

Selective and Efficient Quantum Process Tomography in Arbitrary Finite Dimension

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The characterization of quantum processes is a key tool in quantum information processing tasks for several reasons: on one hand, it allows to acknowledge errors in the implementations of quantum algorithms; on the other, it allows to characterize unknown processes occurring in Nature. In [1, 2] it was introduced a method to selectively and efficiently measure any given coefficient from the matrix description of a quantum channel. However, this method heavily relies on the construction of maximal sets of mutually unbiased bases (MUBs), which are known to exist only when the dimension of the Hilbert space is the power of a prime number. In this article, we lift the requirement on the dimension by presenting two variations of the method that work on arbitrary finite dimensions: one uses tensor products of maximally sets of MUBs, and the other uses a dimensional cutoff of a higher prime power dimension.

I. INTRODUCTION

Being able to efficiently characterize the temporal evolution of a quantum system is one of the main tasks that is needed to accomplish in order to reliably operate large-scale and general purpose quantum devices. In the last few years, there were proposed several methods for this issue, commonly known as quantum process tomography [1–8]. In general, for an unknown evolution, full process tomography is inefficient since it requires resources that scale exponentially with the number of subsystems. However, it has been shown that is possible to efficiently obtain partial and relevant information about the process under consideration [1, 2, 6–8].

The temporal evolution of a quantum system can be described by a linear, completely positive, trace preserving map $\rho_{\text{out}} = \mathcal{E}(\rho_{\text{in}})$. If the system under consideration has dimension d , one can choose a basis $\{E_m, m = 0 \dots d^2 - 1\}$ of operators and write the map as:

$$\mathcal{E}(\rho) = \sum_{mn} \chi_{mn} E_m \rho E_n^\dagger \quad (1)$$

where the trace preserving condition is given by $\sum_{mn} \chi_{mn} E_n^\dagger E_m = \mathbb{1}$ and the matrix χ is Hermitian and positive, making the map completely positive. In [1, 2, 7] the authors present a method that efficiently estimates any coefficient from the χ -matrix of a quantum channel. Their method relies on averaging the survival probability of a particular set of states, namely a state 2-design, when being acted upon by modified versions of the channel \mathcal{E} . The state 2-designs in this case are constructed using maximal sets of mutually unbiased bases (MUBs) [9]. These bases are such that if a system is prepared in any state from one of them, then a measurement with respect to other basis of this set provides maximal uncertainty. The construction of state 2-designs imposes several limitations on the dimension d for which this protocol can

be implemented, since these maximal sets of bases are known to exist only for dimensions that are powers of prime numbers [10–12]. For arbitrary dimensions only approximate constructions are known [13].

In this paper we present two variations of the above protocol that can be implemented in an arbitrary dimension. The first one is based on the construction of tensor products of 2-designs and measure the survival probability over modified channels. This idea not only allows to do selective and efficient quantum process tomography in arbitrary systems, but also requires the preparation of product states in smaller dimensions. The other scheme relies on the preparation of state 2-designs in a higher dimension and then project this design onto the desired dimension.

This paper is organized as follows. In Sec. II, we review some properties of state 2-designs, MUB sets and the results of Ref. [1], specifically, a protocol for selective and efficient quantum process tomography (SEQPT). In Sec. III, we show that tensor products of 2-designs are approximate 2-designs up to local corrections, making them suitable for SEQPT; then we describe our first scheme. In Sec. IV, we introduce our second scheme, that performs SEQPT based on the construction of a state 2-design on a higher dimension.

II. SEQPT IN PRIME POWER DIMENSIONS

In this section we review the fundamental concepts for the protocol developed in [1] that allows to perform selective and efficient quantum process tomography. The method is based on the concept of state 2-designs, that allows to efficiently perform averages of quadratic functions over the Haar measure. Interestingly, the construction of 2-designs is intimately related to the existence of maximal sets mutually unbiased bases.

A. 2–designs and Mutually Unbiased Bases

A finite set of states $\mathcal{X} = \{|\psi_m\rangle, m = 1, \dots, N\}$ is a state 2–design if there is some fixed probability distribution $\{p_i\}_{i=1, \dots, N}$ such that

$$\int_{\mathcal{H}} d\psi f(P_\psi) = \sum_{m=1}^N p_m f(P_{\psi_m}) \quad (2)$$

for any f that is quadratic in $P_\psi = |\psi\rangle\langle\psi|$, and the integral is made over the only normalized unitarily invariant measure on \mathcal{H} , namely, the Haar measure. That is, a state 2–design is a set of states such that the mean value of any quadratic function in P_ψ over such a set gives the same mean value over the set of all possible states. When the probability distribution over the states in the set is uniform, we have $p_i = 1/|\mathcal{X}| = 1/N$ for all $i = 1, \dots, N$ and we refer to the set as a uniform 2–design. In the following, unless explicitly noted, all 2–designs considered will be uniform. An interesting property that we will use extensively is that, for every pair of operators A and B in a Hilbert space of dimension d , it holds that

$$\langle A, B \rangle = \int_{\mathcal{H}} d\psi \text{Tr}[P_\psi A P_\psi B] = \frac{\text{Tr}A \text{Tr}B + \text{Tr}[AB]}{d(d+1)}. \quad (3)$$

