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*Research Article*

# Optimal Portfolios with End-of-Period Target

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We study the estimation of optimal portfolios for a Reserve Fund with an end-of-period target and when the returns of the assets that constitute the Reserve Fund portfolio follow two specifications. In the first one, assets are split into short memory (bonds) and long memory (equity), and the optimality of the portfolio is based on maximizing the Sharpe ratio. In the second, returns follow a conditional heteroskedasticity autoregressive nonlinear model, and we study when the distribution of the innovation vector is heavy-tailed stable. For this specification, we consider appropriate estimation methods, which include bootstrap and empirical likelihood.

## 1. Introduction

The Government Pension Investment Fund (GPIF) of Japan was established in April 1st 2006 as an independent administrative institution with the mission of managing and investing the Reserve Fund of the employees' pension insurance and the national pension (http://www.gpif.go.jp/for more information) [1]. It is the world's largest pension fund ($1.4 trillions in assets under management as of December 2009), and it has a mission of managing and investing the Reserve Funds in *safe and efficient investment with a long-term perspective*. Business management targets to be achieved by GPIF are set by the Minister of Health, Labour, and Welfare based on the law on the general rules of independent administrative agencies. In the actuarial science, "required Reserve Fund" for pension insurance has been investigated for a long time. The traditional approach focuses on the expected value of future obligations and interest rate. Then, the investment strategy is determined for exceeding the expected value of interest rate. Recently, solvency for the insurer is defined in terms of random values of future obligations (e.g., Olivieri and Pitacco [2]). In this paper, we assume that the Reserve Fund is defined in terms of the random interest rate and the expected future obligations. Then, we propose optimal portfolios by optimizing the randomized Reserve Fund.

The GPIF invests in a portfolio of domestic and international stocks and bonds. In this paper, we consider the optimal portfolio problem of the Reserve Fund under two econometric specifications for the asset's returns.

First, we select the optimal portfolio weights based on the maximization of the Sharpe ratio under three different functional forms for the portfolio mean and variance, two of them depending on the Reserve Fund at the end-of-period target (about 100 years). Following the asset structure of the GPIF, we split the assets into cash and domestic and foreign bonds on one hand and domestic and foreign equity on the other. The first type of assets are assumed to be short memory, while the second type are long memory. To obtain the optimal portfolio weights, we rely on bootstrap. For the short memory returns, we use wild bootstrap (WB). Early work focuses on providing first- and second-order theoretical justification for the wild bootstrap in the classical linear regression model (see, e.g., [3]). Gonçalves and Kilian [4] show that WB is applicable for the linear regression model with conditional heteroscedastic such as stationary ARCH, GARCH, and stochastic volatility effects. For the long memory returns, we apply sieve bootstrap (SB). Bühlmann [5] establishes consistency of the autoregressive sieve bootstrap. Assuming that the long memory process can be written as AR($\infty$) and MA($\infty$) processes, we estimate the long memory parameter by means of the Whittle's approximate likelihood [6]. Given this estimator, the residuals are computed and resampled for the construction of the bootstrap samples, from which the optimal portfolio estimated weights are obtained. We study the usefulness of these procedures with an application to the GPIF assets.

Second, we consider the case when the returns are time dependent and follow a heavy-tailed. It is known that one of the stylized facts of financial returns are heavy tails. It is, therefore, reasonable to use the stable distribution, instead of the Gaussian, since it allows for skewness and fat tails. We couple this distribution with the conditional heteroskedasticity autoregressive nonlinear (CHARN) model that nests many well-known time series models, such as ARMA and ARCH. We estimate the parameters and the optimal portfolio by means of empirical likelihood.

The paper is organized as follows. Section 2 sets the Reserve Fund portfolio problem. Section 3 focuses on the first part, that is, estimation in terms of the Sharpe ratio and discusses the bootstrap procedure. Section 4 covers the CHARN model under stable innovations and the estimation by means of empirical likelihood. Section 5 concludes.

## 2. Reserve Funds Portfolio with End-of-Period Target

Let $S_{i,t}$ be the price of the $i$th asset at time $t$ ($i = 1, \ldots, k$), and let $X_{i,t}$ be its log-return. Time runs from 0 to $T$. The paper, we consider that today is $T_0$ and $T$ is the end-of-period target. Hence the past and present observations run for $t = 0, \ldots, T_0$, and the future until the end-of-period target for $t = T_0 + 1, \ldots, T$. The price $S_{i,t}$ can be written as

$$S_{i,t} = S_{i,t-1} \exp\{X_{i,t}\} = S_{i,0} \exp\left(\sum_{s=1}^{t} X_{i,s}\right), \tag{2.1}$$

where $S_{i,0}$ is the initial price. Let $F_{i,t}$ denote the Reserve Fund on asset $i$ at time $t$ and be defined by

$$F_{i,t} = F_{i,t-1} \exp\{X_{i,t}\} - c_{i,t}, \tag{2.2}$$

where $c_{i,t}$ denotes the maintenance cost at time $t$. By recursion, $F_{i,t}$ can be written as

$$\begin{aligned} F_{i,t} &= F_{i,t-1} \exp\{X_{i,t}\} - c_{i,t} \\ &= F_{i,t-2} \exp\left(\sum_{s=t-1}^{t} X_{i,s}\right) - \sum_{s=t-1}^{t} c_{i,s} \exp\left(\sum_{s'=s+1}^{t} X_{i,s'}\right) \\ &= F_{i,0} \exp\left(\sum_{s=1}^{t} X_{i,s}\right) - \sum_{s=1}^{t} c_{i,s} \exp\left(\sum_{s'=s+1}^{t} X_{i,s'}\right), \end{aligned} \tag{2.3}$$

where $F_{i,0} = S_{i,0}$.

