# Second-order fourth-degree Painlevé-type equations related to PIII, PIV, and PV equations

Ayman Sakka
Department of Mathematics, Islamic University of Gaza
P.O.Box 108, Rimal, Gaza, Palestine

e-mail: asakka@mail.iugaza.edu
Fax Number: (+972)(7)2860800

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**Abstract**

Transformations that involve Fuchsian-type equation is used to obtain one-to-one correspondence between the third, fourth, and fifth Pianlevé equations and certain second-order fourth-degree Painlevé-type equations.

# I Introduction

Fokas and Ablowitz [1] developed an algorithmic method to investigating the transformation properties of the Pianlevé equations. However, certain second-order second-degree equations of Painlevé type equations related to PIII and PVI were also discussed. They used the transformation

$$u = \frac{v' + av^2 + bv + c}{dv^2 + ev + f}, \tag{1.1}$$

where $a, b, c, d, e, f$ are functions of $z$ only. The transformation (1.1) is the only transformation that is linear in $v'$ and preserves the Painlevé property. The method can be summarized as follows: Let $v$ be a solution of one of the Painlevé equations, which has the general form

$$v'' = P_2(v, z)(v')^2 + P_1(v, z)v' + P_0(v, z). \tag{1.2}$$

Differentiating (1.1) and using (1.2) to replace $v''$ and (1.1) to replace $v'$ one obtains a polynomial in $v$,

$$A_n(u', u, z)v^n + A_{n-1}(u', u, z)v^{n-1} + \cdots + A_1(u', u, z)v + A_0(u', u, z) = 0. \tag{1.3}$$

In [1] two types of transformations are considered:

(I) Find $a,\ b,\ c,\ d,\ e,\ f$ so that (1.3) reduces to a linear equation for $v$. Then substituting $v$ into (1.1) yields a second-order first-degree Painlevé type equations for $u$.

(II) Find $a,\ b,\ c,\ d,\ e,\ f$ so that (1.3) reduces to a quadratic equation for $v$. Then substituting $v$ into (1.1) yields a second-order second-degree Painlevé type equations for $u$.

Gordoa and Pickering [2], introduced a new type. Namely, find $a,\ b,\ c,\ d,\ e,\ f$ so that (1.3) reduces to a polynomial of degree $m > 2$ for $v$

$$B_m v^m + B_{m-1}v^{m-1} + \ldots + B_1 v + B_0 = 0. \tag{1.4}$$

Then (1.1) and the derivative of (1.4) are used first to obtain the corresponding inverse transformation for $v$ as rational function of $u,\ u'$ and $u''$, and second to obtain the corresponding second-order differential equation for $u$ of degree $> 2$.

A generalization of the algorithm of Fokas an Ablowitiz so that it can be applied to equations of order $> 2$ is given in [3].

As an extension to the method of Fokas and Ablowitz, one may replace (1.1) by transformation of the form

$$u = \frac{(v')^2 + (a_2v^2 + a_1v + a_0)v' + b_4v^4 + b_3v^3 + b_2v^2 + b_1v + b_0}{(c_2v^2 + c_1v + c_0)v' + d_4v^4 + d_3v^3 + d_2v^2 + d_1v + d_0}, \tag{1.5}$$

where $a_j, b_k, c_j, d_k, j = 0,1,2, k = 0,1,2,3,4$ are functions of $z$. Let $A_j := c_j u - a_j, B_k := d_k u - b_k$, $j = 0,1,2, k = 0,1,2,3,4$. Then the transformation (1.5) preserves the Painlevé property if the equation

$$(v')^2 = (A_2 v^2 + A_1 v + A_0)v' + B_4 v^4 + B_3 v^3 + B_2 v^2 + B_1 v + B_0, \tag{1.6}$$

is of Painlevé type. To be more specific, let

$$F(v) := -[f_4 v^4 + f_3 v^3 + f_2 v^2 + f_1 v + f_0], \tag{1.7}$$

where

$$\begin{aligned} &f_4 = 4B_4 + A_2^2, \qquad f_3 = 2(2B_3 + A_1 A_2), \\ &f_2 = 4B_2 + A_1^2 + 2A_0 A_2, \\ &f_1 = 2(2B_1 + A_0 A_1), \qquad f_0 = 4B_0 + A_0^2. \end{aligned} \tag{1.8}$$

