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Thin-layer inertial effects in plasticity and dynamics in the Prandtl problem

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Especially in metal forming, large plastic deformation occurs in thin plates. The problem of compressing dies is analyzed to evaluate the spreading of a thin layer in between. The velocity of dies is a given function in time so that the kinematics of the process is known. This problem can be considered as a generalization of the classical Prandtl problem by taking inertial effects into account and introducing dimensionless parameters as internal variables depending on time. The first parameter is purely geometric corresponding to the thin-layer approximation; the second and the third parameters are dimensionless velocity and acceleration during the dies getting pressed. We use singular asymptotic expansions of unknown functions and study how these parameters vary preceding the dies of moment. Depending on this relation, the dynamic corrections to the quasistatic solution is a part of various terms of the asymptotic series. The corresponding analytical investigation both for general case and for particular typical regimes of plates motion is carried out.

KEYWORDS
analytical mechanics, inertial effects, plasticity, thin layers

Almost a century has passed since the publication of Ludwig Prandtl's pioneering work[1] in *Zeitschrift für Angewandte Mathematik und Mechanik*, which was a groundbreaking study in the theory of plasticity of a thin layer. Among the many generalizations of the classical Prandtl problem that have evolved over the course of this century, is of great interest to take thin-layer inertial effects into account both from the theoretical and from the practical point of view. These effects significantly affect the pressure distribution in the layer and, ultimately, the total force to be applied to the pressure plates.

1 | STATEMENT OF THE DYNAMIC PRANDTL PROBLEM

Let us consider an incompressible plane flow of a material of mass density ρ, described by perfect plasticity with a yield stress σ_s, which is realized in a thin rectangular layer Ω_t, $t \geq 0$,

$$\Omega_t = \{-l(t) < x_1 < l(t), \ -h(t) < x_2 < h(t)\}, \quad h(t) \ll l(t),$$

$$h(0) = h_0, \quad l(0) = l_0, \quad l(t)h(t) \equiv l_0 h_0 = S, \tag{1}$$

where S is constant in time giving the area of the layer moving as a result of the incompressible flow.

The sides $x_2 = \pm h$ of the plastic material are adhering to rigid plates (dies in a metal forming process) along the axis x_1, where both plates move toward each other. Surface roughness of these plates are given by a constant, m_0, we assume it is known, $0 < m_0 \leq 1$. Tied coupling of the plates and the plastic material is achieved by clamping them with $m_0 = 1$. The pressing velocity $V(t)$ of each of the plates is a given positive function in time related to a monotonically decreasing function $h(t)$,

$$h(t) = h_0 - \int_0^t V(\xi)\,d\xi, \quad t \geq 0. \tag{2}$$

We stress that the coupled system of plane theory of perfect plastic flow consists the equations of motion:

$$-p_{,1} + s_{11,1} + s_{12,2} = \rho(v_{1,t} + v_{1,1}v_1 + v_{1,2}v_2), \quad -p_{,2} - s_{11,2} + s_{12,1} = \rho(v_{2,t} + v_{2,1}v_1 + v_{2,2}v_2), \tag{3}$$

where a comma notation indicates a partial derivative with respect to the corresponding variables x_1, x_2, or t. Here p is the pressure, s_{11} and s_{12} are the components of stress deviator tensor (note that $s_{22} = -s_{11}$ because any deviator has zero trace and $s_{33} \equiv 0$ for plane deformation), v_1 and v_2 are the components of velocity vector.

Analogously to Prandtl's work,[1] we confine the analysis to consider tensor linear materials, in other words, the stress deviator tensor is proportional to the strain rate tensor component-wise (co-linearity). In the case of plane deformation, this restriction imposes an independent condition:

$$s_{11}(v_{1,2} + v_{2,1}) = 2s_{12}v_{1,1}. \tag{4}$$

The Mises–Hencky criterion of plasticity,

$$s_{11}^2 + s_{12}^2 = \frac{\sigma_s^2}{2} \equiv \tau_s^2, \tag{5}$$

is used in order employ the constant $\tau_s = \sigma_s/\sqrt{2}$, as shear yield stress; using τ_s is more convenient for calculation in problems on plane deformation than using σ_s.

At last the condition of incompressibility is noted,

$$v_{1,1} + v_{2,2} = 0. \tag{6}$$

The functions p, s_{11}, s_{12}, v_1, and v_2 are the unknowns in the system (3)–(6).

On the surface of pressing plates (dies), which is tantamount to top and bottom of the domain Ω_t, we have both the impenetrability conditions,

$$v_2\big|_{x_2=-h} = V(t), \quad v_2\big|_{x_2=h} = -V(t), \tag{7}$$

and the requirements that the absolute value of the shear stress reaches its supremum,

$$|s_{12}|_{x_2=\pm h} = m_0\tau_s. \tag{8}$$

On the right and left ends of the domain, $x_1 = \pm l(t)$ of Ω_t, exact boundary conditions are not prescribed. Region near these ends are to be interpreted as boundary effect zones at a distance of order h. The system (3)–(6) with the boundary conditions (7), (8) represents the formulation of the dynamic Prandtl problem. One can refer to it as a conditional initial-boundary-value problem, because initial conditions are absent in an explicit form. A solution is sought corresponding to compression along the axis x_2 as well as for arbitrary values on both sides along the axis x_1.

The quasistatic statement differs from (3)–(8) in that the right parts of the equations of motion (3) are equal to zero, i.e., the time, t, becomes a parameter implicitly contained in the solution in terms of h and l. The equations of motion (3) then turn into equilibrium equations.

