# Linear problems and hierarchies of Painlevé equations

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October 29, 2008

**Abstract**

In this paper, we show that the expansion of linear problems of Painlevé equation in powers of the spectral variable can be used to derive hierarchies of ordinary differential equations. We applied this approach to linear problems of the first, second, third, and fourth Painlevé equations. We derived a new hierarchy of the third Painlevé equation and rederived known hierarchies of the other equations. Moreover some special solutions of the hierarchies of second, third, and fourth Painlevé equations are also given.

## 1 Introduction

In the last three decades there has been much interest in searching for higher-order analogues of Painlevé equations. There are several method to derive higher-order analogues of Painlevé equations. Some of these methods are the $\alpha$-method used by Painlevé and his school, the Painlevé test, and the similarity reductions of higher-order completely integrable partial differential equations. The last method leads to the derivation of Painlevé hierarchies, that is, sequences of ordinary differential equations whose first members are Painlevé equations.

One of the important properties of the six Painlevé equations ( PI-PVI) is that each Painlevé equation can be written as a compatibility condition of a linear system

$$\Phi_\lambda(x,\lambda) = A(x,\lambda)\Phi(x,\lambda), \qquad \Phi_x(x,\lambda) = B(x,\lambda)\Phi(x,\lambda), \tag{1}$$

where

$$A(x,\lambda)=\sum_{j=0}^{N+n} A_j\lambda^{N-j}, \qquad B(x,\lambda)=\sum_{j=0}^{L+l} B_j\lambda^{L-j}, \tag{2}$$

and $A_j$ and $B_j$ are matrices with entries depending on the solution $u(x)$ of the Painlevé equation [1, 2, 3, 4, 5]. These linear problems are not unique. For example the second Painlevé equation has two different linear problems, one given by Flaschka and Newell [2] and the other one given by Jimbo and Miwa [1].

Gordoa , Joshi, and Pickering [6] derive Jimbo-Miwa linear problems for second and fourth Painlevé hierarchies. This leads to the observation that these hierarchies can be obtained by expanding the Jimbo-Miwa linear problems of PII and PIV in powers of the spectral variable $\lambda$. In this article, we will show that this approach can be applied to many linear problems of Pianlevé equations. More precisely, given a linear problem (1-2) for a Painlevé equation, we generalize it by replacing the fixed number $N$ in (2) by a parameter $M \geq N$. While the compatibility condition gives the considered Painlevé equation when $M = N$, it gives higher-order analogues of this equation when $M > N$. We illustrate this by application to the linear problem for the first Painlevé equation given by Jimbo and Miwa [1, 3], the linear problem for the second Painlevé equation given by Flaschka and Newell [2], the linear problem for the third Painlevé equation given by Joshi, Kitaev, and Treharne [7], and the linear problem for the fourth Painlevé equation given by Kitaev [4] and Milne, Clarkson, Bassom [5]. It turns out that the resulting hierarchies are the first and the second Painlevé hierarchies given in [8, 9], the fourth Painlevé hierarchy given in [10], and a new third Painlevé hierarchy.

We will also give some special solutions of the second, third, and fourth Painlevé hierarchies. The special solutions of the second Painlevé hierarchy are solved in terms of the first Painlevé hierarchy and the special solutions of the fourth Painlevé hierarchy are solved in terms of the second Painlevé hierarchy. Using the relation between the fourth and the second Painlevé hierarchies, we will obtain a new linear problem for the second Painlevé hierarchy and in particular a new linear problem for the second Painlevé equation.

## 2 First Paninlevé hierarchy

As it is well known, the first Painlevé equation,

$$u_{xx}=6u^2+x, \tag{3}$$

can be obtained as the compatibility condition of the linear system (1), where $A$ and $B$ are the following matrices [1, 3]

$$
\begin{aligned}
&B = B_0\lambda + B_2\lambda^{-1}, \quad A = \sum_{j=0}^{5} A_j\lambda^{4-j}, \\
&B_0 = -i\sigma_3, \quad B_2 = iu(\sigma_3 - i\sigma_2), \quad A_0 = -4i\sigma_3, \quad A_1 = 0, \\
&A_2 = 4u\sigma_2, \quad A_3 = 2u_x\sigma_1, \quad A_4 = -i(2u^2 + x)(\sigma_3 - \sigma_2), \quad A_5 = -\tfrac{1}{2}\sigma_1,
\end{aligned}
\tag{4}
$$

and $\sigma_j,\ j = 1, 2, 3$, denote the Pauli matrices

$$
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5}
$$

As we mentioned in the introduction, we will use a generalization of the linear problem (1, 4) to derive a hierarchy of ordinary differential equations. More precisely, we assume that

$$
B = B_0\lambda + B_2\lambda^{-1}, \quad A = \sum_{j=0}^{2m+3} A_j\lambda^{2m+2-j}, \tag{6}
$$

where $m$ is a positive integer. The compatibility condition $\Phi_{x\lambda} = \Phi_{\lambda x}$ of equation (1) reads

$$
A_x = B_\lambda + [B, A]. \tag{7}
$$

Substituting $A$ and $B$ from (6) into (7), we obtain

$$
\begin{aligned}
&0 = [B_0, A_0], \quad A_{0,x} = [B_0, A_1], \\
&A_{j,x} = [B_0, A_{j+1}] + [B_2, A_{j-1}], \qquad j = 1, 2, \ldots, 2m+1, \\
&A_{2m+2,x} = B_0 + [B_0, A_{2m+3}] + [B_2, A_{2m+1}], \\
&A_{2m+3,x} = [B_2, A_{2m+2}], \quad B_2 = [B_2, A_{2m+3}].
\end{aligned}
\tag{8}
$$

Taking in account the linear problem (1) and (4), we assume that

$$
\begin{aligned}
&B_0 = \sigma_3, \quad B_2 = u(\sigma_3 + i\sigma_2), \quad A_0 = 4\sigma_3, \quad A_1 = 0, \quad A_{2m+3} = \tfrac{1}{2}\sigma_1, \\
&A_j = \begin{pmatrix} a_j & b_j \\ (-1)^{j+1}b_j & -a_j \end{pmatrix}, \quad j = 2, \ldots, 2m+2,
\end{aligned}
\tag{9}
$$

where $a_{2j-1} = 0,\ j = 2, \ldots, m+1$. Then equation (8) gives

$$
\begin{aligned}
&a_{2j,x} = 2ub_{2j-1}, && j = 1, 2, \ldots, m, \\
&a_{2m+2,x} = 1 + 2ub_{2m+1}, && \\
&b_{j,x} = 2b_{j+1} + 2u(b_{j-1} - a_{j-1}), && j = 0, 1, \ldots, 2m+1, \\
&b_{2m+2} - a_{2m+2} = 0. &&
\end{aligned}
\tag{10}
$$

For any positive integer $m$, any $a_0 \neq 0$, and $a_1 = b_1 = 0$, (10) determines $a_{2j}$, $j = 1, 2, \ldots, m+1$, and $b_j$, $j = 2, 3, \ldots, 2m+2$, recursively. Moreover the condition $b_{2m+2} - a_{2m+2} = 0$ gives an ordinary differential equation of order $2m$ for $u$.

Let us be more specific. Define $U_j = b_{2j} - a_{2j},\ j = 0, 1, \ldots, m$, and $U_{m+1} = b_{2m+2} - a_{2m+2} + x$. Then using equation (10), we obtain

$$D_x U_j = 2b_{2j+1}, \qquad j = 1, 2, \ldots, m+1, \tag{11}$$

where $D_x = \frac{d}{dx}$. Using equation (10.c) to substitute $b_{2j+1}$ into (11), we obtain

$$D_x U_j = D_x b_{2j} - 2u b_{2j-1}. \tag{12}$$

Now substitute $b_{2j}$ from (10.c) and $b_{2j-1}$ from (11) into (12), we get

$$D_x U_j = \frac{1}{4}(D_x^3 - 8uD_x - 4u_x)U_{j-1}, \qquad j = 1, \ldots, m+1. \tag{13}$$

Integrating (13), we obtain

$$U_j = \frac{1}{4}(D_x^2 - 8u + 4D_x^{-1}u_x)U_{j-1} - 4^{1-j}K_{2j}, \qquad j = 1, \ldots, m+1, \tag{14}$$

where $K_{2j}$ are constants of integration and $D_x^{-1}$ is the inverse operator of $D_x$. Without loss of generality we will take $K_2 = K_{2m+2} = 0$.

