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THE GLOBAL-IN-TIME EXISTENCE OF A CLASSICAL SOLUTION FOR SOME FREE BOUNDARY PROBLEM

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Abstract: We consider the problem with free (unknown) boundary for the one-dimensional diffusion–convection equation. The unknown boundary is found from the additional condition on the free boundary. A dilation of the variables reduces the problem to an initial-boundary value problem for a strictly parabolic equation with unknown coefficients in the known domain. These coefficients are found from an additional boundary condition, which makes it possible to construct a nonlinear operator whose fixed points define the solution to the initial problem.

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1. Introduction

Free (unknown) boundary problems for differential equations are among the most difficult in the theory of partial differential equations. In these problems, along with solving differential equations, we must find the domain where the solution is sought for. As a rule, this domain (boundary) is determined from an additional boundary condition on the free boundary. In the theory of free boundary problems, well-known are the Stefan problem [1, 2] and the Hele–Shaw problem [3] for the heat equation and the Laplace equation respectively. These problems are rather simply formulated but still the existence is proved only of a classical "local-in-time" solution (excluding some simple cases). As for systems of differential equations, we mention the works by Solonnikov and his students on free boundary problems for the system of Navier–Stokes equations [4–6] and Friedman [7]. But, as in the case of scalar equations, here it is possible to prove only the existence of a classical "local-in-time" solution.

There separately exists a large class of free boundary problems for equations of gas dynamics and the hydrodynamics of an ideal incompressible fluid. These problems are well studied and have a rich history (see [8–10]). A typical representative of problems of this class is given by the Cauchy–Poisson problem about waves on the surface of an ideal incompressible fluid. In such problems, either approximate solutions ("shallow water") or solutions close to exact solutions are studied.

Consider the problem formulated in [11], where the dynamics of the fluid in the sought domain $\Omega_f(t)$ obeys the linear Stokes equations:

$$\alpha_\mu \Delta \boldsymbol{v} - \nabla p = \varrho_f \boldsymbol{f}, \quad \nabla \cdot \boldsymbol{v} = 0. \tag{1.1}$$

Here $\boldsymbol{v}$ is the velocity of the fluid, p is the pressure in the fluid, ϱ_f is the dimensionless density of the fluid related to the density of water in the natural conditions, $\boldsymbol{f}$ is the dimensionless vector of the given mass forces, $\alpha_\mu = \frac{\mu}{TLg\rho^0} = \mu_1 \varepsilon^2$ is the dimensionless viscosity of the fluid, μ is the viscosity of the fluid, $\mu_1 = \text{const}$, $\mu_1 > 0$, T is the characteristic time of the physical process, L is the characteristic size of the physical domain under consideration, g is the gravity acceleration, and ρ^0 is the water density in the natural conditions.

The diffusion and convection of the concentration of the acid c is described in $\Omega_f(t)$ by the diffusion–convection equations

$$\frac{\partial c}{\partial t} + \boldsymbol{v} \cdot \nabla c = \alpha_c \Delta c. \tag{1.2}$$

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Here $\alpha_c = \frac{DT}{L^2}$ is the dimensionless acid diffusion coefficient and D is the acid diffusion coefficient. Equations (1.1) and (1.2) are generally accepted as well as the mass conservation law

$$v_n = -d_n \varrho_0, \varrho_0 = \frac{\varrho_s - \varrho_f}{\varrho_f} \tag{1.3}$$

on the free boundary $\Gamma(t)$ (see [8]). Rather natural on the free boundary is also the condition of the equality to zero of the tangential component of the velocity. Here v_n is the normal velocity of the fluid in the direction of the outward unit normal $\boldsymbol{n}$ to $\Omega_f(t)$, d_n is the normal velocity of the free boundary, ϱ_s is the dimensionless density of the solid skeleton.

