Elsevier Editorial System(tm) for Physics Letters A
Manuscript Draft

Manuscript Number:

Title: On special solutions of second and fourth Painlev\'{e} hierarchies

Article Type: Letter

Section/Category: Mathematical physics

Keywords: Pianlev\'{e} hierarchies; Special solutions

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Manuscript Region of Origin:

On special solutions of second and fourth Painlevé hierarchies

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April 26, 2008

Abstract

In this article, we give special solutions of second and fourth Painlevé hierarchies derived by Gordoa, Joshi, and Pickering. We show that for certain choice of the parameters each n-th member of these hierarchies has a special solution in terms of an n-th order differential equation. Furthermore we derive a relation between these two hierarchies.

Keywords: Pianlevé hierarchies, Special solutions

PACS number : 02.30 hq, 02.30.Gp

1 Introduction

The six Painlevé equations, PI-PVI, were discovered by Painlevé and Gambier about one century ago. They are the only second-order ordinary differential equations that define new functions, the Painlevé transcendents. A remarkable property of these Painlevé equations is that while their general solutions are new transcendental functions, they posses special solutions that can be expressed in terms of the elementary functions. These solutions are either rational or can be expressed in terms of the classical special functions. For example, it is well known that for certain values of the parameters, the second and fourth Painlevé equations admit special solutions in terms of the Airy and Weber-Hermitte functions respectively [1]-[4].

In recent years there is a considerable interest in searching for higher-order analogues of Painlevé equations [5]-[14]. Since Painlevé equations can be obtained as a similarity reductions of completely integrable partial differential equations [15], higher-order analogues of them arise as similarity reductions of higher-order integrable partial differential equations. This was first realized by Airault [5] who derived a second Painlevé hierarchy by similarity reduction of the KdV-mKdV hierarchies. A Pianlevé hierarchy is an infinite sequence of nonlinear ordinary differential equations whose first member is a Pianlevé equation. Gordoa, Joshi, and Pickering [13] have used non-isospectral scattering problems to derive new second and fourth Pianlevé hierarchies.

In this paper, we study special solutions of the second and fourth Pianlevé hierarchies given in [13]. It turns out that for certain restriction on the parameters, the n-th member of these hierarchies, which is an equation of of order $2n$, has a special solution in terms of an equation of order n, $n \geq 1$.

In particular, the first members of the second Pianlevé hierarchy has the Airy solution while the first member of the fourth Pianlevé hierarchy has the Weber-Hermitte solution. Moreover, it turns out that the n-th member of the fourth Pianlevé hierarchy has a special solution in terms of the $(n-1)$-st member of the second Pianlevé hierarchy, $n \geq 2$.

2 The second Painlevé hierarchy

Consider the following second Painlevé hierarchy presented in [13]

$$\begin{pmatrix} \partial_x^{-1} & 0 \\ 0 & \partial_x^{-1} \end{pmatrix} \left[\mathcal{R}^n \mathbf{u}_x + \sum_{j=0}^{n-2} \gamma_j \mathcal{R}^j \mathbf{u}_x \right] + \begin{pmatrix} g_{n+1} x \\ -\delta_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{1}$$

where

$$\begin{aligned} \mathbf{u} &= (u, v)^T, \\ \mathcal{R} &= \tfrac{1}{2} \begin{pmatrix} \partial_x u \partial_x^{-1} - \partial_x & 2 \\ 2v + v_x \partial_x^{-1} & u + \partial_x \end{pmatrix}, \end{aligned} \tag{2}$$

and $g_{n+1} \neq 0$. The second Painlevé hierarchy (1) can be obtained as the compatibility condition of the following linear system of equations [16]

$$\frac{\partial \Phi}{\partial \lambda} = A(\lambda)\Phi(\lambda), \qquad \frac{\partial \Phi}{\partial x} = B(\lambda)\Phi(\lambda), \tag{3}$$

where

$$\begin{aligned} B &= \begin{pmatrix} -\lambda & \frac{\tilde{w}}{2} \\ -2v/\tilde{w} & \lambda \end{pmatrix}, \\ A &= \frac{1}{g_{n+1}} \sum_{j=0}^{n+1} A_j \lambda^{n+1-j}, \\ A_0 &= -2\sigma_3, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad u = -\frac{\tilde{w}_x}{\tilde{w}}. \end{aligned} \tag{4}$$

The form of $A_j,\ 1 \leq j \leq n+1$, can be found in [16].