Since $U_0 = -4$, (14) yields $U_1 = 4u$. Thus using induction, we can write $U_j$ as

$$U_j = 4^{2-j}\big[\mathcal{R}_I^{j-1}u + \sum_{i=2}^{j-1} K_{2i}\mathcal{R}_I^{j-i-1}u\big] - 4^{1-j}K_{2j}, \qquad j = 2, \ldots, m+1, \tag{15}$$

where $\mathcal{R}_I$ is the recursion operator

$$\mathcal{R}_I = D_x^2 - 8u + 4D_x^{-1}u_x. \tag{16}$$

Now $a_{2j},\ j = 1, 2, \ldots, m+1$, and $b_j,\ j = 2, 3, \ldots, 2m+2$, can be determined in terms of $U_j$ as follows

$$\begin{aligned} b_{2j+1} &= \frac{1}{2}D_x U_j, \quad j = 1, 2, \ldots, m, \\ b_{2j} &= \frac{1}{4}(D_x^2 - 4u)U_{j-1}, \quad j = 1, \ldots, m+1, \\ a_{2j} &= (u - D_x^{-1}u_x)U_{j-1} + 4^{1-j}K_{2j}, \quad j = 1, \ldots, m \\ a_{2m+2} &= (u - D_x^{-1}u_x)U_m + x. \end{aligned} \tag{17}$$

The condition

$$b_{2m+2} - a_{2m+2} = 0 \tag{18}$$

yields the equation $U_{m+1} = x$. Substituting $U_{m+1}$ from (15) into $U_{m+1} = x$, we get the following hierarchy of ordinary differential equation of order $2m$ for $u$

$$\mathcal{R}_I^m u + \sum_{i=2}^{m} K_{2i}\mathcal{R}_I^{m-i}u - 4^{m-1}x = 0. \tag{19}$$

It is easy to show that when $m = 1$, (19) gives the first Painlevé equation (3). Thus the hierarchy (19) is a first Painlevé hierarchy. Now we will consider the cases $m = 2$ and $m = 3$.

**Example (1):** $m = 2$

In this example, we consider the case $m = 2$. Equation (19) yields the following fourth order ordinary differential equation

$$u_{xxxx} = 20uu_{xx} + 10u_x^2 - 40u^3 - K_4u + 4x. \tag{20}$$

Equation (20) has a linear problem (1) with $B = \sigma_3\lambda + u(\sigma_3 + i\sigma_2)\lambda^{-1}$ and $A = \sum_{j=0}^{7} A_j\lambda^{6-j}$, where $A_0 = 4\sigma_3$, $A_1 = 0$, $A_7 = \frac{1}{2}\sigma_1$, and $A_j = \begin{pmatrix} a_j & b_j \\ (-1)^{j+1}b_j & -a_j \end{pmatrix}$, $j = 2, 3, 4, 5, 6$, $a_2 = a_3 = a_5 = 0$, $a_{2j}$, $j = 1, 2, 3$, and $b_j$, $j = 2, 3, 4, 5, 6$, are given as follows

$$\begin{aligned} &b_2 = 4u, \quad b_3 = 2u_x, \quad a_4 = 2u^2 + \frac{1}{4}K_4, \quad b_4 = u_{xx} - 4u^2, \\ &b_5 = \frac{1}{2}[u_{xxx} - 12uu_x], \quad a_6 = uu_{xx} - \frac{1}{2}u_x^2 - 4u^3 + x, \\ &b_6 = \frac{1}{4}[u_{xxxx} - 16uu_{xx} - 12u_x^2 + 24u^3 + K_4u]. \end{aligned} \tag{21}$$

The equation (20) was found previously by Cosgrove [11] and it is the second member of the first Painlevé hierarchy [8, 9]. However the linear problem given here is new.

**Example (2):** $m = 3$

As another example we consider the case $m = 3$. Thus (19) gives the following sixth order ordinary differential equation

$$\begin{aligned} u_{xxxxxx} = {} & 28uu_{xxxx} + 56u_xu_{xxx} + 42u_{xx}^2 - (280u^2 + K_4)u_{xx} \\ & - 280uu_x^2 + 280u^4 + 6K_4u^2 - K_6u + 16x. \end{aligned} \tag{22}$$

The linear problem for (22) has the form (1), with $B = \sigma_3\lambda + u(\sigma_3 + i\sigma_2)\lambda^{-1}$ and $A = \sum_{j=0}^{9} A_j\lambda^{8-j}$, where $A_0 = 4\sigma_3$, $A_1 = 0$, $A_9 = \frac{1}{2}\sigma_1$, and $A_j = \begin{pmatrix} a_j & b_j \\ (-1)^{j+1}b_j & -a_j \end{pmatrix}$, $j = 2, 3, \ldots, 8$, $a_2 = a_3 = a_5 = a_7 = 0$, $a_{2j}$, $j = 2, 3, 4$, and $b_j$, $j = 2, 3, \ldots, 8$, are given as follows

$$
\begin{aligned}
&b_2 = 4u, \quad b_3 = 2u_x, \quad a_4 = 2u^2 + \frac{1}{4}K_4, \quad b_4 = u_{xx} - 4u^2,\\
&b_5 = \frac{1}{2}[u_{xxx} - 12uu_x], \quad a_6 = uu_{xx} - \frac{1}{2}u_x^2 - 4u^3 + \frac{1}{16}K_6,\\
&b_6 = \frac{1}{4}[u_{xxxx} - 16uu_{xx} - 12u_x^2 + 24u^3 + K_4 u],\\
&b_7 = \frac{1}{8}[u_{xxxxx} - 20uu_{xxx} - 40u_x u_{xx} + (120u^2 + K_4)u_x],\\
&a_8 = \frac{1}{8}[2uu_{xxxx} - 2u_x u_{xxx} + u_{xx}^2 - 40u^2 u_{xx} + 60u^4 + K_4 u^2 + 8x],\\
&b_8 = \frac{1}{16}\Big[u_{xxxxxx} - 24uu_{xxxx} - 60u_x u_{xxx} - 40u_{xx}^2\\
&\qquad + (200u^2 + K_4)u_{xx} + 280uu_x^2 - 160u^4 - 4K_4u^2 + K_6 u\Big].
\end{aligned}
\tag{23}
$$

Equation (22) is the third member of the first Painlevé hierarchy [8, 9], but the linear problem is new.

Therefor, we have rederived the first Painlevé hierarchy [8, 9]. It should be noted that in [8, 9], the constants of integrations have been chosen to be zero.

# 3 Second Painlevé hierarchy

It is well known that the second Painlevé equation,

$$u_{xx} = 2u^3 + xu + \alpha, \tag{24}$$

can be obtained as the compatibility condition of (1) where $A$ and $B$ are given by [2]

$$
\begin{aligned}
&B = B_0\lambda + B_1, \quad A = \sum_{j=0}^{3} A_j \lambda^{2-j},\\
&B_0 = -i\sigma_3, \quad B_1 = u\sigma_1, \quad A_0 = -4i\sigma_3, \quad A_1 = 4u\sigma_1,\\
&A_2 = -i(2u^2 + x)\sigma_2 - 2u_x\sigma_2, \quad A_3 = -\alpha\sigma_1.
\end{aligned}
\tag{25}
$$

In this case, we will use the following generalization of the linear problem (1, 25)

$$B = B_0\lambda + B_1, \quad A = \sum_{j=0}^{2m+1} A_j \lambda^{2m-j}, \tag{26}$$

where $m$ is a positive integer and

$$
\begin{aligned}
&B_0 = \sigma_3, \quad B_1 = u\sigma_1, \quad A_0 = -4\sigma_3, \quad A_{2m+1} = -\alpha\sigma_1,\\
&A_j = \begin{pmatrix} a_j & b_j \\ (-1)^{j+1} b_j & -a_j \end{pmatrix}, \; j = 1, 2, \ldots, 2m,
\end{aligned}
\tag{27}
$$

with $a_{2j-1} = 0, \quad j = 1, \ldots, m$. The compatibility condition of equation (1) gives

$$\begin{array}{l} A_{2m+1,x} = [B_1, A_{2m+1}], \quad A_{2m,x} = B_0 + [B_0, A_{2m+1}] + [B_1, A_{2m}], \\ A_{j,x} = [B_0, A_{j+1}] + [B_1, A_j], \quad j = 0, 1, 2, \ldots, 2m-1, \quad 0 = [B_0, A_0]. \end{array} \tag{28}$$

Substituting $A_j$ and $B_j$ from (27) into equation (28) yields

$$\begin{array}{ll} a_{2j,x} = -2ub_{2j}, & j = 0, 1, \ldots, m-1, \\ a_{2m,x} = 1 - 2ub_{2m}, & \\ b_{j,x} = 2b_{j+1} - 2ua_j, & j = 1, 2, \ldots, 2m. \end{array} \tag{29}$$

The equations (29) determine $a_{2j}, \ j = 1, \ldots, m$, and $b_j, \ j = 1, 2, \ldots, 2m+1$, recursively. Imposing the condition $b_{2m+1} = -\alpha$, one obtains an ordinary differential equation of order $2m$ for $u$.