Moreover, the two additional conditions are fulfilled on the free boundary:

$$(d_n - v_n)c + \alpha_c \frac{\partial c}{\partial n} = -\beta c, \tag{1.4}$$

$$d_n = \beta\gamma c. \tag{1.5}$$

Problem (1.1)–(1.5) is closed by boundary conditions on the given boundaries of the domain $\Omega_f(t)$ and the initial conditions.

The above mathematical model is strongly nonlinear and, in our opinion, if it is proved only in a local-in-time well-posedness or global-in-time well-posedness, but for one spatial variable. The latter case will be the topic of our publication.

2. The Main Results

In the dimensionless variables

$$x \to \frac{x}{L}, \quad t \to \frac{t}{T}, \quad v \to \frac{T}{L}\, v, \quad p \to p^* p,$$

the behavior of an incompressible fluid in the domain $0 < x < X(t)$ for $t > 0$ for the pressure p, the velocity v of the fluid, and the concentration of the acid c is described by the system of equations

$$\frac{\partial p}{\partial x} = 0, \tag{2.1}$$

$$\frac{\partial v}{\partial x} = 0, \tag{2.2}$$

$$\frac{\partial c}{\partial t} + v \frac{\partial c}{\partial x} = \alpha_c\, \frac{\partial^2 c}{\partial x^2}, \tag{2.3}$$

complemented by the initial and boundary conditions

$$p\,(0,t) = p^+(t), \quad c(0,t) = \ c^+(t), \quad t > 0, \tag{2.4}$$

$$\left(\frac{dX}{dt} + \beta - v\right) c + \alpha_c \frac{\partial c}{\partial x} = 0, \quad x = \ X\,(t)\,, \ t > 0, \tag{2.5}$$

$$v\,(t) = -\sigma \frac{dX}{dt}(t), \quad t > 0, \tag{2.6}$$

$$c(x,0) = \ c_0(x), \ 0 < x < X_0, \quad X\,(0) = X_0, \tag{2.7}$$

$$\frac{dX}{dt} = \beta\gamma c, \quad x = \ X(t), \ t > 0. \tag{2.8}$$

In (2.1)–(2.8), $\alpha_c = \frac{D_c T}{L^2}$, $\alpha_i = \frac{D_i T}{L^2}$, $\sigma = \frac{(\rho_s - \rho_f)}{\rho_f}$, L is the characteristic size of the domain under consideration, T is the characteristic time of the process, ρ_f is the density of the fluid, ρ_c is the density of the rocks, D is the acid diffusion coefficient, D_i is the diffusion coefficient of the impurity c_i, while β, β_i, and γ are given positive constants.

Note [1] that for the existence of a classical solution to the problem having all continuous derivatives caused by the equations and boundary conditions, the agreement conditions must be fulfilled at the points $x = 0$ and $x = X_0$ for $t = 0$:

$$c_0(0) = c^+(0), \quad \frac{dc^+}{dt}(0) + v(0)\frac{dc_0}{dx}(0) = \alpha_c \frac{d^2 c_0}{dx^2}(0), \tag{2.9}$$

$$\left(\frac{dX}{dt}(0) + \beta - v(0)\right) c_0(X_0) + \alpha_c \frac{dc_0}{dx}(X_0) = 0, \tag{2.10}$$

$$\frac{dX}{dt}(0) = \beta\gamma c_0(X_0), \quad v(0) = -\sigma\beta\gamma c_0(X_0). \tag{2.11}$$

Moreover, we will assume that

$$0 < c_* = \text{const} \le c_0(x) \le 1, \quad 0 < x < X_0,\ 0 < c_* \le c^+(t) \le 1,\ t > 0. \tag{2.12}$$

Theorem 1. *Suppose that* $c^+ \in H^{1+\frac{\alpha}{2}}[0,\infty)$, $c_0 \in H^{2+\alpha}[0, X_0]$, *and the agreement conditions* (2.9)–(2.11) *hold. Then, on a sufficiently small interval* $[0, T_*]$, *problem* (2.1)–(2.8) *has the unique classical solution*

$$c \in H^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Omega}_{T_*}), \quad X \in H^{1+\frac{\alpha}{2}}[0, T_*].$$