A simple way to find a state 2–design is to consider a set of mutually unbiased bases (MUB), which automatically form a state 2–design [14, 15]. Two orthonormal basis \mathcal{B}_1 and \mathcal{B}_2 of \mathcal{H} are unbiased if every state of \mathcal{B}_1 has the same overlap with every state from \mathcal{B}_2 and viceversa. A set of orthonormal bases $\{\mathcal{B}_1, \dots, \mathcal{B}_M\}$ will be a mutually unbiased bases set if every pair of bases is unbiased. That is,

$$|\langle \psi_i^J | \psi_m^K \rangle|^2 = \delta_{JK} \delta_{lm} + (1 - \delta_{JK}) \frac{1}{d}, \quad (4)$$

where $|\psi_i^J\rangle$ is the l -th state from basis J . When the dimension d is a power of a prime, the maximum number of mutually unbiased basis is $d+1$ [10, 11], and the $d(d+1)$ states form such set form a state 2–design [14]. For arbitrary dimension d it is not known the maximum number of MUB.

B. SEQPT

First, let us briefly describe the quantum algorithm for QPT introduced in Ref. [1]. The method allows to efficiently obtain the elements of the χ -matrix defined in Eq. (1), and can be easily understood if one considers an orthonormal operator basis $\{E_m\}$ satisfying $\text{Tr}(E_m E_n^\dagger) = d \delta_{m,n}$ (orthonormality), $E_m E_m^\dagger = \mathbb{1}$ (unitarity), and $E_0 = \mathbb{1}$. From Eq. (3) and the orthogonality of the operators, it is straightforward to verify that the mean fidelity or survival probability is given by:

$$\bar{F}(\mathcal{E}) = \int_{\mathcal{H}} d\psi \text{Tr}[P_\psi \mathcal{E}(P_\psi)], \quad (5)$$

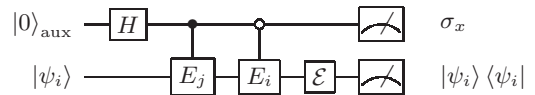


FIG. 1. Circuit depicting the protocol for SEQPT. When averaging over all the states $|\psi_i\rangle$ from a state 2–design, the mean value of σ_x (σ_y) conditioned to the survival of state $|\psi_i\rangle$ yields the real (imaginary) part of χ_{ij} .

and provides information about the diagonal contribution of E_0 . In general, each element χ_{ij} can be obtained by measuring the following averaged survival probability:

$$\bar{F}(\mathcal{E}_{ij}) = \frac{d\chi_{ij} + \delta_{ij}}{d+1}. \quad (6)$$

where $\mathcal{E}_{ij}(\rho) \equiv \mathcal{E}(E_i^\dagger \rho E_j)$. Thus, using a slightly modified channel, it would be possible to measure the elements of the χ -matrix provided one is able to perform the average over the Haar measure. The key point of the method is that, since both expressions are quadratic in P_ψ , this average can be evaluated by considering just a finite set of states from a 2–design. Notice that the modified channel \mathcal{E}_{ij} is not physical, however, it can be implemented via the tomographic protocol depicted in Fig. 1. Then, it is straightforward to show that the real (imaginary) part of χ_{ij} is obtained by measuring the mean value of σ_x (σ_y) conditioned on the survival of the state $|\psi_i\rangle$, where $|\psi_i\rangle$ is randomly chosen from a state 2–design.

III. GENERAL SEQTP WITH TENSOR PRODUCTS OF 2–DESIGNS

A. Tensor products of 2–designs

In this Section we will first describe how tensor products of 2–designs can be used to approximate 2–designs, and then we show that tensor products of maximally MUB sets provide a good approximation for integration purposes.

