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# SOME FIXED POINT RESULTS IN THE GENERALIZED CONVEX METRIC SPACES

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Abstract. In this study, we introduce a new three step iteration process and show that the iteration process converges to the unique fixed point by two theorems under different conditions of contractive mappings on the generalized G- convex metric spaces. Also, we investigate data dependence result for this iterative process in the generalized G- convex metric spaces.

Keywords: Convex G-metric spaces, G-convergence, Fixed point iteration process

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## 1. Introduction and preliminaries

By several mathematicians have been introduced different generalizations of the usual concept of a metric space . In 1963, Gahler [19], Ha et.al. [20], In 1992, Dhage [18], In 2006, Mustafa along with Sims proposed a new concept of generalized metric space called G-metric space [9]. Fixed point theory in these spaces was studied in [4], Banach contraction mapping being the main tool. Mustafa et al. studied many fixed point results for a self-mapping in G-metric space.[3]-[9] can be cited for reference. Takahashi [1] proposed the notion of convex structure in metric spaces and proved some fixed point results. Inspired by this Thangavelu et.al. [29] proposed the notion of convexity structure in D-metric space. They further extended this notion to get strong convex D-metric space, J-convex D-metric spaces, weak convex D-metric spaces and quasi convex D-Metric spaces. Recently, Modi and Bhatt [30] extend to G-metric space by providing different convex structures to D-metric space analogous to Thangavelu et.al. [29]. Therefore we propose a new iteration process and we prove that this fixed point iteration process converges to fixed point of contractive type mapping in the convex G-metric spaces.

An element $x$ is said to be a fixed point of $T$ if $Tx = x$.

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The iterative approximation of a fixed point for certain classes of mappings is one of the main tools in the fixed point theory. There are many studies conducted on this theory. Some of these [11]-[16].

**Definition 1.1.** [9] *Let $X$ be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:*
*G1) $G(x, y, z) = 0$, if $x = y = z$, G2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$,*
*G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,*
*G4) $G(x, y, z) = G(y, x, z) = G(z, y, x) = \cdots$ (symmetry in all three variables),*
*G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).*

*Then the function $G$ is called a generalized metric or a G-metric on $X$ and the pair $(X, G)$ is called a G-metric space.*

The following useful properties of a G-metric are readily derived from the axioms.

**Proposition 1.1.** [9] *Let $X$ be a nonempty set and $G : X \times X \times X \to \mathbb{R}^+$ be a function*
*1) if $G(x, y, z) = 0$, then $x = y = z$,*
*2) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,*
*3) $G(x, y, y) \leq 2G(y, x, x)$,*
*4) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,*
*5) $G(x, y, z) \leq \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$,*
*6) $G(x, y, z) \leq (G(x, a, a) + G(y, a, a) + G(z, a, a))$,*
*7) $|G(x, y, z) - G(x, y, a)| \leq max\{G(a, z, z), G(z, a, a)\}$,*
*8) $|G(x, y, z) - G(x, y, a)| \leq G(x, a, z)$,*
*9) $|G(x, y, z) - G(y, z, z)| \leq max\{G(x, z, z), G(z, x, x)\}$,*
*10) $|G(x, y, y) - G(y, x, x)| \leq max\{G(y, x, x), G(x, y, y)\}$.*

**Proposition 1.2.** [9] *Let $(X, G)$ be $G-$metric space, then for a sequence $(x_n) \subseteq X$ and point $x \in X$ the following are equivalent.*
*1) $(x_n)$ is $G-$convergent to $x$.*
*2) $d_G(x_n, x) \to 0$, as $n \to \infty$ (that is, $(x_n)$ converges to $x$ relative to the metric $d_G$).*
*3) $G(x_n, x_n, x) \to 0$, as $n \to \infty$.*
*4) $G(x_n, x, x) \to 0$, as $n \to \infty$.*
*5) $G(x_m, x_n, x) \to 0$, as $m, n \to \infty$.*

**Definition 1.2.** [1] *Let $(X, d)$ be a metric space and $I = [0, 1]$. A mapping $W : X \times X \times I \to X$ is said to be a convex structure on $X$ if each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$,*

$$d\left(u, W\left(x, y, \lambda\right)\right) \leq \lambda d\left(u, x\right) + \left(1 - \lambda\right) d\left(u, y\right)$$

*If $(X, d)$ is equipped with a Takahashi convex structure, then it is called a convex metric space indicated by $(X, d, W)$. A Banach space , or any convex subset of it is a convex metric space with*

$$W\left(x, y, \lambda\right) = \lambda x + \left(1 - \lambda\right) y.$$

**Definition 1.3.** *Let $X$ be a convex metric space. A nonempty subset $A$ of $X$ is said to be convex if $W(x, y, \lambda) \in A$ whenever $(x, y, \lambda) \in A \times A \times [0, 1]$.*

A Banach space, or any convex subset of it, is a convex metric space with $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$. More generally, if $X$ is a linear space with a translation invariant metric satisfying $d(\lambda x + (1 - \lambda)y, 0) \leq \lambda d(x, 0) + (1 - \lambda)d(y, 0)$, then $X$ is a convex metric space.

**Definition 1.4.** [2] *Let $(X, d)$ be a metric space. A mapping $W : X \times X \times X \times [0, 1] \times [0, 1] \times [0, 1] \to X$ is said to be a generalized convex structure on $X$ if for each $(x, y, z; a, b, c) \in$*

$X \times X \times X \times [0,1] \times [0,1] \times [0,1]$ *and* $u \in X$,

$$d(u, W(x,y,z;a,b,c)) \leq ad(u,x) + bd(u,y) + cd(u,z);$$

$a+b+c=1$. *The metric space* $X$ *together with* $W$ *is called a generalized convex metric space.*

**Definition 1.5.** [10] *Let* $X$ *be a generalized convex metric space. A nonempty subset* $A$ *of* $X$ *is said to be generalized convex if* $W(x,y,z;a,b,c) \in A$ *whenever* $(x,y,z;a,b,c) \in A \times A \times A \times [0,1] \times [0,1] \times [0,1]$.