More precisely, let

$$\begin{array}{l} U_j = b_{2j+1}, \quad j = 1, \ldots, m-1, \\ U_m = b_{2m+1} - xu. \end{array} \tag{30}$$

Then the equation $b_{2m+1} = -\alpha$ can be written as

$$U_m + xu + \alpha = 0. \tag{31}$$

Now note that (29.a) and (29.b) implies

$$\begin{array}{l} a_{2j} = -2D_x^{-1} u b_{2j} + K_{2j}, \quad j = 1, \ldots, m, \\ a_{2m} = -2D_x^{-1} u b_{2m} + x + K_{2m}, \end{array} \tag{32}$$

where $K_{2j}$ are constants of integration. Substituting into equation (29.c), we obtain

$$\begin{array}{l} b_{2j+1} = \frac{1}{2}(D_x - 4uD_x^{-1}u) b_{2j} + K_{2j} u, \quad j = 1, \ldots, m-1, \\ b_{2m+1} = \frac{1}{2}(D_x - 4uD_x^{-1}u) b_{2m} + xu + K_{2m} u. \end{array} \tag{33}$$

Now substituting $b_{2j}$ from (29.c) into (33), we get

$$\begin{array}{l} b_{2j+1} = \frac{1}{4}(D_x^2 - 4u^2 + 4uD_x^{-1}u_x) b_{2j-1} + K_{2j} u \quad j = 1, \ldots, m-1, \\ b_{2m+1} = \frac{1}{4}(D_x^2 - 4u^2 + 4uD_x^{-1}u_x) b_{2m-1} + xu + K_{2m} u. \end{array} \tag{34}$$

Thus we have

$$U_j = \frac{1}{4}(D_x - 4u^2 + 4uD_x^{-1}u_x) U_{j-1} + K_{2j} u, \quad j = 1, \ldots, m. \tag{35}$$

Using (29) with $a_0 = -4, \ b_0 = 0$, we obtain $U_0 = b_1 = -4u$ . Hence using induction we can rewrite (35) in the form

$$U_j = -4^{1-j} (\mathcal{R}_{II}^j u - \sum_{i=1}^{j-1} K_{2i} \mathcal{R}_{II}^{j-i} u) + K_{2j} u, \quad j = 1, \ldots, m, \tag{36}$$

where $\mathcal{R}_{II}$ is the recursion operator

$$\mathcal{R}_{II} = D_x^2 - 4u^2 + 4uD_x^{-1}u_x. \tag{37}$$

Using (36) to calculate $U_j, \quad j = 1, \ldots, m$, we can determine $a_{2j},\ j = 1, \ldots, m$, and $b_j,\ j = 1, 2, \ldots, 2m$, as follows

$$\begin{array}{l} b_{2j+1} = U_j, \quad\quad j = 0, 1, \ldots, m-1, \\ b_{2j} = \frac{1}{2} D_x U_{j-1}, \quad\quad j = 1, 2, \ldots, m, \\ a_{2j} = -(u - D_x^{-1}u_x)U_{j-1} + K_{2j}, \quad\quad j = 1, \ldots, m-1 \\ a_{2m} = -(u - D_x^{-1}u_x)U_{m-1} + x + K_{2m}. \end{array} \tag{38}$$

Without loss of generality we will take $K_{2m} = 0$.

Lastly, the equation (31) yields the following hierarchy

$$\mathcal{R}_{II}^m u - \sum_{i=1}^{m-1} K_{2i}\mathcal{R}_{II}^{m-i} u - 4^{m-1}(xu + \alpha) = 0. \tag{39}$$

In the case $m = 1$, equation (39) reduces to the second Painlevé equation (24). Therefore the hierarchy (39) is a second Painlevé hierarchy. Now we will consider the cases $m = 2$ and $m = 3$.

**Example (1):**
In this example, we consider the case $m = 2$. Hence equation (39) yields the following fourth-order ordinary differential equation for $u$

$$u_{xxxx} = 10u^2u_{xx} + K_2u_{xx} + 10uu_x^2 - 6u^5 - 2K_2u^3 + 4xu + 4\alpha. \tag{40}$$

Equation (40) has the linear problem (1) with $B = \sigma_3\lambda + u\sigma_1$ and $A = \sum_{j=0}^{5} A_j\lambda^{4-j}$, where $A_0 = -4\sigma_3$, $A_5 = -\alpha\sigma_1$, and $A_j = \begin{pmatrix} a_j & b_j \\ (-1)^{j+1}b_j & -a_j \end{pmatrix},\ j = 1, 2, 3, 4,$ $a_1 = a_3 = 0$, $a_{2j},\ j = 1, 2$, and $b_j,\ j = 1, 2, 3, 4$, are given by

$$\begin{array}{l} b_1 = -4u, \quad b_2 = -2u_x, \quad a_2 = 2u^2 + K_2, \quad b_3 = -[u_{xx} - 2u^3 - K_2u], \\ b_4 = -\dfrac{1}{2}[u_{xxx} - 6u^2u_x - K_2u_x], \quad a_4 = \dfrac{1}{2}[2uu_{xx} - u_x^2 - 3u^4 - K_2u^2 + 2x]. \end{array} \tag{41}$$

Equation (40) was found before [11, 12] and the special case $K_2 = 0$ with its linear problem was given in [9]. The linear problem for the full equation (40) is not given before.

**Example (2):**
Let us take $m = 3$. In this case equation (39) yields the following sixth-order ordinary differential equation for $u$

$$\begin{aligned} u_{xxxxxx} = {} & (14u^2 + K_2)u_{xxxx} + 56uu_xu_{xx} \\ & + 42uu_{xx}^2 + 70u_x^2u_{xx} - 2(35u^4 + 5K_2u^2 - 2K_4)u_{xx} \\ & - 10(14u^2 + K_2)uu_x^2 + 20u^7 + 6K_2u^5 - 12K_4u^3 + 16xu + 16\alpha. \end{aligned} \tag{42}$$

Equation (42) has the linear problem (1) with $B = \sigma_3\lambda + u\sigma_1$ and $A = \sum_{j=0}^{7} A_j\lambda^{6-j}$, where $A_0 = -4\sigma_3$, $A_7 = -\alpha\sigma_1$, and $A_j = \begin{pmatrix} a_j & b_j \\ (-1)^{j+1}b_j & -a_j \end{pmatrix}$, $j = 1, \ldots, 6$, $a_1 = a_3 = a_5 = 0$, $a_{2j}$, $j = 1, 2, 3$ and $b_j$, $j = 1, \ldots, 6$, are given by

$$\begin{aligned}
&b_1 = -4u, \quad b_2 = -2u_x, \quad a_2 = 2u^2 + K_2, \quad b_3 = -[u_{xx} - 2u^3 - K_2 u],\\
&b_4 = -\frac{1}{2}[u_{xxx} - 6u^2u_x - K_2u_x], \quad a_4 = \frac{1}{2}[2uu_{xx} - u_x^2 - 3u^4 - K_2u^2 + 2K_4],\\
&b_5 = -\frac{1}{4}[u_{xxxx} - 10u^2u_{xx} - 10uu_x^2 - K_2u_{xx} + 6u^5 + 2K_2u^3 - 4K_4u]\\
&b_6 = -\frac{1}{8}\Big[u_{xxxxx} - (10u^2 + K_2)u_{xxx}\\
&\qquad\qquad - 40uu_xu_{xx} - 10u_x^2 + 2(15u^4 + 3K_2u^2 - 2K_4)u_x\Big]\\
&a_6 = \frac{1}{8}\Big[2uu_{xxxx} - 2u_xu_{xxx} + u_{xx}^2 - 2(10u^2 + K_2)uu_{xx}\\
&\qquad\qquad - (10u^2 - K_2)u_x^2 + 10u^6 + 3K_2u^4 - 6K_4u^2 + 8x\Big].
\end{aligned} \tag{43}$$

Equation (42) is the third member of the second Painlevé hierarchy given in [9] but here we do not take the integration constants to be zeros.

## 3.1 Special solutions

In this subsection, we will study special solutions of the second Painlevé hierarchy (39). It is well known that the second Painlevé equation (24) admits a special solution in terms of the Airy function when $\alpha = \frac{-1}{2}$. This fact can be generalized to the other members of the second Painlevé hierarchy (39).