Theorem 2. *Under the conditions of Theorem* 1, *on an arbitrary time interval* $[0, T]$, *problem* (2.1)–(2.8) *has the unique classical solution*

$$c \in H^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Omega}_T), \quad X \in H^{1+\frac{\alpha}{2}}[0, T].$$

3. Proof of Theorem 1

3.1. Construction of the operator of the problem. Changing the variables and the sought function

$$t = t, \quad y = \frac{x}{X(t)}, \quad u(y,t) = c(x,t)$$

reduce the initial problem in the unknown domain to a problem in the known domain $Q = \{y : 0 < y < 1\}$ for the parabolic equation

$$\frac{\partial u}{\partial t} + \left(\frac{v(t)}{X} - \frac{y}{X}\frac{dX}{dt}\right)\frac{\partial u}{\partial y} = \frac{\alpha_c}{X^2}\frac{\partial^2 u}{\partial y^2}, \quad y \in Q,\ t > 0, \tag{3.1}$$

with unknown coefficients. These coefficients are determined from relations (2.6) and (2.8), which, in the new variables, take the form

$$\frac{dX}{dt}(t) = \beta\gamma u(1,t) \quad \text{for } t > 0, \tag{3.2}$$

$$v(t) = -a u(1,t) \quad \text{for } t > 0, \tag{3.3}$$

where $a = \sigma\beta\gamma$.

The problem of finding $u(y,t)$ is closed by the boundary and initial conditions:

$$\left(\frac{dX}{dt}(t) + \beta - v(t)\right) u(1,t) + \frac{\alpha_c}{X(t)}\frac{\partial u}{\partial y}(1,t) = 0 \quad \text{for } t > 0, \tag{3.4}$$

$$u(0,t) = u^0(t) \equiv c^+(t) \quad \text{for } t > 0, \tag{3.5}$$

$$u(y,0) = u_0(y) \equiv c_0(X_0 y) \quad \text{for } 0 < y < 1, \tag{3.6}$$

$$X(0) = X_0. \tag{3.7}$$

Obviously, problem (3.1)–(3.7) is equivalent to the initial problem (2.1)–(2.8).

For solving the last problem, use the Schauder Fixed Point Theorem [12]. Namely, let

$$\mathfrak{M} = \Big\{ w \in H^{\frac{1+\alpha}{2}}[0,T] : |w - c_0(X_0)|^{(\frac{1+\alpha}{2})}_{[0,T]} \le 1,\ w \ge 0 \Big\},$$

where the Hölder spaces H^l and $H^{l,\frac{l}{2}}$, $l > 1$ is noninteger, are defined in [1] and the time T will be defined below.

Put

$$\bar{u}(1,t) = \Phi(w),$$

where $\bar{u}(y,t)$ is the solution to the linear problem

$$\frac{\partial \bar{u}}{\partial t} + \left(\frac{v(t)}{X} - \frac{y}{X}\frac{dX}{dt} \right) \frac{\partial \bar{u}}{\partial y} = \frac{\alpha_c}{X^2}\frac{\partial^2 \bar{u}}{\partial y^2}, \quad y \in Q,\ 0 < t < T, \tag{3.8}$$

$$\frac{dX}{dt}(t) = \beta\gamma w(1,t), \quad 0 < t < T, \tag{3.9}$$

$$v(t) = -aw(1,t), \quad 0 < t < T, \tag{3.10}$$

$$b(w(1,t))\bar{u}(1,t) + \frac{\alpha_c}{X(t)}\frac{\partial \bar{u}}{\partial y}(1,t) = 0, \quad 0 < t < T, \tag{3.11}$$

$$\bar{u}(0,t) = u^0(t), \quad 0 < t < T, \tag{3.12}$$

$$\bar{u}(y,0) = u_0(y), \quad 0 < y < 1, \tag{3.13}$$

$$X(0) = X_0. \tag{3.14}$$

In (3.11), $b(s) = (\beta\gamma + a)s + \beta$.