We gather the Reserve Funds in the vector $\mathbf{F}_t = (F_{1,t}, \ldots, F_{k,t})$. Let $F_t(\boldsymbol{\alpha}) = \boldsymbol{\alpha}'\mathbf{F}_t$ be a portfolio form by the $k$ Reserve Funds, which depend on the vector of weights $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$. The portfolio Reserve Fund can be expressed as a function of all past returns

$$\begin{aligned} F_t(\boldsymbol{\alpha}) &\equiv \sum_{i=1}^{k} \alpha_i F_{i,t} \\ &= \sum_{i=1}^{k} \alpha_i \left( F_{i,0} \exp\left(\sum_{s=1}^{t} X_{i,s}\right) - \sum_{s=1}^{t} c_{i,s} \exp\left(\sum_{s'=s+1}^{t} X_{i,s'}\right)\right). \end{aligned} \tag{2.4}$$

We are interested in maximizing $F_t(\boldsymbol{\alpha})$ at the end-of-period target $F_T(\boldsymbol{\alpha})$

$$F_T(\boldsymbol{\alpha}) = \sum_{i=1}^{k} \alpha_i \left( F_{i,T_0} \exp\left(\sum_{s=T_0+1}^{T} X_{i,s}\right) - \sum_{s=T_0+1}^{T} c_{i,s} \exp\left(\sum_{s'=s+1}^{T} X_{i,s'}\right)\right). \tag{2.5}$$

It depends on the future returns, the maintenance cost, and the portfolio weights. While the first two are assumed to be constant from $T_0$ to $T$ (the constant return can be seen as the average return over the $T - T_0$ periods), we focus on the optimality of the weights that we denote by $\boldsymbol{\alpha}^{\text{opt}}$.

## 3. Sharpe-Ratio-Based Optimal Portfolios

In the first specification, the estimation of the optimal portfolio weights is based on the maximization of the Sharpe ratio:

$$\boldsymbol{\alpha}^{\text{opt}} = \arg\max_{\boldsymbol{\alpha}} \frac{\mu(\boldsymbol{\alpha})}{\sigma(\boldsymbol{\alpha})}, \tag{3.1}$$

under different functional forms of the expectation $\mu(\boldsymbol{\alpha})$ and the risk $\sigma(\boldsymbol{\alpha})$ of the portfolio. We propose three functional forms, two of them depending on the Reserve Fund. The first one is the traditional based on the returns:

$$\mu(\boldsymbol{\alpha}) = \boldsymbol{\alpha}' E(\mathbf{X}_T), \qquad \sigma(\boldsymbol{\alpha}) = \sqrt{\boldsymbol{\alpha}' V(\mathbf{X}_T)\boldsymbol{\alpha}}, \tag{3.2}$$

where $E(\mathbf{X}_T)$ and $V(\mathbf{X}_T)$ are the expectation and the covariance matrix of the returns at the end-of-period target. The second form for the portfolio expectation and risk is based on the vector of Reserve Funds:

$$\mu(\boldsymbol{\alpha}) = \boldsymbol{\alpha}' E(\mathbf{F}_T), \qquad \sigma(\boldsymbol{\alpha}) = \sqrt{\boldsymbol{\alpha}' V(\mathbf{F}_T)\boldsymbol{\alpha}}, \tag{3.3}$$

where $E(\mathbf{F}_T)$ and $V(\mathbf{F}_T)$ indicate the mean and covariance of the Reserve Funds at time $T$. Last, we consider the case where the portfolio risk depends on the lower partial moments of the Reserve Funds at the end-of-period target:

$$\mu(\boldsymbol{\alpha}) = \boldsymbol{\alpha}' E(\mathbf{F}_T), \qquad \sigma(\boldsymbol{\alpha}) = E\left\{\left(\widetilde{F} - F_T(\boldsymbol{\alpha})\right)\mathbb{I}\left(F_T(\boldsymbol{\alpha}) < \widetilde{F}\right)\right\}, \tag{3.4}$$

where $\widetilde{F}$ is a given value.

Standard portfolio management rules are based on a mean-variance approach, for which risk is measured by the standard deviation of the future portfolio value. However, the variance often does not provide a correct assessment of risk under dependency and non-Gaussianity. To overcome this problem, various optimization models have been proposed such as mean-semivariance model, mean-absolute deviation model, mean-variance-skewness model, mean-(C)VaR model, and mean-lower partial moment model. The mean-lower partial moment model is an appropriate model for reducing the influence of heavy tails.