We note that $\mathcal{R}_{II} = (D_x - 2u)(D_x + 2u - 2D_x^{-1}u_x)$. Thus (39) can be rewritten as

$$(D_x - 2u)\Big\{2(D_x + 2u - 2D_x^{-1}u_x)\Big[\mathcal{R}_{II}^{m-1}u - \sum_{i=1}^{m-1} K_{2i}\mathcal{R}_{II}^{m-1-i}u\Big] + 4^{m-1}x\Big\} - 4^{m-1}(2\alpha + 1) = 0. \tag{44}$$

Therefore, if $2\alpha + 1 = 0$, then the second Painlevé hierarchy (39) admit special solutions satisfying

$$2(D_x + 2u - 2D_x^{-1}u_x)\Big[\mathcal{R}_{II}^{m-1}u - \sum_{i=1}^{m-1} K_{2i}\mathcal{R}_{II}^{m-1-i}u\Big] + 4^{m-1}x = 0. \tag{45}$$

We will show that for any $m \geq 2$, (45) is solvable in terms of the first Painlevé hierarchy (19). Let

$$\mathcal{R} = (D_x + 2u - 2D_x^{-1}u_x)(D_x - 2u), \qquad y = \frac{1}{2}(u_x + u^2). \tag{46}$$

Then we have

$$\mathcal{R} = D_x^2 - 8y + 4D_x^{-1}y_x. \tag{47}$$

Since $\mathcal{R}_{_{II}} = (D_x - 2u)(D_x + 2u - 2D_x^{-1}u_x)$, we have

$$\mathcal{R}_{_{II}}^j = (D_x - 2u)\mathcal{R}^{j-1}(D_x + 2u - 2D_x^{-1}u_x). \tag{48}$$

Thus equation (45) can be written as

$$2\mathcal{R}^{m-1}(D_x+2u-2D_x^{-1}u_x)u-2\sum_{i=1}^{m-1}K_{2i}\mathcal{R}^{m-1-i}(D_x+2u-2D_x^{-1}u_x)u+4^{m-1}x = 0. \tag{49}$$

But $(D_x + 2u - 2D_x^{-1}u_x)u = u_x + u^2 = 2y$. Hence equation (49) becomes

$$\mathcal{R}^{m-1}y - \sum_{i=1}^{m-1}K_{2i}\mathcal{R}^{m-1-i}y + 4^{m-2}x = 0, \tag{50}$$

which is equivalent to the first Painlevé hierarchy (19).

Therefore, we have shown that the solution of (45) is given by $u_x + u^2 = 2y$, where $y$ solves the first Painlevé hierarchy (50). This relation between the first and second Painlevé hierarchies was given before [13]. Using this relation, we can rederive the linear problem for the first Painlevé hierarchy (50) given in [9] from the linear problem (1) and (26) of the second Painlevé hierarchy (39). Thus one can derive the first Painlevé hierarchy (19) starting from the linear problem of the first Painlevé equation given by Fokas, U. Muğan and Zhou [3].

When $m = 2$, equation (45) reads

$$u_{xxx} + 2uu_{xx} - u_x^2 - 6u^2u_x - 3u^4 - K_2(u_x + u^2) + 2x = 0. \tag{51}$$

That is, if $2\alpha + 1 = 0$, then (40) has special solutions satisfying (51). Equation (51) is a special case of Chazy-XI equation (with $N = 3$) [14] and its solution is given by $u_x + u^2 = 2y$, where $y$ solves the first Painlevé equation

$$y_{xx} = 6y^2 + K_2y - x. \tag{52}$$

Similarly, if $2\alpha + 1 = 0$, then (42) has special solutions satisfying

$$\begin{aligned}&u_{xxxxx} + 2uu_{xxxx} - 2(u_x + 5u^2)u_{xxx} + u_{xx}^2 - 20u(2u_x + u^2)u_{xx}\\ &-10u_x^3 - 10u^2u_x^2 + 30u^4u_x + 10u^6 - K_2[u_{xxx} + 2uu_{xx} - u_x^2 - 6u^2u_x - 3u^4]\\ &-K_4(u_x + u^2) + 8x = 0.\end{aligned} \tag{53}$$

The solution of (53) is given by $u_x + u^2 = 2y$, where $y$ solves the second member of first Painlevé hierarchy (50)

$$y_{xxxx} - 20yy_{xx} - 10y_x^2 + 40y^3 - K_2(y_{xx} - 6y^2) - K_4y + 4x = 0. \tag{54}$$

## 4 Third Painlevé Hierarchy

In [7], the third Painlevé equation,

$$u_{xx} = \frac{u_x^2}{u} - \frac{1}{x}u_x + \frac{1}{x}(\alpha u^2 + \beta) + \gamma u^3 + \frac{\delta}{u}, \tag{55}$$

has been written as the compatibility condition of the linear system (1) where

$$\begin{array}{l} B = B_0\lambda + B_1, \quad A = \sum_{j=0}^{2} A_j \lambda^{-j}, \\ B_0 = \frac{1}{2}\sigma_3, \quad B_1 = \frac{1}{x}\begin{pmatrix} 0 & -\tilde{w}_3 \\ w_3 & 0 \end{pmatrix}, \quad A_0 = \frac{x}{2}\sigma_3, \\ A_1 = \begin{pmatrix} -\theta_\infty/2 & -\tilde{w}_3 \\ w_3 & \theta_\infty \end{pmatrix}, \quad A_2 = -\begin{pmatrix} w_2\tilde{w}_2 & w_1 w_2 \\ \tilde{w}_1\tilde{w}_2 & w_1\tilde{w}_1 \end{pmatrix}, \end{array} \tag{56}$$

and $u = \dfrac{\tilde{w}_3}{x w_1 w_2}$.

In this section, we will use the linear problem (1, 56) to obtain a hierarchy of ordinary differential equation, namely a third Painlevé hierarchy. We assume that

$$B = B_0\lambda + B_1, \quad A = \sum_{j=0}^{m+1} A_j \lambda^{m-j-1}, \tag{57}$$

where $m$ is a positive integer. Moreover we set

$$B_0 = \tfrac{1}{2}\sigma_3, \quad B_1 = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}, \quad A_j = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix}, \qquad j = 0, 1, \ldots, m+1. \tag{58}$$

The compatibility condition of equation (1) gives

$$\begin{array}{l} 0 = [B_0, A_0], \quad A_{m+1,x} = [B_1, A_{m+1}], \quad A_{m,x} = [B_0, A_{m+1}] + [B_1, A_m], \\ A_{m-1,x} = B_0 + [B_0, A_n] + [B_1, A_{n-1}], \\ A_{j,x} = [B_0, A_{j+1}] + [B_1, A_j], \qquad j = 0, 1, \ldots, m-2. \end{array} \tag{59}$$

Substituting $A_j$ and $B_j$ from (58) into equation (59), we obtain $A_0 = a_0\sigma_3$

$$\begin{array}{ll} a_{j,x} = pc_j - qb_j, & j = 0, 1, \ldots, m-2, m, \\ a_{m-1,x} = \frac{1}{2} + pc_{m-1} - qb_{m-1}, & \\ b_{j,x} = b_{j+1} - 2pa_j, & j = 0, 1, \ldots, m, \\ c_{j,x} = -c_{j+1} + 2qa_j, & j = 0, 1, \ldots, m, \end{array} \tag{60}$$

and

$$a_{m+1,x} = pc_{m+1} - qb_{m+1}, \quad b_{m+1,x} = -2pa_{m+1}, \quad c_{m+1,x} = 2qa_{m+1}. \tag{61}$$

For any positive integer $m$, the formulas (60) determine $a_j,\ j=0,1,\dots,m$, $b_j,\ j=1,2,\dots,m+1$, and $c_j,\ j=1,2,\dots,m$, recursively. Moreover (61) has the following two first integrals

$$c_m b_{m+1} + b_m c_{m+1} + 2a_m a_{m+1} = \gamma_2, \tag{62}$$

$$b_{m+1}c_{m+1} + a_{m+1}^2 = \gamma_3, \tag{63}$$

where $\gamma_2$ and $\gamma_3$ are constants of integrations. Using the parametrization $a_{m+1}=v,\ b_{m+1}=w$, and $p=uw$, we obtain from (63)

$$c_{m+1} = \frac{-1}{w}(v^2-\gamma_3), \tag{64}$$

and hence (62) gives

$$c_m = \frac{1}{w}\Big[\frac{1}{w}(v^2-\gamma_3)b_m - 2va_m + \gamma_2\Big]. \tag{65}$$

The system (61) yields

$$w_x = -2uvw, \quad q = \frac{-1}{w}[v_x + u(v^2-\gamma_3)]. \tag{66}$$

As a last step we impose the conditions

$$b_{m+1} = w, \quad c_m = \frac{1}{w}\Big[\frac{1}{w}(v^2-\gamma_3)b_m - 2va_m + \gamma_2\Big] \tag{67}$$

to obtain an $m-$th order system for $u$ and $v$. Eliminating one of the two dependent variables $u$ and $v$ between the two equations in the system, one obtains a differential equation of order $2m$ for the other variable.