The first results of analytical investigation of the quasistatic problem are due to Ludwig Prandtl who noticed[1] that the stresses

$$s_{11} = \frac{\tau_s}{h}\sqrt{h^2 - x_2^2}, \quad s_{12} = -\frac{\tau_s}{h}x_2, \quad p = p_0 - \frac{\tau_s}{h}\left(x_1 + \sqrt{h^2 - x_2^2}\right), \tag{9}$$

where p_0 is pressure—simply the hydrostatic pressure effected by the incompressibility—and the stresses comply with equations of equilibrium as well as the criterion of plasticity (1.5), and the boundary conditions (8) if $m_0 = 1$. He also constructed the slip lines in form of two families of cycloids being orthogonal to each other.

Later in 1934, William Prager and Hilda Geiringer described[2] the velocity field

$$v_1 = \frac{V}{h}\left(x_1 + 2\sqrt{h^2 - x_2^2}\right), \quad v_2 = -\frac{V}{h}x_2, \quad V = \text{const} \tag{10}$$

compliant with the Equations (4), (6) by taking (9) as well as with the kinematic conditions (7) into account.

In many aspects the "analytical history" of the Prandtl problem begins with the formulae (9) and (10). Its presentation in monographs and handbooks including various generalizations[3–11] is based on certain natural hypotheses for the force and kinematics such that the shear stress, s_{12}, and the velocity component v_2 both are linear along the thickness (the Prandtl hypotheses). We may ask as to whether asymptotic solutions different from (9) and (10) of this problem exist, where these hypotheses are not fulfilled. Nonlinearity and absence of theorems on the uniqueness do not make it possible to answer this question negatively *a fortiori*. By means of asymptotic analysis, where the natural geometric parameter, h/l, was used, an exact solution—in the sense of a finite number of terms in expansions—was obtained,[12,13] which turned out to be the same as Prandtl's quasistatic solution for an arbitrary coefficient $m_0 \in (0; 1]$. The works[14,15] are devoted to the viscoplastic analogue of the Prandtl problem and transition from a highly viscous fluid to a rigid solid. We also mention the similar generalizing investigations considering plastic flows with damage[16] and cyclic loading.[17]

The formulae for stresses (9) and velocities (10) apply only if $x_1 > 0$. This case follows from the natural requirement of even functions in x_1, specifically p, s_{11}, and v_2; as well as of odd functions in x_1, namely s_{12} and v_1. In addition, by virtue of the layer Ω_t, the functions p, s_{11} and v_1 should be even in x_2, while the functions s_{12} and v_2 should be odd in x_2. By generalizing the solution to the whole layer Ω_t, while observing the mentioned requirements for symmetry and arbitrariness of the roughness coefficient m_0, we can write

$$s_{11} = \frac{\tau_s}{h}\sqrt{h^2 - m_0^2 x_2^2}, \quad s_{12} = -\frac{\tau_s}{h}sm_0 x_2, \quad p = p_0 + \frac{\tau_s}{h}\left(m_0(l - |x_1|) - \sqrt{h^2 - m_0^2 x_2^2}\right),$$

$$v_1 = \frac{V}{h}\left(x_1 + \frac{2s}{m_0}\sqrt{h^2 - m_0^2 x_2^2}\right), \quad v_2 = -\frac{V}{h}x_2, \quad s = \text{sign } x_1, \tag{11}$$

where p_0 is a regular hydrostatic constant not going to infinity if $h \to 0$ unlike p_0 in (9).

The presence of the sign function s confirms the irregularity of the solution (1.11), (1.12) near the middle section $x_1 = 0$ and its validity only at a distance from this section. The first investigations dealing with a deduction of a sufficiently smooth solution for the whole domain Ω_t were carried out by H. Geiringer (for the history of the problem, see the work by R. Hill[3]).

When returning to the dynamic statement (3)–(8), it becomes necessary to consider the importance of taking inertial terms into account, in particular for simulation of high-speed plastic flows in thin layers, as it was repeatedly emphasized by Alexei A. Il'yushin.[18] It is shown[19,20] that in dynamic regimes, the pressure fails to depend on x_1 linearly, as in (9) and (10), but it becomes a quadratic function instead. This change leads to an increase of the total force acting on the rigid plates (dies) that is an important design criteri. The influence of inertia affects mainly the spherical part of stress and considerably less the deviatoric part.

In applications like metal forming or forging, especially for the quasistatic approximation,[21] the range of parameter variation reads

$$\frac{1}{\text{Eu}} \ll \frac{h^2(t)}{l^2(t)} \ll 1, \tag{12}$$

where $\text{Eu} = \tau_s/(\rho V^2)$ is the Euler number, which is equal to ratio of characteristic stress to characteristic (dynamic) pressure. This number plays an important role for scaling the model in dynamic plasticity. If $V = \text{const}$ then $1/\text{Eu}$ is constant with an order of smallness in the geometric parameter, $h^2(t)/l^2(t)$, which increases infinitely over time as $t \to t^* = h_0/V$. Hence even for very low plate velocities there is a threshold in time, t^*, beyond which the first inequality in (12) fails to be satisfied. After this threshold, the influence of the inertial terms becomes significant and need to be incorporated in the analysis.

In this paper we consider a more general case when $V(t)$ is a function in time and $h(t)$ decreases nonlinearly as given by the formula (2). We will investigate whether pressing regimes exist, in which one reaches arbitrarily small thickness, h, while not falling outside the limits of quasistatics.