Let us start with the bipartite case. Consider a Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ of dimension $d = D_1 D_2 = p_1^{n_1} p_2^{n_2}$, with p_1 and p_2 different prime numbers. This ensures that a maximally set of MUBs exists for both dimensions D_1 and D_2 , and that those MUB sets are 2–designs. Let A and B be operators in dimension d , and consider the following integral

$$\langle A, B \rangle_{\otimes} = \int_{\mathcal{H}_1} \int_{\mathcal{H}_2} d\psi_1 d\psi_2 \text{Tr}[P_{\psi_1 \psi_2} A P_{\psi_1 \psi_2} B], \quad (7)$$

where $P_{\psi_1 \psi_2} = |\psi_1\rangle\langle\psi_1| \otimes |\psi_2\rangle\langle\psi_2|$ and the integrations are performed over the Haar measure for \mathcal{H}_1 and \mathcal{H}_2 respectively. It is easy to see that the above expression can be evaluated using 2–designs in \mathcal{H}_1 and \mathcal{H}_2 : $\langle A, B \rangle_{\otimes} = \frac{1}{|X_{\otimes}|} \sum_{\psi \in X_{\otimes}} \text{Tr}[P_\psi A P_\psi B]$, where the sum is taken over

the product states $|\psi\rangle \equiv |\psi_1\rangle |\psi_2\rangle \in X_\otimes \equiv X_1 \otimes X_2$ with $|\psi_1\rangle \in X_1$ and $|\psi_2\rangle \in X_2$, X_1 and X_2 are 2–designs in \mathcal{H}_1 and \mathcal{H}_2 respectively. Now, expanding the operators A and B in a product basis and using the identity of Eq. (3) for each dimension D_i we arrive at:

$$\langle A, B \rangle_\otimes = \frac{1}{d(D_1+1)(D_2+1)} (\text{Tr} A \text{Tr} B + \text{Tr}(AB) + \text{Tr}[A(\mathbb{I}_1 \otimes \text{Tr}_1(B))] + \text{Tr}[A(\text{Tr}_2(B) \otimes \mathbb{I}_2)]) \quad (8)$$

where \mathbb{I}_i is the identity over system i and Tr_i is the partial trace over system i . Notice that the first two terms in Eq. (8) are proportional to the ones that would appear in averages over a 2–design (see Eq. (3)). In fact, for an average like $\langle A, B \rangle_\otimes$, one can use two 2–designs and the resulting expression would be similar to Eq. (3), with the addition of two new terms. Thus,

$$\langle A, B \rangle = \frac{(D_1+1)(D_2+1)}{(d+1)} \langle A, B \rangle_\otimes + \frac{\text{Tr}[A(\mathbb{I}_1 \otimes \text{Tr}_1(B))] + \text{Tr}[A(\text{Tr}_2(B) \otimes \mathbb{I}_2)]}{d(d+1)}. \quad (9)$$

We will see later, in the context of QPT, that the last two terms can either be neglected (depending on the values of D_i) or measured. Thus, Eq. (9) establishes the connection between averages over 2–designs $\langle A, B \rangle$ and averages over tensor product of 2–designs $\langle A, B \rangle_\otimes$ in the bipartite case.

In general, when the dimension d is the product of more than two prime power factors $d = p_1^{n_1} \dots p_N^{n_N}$, we will consider tensor products of N different 2–designs. In this case, one can show that the following identity holds:

$$\langle A, B \rangle_\otimes = \sum_{X \in \{0,1\}^N} \frac{\text{Tr}[A(\mathbb{I}^{(X)} \otimes \text{Tr}_X B)]}{d \prod_{k=1}^N (D_k+1)}, \quad (10)$$

where $\mathbb{I}^{(X)}$ is the identity operator on the subsystems for which the binary N –uple X has a 1, and Tr_X is the partial trace over those subsystems. Hence, this identity is the extension of Eq. (8) when the average is taken over general tensor products of 2–designs.

B. SEQPT with tensor products of 2–designs: the bipartite case

1. Mean survival probabilities

Consider that $d = D_1 D_2$ with $D_1 = p_1^{n_1}$ and $D_2 = p_2^{n_2}$. As it was shown in Sec. II B, the original method relies on the fact that one can efficiently perform averages, like the one in Eq. (6), when the dimension of the system is a prime power. Here we will show that this could also be done by taking averages over tensor product of 2–designs. First, let us expand the channel of Eq. (1) in a product basis of operators acting on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. This basis can be written in terms of two orthogonal basis

of each system: $\{E_{j_1 j_2} \equiv E_{j_1} \otimes E_{j_2}\}_{j_1=0, \dots, D_1-1, j_2=0, \dots, D_2-1}$, where each element E_{j_i} is a unitary matrix. Thus, the channel in this basis is defined as:

$$\mathcal{E}(\rho) = \sum_{\mu_1 \mu_2 \nu_1 \nu_2} \chi_{\nu_1 \nu_2}^{\mu_1 \mu_2} E_{\mu_1 \mu_2} \rho E_{\nu_1 \nu_2}, \quad (11)$$

for some coefficients $\chi_{\nu_1 \nu_2}^{\mu_1 \mu_2}$. We have adopted the convention $E^{j_1 j_2} \equiv E_{j_1 j_2}^\dagger$ and $E^{j_i} \equiv E_{j_i}^\dagger$, and from now on we will also consider $\delta_i^j \equiv \delta_{ij}$.