**Definition 1.6.** [10] *Let* $(X,G)$ *be a* $G-$*metric space. A mapping* $W : X \times X \times X \times [0,1] \times [0,1] \to X$ *is said to be a generalized convex structure on* $X$ *if for each* $(x,y,z;a,b) \in X \times X \times X \times [0,1] \times [0,1]$ , $a \geq b$ *and* $u,v \in X$,

$$G(u,v,W(x,y,z;a,b)) \leq (a-b)\,G(u,v,x) + (1-a)\,G(u,v,y) + bG(u,v,z);$$

**Theorem 1.1.** [4] *The G-metric space* $X$ *together with* $W$ *is called a generalized convex G-metric space. Let* $(X,G)$ *be a complete* $G-$*metric space, and let* $T : X \to X$ *be a mapping satisfying one of the following conditions:*
$G\left(Tx,Ty,Tz\right) \leq a\left(Gx,y,z\right) + bG\left(x,Tx,Tx\right) + cG\left(y,Ty,Ty\right) + dG\left(z,Tz,Tz\right)$
*for all* $x,y,z \in X$ *where* $0 \leq a,b,c,d < 1$, *then* $T$ *has a unique fixed point (say* $u$, *i.e.,* $Tu = u$*), and* $T$ *is G-continuous at* $u$.

**Lemma 1.1.** [17] *If* $\rho$ *is a real number satisfying* $0 \leq \rho < 1$ *and* $(\epsilon_n)_{n\in\mathbb{N}}$ *is a sequence of positive numbers such that* $lim_{n\to\infty}\epsilon_n = 0$, *then for any sequence of positive numbers* $(\epsilon_n)_{n\in\mathbb{N}}$ *satisfying*

$$a_{n+1} \leq \rho a_n + \epsilon_n, n = 1,2,...,$$

*one has*

$$\lim_{n\to\infty} a_n = 0.$$

**Lemma 1.2.** [31]*Let* $\{\psi_n\}$ *be a nonnegative sequence for which one supposes there exists* $n_0 \in \mathbb{N}$, *such that for all* $n \geq n_0$ *one has satisfied the following inequality:*

$$\psi_{n+1} \leq \left(1-\lambda_n\right)\psi_n + \lambda_n\phi_n$$

*where* $\lambda_n \in (0,1)$, $\forall n \in \mathbb{N}$, $\sum\limits_{n=1}^{\infty}\lambda_n = \infty$ *and* $\phi_n \geq 0$, $\forall n \in \mathbb{N}$. *Then*

$$0 \leq \lim_{n\to\infty}\sup\psi_n \leq \lim_{n\to\infty}\sup\phi_n.$$

## 2. MAIN RESULTS

2.1. **convergenge analysis.** We prove two theorems under different two conditions in the generalized convex G-metric spaces.

**Theorem 2.1.** *Let* $K$ *be a nonempty closed convex subset of a* $(X,G,W)$ *complete convex* $G-$*metric space with* $W$ *convex structure and* $T : X \to X$ *be a mapping satisfying the following condition:*

$$G\left(Tx,Ty,Tz\right) \leq aG\left(x,y,z\right) + bG\left(x,Tx,Tx\right) + cG\left(y,Ty,Ty\right) + dG\left(z,Tz,Tz\right) \quad (1)$$

*for all* $x,y,z \in X$ *where* $0 \leq a,b,c,d < 1$ *and let* $\{x_n\}_{n\geq 0}$ *be the iterative scheme defined by*

$$\left\{\begin{array}{c} x_0 \in X, \forall n \in \mathbb{N}, \\ x_{n+1} = W\left(Ty_n, Ty_n, Ty_n : \gamma_n, \gamma_n\right) \\ y_n = W\left(z_n, Tz_n, Tx_n : \alpha_n, \beta_n\right) \\ z_n = W\left(Tx_n, x_n, Tx_n : \theta_n, \theta_n\right) \end{array}\right. \quad (2)$$

*such that* $\lim\limits_{n\to\infty} G(x_n, Tx_n, Tx_n) = 0$ *with* $\{\gamma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\theta_n\} \subset [0,1]$. *Then the sequence* $\{x_n\}_{n\geq 0}$ $G-$*convergence to unique fixed point* $p$ *of* $T$.

*Proof.* Suppose that $T$ satisfies condition (2), we have

$$\begin{aligned} G(x_{n+1}, p, p) &= G(W(Ty_n, Ty_n, Ty_n : \gamma_n, \gamma_n), p, p) \\ &\leq (\gamma_n - \gamma_n) G(Ty_n, p, p) + (1-\gamma_n) G(Ty_n, p, p) + \gamma_n G(Ty_n, p, p) \\ &\leq aG(y_n, p, p) + bG(y_n, Ty_n, Ty_n) + cG(p, p, Tp) + dG(p, p, Tp) \quad (3) \\ &= aG(y_n, p, p) + bG(y_n, Ty_n, Ty_n) + (c+d) G(p, p, Tp) \end{aligned}$$

and

$$\begin{aligned} G(y_n, p, p) &= G(W(z_n, Tz_n, Tx_n : \alpha_n, \beta_n), p, p) \\ &\leq (\alpha_n - \beta_n) G(z_n, p, p) + (1-\alpha_n) G(Tz_n, p, p) + \beta_n G(Tx_n, p, p) \\ &\leq (\alpha_n - \beta_n + (1-\alpha_n) a) G(z_n, p, p) + \beta_n a G(x_n, p, p) \\ &\quad + (1-\alpha_n) bG(z_n, Tz_n, Tz_n) \quad (4) \\ &\quad + \beta_n bG(x_n, Tx_n, Tx_n) + (1 - (\alpha_n - \beta_n))(c+d) G(p, Tp, Tp) \end{aligned}$$

and

$$\begin{aligned} G(z_n, p, p) &= G(W(Tx_n, x_n, Tx_n : \theta_n, \theta_n), p, p) \quad (5) \\ &\leq (1 - \theta_n(1-a)) G(x_n, p, p) + \theta_n bG(x_n, Tx_n, Tx_n) \\ &\quad + \theta_n (c+d) G(p, Tp, Tp) \end{aligned}$$