2 | ASYMPTOTIC EXPANSIONS

Let us introduce three smallness parameters explicitly depending on time as follows:

$$0 < \alpha(t) = \frac{h(t)}{l(t)} \ll 1, \qquad 0 < \varepsilon_1(t) = \frac{\rho V^2(t)}{\tau_s} \ll 1, \qquad 0 \le \varepsilon_2(t) = -\frac{\rho \dot{V}(t) h(t)}{\tau_s} \ll 1. \tag{13}$$

The functions h, l, V uniquely determine all three functions α, ε_1, ε_2. At different time intervals, the order of smallness of α with respect to ε_1 and ε_2 may change. This change influences the significance of inertial terms in the equations of motion (3).

We write the following expansions of five unknown functions in (3)–(6) in asymptotic series by using integer powers of α as follows:

$$v_1(x_1, x_2, t) = V(t) \sum_{n=-1}^{\infty} \alpha^n \bar{v}_1^{\{n\}}(\eta_1, \eta_2, \tau), \quad v_2(x_1, x_2, t) = V(t) \sum_{n=0}^{\infty} \alpha^n \bar{v}_2^{\{n\}}(\eta_1, \eta_2, \tau),$$

$$s_{11;12}(x_1, x_2, t) = \tau_s \sum_{n=0}^{\infty} \alpha^n \bar{s}_{11;12}^{\{n\}}(\eta_1, \eta_2, \tau), \quad p(x_1, x_2, t) = \tau_s \sum_{n=-1}^{\infty} \alpha^n \bar{p}^{\{n\}}(\eta_1, \eta_2, \tau),$$

$$\eta_1 = \frac{x_1}{l(t)} = \frac{\alpha(t)}{h(t)} x_1, \quad \eta_2 = \frac{x_2}{h(t)}, \quad \tau = \frac{V(t)}{h(t)} t. \tag{14}$$

The normalized coefficients, $\bar{v}_1$, $\bar{v}_2$, $\bar{s}_{11}$, $\bar{s}_{12}$, $\bar{p}$, in the series (14) are dimensionless and depend on normalized coordinates, η_1, η_2, and on normalized time τ. The domain of interest, Ω_t, is a square, $\bar{\Omega} = \{-1 < \eta_1 < 1, \ -1 < \eta_2 < 1\}$, constant in time. For non-degeneracy of the change (14) of independent variables $(x_1, x_2, t) \to (\eta_1, \eta_2, \tau)$, it is necessary to require that the function τ gets equal to zero at $t = 0$ and increase monotonically over the entire time interval under consideration. It should be noted that the expansions (14) of the variables v_1 and p are singular when $\alpha \to 0$ whereas the deviatoric part of stress, s_{11} and s_{12}, and the velocity, v_2, remain bounded just as in the classical solution (11).

It follows from (14) that

$$\frac{\partial}{\partial x_1} = \frac{1}{l} \frac{\partial}{\partial \eta_1} = \frac{\alpha}{h} \frac{\partial}{\partial \eta_1}, \quad \frac{\partial}{\partial x_2} = \frac{1}{h} \frac{\partial}{\partial \eta_2}, \quad \frac{\partial \alpha}{\partial t} = -\frac{2V}{h} \alpha,$$

$$\frac{\partial}{\partial t} = \frac{\partial \eta_1}{\partial t} \frac{\partial}{\partial \eta_1} + \frac{\partial \eta_2}{\partial t} \frac{\partial}{\partial \eta_2} + \frac{\partial \tau}{\partial t} \frac{\partial}{\partial \tau} = -\frac{V \eta_1}{h} \frac{\partial}{\partial \eta_1} + \frac{V \eta_2}{h} \frac{\partial}{\partial \eta_2} + \frac{V}{h}(1 + \tau) \frac{\partial}{\partial \tau} + \frac{\dot{V} \tau}{V} \frac{\partial}{\partial \tau}. \tag{15}$$

We substitute the series (14) in the five Equations (3)–(6) within Ω_t and in the boundary conditions (7), (8). By observing the formulae (15), we receive the following system consisting of the equations of motion:

$$-\sum_{n=0}^{\infty} \alpha^n \bar{p}_{,1}^{\{n-1\}} + \sum_{n=0}^{\infty} \alpha^{n+1} \bar{s}_{11,1}^{\{n\}} + \sum_{n=0}^{\infty} \alpha^n \bar{s}_{12,2}^{\{n\}} = \varepsilon_1 \left[\sum_{n=-1}^{\infty} \alpha^n \left(-2n\bar{v}_1^{\{n\}} - \eta_1 \bar{v}_{1,1}^{\{n\}} + \eta_2 \bar{v}_{1,2}^{\{n\}} + (1+\tau)\bar{v}_{1,\tau}^{\{n\}} \right) \right.$$

$$\left. + \sum_{n=0}^{\infty} \alpha^n \bar{v}_{1,1}^{\{n-1\}} \cdot \sum_{n=-1}^{\infty} \alpha^n \bar{v}_1^{\{n\}} + \sum_{n=-1}^{\infty} \alpha^n \bar{v}_{1,2}^{\{n\}} \cdot \sum_{n=0}^{\infty} \alpha^n \bar{v}_2^{\{n\}} \right] - \varepsilon_2 \sum_{n=-1}^{\infty} \alpha^n \left(\bar{v}_1^{\{n\}} + \tau \bar{v}_{1,\tau}^{\{n\}} \right),$$