Writing the hierarchy (1) as a compatibility condition of the linear system (3) leads to another representation of this hierarchy. More precisely, let $A_j = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix}$, $0 \leq j \leq n+1$. Then the compatibility condition of the linear system (3) gives

$$\begin{aligned} a_{j,x} &= \frac{1}{2}\tilde{w} c_j + \frac{2v}{\tilde{w}} b_j, \quad 0 \leq j \leq n, \\ b_{j,x} &= -2b_{j+1} - \tilde{w} a_j, \quad 0 \leq j \leq n, \\ c_{j,x} &= 2c_{j+1} - \frac{4v}{\tilde{w}} a_j, \quad 0 \leq j \leq n, \\ a_{n+1,x} &= \frac{1}{2}\tilde{w} c_{n+1} + \frac{2v}{\tilde{w}} b_{n+1} - g_{n+1}, \\ b_{n+1,x} &= -\tilde{w} a_{n+1}, \\ c_{n+1,x} &= -\frac{4v}{\tilde{w}} a_{n+1}. \end{aligned} \tag{5}$$

Using $A_0 = -2\sigma_3$ and setting $a_1 = 0$, equation (5) determines a_j, b_j, and c_j recursively as follows

$$\begin{aligned}
&a_0 = -2, \quad a_1 = 0, \quad b_0 = c_0 = 0, \\
&b_j = -\frac{1}{2}(b_{j-1,x} + \tilde{w} a_{j-1}), && 1 \le j \le n+1, \\
&c_j = \frac{1}{2} c_{j-1,x} + \frac{2v}{\tilde{w}} a_{j-1}, && 1 \le j \le n+1, \\
&a_j = \frac{1}{4}\sum_{i=1}^{j-1}(b_i c_{j-i} + a_i a_{j-i}) + k_j, && 2 \le j \le n, \\
&a_{n+1} = \frac{1}{4}\sum_{i=1}^{n}(b_i c_{n+1-i} + a_i a_{n+1-i}) - g_{n+1}x + k_{n+1},
\end{aligned} \tag{6}$$

where k_j are constants of integrations. Moreover, equation (5.f) can be integrated to obtain the first integral

$$\sum_{i=1}^{n+1}(b_i c_{n+2-i} + a_i a_{n+2-i}) + K = 0, \tag{7}$$

where K is a constant of integration. Therefor , up to renaming the constants, the second Painlevé hierarchy (1) can be written in the following form

$$\begin{aligned}
&b_{n+1,x} + \tilde{w} a_{n+1} = 0, \\
&\sum_{i=1}^{n+1}(b_i c_{n+2-i} + a_i a_{n+2-i}) + 4\delta_n = 0,
\end{aligned} \tag{8}$$

where $a_j,\ b_j,\ c_j$ are given by (6).

2.1 Special solutions of the second Painlevé hierarchy

The first member of the second Painlevé hierarchy (8) reads

$$u_x - u^2 - 2(v + g_2 x) = 0, \quad v_x + 2uv - 2\delta_1 = 0. \tag{9}$$

Since $g_2 \neq 0$, without loss of generality we set $g_2 = \frac{1}{4}$. Thus eliminating v between (9.a) and (9.b) we obtain the second Painlevé equation.

$$u_{xx} = 2u^3 + xu + 4\delta_1 + \frac{1}{2}. \tag{10}$$

Equation (9) has the special solution $\delta_1 + \frac{1}{4} = 0$, $v = u_x$, and u satisfies the Riccati equation

$$u_x + u^2 + \frac{1}{2}x = 0. \tag{11}$$

The transformation $u = \dfrac{U_x}{U}$ transforms (11) into the well known Airy equation

$$U_{xx} + \frac{1}{2}xU = 0. \tag{12}$$

Consider the second member of the hierarchy (8)

$$\begin{aligned}
u_{xx} &= 3uu_x - u^3 - 6uv + 2K_2 u - 4g_3 x, \\
v_{xx} &= -3uv_x - 3u^2 v - 3v^2 + 2K_2 v + 4\delta_2.
\end{aligned} \tag{13}$$