The $k$ returns are split into $p$- and $q$-dimensional vectors $\{\mathbf{X}_t^S; t \in \mathbb{Z}\}$ and $\{\mathbf{X}_t^L; t \in \mathbb{Z}\}$, where $S$ and $L$ stand for short and long memory, respectively. The short memory returns correspond to cash and domestic and foreign bonds, which we generically denote by bonds. The long memory returns correspond to domestic and foreign equity, which we denote as equity.

Cash and bonds follow the nonlinear model

$$\mathbf{X}_t^S = \boldsymbol{\mu}^S + \mathbf{H}\left(\mathbf{X}_{t-1}^S, \ldots, \mathbf{X}_{t-m}^S\right)\boldsymbol{\epsilon}_t^S, \tag{3.5}$$

where $\boldsymbol{\mu}^S$ is a vector of constants, $\mathbf{H} : \mathbb{R}^{mp} \to \mathbb{R}^p \times \mathbb{R}^p$ is a positive definite matrix-valued measurable function, and $\boldsymbol{\epsilon}_t^S = (\epsilon_{1,t}^S, \ldots, \epsilon_{p,t}^S)$ are i.i.d. random vectors with mean $\mathbf{0}$ and covariance matrix $\boldsymbol{\Sigma}^S$. By contrast, equity returns follow a long memory nonlinear model

$$\mathbf{X}_t^L = \sum_{\nu=0}^{\infty} \phi_\nu \boldsymbol{\epsilon}_{t-\nu}^L, \qquad \boldsymbol{\epsilon}_t^L = \sum_{\nu=0}^{\infty} \psi_\nu \mathbf{X}_{t-\nu}^L, \tag{3.6}$$

where

$$\phi_\nu = \frac{\Gamma(\nu+d)}{\Gamma(\nu+1)\Gamma(d)}, \qquad \psi_\nu = \frac{\Gamma(\nu-d)}{\Gamma(\nu+1)\Gamma(-d)} \tag{3.7}$$

with $-1/2 < d < 1/2$, and $\boldsymbol{\epsilon}_t^L = (\epsilon_{1,t}^L, \dots, \epsilon_{p,t}^L)$ are i.i.d. random vectors with mean $\mathbf{0}$ and covariance matrix $\boldsymbol{\Sigma}^L$.

We estimate the optimal portfolio weights by means of bootstrap. Let the superindexes $(S,b)$ and $(L,b)$ denote the bootstrapped samples for the bonds and equity, respectively, and $B$ the total number of bootstrapped samples. In the sequel, we show the bootstrap procedure for both types of assets.

*Bootstrap Procedure for* $\mathbf{X}_t^{(S,b)}$

*Step 1.* Generate the i.i.d. sequences $\{\boldsymbol{\epsilon}_t^{(S,b)}\}$ for $t = T_0+1, \dots, T$ and $b = 1, \dots, B$ from $N(\mathbf{0}, \mathbf{I}_p)$.

*Step 2.* Let $\mathbf{Y}_t^S = \mathbf{X}_t^S - \widehat{\boldsymbol{\mu}}^S$, where $\widehat{\boldsymbol{\mu}}^S = (1/T_0)\sum_{s=1}^{T_0} \mathbf{X}_s^S$. Generate the i.i.d. sequences $\{\mathbf{Y}_t^{(S,b)}\}$ for $t = T_0+1, \dots, T$ and $b = 1, \dots, B$ from the empirical distribution of $\{\mathbf{Y}_t^S\}$.

*Step 3.* Compute $\{\mathbf{X}_t^{(S,b)}\}$ for $t = T_0+1, \dots, T$ and $b = 1, \dots, B$ as

$$\mathbf{X}_t^{(S,b)} = \widehat{\boldsymbol{\mu}}^S + \mathbf{Y}_t^{(S,b)} \odot \boldsymbol{\epsilon}_t^{(S,b)}, \tag{3.8}$$

where $\odot$ denotes the cellwise product.

*Bootstrap Procedure for* $\mathbf{X}_t^{(L,b)}$

*Step 1.* Estimate $\widehat{d}$ from the observed returns by means of Whittle's approximate likelihood:

$$\widehat{d} = \arg \min_{d \in (0,1/2)} L(d, \Sigma), \tag{3.9}$$

where

$$\begin{aligned}
L(d,\Sigma) &= \frac{2}{T_0} \sum_{j=1}^{(T_0-1)/2} \left\{ \log \det f(\lambda_{j,T_0}, d, \Sigma) + tr\Big( f(\lambda_{j,T_0}, d, \Sigma)^{-1} I(\lambda_{j,T_0}) \Big) \right\}, \\
f(\lambda, d, \Sigma) &= \frac{\left|1 - \exp(i\lambda)\right|^{-2d}}{2\pi} \Sigma, \\
I(\lambda) &= \frac{1}{\sqrt{2\pi T_0}} \left| \sum_{t=1}^{T_0} \mathbf{X}_t^L e^{it\lambda} \right|^2, \\
\lambda_{j,T_0} &= \frac{2\pi j}{T_0}.
\end{aligned} \tag{3.10}$$