$$-\sum_{n=-1}^{\infty} \alpha^n \bar{p}_{,2}^{\{n\}} - \sum_{n=0}^{\infty} \alpha^n \bar{s}_{11,2}^{\{n\}} + \sum_{n=0}^{\infty} \alpha^{n+1} \bar{s}_{12,1}^{\{n\}} = \varepsilon_1 \left[\sum_{n=0}^{\infty} \alpha^n \left(-2n\bar{v}_2^{\{n\}} - \eta_1 \bar{v}_{2,1}^{\{n\}} + \eta_2 \bar{v}_{2,2}^{\{n\}} + (1+\tau)\bar{v}_{2,\tau}^{\{n\}} \right) \right.$$

$$\left. + \sum_{n=0}^{\infty} \alpha^{n+1} \bar{v}_{2,1}^{\{n\}} \cdot \sum_{n=-1}^{\infty} \alpha^n \bar{v}_1^{\{n\}} + \sum_{n=0}^{\infty} \alpha^n \bar{v}_{2,2}^{\{n\}} \cdot \sum_{n=0}^{\infty} \alpha^n \bar{v}_2^{\{n\}} \right] - \varepsilon_2 \sum_{n=0}^{\infty} \alpha^n \left(\bar{v}_2^{\{n\}} + \tau \bar{v}_{2,\tau}^{\{n\}} \right), \tag{16}$$

the condition of proportionality of the stress deviator and the strain rate tensors:

$$\sum_{n=0}^{\infty} \alpha^n \bar{s}_{11}^{\{n\}} \cdot \left(\sum_{n=-1}^{\infty} \alpha^n \bar{v}_{1,2}^{\{n\}} + \sum_{n=0}^{\infty} \alpha^{n+1} \bar{v}_{2,1}^{\{n\}} \right) = 2 \sum_{n=0}^{\infty} \alpha^n \bar{s}_{12}^{\{n\}} \cdot \sum_{n=0}^{\infty} \alpha^n \bar{v}_{1,1}^{\{n-1\}}, \tag{17}$$

the Mises–Hencky criterion of plasticity:

$$\left(\sum_{n=0}^{\infty} \alpha^n \bar{s}_{11}^{\{n\}}\right)^2 + \left(\sum_{n=0}^{\infty} \alpha^n \bar{s}_{12}^{\{n\}}\right)^2 = 1, \tag{18}$$

the condition of incompressibility:

$$\sum_{n=0}^{\infty} \alpha^n \left(\bar{v}_{1,1}^{\{n-1\}} + \bar{v}_{2,2}^{\{n\}}\right) = 0, \tag{19}$$

and the boundary conditions:

$$\eta_2 = \mp 1 : \quad \bar{v}_2^{\{0\}} = \pm 1, \quad \bar{v}_2^{\{1\}} = \bar{v}_2^{\{2\}} = \cdots = 0$$

$$\eta_2 = \mp 1 : \quad \left|\bar{s}_{12}^{\{0\}}\right| = m_0, \quad \bar{s}_{12}^{\{1\}} = \bar{s}_{12}^{\{2\}} = \cdots = 0. \tag{20}$$

For the normalized quantities, comma denotes a partial derivative in normalized coordinates, η_1, η_2, and τ. Also the normalized parameters, $\varepsilon_1(t)$ and $\varepsilon_2(t)$, which contribute to inertial terms, occur only in the equations of motion (16).

The asymptotic integration method relies on sequential solving of the closed system of equations in order to determine $\bar{v}_1^{\{n\}}$, $\bar{v}_2^{\{m\}}$, $\bar{p}^{\{n\}}$, $\bar{s}_{11}^{\{m\}}$, and $\bar{s}_{12}^{\{m\}}$, where $n = -1, 0, 1, \ldots$; $m = 0, 1, \ldots$, by equating the coefficients of integer powers of α in (2.5)–(2.11). Hence it follows immediately from (17) that $\bar{v}_{1,2}^{\{-1\}} = 0$ and then from (19) $\bar{v}_{2,22}^{\{0\}} = 0$. By taking the boundary conditions (20) into account, we have

$$\bar{v}_2^{\{0\}} = -\eta_2, \quad \bar{v}_1^{\{-1\}} = \eta_1. \tag{21}$$

The components (21) of the velocity field correspond to the evident kinematics for biaxial compressing-spreading of a thin plane layer. They can accommodate arbitrary relations of smallness orders in the dimensionless parameters $\alpha(t)$, $\varepsilon_1(t)$ and $\varepsilon_2(t)$.

3 | POWER DEPENDENCE OF THE FUNCTIONS ε_1 AND ε_2 ON α

We suppose that at a certain time interval in the deformation process the following relations for the order of the smallness parameter hold

$$\varepsilon_1(t) = C(t)\alpha^{b_1}(t), \quad \varepsilon_2(t) = D(t)\alpha^{b_2}(t), \quad b_1 > 0, \ b_2 > 0, \tag{22}$$

where $C(t) = O(1)$, $D(t) = O(1)$ for $\alpha \to 0$ and b_1, b_2 are indices (powers) typical for the given time interval. From the analysis of the principal (minimal) powers of α on the left-hand and right-hand sides of the equations of motion (16), if $b_1 > 2$ and $b_2 > 2$ then the quasistatic solution (11) with nonzero coefficients (21) follows such that

$$\bar{p}^{\{-1\}} = m_0(1 - |\eta_1|), \quad \bar{p}^{\{0\}} = -\sqrt{1 - m_0^2 \eta_2^2} + \bar{p}_0^{\{0\}}, \quad \bar{s}_{11}^{\{0\}} = \sqrt{1 - m_0^2 \eta_2^2}, \quad \bar{s}_{12}^{\{0\}} = -sm_0\eta_2, \quad \bar{v}_1^{\{0\}} = \frac{2s}{m_0}\sqrt{1 - m_0^2 \eta_2^2}. \tag{23}$$

The dynamic corrections in this case is only in higher orders of smallness parameters. The inequality $b_1 > 2$ is clearly consistent with the known requirement for quasistatics.