It is known that when $F(v) \neq 0$, there are unique monic polynomials $F_1(v),\ F_2(v)$ such that

$$F(v) = A(z) F_1(v) [F_2(v)]^2, \tag{1.9}$$

where $A(z)$ is an analytic function and $F_1(v)$ has no multiple roots. Then equation (1.6) is of Painlevé type if it satisfies the following conditions [4]

$$\begin{aligned} &i)\ F_1(v) \text{ divides } G_1(v) := -(A_2 v^2 + A_1 v + A_0)\frac{\partial F_1}{\partial v} - 2\frac{\partial F_1}{\partial z}, \\ &ii)\ f_4 = 0 \text{ and } f_3 \neq 0 \text{ imply } A_2 = 0, \\ &iii)\ f_4 = f_3 = f_2 = 0 \text{ and } f_1 \neq 0 \text{ imply } A_2 = 0. \end{aligned} \tag{1.10}$$

In particular, if

$$G(v) := -(A_2 v^2 + A_1 v + A_0)\frac{\partial F}{\partial v} - 2\frac{\partial F}{\partial z} = 0, \tag{1.11}$$

then (1.6) is of Painlevé type.

In [5], the transformation (1.5) is used to obtain one-to-one correspondence between solutions $v$ of PI-PVI equations and solutions $u$ of certain second-order second-degree equations of Pianlevé type.

In [6], the transformation (1.5), with the constraint $G(v) = 0$, is used to obtain one-to-one correspondence between solutions $v$ of PI-PIV equations and solutions $u$ of certain second-order fourth-degree equations of Pianlevé type. It turns out that PV and PVI does not have transformation of this type.

In this article, we will use the transformation (1.5) subject to the condition $F(v)$ has no multiple root. This is a generalization of the condition $G(v) = 0$ which we used in [6]. To see this, assume that

$F(v)$ has no multiple root. Then, to write $F(v)$ in the form of (1.9), we have to take $F_1(v) = \dfrac{1}{A(z)}F(v)$ and $F_2(v) = 1$. Thus the condition $i)$ of (1.10) is satisfied if and only if $F(v)$ divides $G(v)$.

To obtain the second-order fourth-degree equation and the one-to-one correspondence, we proceed as follows: Let $v(z)$ be a solution of one of the Painlevé equations, which has the general form (1.2), and let $u(z)$ be given by the transformation (1.5). Differentiating the equation (1.6), one obtains

$$[2v' - (A_2v^2 + A_1v + A_0)][2v'' - (2A_2v + A_1)v' + \frac{1}{4}\frac{\partial F}{\partial v} - (A_2'v^2 + A_1'v + A_0')] = \frac{1}{4}G(v). \tag{1.12}$$

Since $F(v)$ divides $G(v)$, there exists a polynomial $E(v) = e_1v + e_0$ such that $G(v) = E(v)F(v)$. But equation (1.6) implies that $F(v) = -[2v' - (A_2v^2 + A_1v + A_0)]^2$. Hence equation (1.12) becomes

$$2v'' = [2A_2v + A_1 - \frac{1}{2}E(v)]v' - \frac{1}{4}\frac{\partial F}{\partial v} + \frac{1}{4}E(v)(A_2v^2 + A_1v + A_0) + (A_2'v^2 + A_1'v + A_0'). \tag{1.13}$$

Now using (1.2) to replace $v''$ and (1.6) to replace $(v')^2$ equation (1.13) gives

$$\Phi v' + \Psi = 0, \tag{1.14}$$

where

$$\begin{aligned} \Phi &= 2P_1 - [2A_2v + A_1 - \tfrac{1}{2}E(v)] + 2P_2(A_2v^2 + A_1v + A_0), \\ \Psi &= 2P_2(B_4v^4 + B_3v^3 + B_2v^2 + B_1v + B_0) + \frac{1}{4}\frac{\partial F}{\partial v} + 2P_0 \\ &\quad - (A_2'v^2 + A_1'v + A_0') - \tfrac{1}{4}E(v)(A_2v^2 + A_1v + A_0). \end{aligned} \tag{1.15}$$