In [17], setting $g_3 = 1$, it was shown that (13) has the special solution $\delta_2 + 1 = 0$, $v = u_x$, and u solves

$$u_{xx} + 3uu_x + u^3 - 2k_2 u + 4x = 0. \tag{14}$$

In this article we will generalize the above results. More precisely we will show that for each n, the n-th member of the second Painlevé hierarchy (8) has a special solution $\delta_n + g_{n+1} = 0$, $v = u_x$, and u satisfies the n-th order equation

$$b_{n+1,x} + \tilde{w} a_{n+1} = 0. \tag{15}$$

Define $a_{n+2} = \dfrac{1}{4}\displaystyle\sum_{i=1}^{n+1}(b_i c_{n+2-i} + a_i a_{n+2-i}) + k_{n+2}$, $b_{n+2} = -\dfrac{1}{2}(b_{n+1,x} + \tilde{w} a_{n+1})$, and $c_{n+2} = \dfrac{1}{2} c_{n+1,x} + \dfrac{2u_x}{\tilde{w}} a_{n+1}$. Then a_{n+2} satisfies

$$a_{n+2,x} = \frac{1}{2}\tilde{w} c_{n+2} + \frac{2u_x}{\tilde{w}} b_{n+2}, \tag{16}$$

and (8) can be written as

$$b_{n+2} = 0, \qquad a_{n+2} - K_{n+2} + \delta_n = 0. \tag{17}$$

Since u satisfies (15), the equation (17.a) is satisfied. Thus we only have to prove that equation (17.b) is satisfied.

First of all we will use induction to show that

$$a_j = -2\frac{d}{dx}\left(\frac{b_j}{\tilde{w}}\right) + k_j, \quad 2 \leq j \leq n. \tag{18}$$

Using (6) we find $b_1 = \tilde{w}$, $c_1 = -\dfrac{4u_x}{\tilde{w}}$, and $a_2 = -u_x + k_2$. Thus direct calculations show that $a_2 = -2\dfrac{d}{dx}\left(\dfrac{b_2}{\tilde{w}}\right) + k_2$.

Assume that it is true for $j = m$, $\quad 2 \leq m \leq n-1$. Then using (6.b) we have

$$-2\frac{d}{dx}\left(\frac{b_{m+1}}{\tilde{w}}\right) = \frac{d}{dx}\left[a_m + \frac{1}{\tilde{w}} b_{m,x}\right]. \tag{19}$$

By the induction hypothesis we have

$$\begin{aligned} a_m &= -2\frac{d}{dx}\left(\frac{b_m}{\tilde{w}}\right) + k_m \\ &= -\frac{2}{\tilde{w}}(b_{m,x} + u b_m) + k_m. \end{aligned} \tag{20}$$

Using (20) to substitute $b_{m,x}$ and using equation (5.a) to substitute $a_{m,x}$, equation (19) yields

$$-2\frac{d}{dx}\left(\frac{b_{m+1}}{\tilde{w}}\right) = \frac{1}{2}u(a_m - k_m) + \frac{1}{4}\tilde{w} c_m. \tag{21}$$

Now equation (5.a) implies that $4a_{m+1,x} = 2\tilde{w} c_{m+1} + \dfrac{8u_x}{\tilde{w}} b_{m+1}$. Using (6) to substitute b_{m+1} and c_{m+1}, we find

$$4a_{m+1,x} = \tilde{w} c_{m,x} - \frac{4u_x}{\tilde{w}} b_{m,x}. \tag{22}$$

Using $\tilde{w}_x = -u\tilde{w}$, we find

$$\frac{d}{dx}(\tilde{w} c_m) = \tilde{w} c_{m,x} - u\tilde{w} c_m. \tag{23}$$

Using equation (5.a) to substitute $a_{m,x}$, we get

$$\frac{d}{dx}[2u(a_m - k_m)] = 2u_x(a_m - k_m) + u(\tilde{w}c_m + \frac{4u_x}{\tilde{w}}b_m). \tag{24}$$