3.1 | The case $b_1 = b_2 = 2$

Let the time interval be such that either one or both indices, b_1 and b_2 in (22), are equal to two. For generality, we consider the case $b_1 = b_2 = 2$; if one of the indices is greater than two, it is necessary to put $C = 0$ or $D = 0$ in (22). Therefore, for $b_1 = b_2 = 2$ and by observing (21), the following system of ten equations in $\bar{\Omega}$ follows from (16)–(19),

$$-\bar{p}_{,1}^{\{-1\}} + \bar{s}_{12,2}^{\{0\}} = 0, \quad -\bar{p}_{,1}^{\{0\}} + \bar{s}_{11,1}^{\{0\}} + \bar{s}_{12,2}^{\{1\}} = (2C - D)\eta_1, \bar{p}_{,2}^{\{-1\}} = 0,$$

$$\bar{p}_{,2}^{\{0\}} + \bar{s}_{11,2}^{\{0\}} = 0, \quad \bar{p}_{,2}^{\{1\}} + \bar{s}_{11,2}^{\{1\}} - \bar{s}_{12,1}^{\{0\}} = 0, \quad \left(\bar{s}_{11}^{\{0\}}\right)^2 + \left(\bar{s}_{12}^{\{0\}}\right)^2 = 1, \quad \bar{s}_{11}^{\{0\}}\bar{s}_{11}^{\{1\}} + \bar{s}_{12}^{\{0\}}\bar{s}_{12}^{\{1\}} = 0, \tag{24}$$

$$\bar{s}_{11}^{\{0\}}\bar{v}_{1,2}^{\{0\}} = 2\bar{s}_{12}^{\{0\}}, \quad \bar{s}_{11}^{\{1\}}\bar{v}_{1,2}^{\{0\}} + \bar{s}_{11}^{\{0\}}\bar{v}_{1,2}^{\{1\}} = 2\left(\bar{s}_{12}^{\{1\}} + \bar{s}_{12}^{\{0\}}\bar{v}_{1,1}^{\{0\}}\right), \quad \bar{v}_{1,1}^{\{0\}} + \bar{v}_{2,2}^{\{1\}} = 0$$

with respect to ten unknown coefficients: $\bar{p}^{\{-1\}}, \bar{p}^{\{0\}}, \bar{p}^{\{1\}}, \bar{s}_{11}^{\{0\}}, \bar{s}_{11}^{\{1\}}, \bar{s}_{12}^{\{0\}}, \bar{s}_{12}^{\{1\}}, \bar{v}_1^{\{0\}}, \bar{v}_1^{\{1\}}, \bar{v}_2^{\{0\}}$, as expanded in the series (14). This system differs from the corresponding quasistatic system by the presence of the term $(2C - D)\eta_1$ in (24).

Sequential integration of (24) while observing the boundary conditions (20) permits to obtain nine out of ten of these functions:

$$\bar{p}^{\{-1\}} = m_0(1 - |\eta_1|), \quad \bar{p}^{\{0\}} = -\sqrt{1 - m_0^2\eta_2^2} + \left(C - \frac{D}{2}\right)(1 - \eta_1^2) + \bar{p}_0^{\{0\}},$$

$$\bar{s}_{11}^{\{0\}} = \sqrt{1 - m_0^2\eta_2^2}, \quad \bar{s}_{12}^{\{0\}} = -sm_0\eta_2, \quad \bar{s}_{11}^{\{1\}} = \bar{s}_{12}^{\{1\}} \equiv 0, \tag{25}$$

$$\bar{v}_1^{\{0\}} = \frac{2s}{m_0}\sqrt{1 - m_0^2\eta_2^2}, \quad \bar{v}_1^{\{1\}} = \bar{v}_2^{\{1\}} \equiv 0.$$

The remaining coefficient $\bar{p}^{\{1\}}$ may be found by taking the equations of the next approximation into account

$$-\bar{p}_{,1}^{\{1\}} + \bar{s}_{12,2}^{\{2\}} = (C - D)\bar{v}_1^{\{0\}}, \quad \bar{p}_{,2}^{\{1\}} = 0 \tag{26}$$

with the boundary condition (20) for $\bar{s}_{12}^{\{2\}}$. The expressions for $\bar{p}^{\{1\}}$ and $\bar{s}_{12}^{\{2\}}$ are as follows

$$\bar{p}^{\{1\}} = -\frac{C - D}{m_0^2}\left(\arcsin(m_0) + m_0\sqrt{1 - m_0^2}\right)|\eta_1| + \bar{p}_0^{\{1\}},$$

$$\bar{s}_{12}^{\{2\}} = \frac{s(C - D)}{m_0^2}\left[\arcsin(m_0\eta_2) - \eta_2\arcsin(m_0) + m_0\eta_2\left(\sqrt{1 - m_0^2\eta_2^2} - \sqrt{1 - m_0^2}\right)\right]. \tag{27}$$

3.2 | The case $b_1 = b_2 = 1$

Since the expansions (14) have the form of integer powers of the parameter α we consider one more point $b_1 = b_2 = 1$ with integer coordinates on the plane of the parameters (b_1, b_2) and substitute these indices into (22). As before if $b_1 = 1, b_2 > 1$ or $b_2 = 1, b_1 > 1$ one can formally put in (22) $D = 0$ or $C = 0$.