*Step 2.* Compute $\{\widehat{\boldsymbol{\epsilon}}_t^L\}$ for $t = 1, \dots, T_0$,

$$\widehat{\boldsymbol{\epsilon}}_t^L = \sum_{k=0}^{t-1} \pi_k \mathbf{X}_{t-k}^L, \quad \text{where } \pi_k = \begin{cases} \dfrac{\Gamma\left(k-\widehat{d}\right)}{\Gamma(k+1)\Gamma\left(-\widehat{d}\right)}, & k \le 100, \\ \dfrac{k^{-\widehat{d}-1}}{\Gamma\left(-\widehat{d}\right)}, & k > 100. \end{cases} \tag{3.11}$$

*Step 3.* Generate $\{\boldsymbol{\epsilon}_t^{(L,b)}\}$ for $t = T_0 + 1, \dots, T$ and $b = 1, \dots, B$ from the empirical distribution of $\{\widehat{\boldsymbol{\epsilon}}_t^L\}$.

*Step 4.* Generate $\{\mathbf{X}_t^{(L,b)}\}$ for $t = T_0 + 1, \dots, T$ and $b = 1, \dots, B$ as

$$\mathbf{X}_t^{(L,b)} = \sum_{k=0}^{t-T_0-1} \tau_k \epsilon_{t-k}^{(L,b)} + \sum_{k=t-T_0}^{t-1} \tau_k \widehat{\epsilon}_{t-k}. \tag{3.12}$$

We gather $\mathbf{X}_t^{(S,b)}$ and $\mathbf{X}_t^{*(L,b)}$ into $\mathbf{X}_t^{(b)} = (\mathbf{X}_t^{(S,b)}, \mathbf{X}_t^{(L,b)}) = (X_{1,t}^{*(b)}, \dots, X_{p+q,t}^{*(b)})$ for $t = T_0 + 1, \dots, T$ and $b = 1, \dots, B$. The bootstrapped Reserve Funds $\mathbf{F}_T^{(b)} = (F_{1,T}^{(b)}, \dots, F_{p+q,T}^{(b)})$

$$F_{i,T}^{(b)} = F_{T_0} \exp\left(\sum_{s=T_0+1}^{T} X_{i,s}^{(b)}\right) - \sum_{s=T_0+1}^{T} c_{i,s} \exp\left(\sum_{s'=s+1}^{T} X_{i,s'}^{(b)}\right). \tag{3.13}$$

And the bootstrapped Reserve Fund portfolio is

$$F_T^{(b)}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}' \mathbf{F}_T^{(b)} = \sum_{i=1}^{p+q} \alpha_i F_{i,T}^{(b)}. \tag{3.14}$$

Finally, the estimated portfolio weights that give the optimal portfolio are

$$\widehat{\boldsymbol{\alpha}}^{\text{opt}} = \arg\max_{\boldsymbol{\alpha}} \frac{\mu^{(b)}(\boldsymbol{\alpha})}{\sigma^{(b)}(\boldsymbol{\alpha})}, \tag{3.15}$$

where $\mu^{(b)}(\boldsymbol{\alpha})$ and $\sigma^{(b)}(\boldsymbol{\alpha})$ may take any of the three forms introduced earlier but be evaluated in the bootstrapped returns or Reserve Funds.

### *3.1. An Illustration*

We consider monthly log-returns from January 31 1971 to October 31 2009 (466 observations) of the five types of assets considered earlier: domestic bond (DB), domestic equity (DE), foreign bond (FB), foreign equity (FE), and cash (cash). Cash and bonds are gathered in the short-memory panel $\mathbf{X}_t^S = (X_t^{(\mathrm{DB})}, X_t^{(\mathrm{FB})}, X_t^{(\mathrm{cash})})$ and follow (3.5). Equities are gathered into

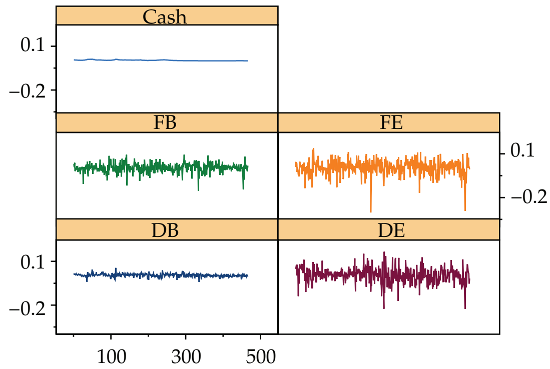


**Figure** 1

**Table 1:** Estimated optimal portfolio weights (Section 3).

| | DB | DE | FB | FE | Cash |
|---|---|---|---|---|---|
| Returns | 0.95 | 0.00 | 0.00 | 0.00 | 0.05 |
| Reserve fund | 0.75 | 0.00 | 0.20 | 0.00 | 0.05 |
| Lowe partial | 0.85 | 0.10 | 0.00 | 0.00 | 0.05 |

the long-memory panel $\mathbf{X}_t^L = (X_t^{(\mathrm{DE})}, X_t^{(\mathrm{FE})})$ and follow (3.6). Figure 1 shows the five assets. Cash is virtually constant, and equities are significantly more volatile than bonds and with averages that are slightly higher than those of bonds.