In order to obtain an arbitrary coefficient $\chi_{i_1 i_2}^{j_1 j_2}$, we can consider the modified channel $\mathcal{E}_{j_1 j_2}^{i_1 i_2}(\rho) \equiv \mathcal{E}(E^{i_1 i_2} \rho E_{j_1 j_2})$. Using this modified channel, the averaged survival probability is equal to:

$$\bar{F}(\mathcal{E}_{j_1 j_2}^{i_1 i_2}) = \frac{d \chi_{j_1 j_2}^{i_1 i_2} + \delta_{j_1}^{i_1} \delta_{j_2}^{i_2}}{d+1}. \quad (12)$$

Which is useful provided one can implement the average using state 2–designs but, as we mentioned before, this is not the case for an arbitrary dimension d . So let us consider the following mean survival probability:

$$\bar{F}_\otimes(\mathcal{E}_{j_1 j_2}^{i_1 i_2}) = \int_{\mathcal{H}_1} \int_{\mathcal{H}_2} d\psi_1 d\psi_2 \text{Tr}[P_{\psi_1 \psi_2} \mathcal{E}_{j_1 j_2}^{i_1 i_2}(P_{\psi_1 \psi_2})].$$

Notice that this expression can be evaluated by averaging over a finite set of states belonging to X_\otimes . Additionally, one can show that the elements of the χ –matrix obey the following identity (see Appendix A):

$$\chi_{j_1 j_2}^{i_1 i_2} = \bar{F}_\otimes(\mathcal{E}_{j_1 j_2}^{i_1 i_2}) \frac{(1+D_1)(1+D_2)}{d} + \frac{\delta_{j_1}^{i_1} \delta_{j_2}^{i_2}}{d} - \bar{F}_1(\mathcal{E}_{j_1 j_2}^{i_1 i_2}) \frac{(1+D_1)}{d} - \bar{F}_2(\mathcal{E}_{j_1 j_2}^{i_1 i_2}) \frac{(1+D_2)}{d}. \quad (13)$$

The last terms in the expression have a clear interpretation: \bar{F}_1 can be thought as a reduced mean survival fidelity over system 1: $\bar{F}_1(\mathcal{E}_{j_1 j_2}^{i_1 i_2}) = \frac{1}{d_2 |X_1|} \sum_{\psi_1 \in X_1} \text{Tr}[(P_{\psi_1} \otimes \mathbb{I}_2) \mathcal{E}_{j_1 j_2}^{i_1 i_2}(P_{\psi_1} \otimes \mathbb{I}_2)]$. In terms of integrals over the Haar measure it can be written as: $\bar{F}_1(\mathcal{E}_{j_1 j_2}^{i_1 i_2}) = \int_{\mathcal{H}_1} d\psi_1 \langle \psi_1 | \text{Tr}_2(\mathcal{E}_{j_1 j_2}^{i_1 i_2}(P_{\psi_1} \otimes \mathbb{I}_2/d_2)) | \psi_1 \rangle$, and similarly for $\bar{F}_2(\mathcal{E}_{j_1 j_2}^{i_1 i_2})$. Thus, the above identity tells us that we do not need to compute Haar integrals in order to evaluate the elements $\chi_{j_1 j_2}^{i_1 i_2}$, it is enough to perform a sampling over a finite set of states from X_\otimes . Notice also that the first term in the right hand side of the last equation is the mean survival probability of the elements of our approximate 2–design over a channel that is non-physical. This feature already appears in the prime power dimension case, and how to circumvent this issue was described in Ref. [1]. Bellow, we describe how to determine each of the terms with a single quantum circuit.

2. Estimation of the elements of the χ –matrix

In order to estimate the coefficients $\chi_{j_1 j_2}^{i_1 i_2}$ we have to determine three different complex terms, as it is shown

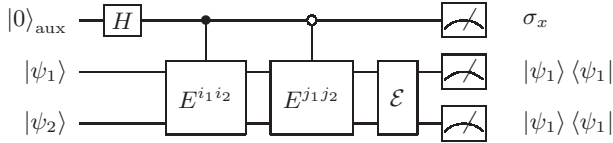


FIG. 2. Circuit that estimates the real part of $\chi_{j_1 j_2}^{i_1 i_2}$. The initial states $|\psi_1\rangle$ and $|\psi_2\rangle$ are uniformly sampled from the set of states that belongs to the 2-designs X_1 and X_2 respectively. A measurement of the expectation value of σ_x (σ_y) conditioned to the results of the measurements at the other two systems allows to estimate the $\text{Re}(\chi_{j_1 j_2}^{i_1 i_2})$ ($\text{Im}(\chi_{j_1 j_2}^{i_1 i_2})$).