Thus equations (22-24) give

$$\frac{d}{dx}[4a_{m+1} - 2u(a_m - k_m) - b_1 c_m] = -2u_x[a_m - k_m + \frac{2}{\tilde{w}}(b_{m,x} + ub_m)]. \tag{25}$$

By the induction hypothesis (20), we see that the right hand side of equation (25) is zero and hence

$$a_{m+1} - k_{m+1} = \frac{1}{2}u(a_m - k_m) + \frac{1}{4}\tilde{w}c_m, \tag{26}$$

where k_{m+1} is the constant of integration. Therefore (26) and (21) imply the result.

Following similar arguments we can show that

$$a_{n+1} = -2\frac{d}{dx}\left(\frac{b_{n+1}}{\tilde{w}}\right) - g_{n+1}x + k_{n+1} \tag{27}$$

and

$$a_{n+2} = -2\frac{d}{dx}\left(\frac{b_{n+2}}{\tilde{w}}\right) + g_{n+1} + k_{n+2}. \tag{28}$$

As the last step substituting $b_{n+2} = 0$ into (28), we have

$$a_{n+2} - k_{n+2} = g_{n+1}. \tag{29}$$

Therefore (29) implies that, if $\delta_n + g_{n+1} = 0$, then (17.b) is satisfied and the proof is completed.

3 The fourth Painlevé hierarchy

We consider the following fourth Pianlevé hierarchy given in [13]

$$\begin{aligned} L_{n,x} &= 2K_n + uL_n + g_n - 2\alpha_n, \\ K_{n,x} &= \frac{1}{L_n}[(K_n + \frac{1}{2}g_n - \alpha_n)^2 - \frac{1}{4}\beta_n^2] - vL_n, \end{aligned} \tag{30}$$

where $\mathbf{K}_n = (K_n, L_n)^T$ is defined recursively as follows:

$$\begin{aligned} &\mathbf{K}_n[\mathbf{u}] = \mathbf{L}_n[\mathbf{u}] + \sum_{j=1}^{n-1}\gamma_j \mathbf{L}_j[\mathbf{u}] + g_n x \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ &\mathbf{u} = (u, v)^T, \qquad \mathbf{L}_1[\mathbf{u}] = (v, u)^T, \\ &\mathcal{B}_1 \mathbf{L}_{j+1}[\mathbf{u}] = \mathcal{B}_2 \mathbf{L}_j[\mathbf{u}], \\ &\mathcal{B}_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}, \qquad \mathcal{B}_2 = \frac{1}{2}\begin{pmatrix} 2\partial_x & \partial_x u - \partial_x^2 \\ u\partial_x + \partial_x^2 & v\partial_x + \partial_x v \end{pmatrix}. \end{aligned} \tag{31}$$

The fourth Painlevé hierarchy (30) can be obtained as the compatibility condition of the following linear system of equations [16]

$$\frac{\partial \Phi}{\partial \lambda} = A(\lambda)\Phi(\lambda), \qquad \frac{\partial \Phi}{\partial x} = B(\lambda)\Phi(\lambda), \tag{32}$$

where

$$\begin{aligned} B &= \begin{pmatrix} -\lambda & w \\ -v/w & \lambda \end{pmatrix}, \\ A &= \frac{1}{g_{n+1}} \sum_{j=0}^{n+1} A_j \lambda^{n-j}, \\ A_0 &= -2\sigma_3, \quad u = -\frac{w_x}{w}. \end{aligned} \tag{33}$$

The precise form of $A_j,\ 1 \leq j \leq n+1$, can be found in [16].