Now the aim is to choose $a_j, b_k, c_j$ and $d_k$ so that $\Phi$ and $\Psi$ are identically zero and the constrained $G(v) = E(v)F(v)$ is reduced to a quadratic equation for $v$,

$$A(u', u, z)v^2 + B(u', u, z)v + C(u', u, z) = 0. \tag{1.16}$$

Then, it is up to solve the equation (1.16) for $v$ and substituting into equation (1.6) one obtains second-order fourth-degree Painlevé-type equations for $u$.

In this work, we will apply the procedure described above to the third, fourth, and fifth Painlevé equations. It turns out that while the fifth Painlevé equations admits two transformations of this type, the sixth Painlevé equation admits only one transformation. We will show that these transformations breaks downs if and only if the Pianlevé equation has one-parameter family of solutions characterized by first-order second-degree differential equations.

One can may look for transformations so that the constrained $G(v) = E(v)F(v)$ reduces to a polynomial of degree $m > 2$ for $v$ and obtain one-to-one correspondence between a given Painlevé equation and second-order Painlevé-type equations of degree $> 4$. Moreover one may applied the method to higher order equations.

Throughout this article $'$ denotes the derivative with respect to $z$ and $\dot{}$ denotes the derivative with respect to $x$.

## II Painlevé III equation

In this section, we will apply the method to the third Painlevé equation, PIII,

$$v'' = \frac{1}{v}(v')^2 - \frac{1}{z}v' + \gamma v^3 + \frac{1}{z}(\alpha v^2 + \beta) + \frac{\delta}{v}. \tag{2.1}$$

For PIII, $P_2 = \frac{1}{v}$, $P_1 = -\frac{1}{z}$, and $P_0 = \gamma v^3 + \frac{1}{z}(\alpha v^2 + \beta) + \frac{\delta}{v}$. Thus using (1.15) we find

$$\Phi = \frac{1}{2}e_1 v^2 + (A_1 - \frac{2}{z} - \frac{1}{2}e_0)v + 2A_0. \tag{2.2}$$

Setting $\Phi = 0$, we obtain $A_0 = e_1 = 0$, $e_0 = \frac{4}{z} - 2A_1$. Now $\Psi$ becomes

$$\Psi = 2(\gamma - B_4)v^4 + (\frac{2\alpha}{z} - B_3)v^3 - (A_1' + \frac{1}{z}A_1)v^2 + (B_0 + \frac{2\beta}{z})v + 2(B_0 + \delta). \tag{2.3}$$

Setting $\Psi = 0$, we obtain

$$A_1 = \frac{k}{z}, \qquad B_4 = \gamma - \frac{1}{2}A_2^2, \qquad B_3 = \frac{2\alpha}{z} - \frac{(k+1)}{z} - A_2', \qquad B_1 = -\frac{2\beta}{z}, \qquad B_0 = -\delta. \tag{2.4}$$

If $d_2 = 0$, then the transformation (1.7) becomes undefined. Thus we should take $d_2 \neq 0$ and hence without loss of generality we may set $B_2 = \dfrac{u}{z^2}$.

The equation $G(v) = E(v)F(v)$ is a fifth degree polynomial in $v$

$$sigma_5 v^5 + \sigma_4 v^4 + \sigma_3 v^3 + \sigma_2 v^2 + \sigma_1 v + \sigma_0 = 0, \tag{2.5}$$

where

$$\begin{aligned}
&\sigma_5 = 4A_2(4\gamma - A_2^2), \qquad \sigma_4 = \frac{8}{z}[(\gamma - A_2^2)(k+2) + 3\alpha A_2 - 2zA_2A_2'],\\
&\sigma_3 = \frac{8}{z^2}[A_2 u - (k+1)A_2 + \alpha(k+2) - zA_2'(k+3) - z^2 A_2''],\\
&\sigma_2 = \frac{8}{z^2}[u' - \beta z A_2], \qquad \sigma_1 = \frac{8\beta}{z^2}(k-2), \qquad \sigma_0 = \frac{8\delta}{z}(k-2).
\end{aligned} \tag{2.6}$$