Substituting (4) and (5) in (3), we obtain

$$\begin{aligned} G(x_{n+1}, p, p) &\leq a \left[ \begin{array}{c} (\alpha_n - \beta_n + (1-\alpha_n) a) \left( \begin{array}{c} (1 - \theta_n(1-a)) G(x_n, p, p) \\ +\theta_n bG(x_n, Tx_n, Tx_n) \\ +\theta_n (c+d) G(p, Tp, Tp) \end{array} \right) \\ +\beta_n a G(x_n, p, p) + (1-\alpha_n) bG(z_n, Tz_n, Tz_n) \\ +\beta_n bG(x_n, Tx_n, Tx_n) + (1-(\alpha_n - \beta_n))(c+d) G(p, Tp, Tp) \end{array} \right] \\ &\quad + bG(y_n, Ty_n, Ty_n) + (c+d) G(p, p, Tp) \\ &= a((\alpha_n - \beta_n + (1-\alpha_n) a)(1 - \theta_n(1-a)) + \beta_n a) G(x_n, p, p) \\ &\quad + bG(y_n, Ty_n, Ty_n) + a((1-\alpha_n) b) G(z_n, Tz_n, Tz_n) \\ &\quad + a((\alpha_n - \beta_n + (1-\alpha_n) a) \theta_n b + \beta_n b) G(x_n, Tx_n, Tx_n) \\ &\quad + \left[ \left( a \left( \begin{array}{c} (\alpha_n - \beta_n + (1-\alpha_n) a) \theta_n (c+d) \\ + (1 - (\alpha_n - \beta_n))(c+d) \end{array} \right) \right) + (c+d) \right] G(p, Tp, Tp) \end{aligned}$$

Since $G(p, Tp, Tp) = 0$, we obtain

$$\begin{aligned} G(x_{n+1}, p, p) &\leq a((\alpha_n - \beta_n + (1-\alpha_n) a)(1 - \theta_n(1-a)) + \beta_n a) G(x_n, p, p) \\ &\quad + bG(y_n, Ty_n, Ty_n) + a((1-\alpha_n) b) G(z_n, Tz_n, Tz_n) \\ &\quad + a((\alpha_n - \beta_n + (1-\alpha_n) a) \theta_n b + \beta_n b) G(x_n, Tx_n, Tx_n) \end{aligned}$$

In order to satisfy the conditions of Lemma 1.1, we take $\delta$, $\varepsilon_n$ and $\kappa_n$ as follows:

$$\begin{aligned} 0 &\leq \delta = a((\alpha_n - \beta_n + (1-\alpha_n) a)(1 - \theta_n(1-a)) + \beta_n a) < 1 \\ \varepsilon_n &= bG(y_n, Ty_n, Ty_n) + a((\alpha_n - \beta_n + (1-\alpha_n) a) \theta_n b + \beta_n b) G(x_n, Tx_n, Tx_n) \\ &\quad + a((1-\alpha_n) b) G(z_n, Tz_n, Tz_n) \\ \kappa_n &= G(x_n, p, p). \end{aligned}$$

Since

$$\lim_{n\to\infty} G(x_n, Tx_n, Tx_n) = \lim_{n\to\infty} G(y_n, Ty_n, Ty_n) = \lim_{n\to\infty} G(z_n, Tz_n, Tz_n) = 0$$

by Lemma 1.1, we have $\lim_{n\to\infty} G(x_n, p, p) = 0$. □

**Theorem 2.2.** *Let $K$ be a nonempty closed convex subset of a $(X, G, W)$ complete convex $G-$metric space with $W$ convex structure and $T : X \to X$ be a mapping satisfying the following condition:*

$$G(Tx, Ty, Tz) \leq a(Gx, y, z) + bG(x, Tx, Tx) + cG(y, Ty, Ty) + dG(z, Tz, Tz)$$

*for all $x, y, z \in X$ where $0 \leq a, b \leq \frac{1}{4}$ , $c, d \in [0, 1)$ and let $\{x_n\}_{n\geq 0}$ be defined by (2) with*
*i) $\{\theta_n\}_{n\geq 0} \subset \left[0, \frac{1}{4}\right)$ ,*
*ii) $\beta_n \leq (1-\alpha_n) a \leq \alpha_n$ and then the sequence $\{x_n\}_{n\geq 0}$ converges to unique fixed point $p$ of $T$.*

*Proof.* Suppose that $T$ satisfies condition (2), we have

$$\begin{aligned} G(x_{n+1}, p, p) &= G(W(Ty_n, Ty_n, Ty_n : \gamma_n, \gamma_n), p, p) \\ &\leq (\gamma_n - \gamma_n) G(Ty_n, p, p) + (1-\gamma_n) G(Ty_n, p, p) + \gamma_n G(Ty_n, p, p) \quad (6) \\ &\leq aG(y_n, p, p) + bG(y_n, Ty_n, Ty_n) + cG(p, p, Tp) + dG(p, p, Tp) \\ &= aG(y_n, p, p) + bG(y_n, Ty_n, Ty_n) + (c+d) G(p, p, Tp) \end{aligned}$$

$$\begin{aligned} G(y_n, p, p) &= G(W(z_n, Tz_n, Tx_n : \alpha_n, \beta_n), p, p) \\ &\leq (\alpha_n - \beta_n) G(z_n, p, p) + (1-\alpha_n) G(Tz_n, p, p) + \beta_n G(Tx_n, p, p) \quad (7) \\ &\leq (\alpha_n - \beta_n + (1-\alpha_n) a) G(z_n, p, p) \\ &\quad + \beta_n aG(x_n, p, p) + (1-\alpha_n) bG(z_n, Tz_n, Tz_n) \\ &\quad + \beta_n bG(x_n, Tx_n, Tx_n) + (1-(\alpha_n - \beta_n))(c+d) G(p, Tp, Tp) \end{aligned}$$

$$\begin{aligned} G(z_n, p, p) &= G(W(Tx_n, x_n, Tx_n : \theta_n, \theta_n), p, p) \quad (8) \\ &\leq (\theta_n - \theta_n) G(Tx_n, p, p) + (1-\theta_n) G(x_n, p, p) + \theta_n G(Tx_n, p, p) \\ &\leq (1-\theta_n(1-a)) G(x_n, p, p) + \theta_n bG(x_n, Tx_n, Tx_n) \\ &\quad + \theta_n (c+d) G(p, Tp, Tp) \end{aligned}$$