in Eq. (13). These terms can be written as mean survival probabilities over non-physical quantum channels. The real part of these terms can be estimated through the circuit represented in Fig. 2. This is a variation of the strategy used in Ref. [1]. The method utilizes one auxiliary qubit whose polarization is measured. The inputs of the circuit are the clean qubit and states randomly chosen from each 2-design. It is easy to show that by measuring the expectation value of $\sigma_x \otimes P_{\psi_1} \otimes P_{\psi_2}$ one estimates $\text{Re}(\bar{F}_{\otimes}(\mathcal{E}_{j_1 j_2}^{i_1 i_2}))$. Analogously, with the expectation value of $\sigma_y \otimes P_{\psi_1} \otimes P_{\psi_2}$ one estimates $\text{Im}(\bar{F}_{\otimes}(\mathcal{E}_{j_1 j_2}^{i_1 i_2}))$.

The other terms, $\bar{F}_1(\mathcal{E}_{j_1 j_2}^{i_1 i_2})$ and $\bar{F}_2(\mathcal{E}_{j_1 j_2}^{i_1 i_2})$, can also be estimated with the same circuit. Thus, for instance, in order to estimate $\text{Re}(\bar{F}_1(\mathcal{E}_{j_1 j_2}^{i_1 i_2}))$ one has to measure the expectation of $\sigma_x \otimes P_{\psi_1} \otimes \mathbb{1}_2$ given the initial state $|0\rangle\langle 0|_{\text{aux}} \otimes P_{\psi_1} \otimes \frac{\mathbb{1}_2}{D_2}$. This is achieved by looking at the statistics of the measurements of the auxiliary qubit and system 1 independently from the results of the measurements at system 2. This is so because the initial state of system 2 is a random state from $(1 + D_2)$ orthogonal bases, which is a possible implementation of $\mathbb{1}_2$ provided the result of the measurement of system 2 is not taken into account. Thus, each run of the circuit in Fig. 2 yields one of eight possible outputs for each polarization measurement. The number of outputs is the result of the product of 2 values for the polarization, 2 values for P_{ψ_1} and 2 values for P_{ψ_2} . In summary, detection of P_{ψ_1} along with P_{ψ_2} contributes to the estimation of \bar{F}_{\otimes} , detection of just P_{ψ_1} to the estimation of \bar{F}_1 , and detection of just P_{ψ_2} to the estimation of \bar{F}_2 .

So far this protocol seems to be inefficient: given that $|X_{\otimes}|$ grows exponentially with n_1 and n_2 , one would expect that also the number of experiments required to estimate $\text{Re}(\bar{F}_{\otimes}(\mathcal{E}_{j_1 j_2}^{i_1 i_2}))$ grows exponentially with n_1 and n_2 . Fortunately, will show that, although the number of experiments required to estimate an element of the χ -matrix depends upon the desired precision, it is independent of n_1 and n_2 . This can easily be understood by looking at Eq. (13) and realising that $\text{Re}(\chi_{j_1 j_2}^{i_1 i_2})$ can be thought of as an average of a random variable which can take eight values. Then, one can relate the number of trials needed to estimate this average, M , with the desired error ϵ and

failure probability p as (see Appendix B):

$$p \leq 2 \exp\left(\frac{-M\epsilon^2}{2(1 - \frac{1}{d})^2}\right). \quad (14)$$

This implies the estimation of an element of the χ -matrix requires a number of experiments of the order $M \geq 2 \ln(2/p)/\epsilon^2$.

C. SEQPT with tensor products of 2-designs: the general case

Now we consider the tensor product approach for a system of arbitrary dimension d . In this case, we can always factorize it as $d = \prod_{a=1}^N p_a^{n_a}$ where p_a are prime numbers and $n_a \geq 1$ their respective multiplicity in the prime number factorization of d . Thus, the Hilbert space of the system is $\mathcal{H} = \bigotimes_{a=1}^N \mathcal{H}_a$ and each subspace has dimension $D_a = p_a^{n_a}$. As it was done before, we will consider unitary orthogonal operator bases for each subsystem $\{E_{i_a}^a\}_{i_a=1, \dots, D_a}$, and the basis of operators for the composite system as $\left\{ \bigotimes_{a=1}^N E_{i_a}^a \equiv E_i \right\}$, where from now on we will assume that $i \equiv i_1 \dots i_N$. Thus, an arbitrary channel \mathcal{E} can be expanded as:

$$\mathcal{E}(\rho) = \sum_{\nu, \mu} \chi_{\nu}^{\mu} E_{\nu} \rho E_{\mu}^{\nu}. \quad (15)$$