Now we will use the linear system (32-33) to rewrite the hierarchy (30) in another form. Let $A_j = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix}$, $0 \leq j \leq n+1$. Then the compatibility condition of the linear system (32) gives

$$\begin{aligned} a_{j,x} &= wc_j + \frac{v}{w} b_j, \quad 0 \leq j \leq n-1, \\ a_{n,x} &= wc_n + \frac{v}{w} b_n - g_n, \\ a_{n+1} &= wc_{n+1} + \frac{v}{w} b_{n+1}, \\ b_{j,x} &= -2b_{j+1} - 2wa_j, \quad 0 \leq j \leq n, \\ c_{j,x} &= 2c_{j+1} - \frac{2v}{w} a_j, \quad 0 \leq j \leq n, \\ b_{n+1,x} &= -2wa_{n+1}, \\ c_{n+1,x} &= -\frac{2v}{w} a_{n+1}. \end{aligned} \tag{34}$$

If $n \geq 2$, then equation (34) determines a_j, b_j, and c_j recursively as follows

$$\begin{aligned} a_0 &= -2, \quad b_0 = c_0 = 0, \\ b_j &= -\tfrac{1}{2} b_{j-1,x} - wa_{j-1}, && 1 \leq j \leq n+1, \\ c_j &= \tfrac{1}{2} c_{j-1,x} + \tfrac{v}{w} a_{j-1}, && 1 \leq j \leq n+1, \\ a_j &= \frac{1}{4} \sum_{i=1}^{j-1} (b_i c_{j-i} + a_i a_{j-i}) + k_j, && 1 \leq j \leq n-1, \\ a_n &= \frac{1}{4} \sum_{i=1}^{n-1} (b_i c_{n-i} + a_i a_{n-i}) - g_n x + k_n, \\ a_{n+1} &= \frac{1}{4} \sum_{i=1}^{n} (b_i c_{n+1-i} + a_i a_{n+1-i}) + \frac{1}{2} g_n a_1 x + \alpha_n - \frac{1}{2} g_n. \end{aligned} \tag{35}$$

If $n = 1$, then $a_1 = -g_n x$, $a_2 = -v + \alpha_1 - \frac{1}{2} g_1$, and $b_j,\ c_j$ are given by (35). Moreover (34.g) can be integrated to obtain the first integral

$$b_{n+1} c_{n+1} + a_{n+1}^2 = K, \tag{36}$$

where K is a constant of integration. Let $K = \frac{1}{4} \beta_n^2$. Then the fourth Painlevé hierarchy (30) can be written as

$$\begin{aligned} b_{n+1,x} + 2wa_{n+1} &= 0, \\ b_{n+1} c_{n+1} &= \frac{1}{4} \beta_n^2 - a_{n+1}^2. \end{aligned} \tag{37}$$

3.1 Special solutions of the fourth Painlevé hierarchy

The first member of the fourth Painlevé hierarchy (37) reads

$$\begin{aligned} u_x &= 2v + u^2 + g_1 x u - 2\alpha_1, \\ v_x &= \frac{1}{(u+g_1 x)}[(v - \alpha_1 - \frac{1}{2}g_1)^2 - \frac{1}{4}\beta_1^2] - v(u + g_1 x). \end{aligned} \tag{38}$$

Eliminating v between these equations and using the transformation $y = u + g_1 x$, yields the PIV equation [16]

$$y_{xx} = \frac{y_x^2}{y} + \frac{3}{2}y^3 - 2g_1 x y^2 + 2(\frac{g_1^2 x^2}{4} - \alpha_1)y - \frac{\beta_1^2}{2y}. \tag{39}$$

Equation (38) has the special solution $(2\alpha_1 + g_1)^2 = \beta_1^2$, $v = x$, and u solves the Riccati equation

$$u_x + u^2 + g_1 x u - 2\alpha_1 = 0. \tag{40}$$

Setting $g_1 = -2$, which can be done without loss of generality, and using the linearizing transformation $u = \frac{U_x}{U}$, (40) is transformed to the Weber-Hermit equation

$$U_{xx} - 2xU_x - 2\alpha_1 U = 0. \tag{41}$$

In [17] it has been shown that the second member of the hierarchy (35, 37), with $g_2 = 1$, has the special solution $2\alpha + \beta + 1 = 0$, $v = u_x$ and u is a solution of the second Painlevé equation

$$u_{xx} = -(3u - k_1)u_x + u^3 + k_1 u^2 - 2xu + \beta - 2\alpha + 1 = 0. \tag{42}$$