We estimate the optimal portfolio weights, $\widehat{\boldsymbol{\alpha}}^{\mathrm{opt1}}$, $\widehat{\boldsymbol{\alpha}}^{\mathrm{opt2}}$, and $\widehat{\boldsymbol{\alpha}}^{\mathrm{opt3}}$, corresponding to the three forms for the expectation and risk of the Sharpe ratio, and we compute the trajectory of the optimal Reserve Fund for $t = T_0 + 1, \ldots, T$. Because of liquidity reasons, the portfolio weight for cash is kept constant to 5%. The target period is fixed to 100 years, and the maintenance cost is based on the 2004 Pension Reform.

Table 1 shows the estimated optimal portfolio weights for the three different choices of the portfolio expectation and risk. The weight of domestic bonds is very high and clearly dominates over the other assets. Domestic bonds are low risk and medium return, which is in contrast with equity that shows higher return but also higher risk, and with foreign bonds that show low return and risk. Therefore, in a sense, domestic bonds are a compromise between the characteristic of the four equities and bonds.

Figure 2 shows the trajectory of the future Reserve Fund for different values of the yearly return (assumed to be constant from $T_0 + 1$ to $T$) ranging from 2.7% to 3.7%. Since the investment term is extremely long, 100 years, the Reserve Fund is quite sensitive to the choice of the yearly return. In the 2004 Pension Reform, authorities assumed a yearly return of the portfolio of 3.2%, which corresponds to the middle thick line of the figure.

## 4. Optimal Portfolio with Time-Dependent Returns and Heavy Tails

In this section, we consider the second scenario where returns follow a dependent model with stable innovations. The theory of portfolio choice is mostly based on the assumption that investors maximize their expected utility. The most well-known utility is the Markowitz's mean-variance function that is optimal under Gaussianity. However, it is widely

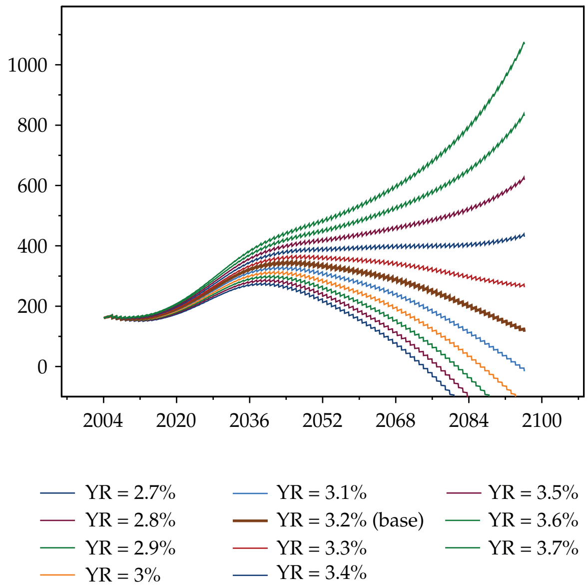


**Figure** 2

acknowledged that financial returns show fat tails and, frequently, skewness. Moreover, the variance may not always be the best risk measure. Since the purpose of GPIF is to avoid making a big loss at a certain point in future, risk measures that summarize the probability that the Reserve Fund is below the prescribed level at a certain future point, such as value at risk (VaR), are more appropriate [7]. In addition, the traditional mean-variance approach considers that returns are i.i.d., which is not realistic as past information may help to explain today's distribution of returns.

We need a specification that allows for heavy tails and skewness and time dependencies. This calls for a general model with location and scale that are a function of past observations and with innovations that are stable distributed. The location-scale model for the returns is the conditional heteroscedastic autoregressive nonlinear (CHARN), which is fairly general and it nests important models such as ARMA and ARCH.

Estimation of the parameters in a stable framework is not straightforward since the density does not have a closed form (Maximum likelihood is feasible for the i.i.d. univariate case thanks to the STABLE packages developed by John Nolan—see Nolan [8] and the website http://academic2.american.edu/~jpnolan/stable/stable.html. For more complicated cases, including dynamics, maximum likelihood is a quite complex task.). We rely on empirical likelihood, which is one of the nonparametric methods, as it has been already studied in this context [9]. Once the parameters are estimated, we simulate samples of the returns, which are used to compute the Reserve Fund at the end-of-period target, and estimate the optimal portfolio weights by means of minimizing the empirical VaR of the Reserve Fund at time $T$.

Suppose that the vector of returns $\mathbf{X}_t \in \mathbb{R}^k$ follows the following CHARN model:

$$\mathbf{X}_t = \mathbf{F}_{\boldsymbol{\mu}}(\mathbf{X}_{t-1}, \ldots, \mathbf{X}_{t-p}) + \mathbf{H}_{\boldsymbol{\sigma}}(\mathbf{X}_{t-1}, \ldots, \mathbf{X}_{t-p})\boldsymbol{\varepsilon}_t, \tag{4.1}$$

where $\mathbf{F}_{\boldsymbol{\mu}} : \mathbb{R}^{kp} \to \mathbb{R}^k$ is a vector-valued measurable function with a parameter $\boldsymbol{\mu} \in \mathbb{R}^{p_1}$ and $\mathbf{H}_{\boldsymbol{\sigma}} : \mathbb{R}^{kq} \to \mathbb{R}^k \times \mathbb{R}^k$ is a positive definite matrix-valued measurable function with a parameter $\boldsymbol{\sigma} \in \mathbb{R}^{p_2}$. Each element of the vector of innovations $\boldsymbol{\varepsilon}_t \in \mathbb{R}^k$ is standardized stable distributed: $\varepsilon_{i,t} \sim S(\alpha_i, \beta_i, 1, 0)$ and $\varepsilon_{i,t}$'s are independent with respect to both $i$ and $t$. We set $\boldsymbol{\theta} = (\boldsymbol{\mu}, \boldsymbol{\sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta})$, where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_k)$.