Substituting (7) and (8) in (6), we have

$$\begin{aligned} G\left(x_{n+1},p,p\right) &\leq a\left[\begin{array}{c} \left(\alpha_n-\beta_n+\left(1-\alpha_n\right)a\right)\left(\begin{array}{c}\left(1-\theta_n\left(1-a\right)\right)G\left(x_n,p,p\right)\\ +\theta_n bG\left(x_n,Tx_n,Tx_n\right)\\ +\theta_n\left(c+d\right)G\left(p,Tp,Tp\right)\end{array}\right)\\ +\beta_n aG\left(x_n,p,p\right)+\left(1-\alpha_n\right)bG\left(z_n,Tz_n,Tz_n\right)+\beta_n bG\left(x_n,Tx_n,Tx_n\right)\\ +\left(1-\left(\alpha_n-\beta_n\right)\right)\left(c+d\right)G\left(p,Tp,Tp\right)\end{array}\right]\\ &\quad +bG\left(y_n,Ty_n,Ty_n\right)+\left(c+d\right)G\left(p,p,Tp\right)\\ &= a\left(\left(\alpha_n-\beta_n+\left(1-\alpha_n\right)a\right)\left(1-\theta_n\left(1-a\right)\right)+\beta_n a\right)G\left(x_n,p,p\right)\\ &\quad +bG\left(y_n,Ty_n,Ty_n\right)+a\left(\left(1-\alpha_n\right)b\right)G\left(z_n,Tz_n,Tz_n\right)\\ &\quad +a\left(\left(\alpha_n-\beta_n+\left(1-\alpha_n\right)a\right)\theta_n b+\beta_n b\right)G\left(x_n,Tx_n,Tx_n\right)\\ &\quad +\left[\left(a\left(\begin{array}{c}\left(\alpha_n-\beta_n+\left(1-\alpha_n\right)a\right)\theta_n\left(c+d\right)\\ +\left(1-\left(\alpha_n-\beta_n\right)\right)\left(c+d\right)\end{array}\right)\right)+\left(c+d\right)\right]G\left(p,Tp,Tp\right) \end{aligned}$$

Since $G\left(p,Tp,Tp\right)=0$

$$\begin{aligned} G\left(x_{n+1},p,p\right) &\leq a\left(\left(\alpha_n-\beta_n+\left(1-\alpha_n\right)a\right)\left(1-\theta_n\left(1-a\right)\right)+\beta_n a\right)G\left(x_n,p,p\right)\\ &\quad +bG\left(y_n,Ty_n,Ty_n\right)+a\left(\left(1-\alpha_n\right)b\right)G\left(z_n,Tz_n,Tz_n\right)\\ &\quad +a\left(\left(\alpha_n-\beta_n+\left(1-\alpha_n\right)a\right)\theta_n b+\beta_n b\right)G\left(x_n,Tx_n,Tx_n\right) \end{aligned} \tag{9}$$

Continuing the process

$$G\left(x_n,Tx_n,Tx_n\right)\leq\left(\frac{1+2a}{1-2b}\right)G\left(x_n,p,p\right) \tag{10}$$

$$\begin{aligned} G\left(z_n,Tz_n,Tz_n\right) &\leq \left(\frac{1+2a}{1-2b}\right)G\left(z_n,p,p\right)\\ &\leq \left(\frac{1+2a}{1-2b}\right)\left(1-\theta_n\left(1-a\right)\right)G\left(x_n,p,p\right)+\left(\frac{1+2a}{1-2b}\right)\theta_n bG\left(x_n,Tx_n,Tx_n\right)\\ &\leq \left(\frac{1+2a}{1-2b}\right)\left[\left(1-\theta_n\left(1-a\right)\right)G\left(x_n,p,p\right)+\left(\frac{1+2a}{1-2b}\right)\theta_n bG\left(x_n,p,p\right)\right] \end{aligned} \tag{11}$$

$$\begin{aligned} G\left(y_n,Ty_n,Ty_n\right) &\leq \left(\frac{1+2a}{1-2b}\right)G\left(y_n,p,p\right)\\ &\leq \left(\frac{1+2a}{1-2b}\right)\left(\begin{array}{c} \left[\left(\alpha_n-\beta_n+\left(1-\alpha_n\right)a\right)\right]\\ \times\left(\left[\begin{array}{c}\left(1-\theta_n\left(1-a\right)\right)G\left(x_n,p,p\right)\\ +\left(\frac{1+2a}{1-2b}\right)\theta_n bG\left(x_n,p,p\right)\end{array}\right]\right)\\ +\beta_n aG\left(x_n,p,p\right)+\beta_n b\left(\frac{1+2a}{1-2b}\right)G\left(x_n,p,p\right)\\ +\left(1-\alpha_n\right)b\left(\frac{1+2a}{1-2b}\right)\left[\begin{array}{c}\left(1-\theta_n\left(1-a\right)\right)G\left(x_n,p,p\right)\\ +\left(\frac{1+2a}{1-2b}\right)\theta_n bG\left(x_n,p,p\right)\end{array}\right]\end{array}\right) \end{aligned} \tag{12}$$