## III Painlevé V equation

In this section, we will consider the fifth Painlevé equation, PV,

$$v'' = \frac{3v-1}{2v(v-1)}(v')^2 - \frac{1}{z}v' + \frac{\alpha}{z^2}v(v-1)^2 + \frac{\beta(v-1)^2}{z^2 v} + \frac{\gamma}{z}v + \frac{\delta v(v+1)}{v-1}. \tag{3.1}$$

For PV, $\Phi$ reads

$$\Phi = (e_1 + 2A_2)v^3 + (2A_2 + 4A_1 - \frac{4}{z} - e_1 + e_0)v^2 + (6A_0 - e_0 + \frac{4}{z})v - 2A_0. \tag{3.2}$$

Setting $\Phi = 0$, we obtain $A_0 = 0$, $A_1 = -A_2$, $e_0 = \frac{4}{z}$ and $e_1 = -2A_2$. Now $\Psi$ reads

$$\begin{aligned}\Psi = {} & [\tfrac{2\alpha}{z^2} - B_4 - \tfrac{1}{2}A_2^2]v^5 + [3B_4 + \tfrac{3}{2}A_2^2 - \tfrac{6\alpha}{z^2} - A_2' - \tfrac{1}{z}A_2]v^4 \\ & + [B_2 + 2B_3 + 2A_2' + \tfrac{2}{z}A_2 - \tfrac{3}{2}A_2^2 + \tfrac{2}{z^2}(3\alpha + \beta) + \tfrac{2\gamma}{z} + 2\delta]v^3 \\ & + [2B_1 + B_2 - A_2' - \tfrac{2}{z}A_2 + \tfrac{1}{2}A_2^2 - \tfrac{2}{z^2}(\alpha + 3\beta) - \tfrac{2\gamma}{z} + 2\delta]v^2 \\ & + 3[B_0 + \tfrac{2\beta}{z^2}]v - [B_0 + \tfrac{2\beta}{z^2}].\end{aligned} \tag{3.3}$$

If we require $\Psi = 0$ identically, we obtain

$$\begin{aligned} & A_2 = \frac{2k}{z}, \qquad B_4 = \frac{2\alpha}{z^2} - \frac{1}{2}A_2^2, \qquad B_0 = -\frac{2\beta}{z^2}, \\ & B_3 = -\frac{1}{2}[B_2 - \frac{3}{2}A_2^2 + \frac{(6\alpha + 2\beta)}{z^2} + \frac{2\gamma}{z} + 2\delta], \\ & B_1 = -\frac{1}{2}[B_2 + \frac{1}{2}A_2^2 - \frac{(2\alpha + 6\beta)}{z^2} - \frac{2\gamma}{z} + 2\delta], \end{aligned} \tag{3.4}$$

where $k$ is an arbitrary constant. It is clear that $d_2 \neq 0$ and hence with out loss of generality we take $B_2 = \dfrac{2u}{z^2}$. Now the equation $G(v) = E(v)F(v)$ is a fourth degree polynomial in $v$,

$$\sigma_4 v^4 + \sigma_3 v^3 + \sigma_2 v^2 + \sigma_1 v + \sigma_0 = 0, \tag{3.5}$$

where

$$\begin{aligned} & \sigma_4 = 2k(2\alpha - k^2), \qquad \sigma_3 = -k[u + \delta z^2 + \gamma z + \beta + 11\alpha - 5k^2], \\ & \sigma_2 = -[zu' - 3ku - (3k-2)\delta z^2 - (3k-1)\gamma z - 3k(\beta + 3\alpha - k^2)], \\ & \sigma_1 = 2zu' - 3ku + k(\delta z^2 - \gamma z - 3\beta - \alpha - k^2), \\ & \sigma_0 = -z[zu' - ku - (k-2)\delta z^2 + (k-1)\gamma z - k(\beta - \alpha + k^2)]. \end{aligned} \tag{3.6}$$

We should consider the two cases $\sigma_4 = 0$ and $\sigma_4 \neq 0$.