Let us consider the case $m=1$ in brief. As we explained above, we set $a_2=v,\ b_2=w,\ p=uw$. Then (64) becomes $c_2=\frac{-1}{w}(v^2-\gamma_3)$ and the formulas (60) give

$$\begin{aligned} &a_0 = \tfrac{1}{2}x, \quad a_1 = \frac{1}{2}\gamma_1, \quad b_1 = xuw, \quad c_1 = \frac{-x}{w}[v_x + u(v^2-\gamma_3)], \\ &b_2 = w[xu_x - 2xu^2v + (\gamma_1+1)u]. \end{aligned} \tag{68}$$

Equation (67) gives

$$xu_x = 2xu^2v - (\gamma_1+1)u + 1, \qquad xv_x = -2xu(v^2-\gamma_3) + \gamma_1 v - \gamma_2. \tag{69}$$

The function $tu(t)$, where $x=t^2$ satisfies the third Painlevé equation (55) with $\alpha=-8\gamma_2$, $\beta=4(\gamma_1+1)$, $\gamma=16\gamma_3$, and $\delta=-4$.

Now we will give explicit forms for the hierarchy (67) when $m \geq 2$. In this case equation (60) gives $a_{0,x} = 0$. Without loss of generality we take $a_0 = \frac{1}{2}$. Introduce the notations

$$\begin{aligned} U_j &= \frac{b_{j+1}}{w}, \quad j = 0, 1, \ldots, m-2, \\ U_{m-1} &= \frac{b_m}{w} - xu, \\ U_m &= \frac{b_{m+1}}{w} - u - x(u_x - 2u^2 v), \\ V_j &= wc_j - (v^2 - \gamma_3)\frac{b_j}{w} + 2va_j, \quad j = 0, 1, \ldots, m-2, \\ V_{m-1} &= wc_{m-1} - (v^2 - \gamma_3)\frac{b_{m-1}}{w} + v(2a_{m-1} - x), \\ V_m &= wc_m - (v^2 - \gamma_3)\frac{b_m}{w} + 2va_m + x[v_x + 2u(v^2 - \gamma_3)]. \end{aligned} \tag{70}$$

Then using (60), we have

$$\begin{pmatrix} U_j \\ V_j \end{pmatrix} = \mathcal{R}_{III} \begin{pmatrix} U_{j-1} \\ V_{j-1} \end{pmatrix} + 2K_j \begin{pmatrix} u \\ v \end{pmatrix}, \qquad j = 1, 2, \ldots, m, \tag{71}$$

where $K_j$ are constants of integration and $\mathcal{R}_{III}$ is the recursion operator

$$\mathcal{R}_{III} = \begin{pmatrix} D_x - 2uv + 2uD_x^{-1}v_x & -2u^2 + 2uD_x^{-1}u_x \\ -2(v^2 - \gamma_3) + 2vD_x^{-1}v_x & -D_x - 2uv + 2vD_x^{-1}u_x \end{pmatrix}. \tag{72}$$

Without loss of generality, we set $K_m = \frac{1}{2}\gamma_1$, and $K_{m-1} = 0$.

Since $U_0 = u,\ V_0 = v$, (71) implies that $U_j,\ V_j,\ j = 1, \ldots, m$, are given by

$$\begin{pmatrix} U_j \\ V_j \end{pmatrix} = \mathcal{R}_{III}^j \begin{pmatrix} u \\ v \end{pmatrix} + 2\sum_{i=1}^{j-2} K_i \mathcal{R}_{III}^{j-i} \begin{pmatrix} u \\ v \end{pmatrix} + 2K_j \begin{pmatrix} u \\ v \end{pmatrix}, \qquad j = 1, 2, \ldots, m. \tag{73}$$

The equations (67) and (70) imply

$$U_m = -x(u_x - 2u^2 v) + 1 - u, \quad V_m = x[v_x + 2u(v^2 - \gamma_3)] + \gamma_2. \tag{74}$$

Therefore the hierarchy reads

$$\begin{aligned} &\mathcal{R}_{III}^m \begin{pmatrix} u \\ v \end{pmatrix} + 2\sum_{i=1}^{m-2} K_i \mathcal{R}_{III}^{m-i} \begin{pmatrix} u \\ v \end{pmatrix} \\ &\quad + x \begin{pmatrix} u_x - 2u^2 v \\ -v_x - 2u(v^2 - \gamma_3) \end{pmatrix} + \gamma_1 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 - u \\ \gamma_2 \end{pmatrix}, \quad m \geq 2. \end{aligned} \tag{75}$$

The hierarchy (75) has a linear problem given by (1) and (57) where

$$\begin{aligned} &B_0 = \tfrac{1}{2}\sigma_3, \qquad B_1 = \begin{pmatrix} 0 & uw \\ \frac{-1}{w}[v_x + u(v^2 - \gamma_3)] & 0 \end{pmatrix}, \qquad A_0 = \tfrac{1}{2}\sigma_3, \\ &A_{m+1} = \begin{pmatrix} v & w \\ \frac{-1}{w}(v^2 - \gamma_3) & -v \end{pmatrix}, \qquad A_j = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix}, \qquad j = 1, \ldots, m, \end{aligned} \tag{76}$$

and $a_j$, $b_j$, and $c_j$, $j=1,2,\ldots,m$, are given by the following formulas

$$
\begin{aligned}
&b_j = wU_{j-1}, \qquad j=1,\ldots,m-1,\\
&b_m = w(U_{m-1}+xu),\\
&c_j = \frac{1}{w}\big[V_j+(v^2-\gamma_3)U_{j-1}-2va_j\big], \qquad j=1,\ldots,m-2,\\
&c_{m-1} = \frac{1}{w}\big[V_{m-1}+(v^2-\gamma_3)U_{m-2}-v(2a_{m-1}-x)\big],\\
&c_m = \frac{1}{w}\big[(v^2-\gamma_3)(U_{m-1}+xu)-2va_m+\gamma_2\big],\\
&a_1 = \begin{cases} \frac{x}{2}+K_1, \quad m=2,\\ K_1, \quad m\neq 2,\end{cases}\\
&a_2 = \begin{cases} -b_1c_1+\frac{x}{2}+K_2, \quad m=3,\\ -b_1c_1+K_2, \quad m\neq 3,\end{cases}\\
&a_j = -\sum_{k=1}^{j-1}(b_kc_{j-k}+a_ka_{j-k})+K_j, \qquad j=3,4,\ldots,m-2,\\
&a_{m-1} = -\sum_{k=1}^{m-2}(b_kc_{m-k-1}+a_ka_{m-k-1})+\frac{x}{2},\\
&a_m = -\sum_{k=1}^{m-1}(b_kc_{m-k}+a_ka_{m-k})+K_1x+\frac{1}{2}\gamma_1.
\end{aligned}
\tag{77}
$$

In the following examples we will consider the cases $m=2$ and $m=3$.

**Example (1):**

As a first example of higher order analogue of the third Painlevé equation let us consider the case $m=2$. In this case, (75) gives the following system for $u$ and $v$

$$
\begin{aligned}
u_{xx} &= (6uv-x)u_x-6u^3v^2+2xu^2v+2\gamma_3u^3-(\gamma_1+1)u+1,\\
v_{xx} &= -(6uv-x)v_x-2u(3uv-x)(v^2-\gamma_3)-\gamma_1v+\gamma_2.
\end{aligned}
\tag{78}
$$

Eliminating $v$, equation (78) gives a fourth-order equation for $u$.

Equation (78) has the linear problem (1) with $B=B_0\lambda+B_1$ and $A=\sum_{j=0}^{3}A_j\lambda^{2-j}$, where $B_0$, $B_1$, and $A_0$ are given by (76), $A_3=\begin{pmatrix} v & w\\ \frac{-1}{w}(v^2-\gamma_3) & -v\end{pmatrix}$, and $A_j=\begin{pmatrix} a_j & b_j\\ c_j & -a_j\end{pmatrix}$, $j=1,2$, where $a_j$, $b_j$, $c_j$ are given as follows

$$
\begin{aligned}
&a_1=\tfrac{1}{2}x, \quad b_1=uw, \quad c_1=\frac{-1}{w}[v_x+u(v^2-\gamma_3)],\\
&a_2=u[v_x+u(v^2-\gamma_3)]+\tfrac{1}{2}\gamma_1, \quad b_2=w[u_x-2u^2v+xu],\\
&c_2=\frac{1}{w}[(u_x-4u^2v+xu)(v^2-\gamma_3)-2uvv_x-\gamma_1v+\gamma_2].
\end{aligned}
\tag{79}
$$

Therefore we have derived a new fourth-order equation together with its linear problem.

**Example (2):**
As another example, we will consider the case $m = 3$. In this case the equation (75) yields the following system for $u$ and $v$

$$\begin{aligned} u_{xxx} &= 2(4uv - K_1)u_{xx} + 6vu_x^2 + (4uv_x - 30u^2v^2 + 6\gamma_3 u^2 + 12K_1uv - x)u_x \\ &\quad + 2u^2v_{xx} + 20u^4v^3 - 12K_1u^3v^2 \\ &\quad - 12\gamma_3u^4v + 4K_1\gamma_3u^3 + 2xu^2v - (\gamma_1 + 1)u + 1, \\ v_{xxx} &= -2(4uv - K_1)v_{xx} - 6uv_x^2 + (4vu_x + 30u^2v^2 - 6\gamma_3u^2 - 8K_1uv + x)v_x \\ &\quad + 2(v^2 - \gamma_3)(u_{xx} + 10u^3v^2 - 6K_1u^2v - 2\gamma_3u^3 + xu) + \gamma_1 v - \gamma_2. \end{aligned} \tag{80}$$

Eliminating $v$, equation (80) gives a sixth-order equation for $u$.