Substituting (10), (11) and (12) in (9), we obtain

$$\begin{aligned} G(x_{n+1},p,p) &\leq a\left(\left(\alpha_n-\beta_n+(1-\alpha_n)a\right)(1-\theta_n(1-a))+\beta_n a\right)G(x_n,p,p) \\ &+b\left(\frac{1+2a}{1-2b}\right)\left(\begin{array}{c} \left[(\alpha_n-\beta_n+(1-\alpha_n)a)\right]\left(\left[\begin{array}{c}(1-\theta_n(1-a)) \\ +\left(\frac{1+2a}{1-2b}\right)\theta_n b\end{array}\right]\right) \\ +\beta_n a G(x_n,p,p)+\beta_n b\left(\frac{1+2a}{1-2b}\right) \\ +(1-\alpha_n)b\left(\frac{1+2a}{1-2b}\right)\left[\begin{array}{c}(1-\theta_n(1-a)) \\ +\left(\frac{1+2a}{1-2b}\right)\theta_n b\end{array}\right]\end{array}\right) \\ &+a\left(\left(\alpha_n-\beta_n+(1-\alpha_n)a\right)\theta_n b+\beta_n b\right)\left(\frac{1+2a}{1-2b}\right) \\ &+a\left((1-\alpha_n)b\right)\left(\frac{1+2a}{1-2b}\right)\left[(1-\theta_n(1-a))+\left(\frac{1+2a}{1-2b}\right)\theta_n b\right] \end{aligned}$$

$$G(x_{n+1},p,p)\leq\left[\begin{array}{c} a\left(\left(\alpha_n-\beta_n+(1-\alpha_n)a\right)(1-\theta_n(1-a))+\beta_n a\right) \\ +b\left(\frac{1+2a}{1-2b}\right)\left(\begin{array}{c} \left[(\alpha_n-\beta_n+(1-\alpha_n)a)\right]\left(\left[\begin{array}{c}(1-\theta_n(1-a)) \\ +\left(\frac{1+2a}{1-2b}\right)\theta_n b\end{array}\right]\right) \\ +\beta_n a G(x_n,p,p)+\beta_n b\left(\frac{1+2a}{1-2b}\right) \\ +(1-\alpha_n)b\left(\frac{1+2a}{1-2b}\right)\left[\begin{array}{c}(1-\theta_n(1-a)) \\ +\left(\frac{1+2a}{1-2b}\right)\theta_n b\end{array}\right]\end{array}\right) \\ +a\left(\left(\alpha_n-\beta_n+(1-\alpha_n)a\right)\theta_n b+\beta_n b\right)\left(\frac{1+2a}{1-2b}\right) \\ +a\left((1-\alpha_n)b\right)\left(\frac{1+2a}{1-2b}\right)\left[(1-\theta_n(1-a))+\left(\frac{1+2a}{1-2b}\right)\theta_n b\right] \end{array}\right]G(x_n,p,p)$$

$$\begin{aligned} G(x_{n+1},p,p) &\leq a\left(\left(\alpha_n-\beta_n+(1-\alpha_n)a\right)(1-\theta_n(1-a))+\beta_n a\right)G(x_n,p,p) \\ &+b\left(\frac{1+2a}{1-2b}\right)\left[(\alpha_n-\beta_n+(1-\alpha_n)a)\right](1-\theta_n(1-a))G(x_n,p,p) \\ &+\left(\frac{1+2a}{1-2b}\right)^2\left[(\alpha_n-\beta_n+(1-\alpha_n)a)\right]\theta_n b^2 G(x_n,p,p) \\ &+b\left(\frac{1+2a}{1-2b}\right)\beta_n a G(x_n,p,p)+\left(\frac{1+2a}{1-2b}\right)^2\beta_n b^2 G(x_n,p,p) \\ &+(1-\alpha_n)b^2\left(\frac{1+2a}{1-2b}\right)^2(1-\theta_n(1-a))G(x_n,p,p) \\ &+\left(\frac{1+2a}{1-2b}\right)^3(1-\alpha_n)\theta_n b^3 G(x_n,p,p) \\ &+a\left(\left(\alpha_n-\beta_n+(1-\alpha_n)a\right)\theta_n b+\beta_n b\right)\left(\frac{1+2a}{1-2b}\right)G(x_n,p,p) \\ &+a\left((1-\alpha_n)b\right)\left(\frac{1+2a}{1-2b}\right)(1-\theta_n(1-a))G(x_n,p,p) \\ &+a\left((1-\alpha_n)b\right)\left(\frac{1+2a}{1-2b}\right)^2\theta_n b G(x_n,p,p) \end{aligned}$$

Since

$$0 \leq \left[ \begin{array}{c} a\left(\left(\alpha_n - \beta_n + \left(1-\alpha_n\right)a\right)\left(1-\theta_n\left(1-a\right)\right) + \beta_n a\right) \\ +b\left(\frac{1+2a}{1-2b}\right) \left( \begin{array}{c} \left[\left(\alpha_n - \beta_n + \left(1-\alpha_n\right)a\right)\right] \left( \left[ \begin{array}{c} \left(1-\theta_n\left(1-a\right)\right) \\ +\left(\frac{1+2a}{1-2b}\right)\theta_n b \end{array} \right] \right) \\ +\beta_n a G\left(x_n, p, p\right) + \beta_n b\left(\frac{1+2a}{1-2b}\right) \\ +\left(1-\alpha_n\right) b\left(\frac{1+2a}{1-2b}\right) \left[ \begin{array}{c} \left(1-\theta_n\left(1-a\right)\right) \\ +\left(\frac{1+2a}{1-2b}\right)\theta_n b \end{array} \right] \end{array} \right) \\ +a\left(\left(\alpha_n - \beta_n + \left(1-\alpha_n\right)a\right)\theta_n b + \beta_n b\right)\left(\frac{1+2a}{1-2b}\right) \\ +a\left(\left(1-\alpha_n\right)b\right)\left(\frac{1+2a}{1-2b}\right)\left[\left(1-\theta_n\left(1-a\right)\right) + \left(\frac{1+2a}{1-2b}\right)\theta_n b\right] \end{array} \right] < 1$$

by Lemma 1.1, we have $\lim_{n\to\infty} G\left(x_n, p, p\right) = 0$. □

2.2. **Data dependency.** Let's prove that the iteration (2) is data-dependent.

