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Coupling and Computation of Electromagnetism and Mechanics

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Accurate coupling of electromagnetism and mechanics is of interest in computations of transducers such as piezoelectric, pyroelectric, electromagnetic sensors and actuators. Balance equations in mechanics as well as the MAXWELL equations for electromagnetism have been established in science. However, if the coupling between these governing equations are necessary, several difficulties arise. Herein we identify the challenges and propose possible solutions for computational analysis.

1 Electromagnetic Momentum

Also from the daily life, we know that mechanical and electric energy are transferable. Even a small electric current generates a significant muscle contraction as a consequence of a piezoelectric coupling in biological tissue. Modeling such a behavior is an engineering task and a constitutive equation relating a deformation measure like strain with electric field is sufficient, see for example applications computed in [1–3]. Another very well known mechanics-electromagnetism coupling is a rotor used for vibrating a smartphone supplied by the battery storing electric energy.

We aim at an accurate simulation of such a multiphysics phenomenon, hence, we investigate the interaction term causing the correct coupling between mechanics and electromagnetism. As easily stated, the path ending at the correct coupling is very challenging, there is a long debate in the literature known as the ABRAHAM–MINKOWSKI controversy. Simply, we fail to know the correct modeling of the interaction term, there exists no consensus in the scientific community, see for example [4–7].

For a clear separation of concepts, we start with a massive particle with a "mechanical" momentum, $\rho\boldsymbol{v}$, with the mass density, ρ, and velocity, $\boldsymbol{v}$, of the deforming matter by fulfilling the balance of mechanical momentum:

$$\rho v_i^{\cdot} - \sigma_{ji,j} - \rho f_i = \mathcal{F}_i \,, \tag{1}$$

where we use EINSTEIN's summation convention over repeated indices, comma notation means a partial derivative in space, the dot identifies an objective rate identical to the material time derivative herein. The known function, $\boldsymbol{f}$, is a specific body force in N/kg because of the gravity. For the CAUCHY stress, $\boldsymbol{\sigma}$ in N/m^2, we need to define a constitutive equation. An engineer models with that term the aforementioned effects, it is even possible to derive the stress by using principles of thermodynamics, we refer to [8] in small deformations and to [9] for large deformations. The interaction is given by the electromagnetic force density, $\boldsymbol{\mathcal{F}}$ in N/m^3, and indeed one of the main challenges relies on the correct definition of this force density. A particular approach for modeling this force density is based on the following identity:

$$\mathcal{F}_i = m_{ji,j} - \frac{\partial \mathcal{G}_i}{\partial t} \,, \tag{2}$$

where we express the vector as a divergence in space-time. This representation is very generally applicable to all quantities, which is also called NOETHER's theorem. Because of the units, $\boldsymbol{m}$ is an electromagnetic stress and $\boldsymbol{\mathcal{G}}$ is an electromagnetic momentum. There is no consensus for "the choice" of the electromagnetic momentum. Hence, we clearly emphasize that any used definition is a model, not the law of physics, moreover, different choices are adequate [10]. In the case of no polarization, POYNTING's momentum, $\boldsymbol{\mathcal{G}} = \boldsymbol{D} \times \boldsymbol{B}$, is often chosen—with the *total* charge potential, $\boldsymbol{D}$ in C/m^2, and magnetic flux (area density), $\boldsymbol{B}$ in T—leading to the MAXWELL stress and LORENTZ force density,

$$m_{ji} = -\frac{1}{2}\delta_{ji}(H_k B_k + D_k E_k) + H_i B_j + D_j E_i \,, \quad \mathcal{F}_i = \rho z E_i + \epsilon_{ijk} J_j B_k \,. \tag{3}$$

In the case of a polarized body, the choice is difficult and there are several possibilities. After a decomposition of the electric charge into free and bound charges, we obtain charge and current potentials separated to potentials because of free charges, $\mathfrak{D}$, $\mathfrak{H}$, as well as effected by bound charges, $\boldsymbol{P}$, $\boldsymbol{\mathcal{M}}$, with the charge potential, $\boldsymbol{D} = \mathfrak{D} - \boldsymbol{P}$, and the current potential, $\boldsymbol{H} = \mathfrak{H} + \boldsymbol{\mathcal{M}}$. For example by choosing MINKOWSKI's momentum density, $\boldsymbol{\mathcal{G}} = \mathfrak{D} \times \boldsymbol{B}$, we obtain the following electromagnetic stress and force density,

$$\begin{aligned} m_{ji} &= -\frac{1}{2}\delta_{ji}(H_k B_k + D_k E_k) + H_i B_j + D_j E_i \,, \\ \mathcal{F}_i &= \rho z E_i + \epsilon_{ijk} J_j B_k - \epsilon_{ijk} \frac{\partial P_j}{\partial t} B_k - \epsilon_{ijk} P_j \frac{\partial B_k}{\partial t} \,. \end{aligned} \tag{4}$$