As a generalization of the above results, we will prove that the fourth Pianlevé hierarchy (35, 37) has the special solutions

$$\begin{aligned} &(2\alpha_n + g_n)^2 = \beta_n^2, \\ &v = u_x, \\ &b_{n+1,x} + 2wa_{n+1} = 0. \end{aligned} \tag{43}$$

In order to prove that (43) is a solution of the hierarchy (37), we only have to prove that

$$b_{n+1}c_{n+1} = \frac{1}{4}\beta_n^2 - a_{n+1}^2. \tag{44}$$

As a first step, we will use induction to prove that

$$a_j - k_j = -\frac{d}{dx}\left(\frac{b_j}{w}\right), \quad 2 \le j \le n-1. \tag{45}$$

Using (35), one finds $a_1 = k_1$, $b_1 = 2w$, $c_1 = -\frac{2u_x}{w}$, $a_2 = -u_x + k_2$, $b_2 = w(u - k_1)$. Therefore direct calculations show that $a_2 - k_2 = -\frac{d}{dx}\left(\frac{b_2}{w}\right)$.

Assume it is true for $j = m,\ 2 \le m \le n-2$; that is, assume that

$$\begin{aligned} a_m - k_m &= -\frac{d}{dx}\left(\frac{b_m}{w}\right) \\ &= -\frac{1}{w}(b_{m,x} + ub_m). \end{aligned} \tag{46}$$

Then equation (34.a) implies that $a_{m+1,x} = wc_{m+1} + \frac{u_x}{w}b_{m+1}$. Using (35) to substitute b_{m+1} and c_{m+1}, we find

$$2a_{m+1,x} = wc_{m,x} - \frac{u_x}{w}b_{m,x}. \tag{47}$$

Using $w_x = -uw$ we find

$$\frac{d}{dx}(wc_m) = wc_{m,x} - uwc_m. \tag{48}$$

Moreover (34.a) yields

$$\frac{d}{dx}[u(a_m - k_m)] = u_x(a_m - k_m) + uwc_m + \frac{uu_x}{w}b_m. \tag{49}$$

Thus (47-49) give

$$\frac{d}{dx}[2a_{m+1} - wc_m - u(a_m - k_m)] = -u_x[a_m - k_m + \frac{1}{w}(b_{m,x} + ub_m)]. \tag{50}$$

The induction hypothesis (46) implies that the right hand side of (50) is zero and hence we get

$$a_{m+1} - k_{m+1} = \frac{1}{2}[u(a_m - k_m) + wc_m]. \tag{51}$$

On the other hand (35) gives $\frac{1}{w}b_{m+1} = -\frac{1}{2w}b_{m,x} - a_m$. Using the induction hypothesis (46) to substitute $b_{m,x}$, we get

$$\frac{1}{w}b_{m+1} = -\frac{1}{2}[a_m - \frac{u}{w}b_m]. \tag{52}$$

Now differentiating (52) and using (34.a) to substitute $a_{m,x}$ and (46) to substitute $b_{m,x}$, we obtain

$$\frac{d}{dx}\Big(\frac{b_{m+1}}{w}\Big) = -\frac{1}{2}[u(a_m - k_m) + wc_m]. \tag{53}$$

Therefore (51) and (53) yield the result.

Using similar arguments we can prove that

$$a_n + g_n x - k_n = -\frac{d}{dx}\Big(\frac{b_n}{w}\Big) \tag{54}$$

and

$$a_{n+1} - k_{n+1} - g_n = -\frac{d}{dx}\Big(\frac{b_{n+1}}{w}\Big). \tag{55}$$

Since $b_{n+1,x} = -2wa_{n+1}$, (55) gives

$$a_{n+1} + k_{n+1} + g_n = \frac{u}{w}b_{n+1}. \tag{56}$$

The equation (34.c) implies $c_{n+1} = \dfrac{1}{w^2}[wa_{n+1,x} - u_x b_{n+1}]$. Using (56) to substitute a_{n+1} we obtain $c_{n+1} = \dfrac{u}{w}\dfrac{d}{dx}\Big(\dfrac{b_{n+1}}{w}\Big)$. Hence (55) gives

$$a_{n+1} - k_{n+1} - g_n = -\frac{w}{u}c_{n+1}. \tag{57}$$

Therefore using (56) and (57) to substituting b_{n+1} and c_{n+1}, we see that (44) is satisfied provided that $(2\alpha_n + g_n)^2 = \beta_n^2$; and this completes the proof.