The linear problem for (80) is given by (1) with $B = B_0\lambda + B_1$ and $A = \sum_{j=0}^{4} A_j\lambda^{2-j}$, where $B_0$, $B_1$, and $A_0$ are given by (76), $A_4 = \begin{pmatrix} v & w \\ \frac{-1}{w}(v^2 - \gamma_3) & -v \end{pmatrix}$,
$A_j = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix}$, $j = 1, 2, 3$, and $a_j,\ b_j,\ c_j$ are given as follows

$$\begin{aligned} &a_1 = K_1, \quad b_1 = uw, \quad c_1 = \frac{-1}{w}[v_x + u(v^2 - \gamma_3)], \\ &a_2 = u[v_x + u(v^2 - \gamma_3)] + \tfrac{1}{2}x, \quad b_2 = w[u_x - 2u^2v + 2K_1u], \\ &c_2 = \frac{1}{w}[v_{xx} + 2(2uv - K_1)v_x + (u_x + 2u^2v - 2K_1u)(v^2 - \gamma_3)], \\ &a_3 = -[uv_{xx} - (u_x - 6u^2v + 2K_1u)v_x + 2u^2(2uv - K_1)(v^2 - \gamma_3)] + \tfrac{1}{2}\gamma_1, \\ &b_3 = w[u_{xx} - 2(3uv - K_1)u_x + 6u^3v^2 - 4K_1u^2v - 2\gamma_3u^3 + xu], \\ &c_3 = \frac{1}{w}\Big[2uvv_{xx} - 2v(u_x - 6u^2v + 2K_1u)v_x - \gamma_1v + \gamma_2 \\ &\quad + \{u_{xx} - 2(3uv - K_1)u_x + 14u^3v^2 - 8K_1u^2v - 2\gamma_3u^3 + xu)\}(v^2 - \gamma_3)\Big]. \end{aligned} \tag{81}$$

The above two examples shows that we can derive a new hierarchy of differential equations (75). Since the first member of this hierarchy is the third Painlevé equation, this hierarchy is a third Painlevé hierarchy.

## 4.1 Special solutions

Let us study some special solutions of the third Painlevé hierarchy (75). Assume $\gamma_1 \neq 0$, $v = \frac{\gamma_2}{\gamma_1}$, and $\gamma_2^2 = \gamma_3\gamma_1^2$. Then (71) gives $V_j = 2K_j\frac{\gamma_2}{\gamma_1}$, $j = 1, 2, \ldots, m$, and

$$U_j = \Big(D_x - 2\frac{\gamma_2}{\gamma_1}u\Big)^j u + 2\sum_{i=1}^{j-2} K_i\Big(D_x - 2\frac{\gamma_2}{\gamma_1}u\Big)^{j-i} u + 2K_ju. \tag{82}$$

The hierarchy (75) becomes

$$\Big(D_x - 2\frac{\gamma_2}{\gamma_1}u\Big)^m u + 2\sum_{i=1}^{m-2} K_i\Big(D_x - 2\frac{\gamma_2}{\gamma_1}u\Big)^{m-i} u + (\gamma_1+1)u + x(u_x - 2\frac{\gamma_2}{\gamma_1}u) = 1. \quad (83)$$

Therefore if $\gamma_1 \neq 0$, $v = \dfrac{\gamma_2}{\gamma_1}$, and $\gamma_2^2 = \gamma_3\gamma_1^2$, then the third Painlevé hierarchy (75) admits special solutions given by (83).

If $\gamma_2 = 0$, then equation (83) is linear. If $\gamma_2 \neq 0$, then the transformation $u = -\dfrac{\gamma_1 y_x}{2\gamma_2 y}$ transforms equation (83) into the linear equation

$$D_x^{m+1}y + 2\sum_{i=1}^{m-2} K_i D_x^{m-i+1}y + xy_{xx} + (\gamma_1+1)y_x + 2\frac{\gamma_2}{\gamma_1}y = 0. \quad (84)$$

Let us give the explicit form of (83) when $m = 2, 3$; that is, the special solutions of (78) and (80).

Equation (78) has a special solution $\gamma_2^2 = \gamma_3\gamma_1^2$, $v = \dfrac{\gamma_2}{\gamma_1}$, and $u$ satisfies

$$u_{xx} = 6\frac{\gamma_2}{\gamma_1}uu_x - 4\frac{\gamma_2^2}{\gamma_1^2}u^3 - x(u_x - 2\frac{\gamma_2}{\gamma_1}u^2) - (\gamma_1+1)u + 1. \quad (85)$$

If $\gamma_2 \neq 0$, then equation (85) is equivalent to equation PVI in the complete list of second-order Painlevé equations (see [15] page 334). The transformation $u = -\dfrac{\gamma_1 y_x}{2\gamma_2 y}$ transforms (85) into the linear equation

$$y_{xxx} = -xy_{xx} - (\gamma_1+1)y_x - 2\frac{\gamma_2}{\gamma_1}y. \quad (86)$$

Equation (80) has a special solution $\gamma_2^2 = \gamma_3\gamma_1^2$, $v = \dfrac{\gamma_2}{\gamma_1}$, and $u$ satisfies

$$\begin{aligned} u_{xxx} = {} & 2\frac{\gamma_2}{\gamma_1}(4uu_{xx} + 3u_x^2) - 24\frac{\gamma_2^2}{\gamma_1^2}u^2u_x + 8\frac{\gamma_2^3}{\gamma_1^3}u^4 \\ & - 2K_1(u_{xx} - 6\frac{\gamma_2}{\gamma_1}uu_x + 4\frac{\gamma_2^2}{\gamma_1^2}u^3) - x(u_x - 2\frac{\gamma_2}{\gamma_1}u^2) - (\gamma_1+1)u + 1. \end{aligned} \quad (87)$$

If $\gamma_2 \neq 0$, then the transformation $u = -\dfrac{\gamma_1 y_x}{2\gamma_2 y}$ transforms (87) into the linear equation

$$y_{xxxx} = -2K_1y_{xxx} - xy_{xx} - (\gamma_1+1)y_x - 2\frac{\gamma_2}{\gamma_1}y. \quad (88)$$

## 5 Fourth Painlevé Hierarchy

The fourth Painlevé equation,

$$u_{xx} = \frac{u_x^2}{2u} + \frac{3}{2}u^3 + 4xu^2 + 2(x^2 - \alpha)u + \frac{\beta}{u}, \tag{89}$$

can be obtained as the compatibility condition of the linear system (1) with the following matrices $A$ and $B$ [4, 5]

$$\begin{aligned}
&B = B_0\lambda^2 + B_1\lambda + B_2, \quad A = \sum_{j=0}^{4} A_j\lambda^{3-j},\\
&B_0 = \tfrac{1}{2}\sigma_3, \quad B_1 = \begin{pmatrix} 0 & iw \\ iv & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix},\\
&A_0 = \tfrac{1}{2}\sigma_3, \quad A_1 = B_1, \quad A_2 = \begin{pmatrix} x+u & 0 \\ 0 & -x-u \end{pmatrix},\\
&A_3 = \begin{pmatrix} 0 & i(w_x + 2xw) \\ i(2xv - v_x) & 0 \end{pmatrix}, \quad A_4 = \gamma_0\sigma_3, \quad u = vw.
\end{aligned} \tag{90}$$

Following the same method as in the previous sections, we take $A$ and $B$ in the following form

$$A = \sum_{j=0}^{2m+2} A_j\lambda^{2m+1-j}, \quad B = B_0\lambda^2 + B_1\lambda + B_2, \tag{91}$$

where $m$ is a positive integer. Further more, we set

$$\begin{aligned}
&B_0 = \tfrac{1}{2}\sigma_3, \quad B_1 = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -pq & 0 \\ 0 & pq \end{pmatrix},\\
&A_0 = \tfrac{1}{2}\sigma_3, \quad A_{2j} = a_{2j}\sigma_3, \ j = 1, \dots, m,\\
&A_{2j+1} = \begin{pmatrix} 0 & b_j \\ c_j & 0 \end{pmatrix}, \ j = 0, 1, \dots, m, \quad A_{2m+2} = \gamma_0\sigma_3.
\end{aligned} \tag{92}$$

