = \epsilon(fw^2 - g), \quad \epsilon = \pm 1. \tag{108}$$

The transformation

$$w = -\frac{2\epsilon}{f}[\frac{z(z-1)y'}{y} + n(z-1) + mz], \tag{109}$$

where $4n^2 = (b-a)(c-a)$, $4m^2 = a^2$, transforms equation (108) into the following linear equation

$$\begin{aligned} y'' &= -[\tfrac{2n+1}{z} + \tfrac{2m+1}{z-1} - \tfrac{b}{bz-b+a}]y' \\ &- \tfrac{1}{4z(z-1)(bz-b+a)}[b(2a^2+8nm-ab-ac)z - (b-a)(2a^2+8nm-ab-ac) - 4m(b-a) + 4an]y. \end{aligned} \tag{110}$$

Equation (110) is known as Huen's equation [19].

Case 4. $A_0 = 0, A_2 \neq 0$: In this case, equation (84), and $\phi_6 = \phi_0 = 0$ give $B_4 = -\frac{2\alpha}{z^2(z-1)^2}$, $A_2^2 = \frac{8\alpha}{z^2(z-1)^2}$, $B_0 = -\frac{2\beta}{(z-1)^2}$ respectively. Solving the equations $\psi_7 = 0$, and $\phi_j = 0,\ j = 3, 4$ give $B_3 = -\frac{1}{2}A_2A_1$, and

$$\begin{aligned} B_2 &= -\{\tfrac{1}{4}A_2(z^2+z+1) + \tfrac{1}{2}(z+1)A_2A_1 - A_1' - \tfrac{(2z-1)}{z(z-1)}A_1 + \tfrac{1}{2}A_1^2 - A_2 \\ &\quad + \tfrac{2}{z^2(z-1)^2}[\beta z + \gamma(z-1) + \delta z(z-1)]\}, \\ B_1 &= -\tfrac{1}{2}\{(z+1)A_1' + \tfrac{(z^2+2z-1)}{z(z-1)}A_1 - zA_1A_2 + A_2 - \tfrac{1}{2}z(z+1)A_2^2 \\ &\quad - \tfrac{4}{z(z-1)^2}[\beta(z+1) + (\gamma+\delta)(z-1)]. \end{aligned} \tag{111}$$

Then, $\phi_2 = 0$, yields

$$z(z-1)A_1' + (z-1)A_1 - zA_2 = 0. \tag{112}$$

and $\phi_5 = \phi_1 = \psi_2 = \psi_8 = \psi_7 = 0$ identically. Let $A_2 = \frac{a_2}{(z-1)}$, where $a_2^2 = 8\alpha$, then equation (112) implies that $A_1 = \frac{a_1}{z} - \frac{a_2}{z(z-1)}$, where a_1 is a constant. The equation $\psi_2 = 0$ now gives

$$a_1[a_2^2+2a_2a_1-4a_1-4a_2+8(\gamma+\delta-\beta)] = 0, \quad (a_2+a_1-2)[a_2^2+2a_2a_1-4a_1-4a_2+8(\gamma+\delta+\beta)] = 0. \tag{113}$$

So, there are four subcases to be considered: **i.** $a_1 \neq 0,\ a_2 + a_1 \neq 2$; **ii.** $a_1 = 0,\ a_2 \neq 2$; **iii.** $a_1 \neq 0,\ a_1 = 2 - a_2$; and **iv.** $a_1 = 0,\ a_2 = 2$. If $a_1 = 2 - a_2, a_1 \neq 0$, then one can not choose $B_k,\ k = 0, ..., 4$ so that $\psi_j = 0, j = 2, ..., 5$. When $a_1 = 0, a_2 \neq 2, \psi_j = 0, j = 2, ..., 5$ only if $\alpha + \delta = (2\alpha)^{1/2}, \gamma = \beta = 0$. In this case, v satisfies the following Riccati equation

$$z(z-1)v' = (2\alpha)^{1/2}v(v-1), \tag{114}$$

which is the special case, $\gamma = \beta = 0$, of the equation (87).

When $a_1 \neq 0,\ a_2 + a_1 - 2 \neq 0$, then $\psi_j = 0, j = 2, ..., 5$ only if $\beta = 0$ and $(2\alpha)^{1/2} - (2\gamma)^{1/2} - (1-2\delta)^{1/2} - 1 = 0$. Then v satisfies the following Riccati equation:

$$z(z-1)v' = (2\alpha)^{1/2}v^2 - [(2\gamma)^{1/2}(z-1) + (2\alpha)^{1/2}]v. \tag{115}$$

Equation (115) is the special case, $\beta = 0$, of (87).