Our objective is to estimate efficiently an arbitrary coefficient χ_j^i . The crucial objects we need to compute are integrals of the form $\langle M, Q \rangle = \int_{\mathcal{H}} d\psi \text{Tr}[P_{\psi} M P_{\psi} Q]$. Let us consider the 2-designs $X_a \in \mathcal{H}_a$ for $a = 1, \dots, N$, which we know how to construct because each \mathcal{H}_a has prime power dimension. Now we can define our approximate 2-design $X_{\otimes} \in \mathcal{H}$ as $X_{\otimes} = \bigotimes_{a=1}^N X_a$. It is easy to check that, if $M = \bigotimes_{a=1}^N M_a$ and $Q = \bigotimes_{a=1}^N Q_a$:

$$\langle M, Q \rangle_{\otimes} = \prod_{a=1}^N (\text{Tr} M_a \text{Tr} Q_a + \text{Tr}[M_a Q_a]). \quad (16)$$

If we expand the right hand side of the last equation, we obtain 2^N terms. A convenient way to refer to each of those terms is the following:

$$f_{MQ}(m_1, \dots, m_N) \equiv \prod_{a=1}^N \text{Tr}[M_a Q_a]^{m_a \oplus 1} (\text{Tr} M_a \text{Tr} Q_a)^{m_a},$$

where $m_1 \dots m_N$ take values in $\{0, 1\}$ and \oplus is the addition modulo-2. Note that the sum of two of these terms is given by:

$$f_{MQ}(0, \dots, 0) + f_{MQ}(1, \dots, 1) = d(d+1) \langle M, Q \rangle. \quad (17)$$

Therefore, the integral we are interested in appears in equation Eq. (16), but we have to deal with some extra terms in order perform the process tomography. We will

show that the circuit for the estimation of non-diagonal coefficients is analogous to the one presented for $N = 2$ and it is shown in Fig. 3.

The procedure we can use to obtain the elements of the χ -matrix in terms of mean survival probabilities is analogous as the one we follow in the bipartite case (see Appendix A). Thus, one can show that the elements of the χ -matrix obey:

$$\chi_j^i = \bar{F}_\otimes(\mathcal{E}_j^i) \frac{\prod_{a=1}^N (D_a + 1)}{d} + \frac{(2^N - 3) \prod_{a=1}^N \delta_{j_a}^{i_a}}{d} - \sum_{\vec{m} \neq \{\vec{0}, \vec{1}\}} \bar{F}_{\vec{m}}(\mathcal{E}_j^i) \frac{\prod_{a/m_a=1} (D_a + 1)}{d}, \quad (18)$$

where \vec{m} is a vector with N components each of which take values in $\{0, 1\}$. In this case, the reduced mean survival probabilities are defined as:

$$\bar{F}_{\vec{m}}(\mathcal{E}_j^i) = \frac{\prod_{a/m_a=0} (D_a + 1)}{|X_\otimes|} \sum_{\psi \in X_{\vec{m}}} \text{Tr}[\rho_{\psi}^{\vec{m}} \mathcal{E}_j^i(\rho_{\psi}^{\vec{m}})],$$

where the averages are taken over the states in $X_{\vec{m}} \equiv \otimes_{a/m_a=1} X_a$, and states defined as:

$$\rho_{\psi}^{\vec{m}} = \left(\bigotimes_{a/m_a=0} \frac{\mathbb{1}^{(a)}}{D_a} \right) \otimes P_{\psi} \text{ with } \psi \in X_{\vec{m}}. \quad (19)$$

These are completely mixed states in the subsystems labelled by $m_a = 0$, and elements of a 2-design for the rest.

The circuit that estimates the elements of the χ -matrix is shown in Fig. 3. There, as in the bipartite case, the real and imaginary part of the fidelities are estimated by measuring mean values associated to σ_x and σ_y , respectively. Furthermore, all the terms in Eq. (18) can be evaluated by considering many runs of the same circuit, where the initial states $|\psi_a\rangle$ are sampled from the state 2-design in \mathcal{H}_a . In this case, the circuit has 2^{N+1} outputs: 2 for the auxiliary qubit, and 2 for each of the N subsystems (the projective measurement gives binary results). Thus, one can discriminate between these results to evaluate all these fidelities. The efficiency of this protocol is discussed in Appendix B.