Note that equation (3.8) is uniformly parabolic since

$$X_0 \le X(t) \le X_0 + \beta\gamma(c_0(X_0) + T).$$

By this and the corresponding smoothness of the coefficients of the equation, the boundary conditions, and the data of problem (3.8)–(3.14), we have

$$w(1,.), \frac{dX}{dt}, v \in H^{\frac{1+\alpha}{2}}[0,T], \quad |c^+|^{(1+\frac{\alpha}{2})}_{[0,T]}, |c_0|^{(2+\alpha)}_{[0,X_0]} \le C_0,$$

$$\bar{u}(1,.) \in H^{(1+\frac{\alpha}{2})}[0,T], \quad \bar{u} \in H^{(2+\alpha,1+\frac{\alpha}{2})}(Q_T),$$

and

$$|\bar{u}(1,.)|^{(1+\frac{\alpha}{2})}_{[0,T]}, |\bar{u}|^{(2+\alpha,1+\frac{\alpha}{2})}_{[Q_T]} \le C_1,$$

where $Q_T = Q \times (0,T)$ and C_1 depends only on C_0 and T, $C_1 < \infty$ for all $T < \infty$ (see [1, Chapter IV, § 5, Theorem 5.1]).

Thus, the operator $\Phi(w)$ is compact (i.e., $\Phi(w)$ sends every bounded set into a compact set) since the embedding $H^{1+\frac{\alpha}{2}}[0,T] \to H^{\frac{1+\alpha}{2}}[0,T]$ is compact.

It remains to show that $\Phi(w)$ is continuous and sends $\mathfrak{M}$ into itself.

3.2. Continuity of the operator $\Phi(w)$. The continuity of $\Phi(w)$ follows from the continuous dependence of the solution to a parabolic equation on the coefficients of the equation. The uniform continuity of the solution on the coefficients of the equation is a sufficiently known fact but it is practically impossible to refer to a particular work. For proving our assertion, consider the initial-boundary value problem

$$\frac{\partial v}{\partial t} + d(x,t)\,\frac{\partial v}{\partial y} = f(x,t)\,\frac{\partial^2 v}{\partial y^2}, \quad y \in Q,\ t > 0, \tag{3.15}$$

$$\frac{\partial v}{\partial y}(1,t) + g(t)\,v(1,t) = 0, \quad t > 0, \tag{3.16}$$

$$v(0,t) = v^0(t), \quad t > 0, \tag{3.17}$$

$$v(y,0) = v_0(y), \quad 0 < y < 1. \tag{3.18}$$

Prove the continuity of the solution to problem (3.15)–(3.18) with respect to the coefficient f under the assumption of its boundedness from below and above:

$$0 < \beta \le f(x,t) \le \beta^{-1}.$$

The case of the coefficient d is considered similarly. Since we consider classical solutions, assume in addition that

$$|d|^{(\alpha)}_{Q_T} + |f|^{(\alpha)}_{Q_T} + |g|^{(\frac{1+\alpha}{2})}_{[0,T]} + |v^0|^{(1+\frac{\alpha}{2})}_{[0,T]} + |v_0|^{(2+\alpha)}_{Q} \le C_0.$$

Then (see [1, Chapter IV, § 5, Theorem 5.1]) $|v|^{(2+\alpha)}_{Q_T} \le C^*$, where the constant C^* depends only on C_0 and β.