If $a_1 = 0,\ a_2 = 2$, then one has $\alpha = \frac{1}{2}$, $B_4 = \frac{-1}{z^2(z-1)^2}$, $B_3 = \frac{2}{z^2(z-1)^2}$,

$$\begin{aligned} B_2 &= \frac{-1}{z^2(z-1)^2}[2\beta z + 2\gamma(z-1) + 2\delta z(z-1) - z^2 + z + 1], \\ B_1 &= \frac{1}{z(z-1)^2}[(2\gamma+2\delta-1)(z-1) + 2\beta(z+1)]. \end{aligned} \tag{116}$$

Now, the equations $\psi_j = 0,\ j = 3, 4, 5, 6$ are satisfied identically. Thus, PVI admits a one-parameter family of solution characterized by

$$[z(z-1)v' - v(v-1)]^2 + 2[\lambda z^2 + (\gamma+\beta-\lambda)z - \gamma]v^2 - 2z[(\lambda+\gamma+\beta)z + \beta - \gamma - \lambda]v + 2\beta z^2 = 0, \tag{117}$$

where $\lambda = \delta - \frac{1}{2}$, if and only if $\alpha = \frac{1}{2}$. The Lie point-discrete symmetry $\bar{v} = \frac{1}{v},\ \bar{\alpha} = -\beta,\ \bar{\beta} = -\alpha,\ \bar{\gamma} = \gamma,\ \bar{\delta} = \delta,\ \bar{z} = \frac{1}{z}$ of PVI [20] transforms solutions $v(z; \frac{1}{2}, \beta, \gamma, \delta)$ of equation (117) into solutions $\bar{v}(\bar{z}; \bar{\alpha}, -\frac{1}{2}, \bar{\gamma}, \bar{\delta})$ of equation (100).

7 Conclusion

It is well known that for certain choice of parameters the Painlevé equations, PII-PVI, admit one parameter family of solutions, rational, algebraic and expressible in terms of the classical transcendental functions such as Airy, Bessel, Weber-Hermite, Whitteker, hypergeometric functions respectively. All these known one parameter family of solutions satisfy the Riccati equations. In this article, we investigated the first-order second-degree equations satisfying the Fuchs theorem concerning the absence of movable singular points except the poles, related with the Painlevé equations PII-PVI. By using these first-order second-degree equations, one parameter family of solutions of PII-PVI are also obtained. For the sake of completeness, we examine all possible cases, and hence some of the well known results as well as the new ones are obtained.

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References

[1] E.L. Ince: Ordinary Differential Equations, *Dover: New York*(1956)

[2] Mark J. Ablowitz, Alfred Ramani, Harvey Segur: A connection between nonlinear evolution equations and ordinary differential equations of P-type. I, *J. Math. Phys.*, 21(4), 715-721 (1980)

[3] John Weiss, M. Tabor, George Carnevale: The Painlevé property for partial differential equations, *J. Math. Phys.*, 24(3), 522-526 (1983)

[4] Jiuli Yin: Painlevé integrability, Bäcklund transformation and solitary solutions satability of modified DGH equation, *International Journal of Nonlinear Science*, 2(3), 183-187 (2006)

[5] Suping Qian: Painlevé analysis and symmetry reductions to the strong dispersive DGH equation, *International Journal of Nonlinear Science*, 1(2), 119-123 (2006)

[6] Roger Chalkley: New contributions to the related work of Paul Appell, Lazarus Fuchs, Georg Hamel, and Paul Painlev on nonlinear differential equations whose solutions are free of movable branch points, *J. Diff. Eq.*, 68, 72-117 (1987)

[7] Uğurhan Muğan, Fahd Jrad: Painlevé test and the first Painlevé hierarchy, *J. Phys. A: Math. Gen.*, 32(45), 7933-7952 (1999)

[8] Uğurhan Muğan, Ayman Sakka: Second-order Second-degree Painlevé Equations Related with Painlevé I-VI Equations and Fuchsian Type Transformations, *J. Math. Phys.*, 40(7), 3569-3587 (1999)

[9] Ayman Sakka: Second-order fourth-degree Painlevé-type equations, *J. Phys. A: Math. Gen.*, 34(3), 623-631 (2001)

[10] Athanassios S. Fokas, Mark J. Ablowitz: On a unified approach to transformations and elementary solutions of Painlevé equations, *J. Math. Phys.*, 23, 2033-2042 (1982)

[11] Nikolai A. Lukashevich: Elementary solutions of certain Painlevé equations, *Diff. Urav.*, 1(6), 731-735 (1965)

[12] Nikolai A. Lukashevich: Theory of the fourth Painlevé equation, *Diff. Urav.*, 3(5), 771-780 (1967)

[13] Kazuo Okamoto: Studies on the Painlevé equations .3. 2nd and 4th Painlevé equations, PII and PIV, *Math. Ann.*, 275(2), 221-255 (1986)

[14] David W. Albrecht, Elizabeth L. Mansfield and Alice E. Milne: Algorithms for special integrals of ordinary differential equations, *J. Phys. A: Math. Gen.*, 29(5), 973-992 (1996)

[15] Valerii I. Gromak: Solutions of Painlevé's fifth problem, *Diff. Urav.*, 12(4), 740-742 (1976)

[16] Ayman Sakka, Uğurhan Muğan: Second-order second-degree Painlevé equations related to Painlevé IV,V,VI equations, *J. Phys. A: Math. Gen.*, 31(10), 2471-2490 (1998)

[17] Nikolai A. Lukashevich, A.I. Yablonskii: On a set of solutions of the sixth Painlevé equation, *Diff. Urav.*, 3(3), 520-523 (1967)

[18] Uğurhan Muğan, Ayman Sakka: Schlesinger Transformations For Painlevé VI Equation *J. Math. Phys.*, 36, 1284-1298 (1995)

[19] E.T. Whittaker, G.N. Watson: A Course of Modern Analysis, *MacMillan: New York* (1943)

[20] Marta Mazzocco: Rational solutions of the Painlevé VI equation *J. Phys. A: Math. Gen.*, 34(11), 2281-2294 (2001)