Five equations in (24) out of ten as given in (24) of the system within $\bar{\Omega}$ show no changes *a fortiori*, since the small parameters, ε_1 and ε_2 as in (17)–(19), are missing. Instead of the five equations (24), one can now reformulate as follows:

$$-\bar{p}_{,1}^{\{-1\}} + \bar{s}_{12,2}^{\{0\}} = (2C - D)\eta_1, \quad -\bar{p}_{,1}^{\{0\}} + \bar{s}_{11,1}^{\{0\}} + \bar{s}_{12,2}^{\{1\}} = (C - D)\bar{v}_1^{\{0\}}$$

$$\bar{p}_{,2}^{\{-1\}} = 0, \quad \bar{p}_{,2}^{\{0\}} + \bar{s}_{11,2}^{\{0\}} = 0, \quad \bar{p}_{,2}^{\{1\}} + \bar{s}_{11,2}^{\{1\}} - \bar{s}_{12,1}^{\{0\}} = 0 \tag{28}$$

It is obvious that during the transition from the point $b_1 = b_2 = 2$ to $b_1 = b_2 = 1$ three equations (24) remain the same (see (28)).

The exact solutions of the system (24), (28) for the coefficients with superscripts $\{-1\}$ and $\{0\}$ read

$$\bar{p}^{\{-1\}} = m_0(1 - |\eta_1|) + \left(C - \frac{D}{2}\right)(1 - \eta_1^2),$$

$$\bar{p}^{\{0\}} = -\sqrt{1 - m_0^2\eta_2^2} - \frac{C - D}{m_0^2}\left(\arcsin(m_0) + m_0\sqrt{1 - m_0^2}\right)|\eta_1| + \bar{p}_0^{\{0\}}, \tag{29}$$

$$\bar{s}_{11}^{\{0\}} = \sqrt{1 - m_0^2\eta_2^2}, \quad \bar{s}_{12}^{\{0\}} = -sm_0\eta_2, \quad \bar{v}_1^{\{0\}} = \frac{2s}{m_0}\sqrt{1 - m_0^2\eta_2^2}.$$

By comparing the solution (25) with (29) we see that the term $(C - D/2)(1 - \eta_1^2)$ moves from $\bar{p}^{\{0\}}$ to the principal coefficient $\bar{p}^{\{-1\}}$ by transition from the point $b_1 = b_2 = 2$ to $b_1 = b_2 = 1$. This means that the mentioned term becomes of order $O(\alpha^{-1})$ when $\alpha \to 0$. Another term appearing in (3.10) moves from $\bar{p}^{\{1\}}$ to $\bar{p}^{\{0\}}$, i.e., it becomes of the order $O(1)$ when $\alpha \to 0$.

Let us substitute the coefficients (21), (25), (27) to the series (14) (by this $C = \varepsilon_1/\alpha^2$, $D = \varepsilon_2/\alpha^2$), and after that, we insert the coefficients (21), (29) into (14) (now already $C = \varepsilon_1/\alpha$, $D = \varepsilon_2/\alpha$). By comparing the resulting dimension functions, we see that they coincide up to $O(\alpha^2)$. The values s_{11}, s_{12}, v_1 and v_2 do not differ from their quasistatic expressions (11) up to the same precision. The pressure $p(x_1, x_2, t)$ has the form

$$p = \frac{\tau_s}{h}\left(m_0(l - |x_1|) - \sqrt{h^2 - m_0^2 x_2^2}\right) + \rho\left(\frac{V^2}{h^2} + \frac{\dot{V}}{2h}\right)(l^2 - x_1^2)$$
$$- \frac{\rho}{m_0^2}\left(\frac{V^2}{h} + \dot{V}\right)\left(\arcsin(m_0) + m_0\sqrt{1 - m_0^2}\right)|x_1| + O(\alpha^2). \tag{30}$$

The same form of the dimension functions, in particular (30), confirms the absolute joining of asymptotic expansions on the time intervals corresponding to various pairs of the indices b_1 and b_2 in (22). Smallness orders of the terms in (30) containing τ_s on the one hand and V and $\dot{V}$ on the other change in relation to each other as the rigid plates move.

4 | TOTAL FORCE ACTING ON DIES

Let us find the total force along the axis x_2 acting on the material from the die at $x_2 = h$. The component F_2 of reads

$$F_2(t) = \int_{-l(t)}^{l(t)} (-p - s_{11})(x_1, h, t)\,\mathrm{d}x_1. \tag{31}$$

Hence, for the quasistatic solution substitution of the expressions (11) to (31) leads to the value

$$F_2 = -\left(m_0\tau_s\frac{l}{h} + 2p_0\right)l. \tag{32}$$

In order to demonstrate the role of power orders, α, used in terms in (31), we employ the expansions (14) such that the dimensionless force reads

$$\bar{F}_2(t) = -\frac{\sqrt{S}\,\tau_s}{\sqrt{\alpha}}\int_{-1}^{1}\left(\frac{\bar{p}^{\{-1\}}}{\alpha} + \bar{p}^{\{0\}} + \bar{s}_{11}^{\{0\}} + \alpha\left(\bar{p}^{\{0\}} + \bar{s}_{11}^{\{0\}}\right) + \dots\right)(\eta_1, 1, \tau)\,\mathrm{d}\eta_1 \tag{33}$$

where $\bar{F}_2 = O(\alpha^{-3/2})$, $\alpha \to 0$.