Let v_1 and v_2 be two solutions to problem (3.15)–(3.18) corresponding to two different coefficients f_1 and f_2. The difference $\bar v = v_1 - v_2$ satisfies the initial-boundary value problem

$$\frac{\partial \bar v}{\partial t} + d(x,t)\,\frac{\partial \bar v}{\partial y} = f_1(x,t)\,\frac{\partial^2 \bar v}{\partial y^2} + (f_1(x,t) - f_2(x,t))\frac{\partial^2 v_2}{\partial y^2}, \tag{3.19}$$

$$\frac{\partial \bar v}{\partial y}(1,t) + g(t)\,\bar v(1,t) = 0, \tag{3.20}$$

$$\bar v(0,t) = 0, \tag{3.21}$$

$$\bar v(y,0) = 0, \tag{3.22}$$

in the domain $Q_T = Q \times (0,T)$.

In accordance with [1, Chapter IV, § 5, Theorem 5.1],

$$|\bar v|^{(2+\alpha)}_{Q_T} \le C|f_1 - f_2|^{(\alpha)}_{Q_T}, \tag{3.23}$$

where the constant C depends only on C_0 and β.

The last estimate proves the continuity of $\Phi(w)$.

3.3. The operator $\Phi(w)$ sends $\mathfrak{M}$ into itself. Prove this assertion for a sufficiently small time interval T. The first estimate (the maximum principle) holds for all values of T. Indeed, a positive maximum (a negative minimum) cannot be attained inside the domain (see [1, Chapter I, § 2, Theorem 2.2]). If a positive maximum of the solution is attained on the boundary $y = 1$ then, on it,

$$b(w(1,t)) > 0, \quad \bar u(1,t) > 0, \quad \frac{\partial \bar u}{\partial y}(1,t) > 0,$$

which contradicts (3.11). Thus, a positive maximum of the solution to problem (3.8)–(3.14) is attained either at $t = 0$ or on the boundary $y = 0$. Similarly, a negative minimum can be attained only for $t = 0$ or on the boundary $y = 0$, where the solution is nonnegative. We finally have

$$0 \le \bar u(y,t) \le c^* = \max\{\sup_{0<x<X_0} c_0(x),\ \sup_{0<t<T} c^+(t)\}. \tag{3.24}$$

By the smoothness of $\bar{u}$, its time derivative is bounded: $\frac{\partial \bar{u}}{\partial t}(1,t) \le C_1$,

$$\frac{|\bar{u}(1,t_1) - \bar{u}(1,t_3)|}{|t_1 - t_2|^{\frac{1+\alpha}{2}}} \le \frac{|\bar{u}(1,t_1) - \bar{u}(1,t_3)|}{|t_1 - t_2|} |t_1 - t_2|^{\frac{1-\alpha}{2}} \le C_1 |t_1 - t_2|^{\frac{1-\alpha}{2}}.$$

Consequently,

$$|\bar{u}(1,.) - c_0(X_0)|^{(\frac{1+\alpha}{2})}_{[0,T]} = |\bar{u}(1,.) - \bar{u}(1,0)|^{(\frac{1+\alpha}{2})}_{[0,T]}$$

$$= \max_{0<t<T} |\bar{u}(1,t) - \bar{u}(1,0)| + \max_{0<t<T} \frac{|\bar{u}(1,t) - \bar{u}(1,0)|}{t^{(\frac{1+\alpha}{2})}}$$

$$\le \max_{0<t<T} \frac{|\bar{u}(1,t) - \bar{u}(1,0)|}{t} \cdot \max_{0<t<T} t$$

$$+ \max_{0<t<T} \frac{|\bar{u}(1,t) - \bar{u}(1,0)|}{t} \max_{0<t<T} \cdot t^{(\frac{1-\alpha}{2})} \le C_1 (T + T^{\frac{1-\alpha}{2}})$$

or

$$|\bar{u}(1,.) - c_0(X_0)|^{(\frac{1+\alpha}{2})}_{[0,T]} \le C_1 (T + T^{\frac{1-\alpha}{2}}).$$

For sufficiently small T_* such that

$$C_1 (T_* + T_*^{\frac{1-\alpha}{2}}) \le 1, \tag{3.25}$$

the operator $\mathbf{\Phi}$ takes $\mathfrak{M}$ into itself, and the Schauder Theorem ensures the existence of at least one fixed point for $\mathbf{\Phi}$, which defines the solution to problem (3.8)–(3.14).