The compatibility condition of equation (1) gives

$$\begin{aligned}
&A_{2m+2,x} = [B_2, A_{2m+2}], \quad A_{2m+1,x} = B_1 + [B_1, A_{2m+2}] + [B_2, A_{2m+1}],\\
&A_{2m,x} = 2B_0 + [B_0, A_{2m+2}] + [B_1, A_{2m+1}] + [B_2, A_{2m}],\\
&A_{j,x} = [B_0, A_{j+2}] + [B_1, A_{j+1}] + [B_2, A_j], \quad j = 0, 1, 2, \dots, 2m-1,\\
&0 = [B_0, A_1] + [B_1, A_0], \quad 0 = [B_0, A_0].
\end{aligned} \tag{93}$$

Substituting $A_j$ and $B_j$ from (92) into (93), we find that $A_1 = B_1$ and $A_j$, $j = 2, 3, \dots, 2m+1$, can be determined by the following formulas

$$\begin{aligned}
&a_{2j,x} = pc_{2j+1} - qb_{2j+1}, && j = 1, 2, \dots, m-1,\\
&b_{2j-1,x} = b_{2j+1} - 2pa_{2j} - 2pqb_{2j-1}, && j = 1, 2, \dots, m,\\
&c_{2j-1,x} = -c_{2j+1} + 2qa_{2j} + 2pqc_{2j-1}, && j = 1, 2, \dots, m,\\
&a_{2m,x} = 1 + pc_{2m+1} - qb_{2m+1},
\end{aligned} \tag{94}$$

$$c_{2m+1,x} - q(2pc_{2m+1} + 2\gamma_0 + 1) = 0, \tag{95}$$

and

$$b_{2m+1,x} + p(2qb_{2m+1} + 2\gamma_0 - 1) = 0. \tag{96}$$

The system (95-96) has the following first integral

$$\sum_{j=1}^{m+1} b_{2j-1}c_{2m+3-2j} + \sum_{j=1}^{m} a_{2j}a_{2m-2j+2} = 2x(a_2 + pq) + \gamma_1, \tag{97}$$

where $\gamma_1$ is a constant of integration.

In order to derive a hierarchy of ordinary differential equation, we prossed as follows. Define $u = -pq$, $v = \dfrac{p_x}{p}$, and introduce the notation $U_j,\ V_j,\ j = 0, 1, \ldots, m$, as follows

$$\begin{aligned}
&U_j = a_{2j+2} - K_{2j+2}, \quad j = 0, 1, \ldots, m-2,\\
&U_{m-1} = a_{2m} - x,\\
&U_m = 2x(a_2 - 2u) - \sum_{j=1}^{m+1} b_{2j-1}c_{2m+3-2j} - \sum_{j=1}^{m} a_{2j}a_{2m-2j+2},\\
&V_j = \frac{1}{p}b_{2j+3} - 2K_{2j+2}, \quad j = 0, 1, \ldots, m-2,\\
&V_{m-1} = \frac{1}{p}b_{2m+1} - 2x,\\
&V_m = \frac{1}{p}b_{2m+1,x} + 2qb_{2m+1} + 2U_m + 2x(2u - v) - 2,
\end{aligned} \tag{98}$$

where $K_j$ are constants. Then equations (96) and (97) can be written in the form

$$U_m + 2xu + \gamma_1 = 0, \qquad V_m + 2xv + 2\gamma_0 + 2\gamma_1 + 1 = 0. \tag{99}$$

Equation (94) implies that $U_j,\ V_j,\ j = 0, 1, \ldots, m$, satisfy

$$\begin{pmatrix} U_j \\ V_j \end{pmatrix} = \mathcal{R}_{IV} \begin{pmatrix} U_{j-1} \\ V_{j-1} \end{pmatrix} + 2K_{2j} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{100}$$

where $\mathcal{R}_{IV}$ is the recursion operator

$$\mathcal{R}_{IV} = \begin{pmatrix} -D_x - 2u + v + D_x^{-1}(2u_x - v_x) & 2u - D_x^{-1}u_x \\ -2D_x - 4u + 2v + 2D_x^{-1}(2u_x - v_x) & D_x + 2u + v - 2D_x^{-1}u_x \end{pmatrix}. \tag{101}$$

Using (94), we find

$$\begin{aligned}
&a_2 = \begin{cases} u + x, & m = 1,\\ u + K_2, & m \neq 1, \end{cases}\\
&b_3 = p(v + 2a_2 - 2u),
\end{aligned} \tag{102}$$

and hence we have $U_0 = u$ and $V_0 = v$. Thus (100) implies that

$$\begin{pmatrix} U_j \\ V_j \end{pmatrix} = \mathcal{R}_{IV}^j \begin{pmatrix} u \\ v \end{pmatrix} + 2\sum_{i=1}^{j-1} K_{2i}\mathcal{R}_{IV}^{j-i} \begin{pmatrix} u \\ v \end{pmatrix} + 2K_{2j} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{103}$$

Therefore equation (99) can be written as

$$\mathcal{R}_{IV}^m \begin{pmatrix} u \\ v \end{pmatrix} + 2\sum_{i=1}^{m-1} K_{2i}\mathcal{R}_{IV}^{m-i} \begin{pmatrix} u \\ v \end{pmatrix} + 2x \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \gamma_1 \\ \gamma_0 + \gamma_1 + 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{104}$$

The coefficients $A$ and $B$ in the linear problem (1) of the hierarchy (104) has the form (91), where

$$\begin{array}{l} B_0 = \frac{1}{2}\sigma_3, \quad B_1 = \begin{pmatrix} 0 & p \\ -\frac{u}{p} & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix}, \\ A_0 = \frac{1}{2}\sigma_3, \quad A_1 = B_1, \quad A_{2j} = a_{2j}\sigma_3, \; j = 1, \ldots, m, \\ A_{2j+1} = \begin{pmatrix} 0 & b_j \\ c_j & 0 \end{pmatrix}, \; j = 1, \ldots, m, \quad A_{2m+2} = \gamma_0\sigma_3, \end{array} \tag{105}$$

$p$ satisfies $p_x = pv$, $a_{2j}$, $b_{2j+1}$, and $c_{2j+1}$, $j = 1, 2, \ldots, m$, are given by

$$\begin{array}{l} a_{2j} = U_{j-1} + K_{2j}, \quad j = 1, \ldots, m-1, \\ a_{2m} = U_{m-1} + x, \\ b_{2j+1} = p(V_{j-1} + 2K_{2j}), \quad j = 1, \ldots, m-1, \\ b_{2m+1} = p(V_{m-1} + 2x), \\ c_{2j+1} = \dfrac{1}{p}[D_x U_{j-1} - u(V_{j-1} + 2K_{2j})], \quad j = 1, \ldots, m-1, \\ c_{2m+1} = \dfrac{1}{p}[D_x U_{m-1} - u(V_{m-1} + 2x)]. \end{array} \tag{106}$$

As usual, when $m = 1$, $u$ satisfies the fourth Painlevé equation (89). Next we study the case $m = 2$.

**Example (1):**
When $m = 2$, (104) gives the following system for $u$ and $v$

$$\begin{array}{l} u_{xx} = (3v + 2K_2)u_x - 3uv^2 - 4K_2uv + 2u^3 + 2K_2u^2 - 2xu - \gamma_1, \\ v_{xx} = -(3v + 2K_2)v_x + 2(3v + 2K_2)u_x - v^3 - 6uv^2 - 2K_2v^2 - 2xv \\ \qquad + 6u^2v - 4K_2uv + 4K_2u^2 - (2\gamma_0 + 2\gamma_1 + 1). \end{array} \tag{107}$$

The elimination of $v$ between (107.a) and (107.b) gives a fourth order equation for $u$.

The linear system for (107) is given by (1) with $B = B_0\lambda^2 + B_1\lambda + B_2$, $A = \sum_{j=0}^{6} A_j\lambda^{5-j}$, where $B_j,\ j = 0, 1, 2$, and $A_j,\ j = 0, 1$, are given by (105) and

$$
\begin{aligned}
&A_2 = (u + K_2)\sigma_3, \qquad A_3 = \begin{pmatrix} 0 & p(v + 2K_2) \\ \frac{1}{p}(u_x - uv - 2K_2u) & 0 \end{pmatrix}, \\
&A_4 = -(u_x + u^2 - 2uv - 2K_2u - x)\sigma_3, \qquad A_5 = \begin{pmatrix} 0 & b_5 \\ c_5 & 0 \end{pmatrix}, \qquad A_6 = \gamma_0\sigma_3, \\
&b_5 = p(v_x - 2u_x + 2uv - 2u^2 + 2K_2v + 2x), \\
&c_5 = \frac{-1}{p}[u_{xx} - 2(v + K_2)u_x - uv_x - 2u^3 + 2u^2v + uv^2 + 2K_2uv + 2xu].
\end{aligned} \tag{108}
$$

Once again we can derive a hierarchy of differential equations, a fourth Painlevé hierarchy. This hierarchy was given before [10]. In deed the transformation $y = -u_x + uv - u^2$, $w = -v$ transforms the system (107) into the system

$$
\begin{aligned}
y_{xx} = {} & \frac{[y_x + 2y(w - k_2) - \gamma_1 - \gamma_0 + \frac{1}{2}]^2 - (\gamma_0 - \frac{1}{2})^2}{[2y - w_x + w^2 - 2K_2w + 2x]} \\
& - 2(yw)_x + 2K_2y_x - y[2y - w_x + w^2 - 2K_2w + 2x], \\
w_{xx} = {} & (3w - 2K_2)w_x - 2y(3w - 2K_2) \\
& - w^3 + 2K_2w^2 - 2xw + (2\gamma_0 + 2\gamma_1 + 1).
\end{aligned} \tag{109}
$$

The system (109) is the second member of the fourth Painlevé hierarchy given in [6]. Therefore we have another linear problem for the fourth Pianlevé hierarchy given in [6].