The stable distribution is often represented by its characteristic function:

$$\phi(\nu) = E\left[\exp(i\nu\varepsilon_{i,t})\right] = \exp\left(-\gamma^\alpha|\nu|^\alpha\left(1 + i\beta\ \operatorname{sgn}(\nu)\tan\frac{\pi\alpha}{2}\left(\left|\gamma\nu\right|^{1-\alpha} - 1\right)\right) + i\nu\delta\right), \tag{4.2}$$

where $\delta \in \mathbb{R}$ is a location parameter, $\gamma > 0$ is a scale parameter, $\beta \in [-1, 1]$ is a skewness parameter, and $\alpha \in (0, 2]$ is a characteristic exponent that captures the tail thickness of the distribution: the smaller the $\alpha$ the heavier the tail. The distributions with $\alpha = 2$ correspond to the Gaussian. The existence of moments is given by $\alpha$: moments of order higher than $\alpha$ do not exist, with the case of $\alpha = 2$ being an exception, for which all moments exist.

The lack of important moments may, in principle, render estimation by the method of moments difficult. However, instead of matching moments, it is fairly simple to match the theoretical and empirical characteristic function evaluated at a grid of frequencies [9]. Let

$$\boldsymbol{\epsilon}_t = \mathbf{H}_{\sigma}^{-1}\left(\mathbf{X}_t - \mathbf{F}_{\mu}\right) \tag{4.3}$$

be the residual of the CHARN model. If the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are the true ones, the residuals $\epsilon_{i,t}$ should be independently and identically distributed to $S(\alpha_i, \beta_i, 1, 0)$. So the aim is to find the estimated parameters such that the residuals are i.i.d. and stable distributed, meaning that their probability law is expressed by the above characteristic function. Or, in other words, estimate the parameters by matching the empirical and theoretical characteristic functions and minimizing their distance. Let $J$ be the number of frequencies at which we evaluate the characteristic function: $\nu_1, \ldots, \nu_J$. That makes, in principle, a system of $J$ matching equations. But since the characteristic function can be split into the real and imaginary parts, $\phi(\nu) = E[\cos(\nu\varepsilon_{i,t})] + iE[\sin(\nu\varepsilon_{i,t})]$, we double the dimension of the system by matching these parts. Let $\mathfrak{Re}(\phi(\nu))$ and $\mathfrak{Im}(\phi(\nu))$ be the real and imaginary parts of the theoretical characteristic function, and $\cos(\nu\epsilon_{i,t})$ and $\sin(\nu\epsilon_{i,t})$ the empirical counterparts. The estimating functions are

$$\boldsymbol{\psi}_{\boldsymbol{\theta}}(\epsilon_{i,t}) = \begin{pmatrix} \cos(\nu_1\epsilon_{i,t}) - \mathfrak{Re}\left(\phi(\nu_1)\right) \\ \vdots \\ \cos\left(\nu_J\epsilon_{i,t}\right) - \mathfrak{Re}\left(\phi(\nu_J)\right) \\ \sin(\nu_1\epsilon_{i,t}) - \mathfrak{Im}\left(\phi(\nu_1)\right) \\ \vdots \\ \sin\left(\nu_J\epsilon_{i,t}\right) - \mathfrak{Im}\left(\phi(\nu_J)\right) \end{pmatrix}, \tag{4.4}$$

for each $i = 1, \ldots, k$, and gather them into the vector

$$\boldsymbol{\psi}_{\boldsymbol{\theta}}(\boldsymbol{\epsilon}_t) = \left(\boldsymbol{\psi}_{\boldsymbol{\theta}}(\epsilon_{1,t}), \ldots, \boldsymbol{\psi}_{\boldsymbol{\theta}}(\epsilon_{k,t})\right). \tag{4.5}$$

The number of frequencies $J$ and the frequencies themselves are chosen arbitrary. Feuerverger and McDunnough [10] show that the asymptotic variance can be made arbitrarily close to the Cramér-Rao lower bound if the number of frequencies is sufficiently large and the grid is sufficiently fine and extended. Similarly, Yu [11, Section 2.1] argues that, from the viewpoint of the minimum asymptotic variance, many and fine frequencies are the appropriate. However, Carrasco and Florens [12] show that too fine frequencies lead to a singular asymptotic variance matrix and we cannot calculate its inverse.