4. Proof of Theorem 2

Continuing the process of constructing a classical solution to (3.8)–(3.14), find the maximal time interval $[0, T_\infty)$, $T_\infty \le \infty$, such that

$$\text{either } |u|^{(1+\frac{\alpha}{2})}_{[0,T]} \to \infty \text{ as } T \to T_\infty < \infty \quad \text{or} \quad T_\infty = \infty. \tag{4.1}$$

Theorem 2 is equivalent to the equality $T_\infty = \infty$ or to the fact that

$$|u|^{(1+\frac{\alpha}{2})}_{[0,T]} \le C(T), \tag{4.2}$$

where $C(T) < \infty$ for $T < \infty$.

4.1. A priori estimates of solutions in $W_2^{2,1}(Q_T)$ and $W_p^{1,0}(Q_T)$. Using the boundary conditions, rewrite problem (3.1)–(3.7) as

$$\frac{\partial u}{\partial t} + A \frac{\partial u}{\partial y} = B(t) \frac{\partial^2 u}{\partial y^2}, \quad y \in Q, \; t > 0, \tag{4.3}$$

$$\frac{1}{X(t)} b(u) u + B(t) \frac{\partial u}{\partial y} = 0, \quad y = 1, \; t > 0, \tag{4.4}$$

$$A = \frac{v}{X} \left(1 + \frac{\beta}{a} \gamma y\right), \quad B = \frac{\alpha_c}{X^2}, \tag{4.5}$$

$$\frac{dX}{dt}(t) = \beta \gamma u(1,t), \quad v(t) = -a u(1,t), \quad t > 0, \tag{4.6}$$

$$u(0,t) = u^0(t), \quad v(t) = -a u(1,t), \quad t > 0, \tag{4.7}$$

$$u(y,0) = u_0(y), \quad X(0) = X_0. \tag{4.8}$$

In (4.3)–(4.8),

$$|A| < A_0 = \text{const} < \infty, \quad 0 < B_0^{-1}(T) < B < B_0(T), \tag{4.9}$$

where $B_0(T) < \infty$ for $T < \infty$.

Observe first of all that, by the maximum principle,

$$0 < c_* \le u(y,t) \le U_0 = \max\{\max_{0<y<1} u_0(y), \max_{0<t<T} u^0(t)\}, \left|\frac{dX}{dt}(t)\right| \le C(U_0). \tag{4.10}$$

Multiplying (4.3) by $\frac{\partial^2 u}{\partial y^2}$ and integrating by parts leads to the equality

$$I_0 \equiv B(t) \int_Q \left|\frac{\partial^2 u}{\partial y^2}(y,t)\right|^2 dy + \frac{1}{2}\frac{d}{dt}\int_Q \left|\frac{\partial u}{\partial y}(y,t)\right|^2 dy$$

$$= \int_Q A\frac{\partial^2 u}{\partial y^2}(y,t)\frac{\partial u}{\partial y}(y,t)\,dy - \frac{\partial u}{\partial y}(1,t)\frac{\partial u}{\partial t}(1,t) + \frac{\partial u}{\partial y}(0,t)\frac{du^0}{dt}(t) \equiv I_1 + I_2 + I_3,$$

where

$$I_1 \le \frac{\varepsilon}{2}A_0 \int_Q \left|\frac{\partial^2 u}{\partial y^2}(y,t)\right|^2 dy + \frac{A_0}{2\varepsilon}\int_Q \left|\frac{\partial u}{\partial y}(y,t)\right|^2 dy,$$