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In this case, a thermodynamical study is possible to obtain the correct modeling of constitutive equations, including stresses, $\boldsymbol{\sigma}$, $\boldsymbol{m}$, electric current, $\boldsymbol{J}^{\text{fr.}}$, magnetic polarization, $\boldsymbol{\mathcal{M}}$, electric polarization, $\boldsymbol{P}$, we refer to [11, Chap.3]. By choosing ABRAHAM's momentum density, $\boldsymbol{\mathcal{G}} = \boldsymbol{E} \times \boldsymbol{\mathfrak{H}}$, the same type of study has been obtained in [12, Chap. XIII].

2 Computation of Fields

For a successful computation of deformation and electromagnetic fields, several computational algorithms exist in the literature. Especially the computation of electromagnetic fields is difficult as there are two possible ways, both having their own numerical challenges. One approach is solving the MAXWELL equations numerically to obtain the electromagnetic fields, $\boldsymbol{E}$ and $\boldsymbol{B}$, see [13, Sect. 17] for a thorough discussion. Another approach is solving two MAXWELL equations directly by using ansatz functions leading to electromagnetic potentials, ϕ in V and $\boldsymbol{A}$ in T m. In this case, gauges of these electromagnetic potentials need to be selected by giving the value of divergence of magnetic potential, $A_{i,i}$, and the rate of electric potential, $\partial\phi/\partial t$. The choice is arbitrary such that in analytical mechanics the GAUSS gauge is often used by setting both zero. From the numerical point of view, the choice of gauge condition is of significance. By using the LORENZ gauge:

$$\frac{\partial \phi}{\partial t} = -c^2 A_{i,i} \,, \quad \text{with } c^2 = \frac{1}{\mu_0 \varepsilon_0} \,, \tag{5}$$

a more robust computational algorithm is possible, we refer to [14] for a comparison of an analytical solution to this computational approach based on the finite element method in space and finite difference method in time—with a time step, Δt and variables from the last time step denoted by ^{0}. For computing displacement, $\boldsymbol{u}$, electromagnetic potentials, ϕ, $\boldsymbol{A}$, in [14], the following weak forms have been used by summing the global forms over each finite element, Ω. Especially at the interface between finite elements, $\partial\Omega^I$, the jump conditions in electromagnetism are of paramount importance leading to the terms in jump brackets $[\![\cdot]\!]$ as the difference of functions evaluated in neighboring elements. The weak form $\mathrm{F} = \mathrm{F}_{\boldsymbol{u}} + \mathrm{F}_{\phi} + \mathrm{F}_{\boldsymbol{A}}$ is nonlinear and coupled given by

$$\begin{aligned}
\mathrm{F}_{\boldsymbol{u}} &= \sum_{\text{ele}} \int_{\Omega} \left(\rho \frac{u_i - 2u_i^0 + u_i^{00}}{\Delta t^2} \delta u_i + \sigma_{ji} \delta u_{i,j} - \rho f_i \delta u_i - \mathcal{F}_i \delta u_i \right) \mathrm{d}V + \int_{\partial\Omega^I} n_j [\![m_{ji}]\!] \delta u_i \, \mathrm{d}A \,, \\
\mathrm{F}_{\phi} &= \sum_{\text{ele}} \int_{\Omega} \Big(-(\mathfrak{D}_i - \mathfrak{D}_i^0) \delta\phi_{,i} - \Delta t J_i^{\text{fr.}} \delta\phi_{,i} - \Delta t \epsilon_{ijk} \mathcal{M}_{k,j} \delta\phi_{,i} \Big) \mathrm{d}V + \int_{\partial\Omega^I} \Delta t n_i \big([\![J_i^{\text{fr.}}]\!] + \epsilon_{ijk} [\![\mathcal{M}_{k,j}]\!] \big) \delta\phi \, \mathrm{d}A \,, \\
\mathrm{F}_{\boldsymbol{A}} &= \sum_{\text{ele}} \int_{\Omega} \left(\varepsilon_0 \frac{A_i - 2A_i^0 + A_i^{00}}{\Delta t^2} \delta A_i + \frac{1}{\mu_0} A_{i,j} \delta A_{i,j} - J_i^{\text{fr.}} \delta A_i - \frac{P_i - P_i^0}{\Delta t} \delta A_i + \epsilon_{ijk} \mathcal{M}_k \delta A_{i,j} \right) \mathrm{d}V \,.
\end{aligned} \tag{6}$$

3 Conclusion

For an accurate computation of coupled electromagnetism and mechanics, governing equations are not uniquely defined leading to several methodologies existing in the literature. A systematic and transparent study of taken assumptions is expected to increase the understanding as well as to deliver robust and accurate methods.

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