In the case $b_1 = b_2 = 2$, which was considered in the Section 3.1, after substitution of the expressions (25) in (33) and integration by η_1, we obtain

$$\bar{F}_2 = -\sqrt{S}\,\tau_s\left(\frac{m_0}{\alpha\sqrt{\alpha}} + \left(C - \frac{D}{2}\right)\frac{4}{3\sqrt{\alpha}} + \frac{2\bar{p}^{\{0\}}}{\sqrt{\alpha}} + O\left(\sqrt{\alpha}\right)\right), \quad \alpha \to 0. \tag{34}$$

In another case to be considered, namely $b_1 = b_2 = 1$ (see Section 3.2) the expressions (29) may be used

$$\bar{F}_2 = -\sqrt{S}\,\tau_s \left[\left(m_0 + \frac{4}{3}\left(C - \frac{D}{2}\right)\right)\frac{1}{\alpha\sqrt{\alpha}} + \left(\bar{p}^{\{0\}} - \frac{C - D}{m_0^2}\left(\arcsin(m_0) + m_0\sqrt{1 - m_0^2}\right)\right)\frac{1}{\sqrt{\alpha}} + O\left(\sqrt{\alpha}\right)\right], \quad \alpha \to 0. \tag{35}$$

The coefficients related to $\alpha^{-3/2}$ and $\alpha^{-1/2}$ in (34) and (35) provide information about the inertial effects at different time stages of the pressing process. As for the pressure (30), one obtains the following expression:

$$F_2 = -\left(m_0\tau_s\frac{l}{h} + 2p_0\right)l - \rho\left(\frac{V^2}{h} + \frac{\dot{V}}{2}\right)\frac{4l^3}{3h} + \frac{\rho}{m_0^2}\left(\frac{V^2}{h} + \dot{V}\right)\left(\arcsin(m_0) + m_0\sqrt{1 - m_0^2}\right)l^2, \tag{36}$$

for the force acting on dies in N, F_2, including the dynamical effects.

5 | VARIOUS TYPICAL REGIMES OF PLASTIC PRESSING

Let us consider some typical regimes of plates in motion and the inertial effects arising from the thin-layer following the previous analysis.

5.1 | Uniform motion of dies

For this classical case of the Prandtl problem[13] we have according to (2) and (13)

$$V = V_0 = \text{const}, \quad h(t) = h_0 - V_0 t, \quad \tau = \frac{V_0 t}{h_0 - V_0 t}, \quad 0 \le t < t_* = \frac{h_0}{V_0},$$

$$\alpha(t) = \frac{1}{S}(h_0 - V_0 t)^2, \quad \varepsilon_1(t) = \frac{\rho V_0^2}{\tau_s} = \text{const}, \quad \varepsilon_2(t) \equiv 0, \tag{37}$$

where t_* corresponds to the moment when the upper and lower plates (dies) are getting in contact, which is obviously not under consideration. It should be noted that the dimensionless time τ in (37) is a monotonically increasing branch of an hyperbola, which maps the interval $[0, t_*)$ into $[0, \infty)$.

Because $\varepsilon_2(t) \equiv 0$ it becomes formally valid that $b_2 = \infty$ in (22). The index b_1 with time $t \in [0, t_*)$ (or $\tau \in [0, \infty)$) monotonically decreases and in the infinite limits to zero by $t \to t_*$. For the time intervals where the index b_1 at first equals to 2 and then to 1, the solution is obtained above in Section 3. There it is necessary to substitute $D = 0$ as well as to put $\dot{V} = 0$ in the expression (30) for the pressure.

In the time range

$$t_* - t = O\left(\left(\frac{\rho S^2}{\tau_s V_0^2}\right)^{1/4}\right), \quad \alpha \to 0 \tag{38}$$

the dynamic term $C(1 - \eta_1^2)$ is present in the coefficient $\bar{p}^{\{0\}}$ in (25). In the process of plastic layer thinning when

$$t_* - t = O\left(\sqrt{\frac{\rho S}{\tau_s}}\right), \quad \alpha \to 0 \tag{39}$$

this term moves to the principal (singular) coefficient $\bar{p}^{\{-1\}}$ in (3.14). A further approach of t to t_* when

$$t_* - t = O\left(\left(\frac{\rho}{\tau_s}\right)^{1/(2b_1)}\sqrt{S}\,V_0^{(1-b_1)/b_1}\right), \quad \alpha \to 0 \tag{40}$$

TABLE 1 Geometric and material parameters used for analyzing forces acting on dies based on the following units: m for length, kg for mass, s for time

Parameter	Variable	Value	Unit
Pressing time	t_1	10	s
Thickness	h_0	0.001	m
Length	l	1	m
Mass density	ρ	8000	kg/m^3
Yield stress	σ_s	400×10^6	Pa
Hydrostatic pressure	p_0	10^5	Pa
Parameter	m_0	0.8	

where $0 < b_1 < 1$, the dynamic term $(\rho V_0^2 / \tau_s)(l^2 - x_1^2)/h^2$ in (3.15) is of the order $\alpha^{b_1 - 2}$ by $\alpha \to 0$. Hence it plays a predominant role in the pressure distribution within the layer.