$$I_2(t) = -\frac{1}{X(t)B(t)}u(1,t)b(u(1,t))\frac{\partial u}{\partial t}(1,t) = -\frac{d}{dt}\left(\frac{1}{X(t)B(t)}\int_0^u sb(s)\,ds\right)$$

$$+\int_0^u sb(s)\,ds\frac{d}{dt}\left(\frac{1}{X(t)B(t)}\right) \le -\frac{d}{dt}\left(\frac{1}{X(t)B(t)}\int_0^u sb(s)\,ds\right) + C_1(U_0),$$

$$\left|\int_0^T I_2(t)\,dt\right| \le C_2(U_0,T),$$

$$I_3 = \frac{du^0}{dt}(t)\left(\int_0^y \frac{\partial^2 u}{\partial y^2}(s,t)\,ds\right) \le \frac{\varepsilon}{2}C_3(T)\int_Q \left|\frac{\partial^2 u}{\partial y^2}(y,t)\right|^2 dy + \frac{C_3(T)}{2\varepsilon}.$$

Thus, for $\varepsilon > 0$ sufficiently small,

$$B(t)\int_Q \left|\frac{\partial^2 u}{\partial y^2}(y,t)\right|^2 dy + \frac{d}{dt}\int_Q \left|\frac{\partial u}{\partial y}(y,t)\right|^2 dy \le C_4(U_0,T)\int_Q \left|\frac{\partial u}{\partial y}(y,t)\right|^2 dy + C_5(U_0,T).$$

The lass inequality, Gronwall's inequality [1], and (4.3) give the basic a priori estimates

$$\int_{Q_T}\left(\left|\frac{\partial^2 u}{\partial y^2}(y,t)\right|^2 + \left|\frac{\partial u}{\partial t}(y,t)\right|^2\right) dydt + \max_{0<t<T}\int_Q \left|\frac{\partial u}{\partial y}(y,t)\right|^2 dy \le C_6(U_0,T), \tag{4.11}$$

where $C_6(U_0,T) < \infty$ for $T < \infty$.

Turning to the embedding theorem of $\mathbb{W}_2^{2,1}(Q_T)$ into $\mathbb{W}_p^{1,0}(Q_T)$ [1, Chapter I, § 3, Lemma 3.3], we finally get

$$\int_{Q_T}\left|\frac{\partial u}{\partial y}(y,t)\right|^p dydt \le C_7(U_0,T), \tag{4.12}$$

where $p \le 6$ and $C_7(U_0,T) < \infty$ for $T < \infty$.

Let us formulate the obtained result as a lemma:

Lemma 1. *Under the conditions of Theorem* 1, *the solution to problem* (4.3)–(4.8) *satisfies* (4.11) *and* (3.12).

4.2. A priori estimates of solutions in $\mathbb{H}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)$. First if all, introduce the new sought function

$$w(y,t) = \int\limits_{c_*}^{u(y,t)} \frac{ds}{sb(s)}, \quad u(y,t) = F(w(y,t)). \tag{4.13}$$

By construction,

$$0 \le w(y,t) \le w^* = \int\limits_{c_*}^{U_0} \frac{ds}{sb(s)}, \tag{4.14}$$

$$0 < F_* = c_* b(c_*) \le \frac{dF}{dw}(w(y,t)), \quad \left|\frac{dF}{dw}(w(y,t)\right|, \left|\frac{d^2F}{dw^2}(w(y,t)\right| \le C_7(U_0). \tag{4.15}$$