**Case(1)** $\sigma_4 = 0$**:**

If $k \neq 0$, then we have $f_4 = 0$, $f_3 = -[4u + \frac{1}{z^2}(12\alpha + 4\beta + k^2) + \frac{4\gamma}{z} + 4\delta] \neq 0$ and hence $k = 0$ (see the conditions (1.10)) which is a contradiction. Therefore, we must have $k = 0$. With this choice, we have $\sigma_3 = 0$ and (3.5) reduces to the following quadratic equation for $v$

$$(u' + 2\delta z + \gamma)v^2 - 2u'v + u' + 2\delta z - \gamma = 0. \tag{3.7}$$

The transformation (1.5) reads

$$z^2(v')^2 = 2\alpha v^4 - (u + \delta z^2 + \gamma z + 3\alpha + \beta)v^3 + 2uv^2 - (u + \delta z^2 - \gamma z - \alpha - 3\beta)v - 2\beta. \tag{3.8}$$

Assume that $\delta \neq 0$ and let $u = -\frac{1}{2}\delta[y + x - (r-1)^2 x \ln x]$, $z = \sqrt{x}$, where $r = \dfrac{\gamma}{2\delta\sqrt{x}} + 1$. Then $y(x)$ satisfies the following second-order fourth-degree equation

$$[4x^3\ddot{y}^2 + 2\delta x\dot{y}^3 - Q_1(y)\dot{y}^2 - Q_2(y)\dot{y}]^2 = \dot{y}[4x^3(r-1)\ddot{y} + Q_3(y)\dot{y}]^2, \tag{3.9}$$

where

$$\begin{aligned} Q_1(y) &= \tfrac{1}{2}\delta[y - x\ln x(r-1)^2 + x(6r^2 - 17r + 12)] + 3(\alpha - \beta),\\ Q_2(y) &= \tfrac{1}{8}\big\{\delta r[(r-2)y - x\ln x(r-2)(r-1)^2 + x(r-1)(3r^2 - 9r + 8)]\\ &\qquad - 8x(r-1)^2 + 2\alpha(3r^2 - 10r + 8) - 2\beta r(3r-2)\big\},\\ Q_3(y) &= \tfrac{1}{2}\delta(r-1)[y - (r-1)^2 x\ln x + x(4r^2 - 8r + 3)] + \alpha(3r-5) - \beta(3r-1). \end{aligned} \tag{3.10}$$

If $\delta = 0$, then equation (3.7), after dividing it by $v - 1$, gives

$$(u' + \gamma)v - (u' - \gamma) = 0. \tag{3.11}$$

Moreover, equation (3.8) becomes

$$z^2(v')^2 = 2\alpha v^4 - (u + \gamma z + 3\alpha + \beta)v^3 + 2uv^2 - (u - \gamma z - \alpha - 3\beta)v - 2\beta. \tag{3.12}$$

If $\gamma = 0$, then one get the trivial solution $v(z;\alpha,\beta,0,0) = 1$ of PV. If $\gamma \neq 0$, then solving equation (3.11) for $v$ and substituting in equation (3.12), we obtain the following second-order second-degree equation for $y = -\frac{1}{4}(u - 3\beta + 3\alpha)$

$$z^2(y'')^2 = -4(y')^2(zy' - y) + \lambda_1(zy' - y)^1 + \lambda_2(zy' - y) + \lambda_3 y' + \lambda_4, \tag{3.13}$$

where

$$\lambda_1 = 0, \quad \lambda_2 = \frac{1}{4}\gamma^2, \quad \lambda_3 = \gamma(\alpha + \beta), \quad \lambda_4 = \frac{1}{4}\gamma^2(\alpha - \beta). \tag{3.14}$$

The equation (3.13) was first obtained in [7] and labeled as SD-I.b and solved in terms of the third Painlevé equation, PIII. Equation (3.13) and its solution in terms of PIII were rederived in [5].