IV. GENERAL SEQPT USING LOW DIMENSIONAL PROJECTION

In this section we show a different strategy that allows to construct a non-uniform 2-design in any dimension d different from a prime power. The method is based on taking a 2-design in a dimension greater than d , and then project this set over dimension d .

Let us consider D an integer such that $D > d$ and $D = p^n$ for some prime number p and natural number n . D could be, for instance, the smallest dimension

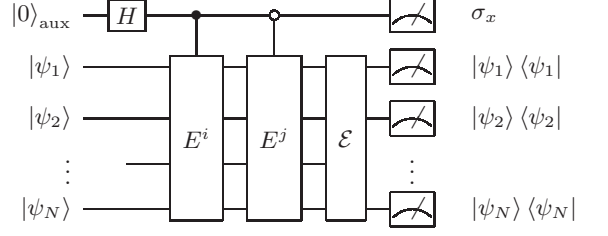


FIG. 3. Circuit that estimates the real part of χ_j^i for a process that acts on a system of arbitrary dimension $d = \prod_{a=1}^N p_a^{n_a}$. The initial states $|\psi_i\rangle$ are uniformly sampled from the set of states that belongs to the 2-designs X_1, \dots, X_N . A measurement of the expectation value of σ_x (σ_y) conditioned to the results of the measurements at the other systems allows to estimate the $\text{Re}(\chi_j^i)$ ($\text{Im}(\chi_j^i)$).

greater than d for which we know how to generate 2-designs. We assume that in dimension D a maximal MUB set is made up of the bases $\mathcal{B}_0, \dots, \mathcal{B}_D$. The basis $\mathcal{B}_0 = \{|\psi_0^0\rangle, \dots, |\psi_{D-1}^0\rangle\}$ can be arbitrary chosen, and the states from the other bases of the MUB set can be written as

$$|\psi_k^M\rangle = \sum_{j=0}^{D-1} \frac{e^{i\alpha_{jk}^M}}{\sqrt{D}} |\psi_j^0\rangle, \quad (20)$$

where $|\psi_j^M\rangle$ denotes the j -th state from basis \mathcal{B}_M [11]. The set of states $X_D = \{|\psi_j^M\rangle\}_{j=0, \dots, D-1}^{M=0, \dots, D}$ form a state 2-design in dimension D [14].

The key point of the method is, as before, the efficient estimation of Haar integrals of quadratic forms. Given two arbitrary operators A and B in \mathcal{H}_d , and their expansion in an orthonormal basis $\{|j\rangle\}_{j=0, \dots, d-1}$:

$$A = \sum_{j,k=0}^{d-1} a_{jk} |j\rangle \langle k|, \quad B = \sum_{j,k=0}^{d-1} b_{jk} |j\rangle \langle k|, \quad (21)$$

we can define two operators that can be considered as their extensions to \mathcal{H}_D :

$$\tilde{A} \equiv \sum_{j,k=0}^{d-1} a_{jk} |\psi_j^0\rangle \langle \psi_k^0|, \quad \tilde{B} \equiv \sum_{j,k=0}^{d-1} b_{jk} |\psi_j^0\rangle \langle \psi_k^0|. \quad (22)$$

Then, as X_D is a 2-design in \mathcal{H}_D , we have that:

$$\frac{1}{|X_D|} \sum_{j,M} \text{Tr}[\tilde{A} P_{\psi_j^M}] \text{Tr}[\tilde{B} P_{\psi_j^M}] = \frac{\text{Tr} \tilde{A} \text{Tr} \tilde{B} + \text{Tr}[\tilde{A} \tilde{B}]}{D(D+1)} \quad (23)$$

with $P_{\psi_j^M} = |\psi_j^M\rangle \langle \psi_j^M|$. We can define the projector $\Pi_d^D = \sum_{j=0}^{d-1} |\psi_j^0\rangle \langle \psi_j^0|$ and the following set of unnormalized vectors $\{|\tilde{\phi}_j^M\rangle \equiv \Pi_d^D |\psi_j^M\rangle\}_{j=0, \dots, D-1}^{M=0, \dots, D}$. Since both \tilde{A} and \tilde{B} act on the subspace spanned by the

first d elements of \mathcal{B}_0 , they satisfy $\tilde{A} = \Pi_d^D \tilde{A} \Pi_d^D$ and $\tilde{B} = \Pi_d^D \tilde{B} \Pi_d^D$, then:

$$\begin{aligned} \text{Tr}[\tilde{A} P_{\psi_j^M}] &= \langle \tilde{\phi}_j^M | \tilde{A} | \tilde{\phi}_j^M \rangle \\ \text{Tr}[\tilde{B} P_{\psi_j^M}] &= \langle \tilde{\phi}_j^M | \tilde{B} | \tilde{\phi}_j^M \rangle. \end{aligned} \quad (24)$$