3.2 Relation between the second and fourth Painlevé hierarchies

In this subsection, we will show that the fourth Painlevé hierarchy (37) have special solutions in terms of the second Painlevé hierarchy (8). Thus we will derive a relation between the second Painlevé hierarchy (8) and the fourth Painlevé hierarchy (37).

Let u and v be solutions of the hierarchy (37) of order $n = m+1$ such that $a_1 = 0$, $\beta_{m+1} = 0$ and $b_{m+1} = 0$. Then the equations (35) and (37) become

$$\begin{aligned}
&a_0 = -2, \quad a_1 = 0, \quad b_0 = c_0 = 0, \\
&b_j = -\tfrac{1}{2} b_{j-1,x} - w a_{j-1}, && 1 \le j \le m+2, \\
&c_j = \tfrac{1}{2} c_{j-1,x} + \frac{v}{w} a_{j-1}, && 1 \le j \le m+2, \\
&a_j = \frac{1}{4} \sum_{i=1}^{j-1} (b_i c_{j-i} + a_i a_{j-i}) + k_j, && 2 \le j \le m \\
&a_{m+1} = \frac{1}{4} \sum_{i=1}^{m} (b_i c_{m+1-i} + a_i a_{m+1-i}) - g_{m+1} x, \\
&a_{m+2} = \frac{1}{4} \sum_{i=1}^{m+1} (b_i c_{m+2-i} + a_i a_{m+2-i}) + \alpha_{m+1} - \frac{1}{2} g_{m+1},
\end{aligned} \tag{58}$$

and

$$a_{m+2} = 0. \tag{59}$$

Setting $2w = \tilde{w}$ and using $a_1 = 0$, then we have $u = -\dfrac{\tilde{w}_x}{\tilde{w}}$ and equation (58) yields

$$\begin{aligned}
&a_0 = -2, \quad a_1 = 0, \quad b_0 = c_0 = 0, \\
&b_j = -\tfrac{1}{2} (b_{j-1,x} - \tilde{w} a_{j-1}), && 1 \le j \le m+1, \\
&c_j = \tfrac{1}{2} c_{j-1,x} + \frac{2v}{\tilde{w}} a_{j-1}, && 1 \le j \le m+1, \\
&a_j = \frac{1}{4} \sum_{i=1}^{j-1} (b_i c_{j-i} + a_i a_{j-i}) + k_j, && 2 \le j \le m \\
&a_{m+1} = \frac{1}{4} \sum_{i=1}^{m} (b_i c_{m+1-i} + a_i a_{m+1-i}) - g_{m+1} x.
\end{aligned} \tag{60}$$

Moreover, since $a_{m+2} = b_{m+1} = 0$, (58) gives

$$\begin{aligned}
&\sum_{i=1}^{m+1} (b_i c_{m+2-i} + a_i a_{m+2-i}) + 4(\alpha_{m+1} - \frac{1}{2} g_{m+1}) = 0, \\
&b_{m+1,x} = -\tilde{w} a_{m+1}.
\end{aligned} \tag{61}$$

Thus, u and v are solutions of the second Painlevé hierarchy (8) of order $n = m$ and with parameter $\delta_m = \alpha_{m+1} - \frac{1}{2} g_{m+1}$.

Therefore we have shown that any n-th member of the fourth Painlevé hierarchy (37), $n \ge 2$, has a special solution $a_1 = \beta_n = 0$ in terms of the $(n-1)$-st member of the second Painlevé hierarchy (8).

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Prof. A.P. Fordy
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Dear Prof. Fordy:

I am submitting a manuscript entitled " On special solutions of second and fourth Painlevé hierarchies" for possible publication in Physics Letters A.

The subject area of my article may be identified with the PACS number 02.30.hq; 02.30.Gp.

If you have any question about the submission, please do not hesitate to contact me.

Yours sincerely,

Ayman Sakka