5.2 | Exponential decreasing of thickness $h(t)$

Let us prescribe $h(t) = h_0 e^{-t/t_1}$, $t_1 = \text{const} > 0$, $t > 0$, so that

$$V(t) = \frac{h_0}{t_1} e^{-t/t_1}, \quad \tau = \frac{t}{t_1}, \quad \alpha(t) = \frac{h_0}{l_0} e^{-2t/t_1},$$

$$\varepsilon_1(t) = \varepsilon_2(t) = \frac{\rho h_0^2}{\tau_s t_1^2} e^{-2t/t_1} = C\alpha, \quad C = D = \frac{\rho S}{\tau_s t_1^2} = \text{const}. \tag{41}$$

The indices b_1 and b_2 in (22) are equal to 1 over the entire infinite interval $t > 0$. This suggests the need to take dynamic effects from $t = 0$ into account. The asymptotic analysis of the case $b_1 = b_2 = 1$ was carried out in Section 3.2.

5.3 | Power-law decreasing of thickness $h(t)$

Let $h(t) = h_0(1 + t/t_1)^{-q}$, $t_1 = \text{const} > 0$, $q = \text{const} > 0$, $t > 0$. Then

$$V(t) = \frac{qh_0}{t_1}\left(1 + \frac{t}{t_1}\right)^{-q-1}, \quad \tau = \frac{qt}{t + t_1}, \quad \alpha(t) = \frac{h_0}{l_0}\left(1 + \frac{t}{t_1}\right)^{-2q},$$

$$\varepsilon_1(t) = \frac{q^2 \rho h_0^2}{\tau_s t_1^2}\left(1 + \frac{t}{t_1}\right)^{-2q-2}, \quad \varepsilon_2(t) = \frac{q+1}{q}\varepsilon_1(t). \tag{42}$$

Equating of powers of t in the relations (22), where the functions from (36) should be substituted, gives the following values:

$$b_1 = b_2 = \frac{q+1}{q}, \quad C = \frac{q^2 \rho h_0^2}{\tau_s t_1^2}\left(\frac{l_0}{h_0}\right)^{(q+1)/q} = \text{const}, \quad D = \frac{q+1}{q}C = \text{const}. \tag{43}$$

When the motion of the plates is slow enough ($b_1 = b_2 > 2$ or, in the other words, $0 < q < 1$) the quasistatic approximation of solution remains principal. If $b_1 = b_2 = 2$, i.e. $q = 1$, then it follows from (43) that $C = D/2$ and the dynamic term $(C - D/2)(1 - \eta_1^2)$ in the coefficient $\bar{p}^{\{0\}}$ in (3.8) vanishes. The quasistatic approximation is inadequate for $q > 1$, then the motion of dies necessitates a corresponding correction of the quasistatic solution by the inertial terms. In order to comprehend the effect of the chosen cases, we use realistic values for a typical metal forming process compiled in Table 1. By using uniform, power-law, and exponential velocities, we obtain force evolution in time demonstrated as in Figure 1. It is interesting to note that the power-law motion creates a nearly linear force increase being an important factor on design.
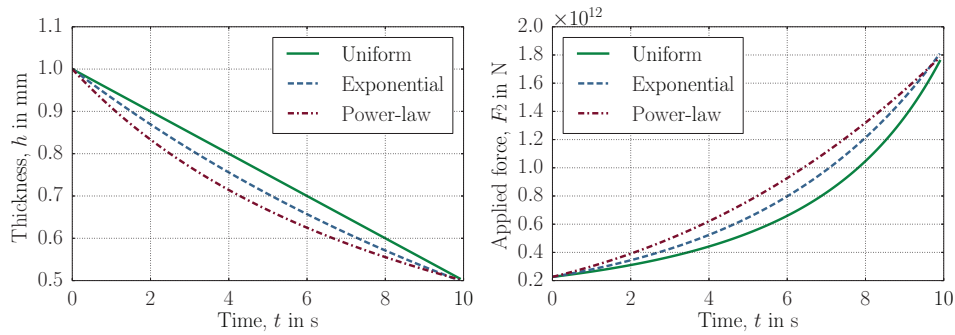
FIGURE 1 Forces (right) acting on dies during the metal pressing by decreasing the thickness (left) with different velocities: uniform given by (37), exponential with (41), power-law as in (42)

6 | CONCLUSIONS

Asymptotic analysis with the small geometric parameter $\alpha = h(t)/l(t)$ shows that in thin-layer approximation the inertial effects in the form of dynamic corrections to the quasistatic Prandtl solution play a different role at various time intervals before upper and lower plates (dies) get into contact. The presence of these dynamic corrections in certain terms of the asymptotic series depends on the orders of smallness of the dimensionless parameters, namely $\varepsilon_1 = \rho V^2(t)/\tau_s \ll 1$ and $\varepsilon_2 = -\rho \dot{V}(t)h(t)/\tau_s \ll 1$ in relation to α. This smallness is described by the positive powers (indices) $b_1(t)$ and $b_2(t)$ in Equations (22). Motion of dies toward each other is tantamount to the motion of a mass point with coordinates (b_1, b_2) in the direction of the point (0,0). This motion is defined concretely for three typical regimes used in metal pressing. Realizing one of them leads to the motion of dies with a power-law thickness reduction given in (42) with $0 < q \leq 1$, where one can reach an arbitrary small thickness of the layer:

- in a finite time;
- remaining within the quasistatic limits,

providing help for designing dies regarding the force applied on them.

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