**Case(2)** $\sigma_4 \neq 0$**:**

In this case, we write (3.5) as

$$(v^2 + gv + h)(Av^2 + Bv + C) = 0, \tag{3.15}$$

where $g$ and $h$ are functions of $z$. To achieve this factorization, we should take $k = h = 1$ and $g = -2$. The equation $Av^2 + Bv + C = 0$ reads

$$2\mu v^2 - [u + \delta z^2 + \gamma z + \beta + 3\alpha - 1]v - [zu' - u + \delta z^2 - \beta + \alpha - 1] = 0, \tag{3.16}$$

where $\mu = 2\alpha - 1$ and the transformation (1.6) becomes

$$[zv'-v(v-1)]^2 = \mu v^4-(u+\delta z^2+\gamma z+3\alpha+\beta-1)v^3+(2u+1)v^2-(u+\delta z^2-\gamma z-\alpha-3\beta+1)v-2\beta. \quad (3.17)$$

Assume that $\mu \neq 0$ and let $y = \dfrac{1}{4\mu}(u + \delta z^2 + \gamma z + \beta + 3\alpha - 1)$ and $x = \frac{1}{2}\ln z$. Then $y(x)$ satisfies the following second-order fourth-degree equation

$$\begin{aligned}&\Big\{[\ddot{y} - 2(3y-1)\dot{y} - 4(2y-1)(y-1)^2]^2\\ &\qquad - 4[\dot{y} + (y-1)^2]\Big((4\mu-1)[\dot{y} + (y-1)^2]^2 - 2Q_1(y)[\dot{y} + (y-1)^2] - Q_2(y)\Big)\Big\}^2 =\\ &16[\dot{y} + (y-1)^2]\Big\{[\dot{y} + 2(y-1)(2y-1)][\ddot{y} - 2(3y-1)\dot{y} - 4(2y-1)(y-1)^2]\\ &\qquad - 4[\dot{y} + (y-1)^2]Q_3(y)\Big\}^2\end{aligned} \quad (3.18)$$

where

$$\begin{aligned}p_1 &= 2(\gamma z + 2\beta + \mu),\\ p_2 &= 2\beta + 3\mu + 2\gamma z + 2\delta z^2,\\ Q_1(y) &= 3(4\mu+1)y^2 - 4(4\mu+1)y + 2p_2 + 1,\\ Q_2(y) &= 3(4\mu+3)y^4 - 8(4\mu+3)y^3 + 2(2p_2 + 8\mu + 11)y^2\\ &\quad - 4(p_1+2)y + (8\beta+1),\\ Q_3(y) &= 8\mu y^3 - 16\mu y^2 + 2(p_2 + 2\mu)y - p_1.\end{aligned} \quad (3.19)$$

When $\mu = 0$, then equation (3.16) and (3.17) become

$$(u + \delta z^2 + \gamma z + \beta + \frac{1}{2})v + (zu' - u + \delta z^2 - \beta - \frac{1}{2}) = 0 \quad (3.20)$$

and

$$[zv' - v(v-1)]^2 = -(u + \delta z^2 + \gamma z + \beta + \frac{1}{2})v^3 + (2u+1)v^2 - (u + \delta z^2 - \gamma z - 3\beta + \frac{1}{2})v - 2\beta \quad (3.21)$$

respectively. Solving equation (3.20) for $v$ and substituting in (3.21) we get the second-order second-degree equation (3.13) for $y = -\frac{1}{8}(2u + 2\delta z^2 + 6\beta + 1)$ with

$$\lambda_1 = -2\delta, \quad \lambda_2 = \frac{1}{4}(\gamma^2 + 8\beta\delta), \quad \lambda_2 = \beta\gamma, \quad \lambda_4 = -\frac{1}{4}\beta(\gamma^2 + 2\beta\delta). \quad (3.22)$$

The transformation (3.20) breaks down if and only if $u = -(\delta z^2 + \gamma z + \beta + \frac{1}{2})$. In this case, the transformation (3.21) reduce to the following first-order second-degree equation for $v$

$$[zv' - v(v-1)]^2 = -2(\delta z^2 + \gamma z + \beta)v^2 + 2(\gamma z + 2\beta)v - 2\beta. \quad (3.23)$$

Therefore, if $\alpha = \frac{1}{2}$, then PV has a one-parameter family of solutions characterized by equation (3.23).

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