**Definition 2.1.** *Let $T, \check{T}, \widetilde{T} : X \to X$ be three operators. We say that $\check{T}$ and $\widetilde{T}$ are the approximate operators of $T$ if for all $x \in X$ and for a fixed $\epsilon > 0$, we have*

$$G\left(Tx, \check{T}x, \widetilde{T}x\right) = \max\left\{\left\|Tx - \check{T}x\right\|, \left\|\check{T}x - \widetilde{T}x\right\|, \left\|\widetilde{T}x - Tx\right\|\right\} \leq \epsilon.$$

**Theorem 2.3.** *Let $K$ be a nonempty closed convex subset of a $(X, G, W)$ complete convex $G-$metric space with $W$ convex structure and $, \check{T}, \widetilde{T} : K \to X$ be the approximate operators of the $T : K \to X$ satisfying the mapping (1) and $\{x_n\}_{n\geq 0}$, $\{u_n\}_{n\geq 0}$ and $\{k_n\}_{n\geq 0}$ three iteration schemes associated to $T, \check{T}$ and $\widetilde{T}$ defined by*

$$\left\{ \begin{array}{c} x_{n+1} = Ty_n \\ y_n = \left(\alpha_n - \beta_n\right) z_n + \left(1-\alpha_n\right) Tx_n + \beta_n Tz_n \\ z_n = \left(1-\theta_n\right) x_n + \theta_n Tx_n, \forall n \in \mathbb{N}, \end{array} \right. \tag{13}$$

$$\left\{ \begin{array}{c} u_{n+1} = \check{T}v_n \\ v_n = \left(\alpha_n - \beta_n\right) w_n + \left(1-\alpha_n\right) \check{T}u_n + \beta_n \check{T}w_n \\ w_n = \left(1-\theta_n\right) u_n + \theta_n \check{T}u_n, \forall n \in \mathbb{N} \end{array} \right. \tag{14}$$

*and*

$$\left\{ \begin{array}{c} k_{n+1} = \widetilde{T}r_n \\ r_n = \left(\alpha_n - \beta_n\right) s_n + \left(1-\alpha_n\right) \widetilde{T}k_n + \beta_n \widetilde{T}s_n \\ s_n = \left(1-\theta_n\right) k_n + \theta_n \widetilde{T}k_n, \forall n \in \mathbb{N}, \end{array} \right. \tag{15}$$

*respectively, where $\{\alpha_n\}$ , $\{\beta_n\}$ and $\{\theta_n\}$ are real sequences in $[0,1]$ satisfying $\alpha_n \geq \beta_n$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. Let $Tx^* = x^*$, $\check{T}k^* = k^*$ and $\widetilde{T}u^* = u^*$ with $\|Tx - Ty\| \leq a\|x - y\| + b\|x - Tx\| + c\|y - Ty\|$. Then for $25b > 0$, we have the following estimate:*

$$G\left(x^*, k^*, u^*\right) \leq \frac{11\epsilon}{1-a}$$

*Proof.* Using iterative schemes (13), (14) and (15) yield the following inequalites:

$$G\left(x_{n+1}, k_{n+1}, u_{n+1}\right) = \max\left\{\left\|x_{n+1} - k_{n+1}\right\|, \left\|k_{n+1} - u_{n+1}\right\|, \left\|u_{n+1} - x_{n+1}\right\|\right\} \tag{16}$$

$$\left\|x_{n+1} - k_{n+1}\right\| = \left\|Ty_n - \check{T}r_n\right\| \leq \epsilon + a\left\|y_n - r_n\right\| + b\left\|y_n - Ty_n\right\| + c\left\|r_n - Tr_n\right\|, \tag{17}$$

$$\begin{aligned} \|y_n - r_n\| &\leq (\alpha_n - \beta_n) \|z_n - s_n\| + (1-\alpha_n) \left\|Tx_n - \check{T}k_n\right\| + \beta_n \left\|Tz_n - \check{T}s_n\right\| \\ &\leq (\alpha_n - \beta_n) \|z_n - s_n\| + (1-\alpha_n)\epsilon + \beta_n \epsilon + a(1-\alpha_n)\|x_n - k_n\| \\ &\quad +b(1-\alpha_n)\|x_n - Tx_n\| + c(1-\alpha_n)\|k_n - Tk_n\| \\ &\quad +a\beta_n \|z_n - s_n\| + b\beta_n \|z_n - Tz_n\| + c\beta_n \|s_n - Ts_n\| \end{aligned} \tag{18}$$

and

$$\begin{aligned} \|z_n - s_n\| &= (1-\theta_n)\|x_n - k_n\| + \theta_n \left\|Tx_n - \check{T}k_n\right\| \\ &\leq (1-\theta_n)\|x_n - k_n\| + \theta_n \epsilon + \theta_n a \|x_n - k_n\| \\ &\quad +\theta_n b\|x_n - Tx_n\| + \theta_n c\|k_n - Tk_n\|. \end{aligned} \tag{19}$$

Substituting (19) in (18) and (18) in (17), we have

$$\begin{aligned} \|x_{n+1} - k_{n+1}\| &\leq \epsilon_n + a \left[ \begin{array}{c} (\alpha_n - \beta_n(1-a)) \left[ \begin{array}{c} (1-\theta_n(1-a))\|x_n - k_n\| \\ +\theta_n\epsilon_n + \theta_n b\|x_n - Tx_n\| + \theta_n c\|k_n - Tk_n\| \end{array} \right] \\ +(1-\alpha_n)\epsilon_n + \beta_n\epsilon_n + a(1-\alpha_n)\|x_n - k_n\| \\ +b(1-\alpha_n)\|x_n - Tx_n\| + c(1-\alpha_n)\|k_n - Tk_n\| \\ +b\beta_n\|z_n - Tz_n\| + c\beta_n\|s_n - Ts_n\| \end{array} \right] \\ &\quad +b\|y_n - Ty_n\| + c\|r_n - Tr_n\| \\ &\leq a\left[(\alpha_n - \beta_n(1-a))(1-\theta_n(1-a)) + a(1-\alpha_n)\right]\|x_n - k_n\| \\ &\quad +a\left[(\alpha_n - \beta_n(1-a))\theta_n b + b(1-\alpha_n)\right]\|x_n - Tx_n\| \\ &\quad +a\left[(\alpha_n - \beta_n(1-a))\theta_n c + c(1-\alpha_n)\right]\|k_n - Tk_n\| \\ &\quad +a\left[(\alpha_n - \beta_n(1-a))\theta_n + (1-\alpha_n) + \beta_n + 1\right]\epsilon_n \\ &\quad +ab\beta_n\|z_n - Tz_n\| + ac\beta_n\|s_n - Ts_n\| + b\|y_n - Ty_n\| + c\|r_n - Tr_n\|. \end{aligned} \tag{20}$$