Given the estimating functions (4.5), the natural estimator is constructed by GMM:

$$\widehat{\boldsymbol{\theta}} = \arg\min_{\theta} E\left[\boldsymbol{\psi}_{\boldsymbol{\theta}}(\boldsymbol{\epsilon}_t)'\right]\mathbf{W}E\left[\boldsymbol{\psi}_{\boldsymbol{\theta}}(\boldsymbol{\epsilon}_t)\right], \tag{4.6}$$

where $\mathbf{W}$ is a weighting matrix defining the metric (its optimal choice is typically the inverse of the covariance matrix of $\boldsymbol{\psi}_{\boldsymbol{\theta}}(\boldsymbol{\epsilon}_t)$) and the expectations are replaced by sample moments. GMM estimator can be generalized to the empirical likelihood estimator, which was originally proposed by Owen [13] as nonparametric methods of inference based on a data-driven likelihood ratio function (see also [14], for a review and applications). It produces a better variance estimate in one step, while, in general, the optimal GMM requires a preliminary step and a preliminary estimation of an optimal $\mathbf{W}$ matrix. We define the empirical likelihood ratio function for $\boldsymbol{\theta}$ as

$$\mathcal{R}(\boldsymbol{\theta}) = \max_{\mathbf{p}}\left\{\prod_{t=1}^{T_0} T_0 p_t \mid \sum_{t=1}^{T_0} p_t \boldsymbol{\psi}_{\boldsymbol{\theta}}(\boldsymbol{\epsilon}_t) = \mathbf{0},\ \sum_{t=1}^{T_0} p_t = 1,\ p_t \geq 0\right\}, \tag{4.7}$$

where $\mathbf{p} = (p_1, \ldots, p_{T_0})$ and the maximum empirical likelihood estimator is

$$\widetilde{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} \mathcal{R}(\boldsymbol{\theta}). \tag{4.8}$$

Qin and Lawless [15] show that this estimator is consistent, asymptotically Gaussian, and with covariance matrix $(B'_{\boldsymbol{\theta}_0} A^{-1}_{\boldsymbol{\theta}_0} B_{\boldsymbol{\theta}_0})^{-1}$, where

$$B_{\boldsymbol{\theta}} = E\left[\frac{\partial \boldsymbol{\psi}_{\boldsymbol{\theta}}}{\partial \boldsymbol{\theta}'}\right], \qquad A_{\boldsymbol{\theta}} = E\left[\boldsymbol{\psi}_{\boldsymbol{\theta}}(\boldsymbol{\epsilon}_t)\boldsymbol{\psi}_{\boldsymbol{\theta}}(\boldsymbol{\epsilon}_t)'\right]. \tag{4.9}$$

Once the parameters are estimated, we compute the optimal portfolio weights and the portfolio Reserve Fund at the end-of-period target. Because of a notational conflict, the weights are now denoted by $\mathbf{a} = (a_1, \ldots, a_k)$. And, for simplicity, we assume that there is no maintenance cost, so (2.5) simplifies to

$$F_T(\mathbf{a}) = \sum_{i=1}^{k} a_i F_{i,T_0} \exp\left(\sum_{t=T_0+1}^{T} X_{i,t}\right). \tag{4.10}$$

The procedure to estimate the optimal portfolio weights is as follows.

*Step 1.* For each asset $i = 1, \ldots, k$, we simulate the innovation process

$$\tilde{\varepsilon}_{i,t} \overset{\text{i.i.d.}}{\sim} S\left(\hat{\alpha}_i, \hat{\beta}_i, 1, 0\right), \quad t = T_0 + 1, \ldots, T \tag{4.11}$$

based on the maximum empirical likelihood estimator $(\hat{\alpha}_i, \hat{\beta}_i)$.

*Step 2.* We calculate the predicted log-returns

$$\tilde{\mathbf{X}}_t = \mathbf{F}_{\tilde{\boldsymbol{\mu}}}\left(\tilde{\mathbf{X}}_{t-1}, \ldots, \tilde{\mathbf{X}}_{t-p}\right) + \mathbf{H}_{\tilde{\boldsymbol{\sigma}}}\left(\tilde{\mathbf{X}}_{t-1}, \ldots, \tilde{\mathbf{X}}_{t-p}\right)\tilde{\boldsymbol{\varepsilon}}_t \tag{4.12}$$

for $t = T_0 + 1, \ldots, T$ and based on the estimators $(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\sigma}})$ and the simulated $\tilde{\boldsymbol{\varepsilon}}_t$ obtained in Step 1.

*Step 3.* For a given portfolio weight $\mathbf{a}$, we calculate the predicted values of fund at time $T$, $F_T(\mathbf{a})$, with (4.10).

*Step 4.* We repeat Step 1–Step 3 $M$ times and save $F_T^{(1)}(\mathbf{a}), \ldots, F_T^{(M)}(\mathbf{a})$. Then we calculate the proportion that the predicted values fail below the prescribed level $F$, that is,

$$g(\mathbf{a}) = \frac{1}{M}\sum_{m=1}^{M} \mathbb{I}_{\{F_T^{(m)}(\mathbf{a}) < F\}}. \tag{4.13}$$

*Step 5.* Minimize $g(\mathbf{a})$ with respect to $\mathbf{a}$: $\mathbf{a}^* = \arg\min_{\mathbf{a}} g(\mathbf{a})$.