The function $w(y,t)$ satisfies the following initial-boundary value problem in Q_T:

$$\frac{\partial w}{\partial t} + A\frac{\partial w}{\partial y} = B(t)\frac{\partial^2 w}{\partial y^2} + C\left(\frac{\partial w}{\partial y}\right)^2, \quad y \in Q_T, \tag{4.16}$$

$$\frac{\partial w}{\partial y}(1,t) = -\frac{1}{B(t)X(t)}, \quad 0 < t < T, \tag{4.17}$$

$$w(1,t) = u^0(t),\ 0 < t < T, \quad w(y,0) = u_0(y),\ 0 < y < 1. \tag{4.18}$$

In (4.16),

$$C = B\frac{d^2F}{dw^2}(w)\left(\frac{dF}{dw}(w)\right)^{-1}, \quad |C| < C_8(c_*, U_0). \tag{4.19}$$

Finally, the new function

$$z = w + \frac{y}{B(t)X(t)}$$

satisfies in Q_T to the equation

$$\frac{\partial z}{\partial t} + A\frac{\partial z}{\partial y} = B(t)\frac{\partial^2 z}{\partial y^2} + f \tag{4.20}$$

with the right-hand side

$$f = C\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left(\frac{y}{B(t)X(t)}\right) + \frac{\partial}{\partial t}\left(\frac{y}{B(t)X(t)}\right)$$

belonging (see estimate (4.12)) to $\mathbb{L}_3(Q_T)$, the boundary conditions, the homogeneous boundary condition

$$z(0,t) = u^0(t), \quad \frac{\partial z}{\partial y}(1,t) = 0, \quad 0 < t < T, \tag{4.21}$$

and the initial condition

$$\frac{\partial z}{\partial y}(1,t) = 0, \quad 0 < y < 1. \tag{4.22}$$

Regarding (4.21), (4.22) as a mixed initial-boundary value problem in $\mathbb{W}_3^{2,1}(Q_T)$, we see from [1, Chapter VI, § 9, Theorem 9.1] that $z \in \mathbb{W}_3^{2,1}(Q_T)$. The embedding theorem of $\mathbb{W}_3^{2,1}(Q_T)$ into $\mathbb{W}_p^{1,0}(Q_T)$

(see [1, Chapter I, § 3, Lemma 3.3]) shows that $\mathbb{W}_p^{1,0}(Q_T) \subset \mathbb{W}_3^{2,1}(Q_T)$ for any $1 < p < \infty$. Thus, $f \in \mathbb{L}_p(Q_T)$ for any $1 < p < \infty$. This implies the embedding $z \in \mathbb{W}_p^{2,1}(Q_T)$ with arbitrary p, $1 < p < \infty$ (see [1, Chapter VI, § 9, Theorem 9.1]. In particular, for $p > \frac{3}{1-\lambda}$, the space $\mathbb{H}^{1+\lambda,\frac{1+\lambda}{2}}(\overline{Q}_T)$ is included in the space $\mathbb{W}_p^{2,1}(Q_T)$ (see [1, Chapter I, § 3, Lemma 3.3]), which means that $\frac{\partial z}{\partial y} \in \mathbb{H}^{\lambda,\frac{\lambda}{2}}(\overline{Q}_T)$, $z(1,.) \in \mathbb{H}^{\frac{\lambda}{2}}[0,T]$, and $X \in \mathbb{H}^{1+\frac{\lambda}{2}}[0,T]$ for all $\lambda < 1$.

Returning to problem (4.20)–(4.22) and reckoning with the results of [1, Chapter IV, § 5, Theorem 5.1] on the solvability of the main initial-boundary value problems for parabolic equations in the Hölder spaces $\mathbb{H}^{2+\lambda,1+\frac{\lambda}{2}}(\overline{Q}_T)$, we verify that, both in the mixed initial-boundary value problem with right-hand side f and coefficients A and B from $\mathbb{H}^{\lambda,\frac{\lambda}{2}}(\overline{Q}_T)$ for $\lambda = \alpha$, the solution z belongs to $\mathbb{H}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)$.

By (4.13) and the smoothness of F, we finally have $u \in \mathbb{H}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)$. Thus, $T_\infty = \infty$, which proves the theorem.

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