In the similar way, we obtain

$$\|k_{n+1} - u_{n+1}\| \leq \epsilon_n + a \left[ \begin{array}{c} (\alpha_n - \beta_n(1-a)) \left[ \begin{array}{c} (1-\theta_n(1-a)) \\ \times\|k_n - u_n\| + \theta_n\epsilon_n \\ +\theta_n b\left\|k_n - \check{T}k_n\right\| \\ +\theta_n c\left\|u_n - \check{T}u_n\right\| \end{array} \right] \\ +(1-\alpha_n)\epsilon_n + \beta_n\epsilon_n + a(1-\alpha_n)\|k_n - u_n\| \\ +b(1-\alpha_n)\left\|k_n - \check{T}k_n\right\| + c(1-\alpha_n)\left\|u_n - \check{T}u_n\right\| \\ +b\beta_n\left\|r_n - \check{T}r_n\right\| + c\beta_n\left\|w_n - \check{T}w_n\right\| \end{array} \right] \tag{21}$$

$$+b\left\|r_n - \check{T}r_n\right\| + c\left\|v_n - \check{T}v_n\right\|$$

$$\begin{aligned} &\leq a\left[(\alpha_n - \beta_n(1-a))(1-\theta_n(1-a)) + a(1-\alpha_n)\right]\|k_n - u_n\| \\ &\quad +a\left[(\alpha_n - \beta_n(1-a))\theta_n b + b(1-\alpha_n)\right]\left\|k_n - \check{T}k_n\right\| \\ &\quad +a\left[(\alpha_n - \beta_n(1-a))\theta_n c + c(1-\alpha_n)\right]\left\|u_n - \check{T}u_n\right\| \\ &\quad +a\left[(\alpha_n - \beta_n(1-a))\theta_n + (1-\alpha_n) + \beta_n + 1\right]\epsilon \\ &\quad +ab\beta_n\left\|s_n - \check{T}s_n\right\| + ac\beta_n\left\|w_n - \check{T}w_n\right\| \\ &\quad +b\left\|r_n - \check{T}r_n\right\| + c\left\|v_n - \check{T}v_n\right\| \end{aligned}$$

and

$$\|u_{n+1}-x_{n+1}\| \leq \epsilon_n + a\begin{bmatrix} (\alpha_n-\beta_n(1-a))\begin{bmatrix} (1-\theta_n(1-a)) \\ \times\|u_n-x_n\|+\theta_n\epsilon_n \\ +\theta_n b\left\|u_n-\widetilde{T}u_n\right\| \\ +\theta_n c\left\|x_n-\widetilde{T}x_n\right\| \end{bmatrix} \\ +(1-\alpha_n)\epsilon_n+\beta_n\epsilon_n+a(1-\alpha_n)\|u_n-x_n\| \\ +b(1-\alpha_n)\left\|u_n-\widetilde{T}u_n\right\|+c(1-\alpha_n)\left\|x_n-\widetilde{T}x_n\right\| \\ +b\beta_n\left\|w_n-\widetilde{T}w_n\right\|+c\beta_n\left\|z_n-\widetilde{T}z_n\right\| \end{bmatrix} \tag{22}$$
$$+b\left\|v_n-\widetilde{T}v_n\right\|+c\left\|y_n-\widetilde{T}y_n\right\|$$

$$\begin{aligned} \leq\ & a\left[(\alpha_n-\beta_n(1-a))(1-\theta_n(1-a))+a(1-\alpha_n)\right]\|u_n-x_n\| \\ & +a\left[(\alpha_n-\beta_n(1-a))\theta_n b+b(1-\alpha_n)\right]\left\|u_n-\widetilde{T}u_n\right\| \\ & +a\left[(\alpha_n-\beta_n(1-a))\theta_n c+c(1-\alpha_n)\right]\left\|x_n-\widetilde{T}x_n\right\| \\ & +a\left[(\alpha_n-\beta_n(1-a))\theta_n+(1-\alpha_n)+\beta_n+1\right]\epsilon_n \\ & +ab\beta_n\left\|w_n-\widetilde{T}w_n\right\|+ac\beta_n\left\|z_n-\widetilde{T}z_n\right\| \\ & +b\left\|v_n-\widetilde{T}v_n\right\|+c\left\|y_n-\widetilde{T}y_n\right\|. \end{aligned}$$