### *4.1. An Illustration*

In this section, we apply the above procedure to real financial data. We consider the same monthly log-returns data in Section 3.1. domestic bond (DB), domestic equity (DE), foreign bond (FB), and foreign equity (FE) are assumed to follow the following ARCH(1) model:

$$X_t = \sigma_t \varepsilon_t, \qquad \sigma^2 = b X_{t-1}^2, \tag{4.14}$$

respectively. Here $b > 0$ and $\varepsilon_t \overset{\text{i.i.d.}}{\sim} S(\alpha, \beta, 1, 0)$. Cash is virtually constant so we assume the log-return of cash as 0, permanently. Set the present Reserve Fund $F_{T_0} = 1$ and the target period is fixed to half years.

Table 2 shows the estimated optimal portfolio weights for the different prescribed level $F$. The weights of domestic and foreign bonds tend to be high when $F$ is small. Small $F$ implies that we want to avoid the loss. On the contrary, the weights of equities become higher when $F$ is larger. Large $F$ implies that we do not want to miss the chance of big gain. This result seems to be natural because bonds are lower risk (less volatile) than equities.

## 5. Conclusions

In this paper, we study the estimation of optimal portfolios for a Reserve Fund with an end-of-period target in two different settings. In the first setting, one assets are split into short

**Table 2:** Estimated optimal portfolio weights (Section 4).

| $F$ | DB | DE | FB | FE | Cash |
|---|---|---|---|---|---|
| 0.90 | 0.79 | 0.03 | 0.07 | 0.05 | 0.05 |
| 0.95 | 0.49 | 0.03 | 0.38 | 0.05 | 0.05 |
| 1.00 | 0.39 | 0.01 | 0.54 | 0.01 | 0.05 |
| 1.05 | 0.17 | 0.36 | 0.36 | 0.06 | 0.05 |
| 1.10 | 0.37 | 0.27 | 0.26 | 0.05 | 0.05 |

memory (bonds) and long memory (equity), and the optimality of the portfolio is based on maximizing the Sharpe ratio. The simulation result shows that the portfolio weight of domestic bonds is quite high. The reason is that the investment term is extremely long (100 years). Because the investment risk for the Reserve Fund is exponentially amplified year by year, the portfolio selection problem for the Reserve Fund is quite sensitive to the year-based portfolio risk. In the second setting, returns follow a conditional heteroskedasticity autoregressive nonlinear model, and we study when the distribution of the innovation vector is heavy-tailed stable. Simulation studies show that we prefer the bonds when we want to avoid the big loss in the future. The result seems to be natural because the bonds are less volatile than the equities.

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## References

[1] *GPIF Home Page*, http://www.gpif.go.jp/eng/index.html.

[2] A. Olivieri and E. Pitacco, "Solvency requirements for life annuities," in *Proceedings of the Proceedings of the Actuarial Approach for Financial Risks (AFIR '00)*, Colloquium, 2000.

[3] C.-F. J. Wu, "Jackknife, bootstrap and other resampling methods in regression analysis," *The Annals of Statistics*, vol. 14, no. 4, pp. 1261–1350, 1986.

[4] S. Gonçalves and L. Kilian, "Bootstrapping autoregressions with conditional heteroskedasticity of unknown form," *Journal of Econometrics*, vol. 123, no. 1, pp. 89–120, 2004.

[5] P. Bühlmann, "Sieve bootstrap for time series," *Bernoulli*, vol. 3, no. 2, pp. 123–148, 1997.

[6] J. Beran, *Statistics for Long-Memory Processes*, vol. 61 of *Monographs on Statistics and Applied Probability*, Chapman and Hall, New York, NY, USA, 1994.

[7] T. Shiohama, M Hallin, D. Veredas, and M. Taniguchi, "Dynamic portfolio optimization using generalized conditional heteroskedasticity factor models," ECARES WP 2010-30, 2010.

[8] J. P. Nolan, "Numerical calculation of stable densities and distribution functions," *Communications in Statistics—Stochastic Models*, vol. 13, no. 4, pp. 759–774, 1997.

[9] H. Ogata, "Empirical likelihood estimation for a class of stable processes," *Mimeo*, 2009.

[10] A. Feuerverger and P. McDunnough, "On the efficiency of empirical characteristic function procedures," *Journal of the Royal Statistical Society B*, vol. 43, no. 1, pp. 20–27, 1981.

[11] J. Yu, "Empirical characteristic function estimation and its applications," *Econometric Reviews*, vol. 23, no. 2, pp. 93–123, 2004.

[12] M. Carrasco and J. Florens, "Efficient GMM estimation using the empirical characteristic function," IDEI Working Papers 140, Institut d'Economie Industrielle (IDEI), Toulouse, France, 2002.

[13] A. B. Owen, "Empirical likelihood ratio confidence intervals for a single functional," *Biometrika*, vol. 75, no. 2, pp. 237–249, 1988.

[14] A. B. Owen, *Empirical Likelihood*, Chapman-Hall/CRC, Boca Raton, Fla, USA, 2001.

[15] J. Qin and J. Lawless, "Empirical likelihood and general estimating equations," *The Annals of Statistics*, vol. 22, no. 1, pp. 300–325, 1994.