## 5.1 Special solutions

In this subsection, we will show that the fourth Painlevé hierarchy ( 104) admit special solutions in terms of the second Painlevé hierarchy (39).

Suppose that $m = 2n$, $p = 1$, $2\gamma_1 + 2\gamma_0 + 1 = 0$, $K_{4j-2} = 0,\ j = 1, 2, \ldots, n$. Then $u = -q,\ v = 0$, and the hierarchy (99) reduces to the following hierarchy

$$
U_{2n} + 2xu + \gamma_1 = 0, \quad V_{2n} = 0. \tag{110}
$$

The operator (101) becomes

$$
\mathcal{R} = \begin{pmatrix} -D_x - 2u + 2D_x^{-1}u_x & 2u - D_x^{-1}u_x \\ -2D_x - 4u + 4D_x^{-1}u_x & D_x + 2u - 2D_x^{-1}u_x \end{pmatrix}. \tag{111}
$$

Now we will use induction to prove that

$$
\begin{aligned}
&U_{2j} = (D_x - 2u)(D_x + 2u - 2D_x^{-1}u_x)U_{2j-2} + 2K_{4j}u, && j = 1, 2, \ldots, n, \\
&V_{2j} = 0, && j = 1, 2, \ldots, n, \\
&U_{2j+1} = -(D_x + 2u - 2D_x^{-1}u_x)U_{2j}, && j = 1, 2, \ldots, n-1, \\
&V_{2j+1} = 2U_{2j+1}, && j = 1, 2, \ldots, n-1.
\end{aligned} \tag{112}
$$

Firstly, we note that $U_0 = u$, $V_0 = 0$. Thus (100) gives $U_1 = -(D_x + 2u - 2D_x^{-1}u_x)U_0$, $V_1 = 2U_1$, $V_2 = (D_x + 2u - 2D_x^{-1}u_x)(V_1 - 2U_1) = 0$, and

$$\begin{aligned} U_2 &= -(D_x - 2u)U_1 + 2K_4 u \\ &= (D_x - 2u)(D_x + 2u - 2D_x^{-1}u_x)U_0 + 2K_4 u. \end{aligned} \tag{113}$$

Hence the formulas (112) are true when $j = 1$.

Assume that (112) is true for $j = k, \quad 1 \leq k \leq n-1$. Then substituting $V_{2k} = 0$ into (100) implies that $U_{2k+1} = -(D_x + 2u - 2D_x^{-1}u_x)U_{2k}$ and $V_{2k+1} = -2(D_x + 2u - 2D_x^{-1}u_x)U_{2k} = 2U_{2k+1}$. Since $V_{2k+2} = (D_x + 2u - 2D_x^{-1}u_x)(V_{2k+1} - 2U_{2k+1})$, we get $V_{2k+2} = 0$. Using (100), $V_{2k+1} = 2U_{2k+1}$, and $U_{2k+1} = -(D_x + 2u - 2D_x^{-1}u_x)U_{2k}$, we get

$$\begin{aligned} U_{2k+2} &= -(D_x + 2u - 2D_x^{-1}u_x)U_{2k+1} + (2u - D_x^{-1}u_x)V_{2k+1} + 2K_{4k+4}u \\ &= -(D_x - 2u)U_{2k+1} + 2K_{4k+4}u \\ &= (D_x - 2u)(D_x + 2u - 2D_x^{-1}u_x)U_{2k} + 2K_{4k+4}u. \end{aligned} \tag{114}$$

This ends the proof.

Now using $(D_x - 2u)(D_x + 2u - 2D_x^{-1}u_x) = D_x^2 - 4u^2 + 4uD_x^{-1}u_x = \mathcal{R}_{II}$, $U_0 = u$, we obtain

$$U_{2j} = \mathcal{R}_{II}U_{2j-2} + 2K_{4j}u, \tag{115}$$

and hence

$$U_{2j} = \mathcal{R}_{II}^{j}u + 2\sum_{i=1}^{j-1} K_{4i}\mathcal{R}_{II}^{j-i}u + 2K_{4j}u. \tag{116}$$

Thus equation (110) yields

$$\mathcal{R}_{II}^{n}u + 2\sum_{i=1}^{n-1} K_{4i}\mathcal{R}_{II}^{n-i}u + 2xu + \gamma_1 = 0. \tag{117}$$

The hierarchy (117) is equivalent to the second Painlevé hierarchy (39). Therefore, if $u$ is a solution of the $2n$-th member of the fourth Painlevé hierarchy (104) with $v = 0$, $2\gamma_1 + 2\gamma_0 + 1 = 0$, $K_{4j-2} = 0$, $j = 1, 2, \ldots, n$, then $u$ satisfies the $n$-th member of the second Painlevé hierarchy (117).

The linear problem for the second Painlevé hierarchy (117) is given by

$$\begin{aligned} &B = B_0\lambda^2 + B_1\lambda + B_2, \quad A = \sum_{j=0}^{4n+2} A_j\lambda^{4n+1-j}, \\ &B_0 = \tfrac{1}{2}\sigma_3, \quad B_1 = \begin{pmatrix} 0 & 1 \\ -u & 0 \end{pmatrix}, \quad B_2 = u\sigma_3, \quad A_0 = \tfrac{1}{2}\sigma_3, \quad A_1 = B_1 \\ &A_{2j} = a_{2j}\sigma_3, \quad j = 1, \ldots, 2n, \\ &A_{2j+1} = \begin{pmatrix} 0 & b_j \\ c_j & 0 \end{pmatrix}, \; j = 1, \ldots, 2n, \quad A_{4n+2} = \gamma_0\sigma_3, \end{aligned} \tag{118}$$

where $a_j$, $b$, and $c_j$ are given by

$$\begin{aligned}
&a_{4j}=-(D_x+2u-2D_x^{-1}u_x)U_{2j-2}+K_{4j}, \quad j=1,\ldots,n-1,\\
&a_{4j-2}=U_{2j-2}, \quad j=1,\ldots,n,\\
&a_{4n}=-(D_x+2u-2D_x^{-1}u_x)U_{2n-2}+x,\\
&c_{4j-1}=D_xU_{2j-2}, \quad j=1,\ldots,n,\\
&c_{4j+1}=-[\mathcal{R}_{_{II}}U_{2j-2}+2K_{4j}u], \quad j=1,2,\ldots,n-1,\\
&c_{4n+1}=-[\mathcal{R}_{_{II}}U_{2n-2}+2xu], \quad j=1,2,\ldots,n,\\
&b_{4j-1}=0, \quad b_{4j+1}=2a_{4j}, \quad j=1,2,\ldots,n.
\end{aligned} \tag{119}$$

Therefor the above relation between the fourth Painlevé hierarchy (104) and second Painlevé hierarchy (117) gives rise to the new linear problem (118) for the second Painlevé hierarchy.

For example, the second member of the fourth Painlevé hierarchy (104), that is equation (107), has the special solution $K_2=0$, $2\gamma_0+2\gamma_1+1=0$, $v=0$, and $u$ satisfies the second Painlevé equation

$$u_{xx}=2u^3-2xu+\gamma_0+\frac{1}{2}. \tag{120}$$

The second Painlevé equation (120) has the following new linear problem

$$\begin{aligned}
&B=B_0\lambda^2+B_1\lambda+B_2, \quad A=\sum_{j=0}^{6}A_j\lambda^{5-j},\\
&B_0=\tfrac{1}{2}\sigma_3, \quad B_1=\begin{pmatrix}0 & 1\\ -u & 0\end{pmatrix}, \quad B_2=u\sigma_3, \quad A_0=\tfrac{1}{2}\sigma_3,\\
&A_1=B_1, \quad A_2=u\sigma_3, \quad A_3=\begin{pmatrix}0 & 0\\ u_x & 0\end{pmatrix}, \quad A_4=-(u_x+u^2-x)\sigma_3,\\
&A_5=\begin{pmatrix}0 & -2(u_x+u^2-x)\\ -u_{xx}+2u^3-2xu & 0\end{pmatrix}, \quad A_6=\gamma_0\sigma_3.
\end{aligned} \tag{121}$$

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