Substituting (20), (21) and (22) in (16), we have

$$G(x_{n+1},k_{n+1},u_{n+1}) \leq \max\left\{\begin{array}{c} a\left[(\alpha_n-\beta_n(1-a))(1-\theta_n(1-a))+a(1-\alpha_n)\right]\|x_n-k_n\| \\ +a\left[(\alpha_n-\beta_n(1-a))\theta_n b+b(1-\alpha_n)\right]\|x_n-Tx_n\| \\ +a\left[(\alpha_n-\beta_n(1-a))\theta_n c+c(1-\alpha_n)\right]\left\|k_n-\check{T}k_n\right\| \\ +a\left[(\alpha_n-\beta_n(1-a))\theta_n+(1-\alpha_n)+\beta_n+1\right]\epsilon+ab\beta_n\|z_n-Tz_n\| \\ +ac\beta_n\left\|s_n-\check{T}s_n\right\|+b\|y_n-Ty_n\|+c\left\|r_n-\check{T}r_n\right\| \\ ,\left[(\alpha_n-\beta_n(1-a))(1-\theta_n(1-a))+a(1-\alpha_n)\right]\|k_n-u_n\| \\ +a\left[(\alpha_n-\beta_n(1-a))\theta_n b+b(1-\alpha_n)\right]\left\|k_n-\check{T}k_n\right\| \\ +a\left[(\alpha_n-\beta_n(1-a))\theta_n c+c(1-\alpha_n)\right]\left\|u_n-\widetilde{T}u_n\right\| \\ +a\left[(\alpha_n-\beta_n(1-a))\theta_n+(1-\alpha_n)+\beta_n+1\right]\epsilon+ab\beta_n\left\|s_n-\check{T}s_n\right\| \\ +ac\beta_n\left\|w_n-\widetilde{T}w_n\right\|+b\left\|r_n-\check{T}r_n\right\|+c\left\|v_n-\widetilde{T}v_n\right\| \\ ,a\left[(\alpha_n-\beta_n(1-a))(1-\theta_n(1-a))+a(1-\alpha_n)\right]\|u_n-x_n\| \\ +a\left[(\alpha_n-\beta_n(1-a))\theta_n b+b(1-\alpha_n)\right]\left\|u_n-\widetilde{T}u_n\right\| \\ +a\left[(\alpha_n-\beta_n(1-a))\theta_n c+c(1-\alpha_n)\right]\|x_n-Tx_n\| \\ +a\left[(\alpha_n-\beta_n(1-a))\theta_n+(1-\alpha_n)+\beta_n+1\right]\epsilon+ab\beta_n\left\|w_n-\widetilde{T}w_n\right\| \\ +ac\beta_n\|z_n-Tz_n\|+b\left\|v_n-\widetilde{T}v_n\right\|+c\|y_n-Ty_n\|. \end{array}\right\} \tag{23}$$

Let $1-\beta_n \leq \beta_n$, then the inequality (23) is rearranged as follows:

$$G\left(x_{n+1},k_{n+1},u_{n+1}\right) \leq \max \left\{ \begin{array}{c} \left[1-\beta_n\left(1-a\right)\right]\left\|x_n-k_n\right\| + a2\beta_n\left\|x_n-Tx_n\right\| + a2\beta_n\left\|k_n-\check{T}k_n\right\| \\ +11\beta_n\epsilon + ab\beta_n\left\|z_n-Tz_n\right\| + ac\beta_n\left\|s_n-\check{T}s_n\right\| + b2\beta_n\left\|y_n-Ty_n\right\| \\ +c2\beta_n\left\|r_n-\check{T}r_n\right\|, \\ \left[1-\beta_n\left(1-a\right)\right]\left\|k_n-u_n\right\| + a2\beta_n\left\|k_n-\check{T}k_n\right\| \\ +a2\beta_n\left\|u_n-\widetilde{T}u_n\right\| + 11\beta_n\epsilon + ab\beta_n\left\|s_n-\check{T}s_n\right\| \\ +ac\beta_n\left\|w_n-\widetilde{T}w_n\right\| + b2\beta_n\left\|r_n-\check{T}r_n\right\| + b2\beta_n\left\|v_n-\widetilde{T}v_n\right\| \\ ,\left[1-\beta_n\left(1-a\right)\right]\left\|u_n-x_n\right\| + a2\beta_n\left\|u_n-\widetilde{T}u_n\right\| \\ +a2\beta_n\left\|x_n-Tx_n\right\| + 11\beta_n\epsilon + ab\beta_n\left\|w_n-\widetilde{T}w_n\right\| \\ +ac\beta_n\left\|z_n-Tz_n\right\| + b2\beta_n\left\|v_n-\widetilde{T}v_n\right\| + b2\beta_n\left\|y_n-Ty_n\right\|. \end{array} \right\} \tag{24}$$

If simplifications are made in the (24), we arrive at

$$\begin{aligned} \left\|k_{n+1}-u_{n+1}\right\| &\leq \left[1-\beta_n\left(1-a\right)\right]\left\|k_n-u_n\right\| \\ &+\beta_n\left(1-a\right)\frac{\begin{array}{c} a2\left\|k_n-\check{T}k_n\right\| + a2\left\|u_n-\widetilde{T}u_n\right\| + 11\epsilon + ab\left\|s_n-\check{T}s_n\right\| \\ +ac\left\|w_n-\widetilde{T}w_n\right\| + b2\left\|r_n-\check{T}r_n\right\| + b2\left\|v_n-\widetilde{T}v_n\right\| \end{array}}{\left(1-a\right)} \end{aligned}$$

$$\begin{aligned} \left\|u_{n+1}-x_{n+1}\right\| &\leq \left[1-\beta_n\left(1-a\right)\right]\left\|u_n-x_n\right\| \\ &+\beta_n\left(1-a\right)\frac{\begin{array}{c} a2\left\|u_n-\widetilde{T}u_n\right\| + a2\left\|x_n-Tx_n\right\| + 11\epsilon + ab\left\|w_n-\widetilde{T}w_n\right\| \\ +ac\left\|z_n-Tz_n\right\| + b2\left\|v_n-\widetilde{T}v_n\right\| + b2\left\|y_n-Ty_n\right\| \end{array}}{\left(1-a\right)}. \end{aligned}$$

Define

$$\begin{aligned} \lambda_n &:= \beta_n\left(1-a\right) \\ \phi_n &:= \frac{\begin{array}{c} a2\left\|u_n-\widetilde{T}u_n\right\| + a2\left\|x_n-Tx_n\right\| + 11\epsilon + ab\left\|w_n-\widetilde{T}w_n\right\| \\ +ac\left\|z_n-Tz_n\right\| + b2\left\|v_n-\widetilde{T}v_n\right\| + b2\left\|y_n-Ty_n\right\| \end{array}}{\left(1-a\right)} \end{aligned}$$

By Lemma 1.2, we obtain

$$\left\|x^*-k^*\right\| \leq \frac{11\epsilon}{1-a}, \left\|k^*-u^*\right\| \leq \frac{11\epsilon}{1-a} \text{ and } \left\|u^*-x^*\right\| \leq \frac{11\epsilon}{1-a}.$$

Then

$$G\left(x^*,k^*,u^*\right) = \max\left\{\left\|x^*-k^*\right\|, \left\|k^*-u^*\right\|, \left\|u^*-x^*\right\|\right\} \leq \frac{11\epsilon}{1-a}.$$

□

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