Now, let us consider the set $X_d^D = \{|\phi_j^M\rangle\} \in \mathcal{H}_d$ defined by the following states:

$$|\phi_j^M\rangle = \begin{cases} |j\rangle & \text{if } M = 0 \text{ and } 0 \leq j \leq d-1 \\ \sum_{k=0}^{d-1} \frac{e^{i\alpha_{kj}^M}}{\sqrt{d}} |k\rangle & \text{if } 1 \leq M \leq (D+1) \\ & \text{and } 0 \leq j \leq D-1. \end{cases}$$

This set is composed by $|X_d^D| = d + D^2$ states, and it is easy to check that:

$$\text{Tr}[\tilde{A} P_{\psi_j^M}] = \begin{cases} \text{Tr}[A P_{\phi_j^M}] & \text{if } M = 0 \text{ and } j \leq d-1 \\ 0 & \text{if } M = 0 \text{ and } j \geq d \\ \frac{d}{D} \text{Tr}[A P_{\phi_j^M}] & \text{if } 1 \leq M \leq (D+1) \\ & \text{and } 0 \leq j \leq D-1, \end{cases}$$

in the same way for the operator B . Taking into account the fact that, by definition, $\text{Tr}\tilde{A} = \text{Tr}A$, $\text{Tr}\tilde{B} = \text{Tr}B$ and $\text{Tr}[\tilde{A}\tilde{B}] = \text{Tr}[AB]$, we obtain:

$$\sum_{\varphi \in X_d^D} c_\varphi \text{Tr}[A P_\varphi] \text{Tr}[B P_\varphi] = \text{Tr}A \text{Tr}B + \text{Tr}[AB], \quad (25)$$

where $c_\varphi = 1$ if $|\varphi\rangle$ is one of the d states with $M = 0$, and $c_\varphi = d^2/D^2$ otherwise. Defining $Z \equiv \sum c_\varphi = d(d+1)$ and the probabilities $p_\varphi \equiv c_\varphi/Z$, we can rewrite the last equation as:

$$\sum_{\varphi \in X_d^D} p_\varphi \langle \varphi | A | \varphi \rangle \langle \varphi | B | \varphi \rangle = \int_{\mathcal{H}_d} d\psi \langle \psi | A | \psi \rangle \langle \psi | B | \psi \rangle, \quad (26)$$

which shows that the set X_d^D is indeed a proper 2-design in \mathcal{H}_d . As we said before, unlike a complete set of mutually unbiased bases in prime power dimension, this set is

not a uniform 2-design. The set X_d^D splits in two subsets whose weights are different when it comes to estimate Haar integrals. For the d states with $M = 0$ we have $p_\varphi = \frac{1}{Z}$, and for the remaining D^2 states $p_\varphi = \frac{d^2}{ZD^2}$.

Finally, the application for SEQPT is direct: the protocol is exactly the same as the one in Ref. [1], but the sampling of the states from the 2-design X_d^D is done according to the probability distribution p_φ . The efficiency of the protocol is also the same, and can also be obtained as the efficiency of the method in dimension D , as shown in [1]. That is, to obtain an error lower than ϵ with a probability of failure lower than p one has to perform M experiments, where $M \geq \ln(2/p)/2\epsilon^2$. The efficiency of the method comes from the fact that M does not depend on the dimension d (nor it depends on D).

V. SUMMARY

Quantum process tomography is a key tool for quantum information processing. In general, full quantum tomography is an inefficient task and protocols for partial tomography provide essential information about a given process. In this article we showed two schemes that allow to perform efficient selective process tomography in arbitrary dimensions. These protocols are based on the idea presented in Ref. [1], and are extended to arbitrary finite dimensions. Our approach requires the estimation of Haar integrals of quadratic forms in dimensions where the construction of uniform 2-designs is not known. Thus, we showed how one can achieve this task using two methods: on one hand, an approximate 2-design can be constructed using tensor product of 2-designs and, on the other, a non-uniform 2-design can be obtained by projecting 2-designs over smaller dimensions.

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Appendix A: Estimating the elements of the χ -matrix with tensor product of 2-designs

Here we sketch a derivation of Eq. (13). In this case, the action of the channel $\mathcal{E}_{j_1 j_2}^{i_1 i_2}$ over states of the 2-designs X_1 and X_2 is given by:

$$\mathcal{E}_{j_1 j_2}^{i_1 i_2}(P_{\psi_1} \otimes P_{\psi_2}) = \sum_{\substack{\mu_1 \nu_1 \\ \mu_2 \nu_2}} \chi_{\nu_1 \nu_2}^{\mu_1 \mu_2} (E_{\mu_1} E^{i_1} P_{\psi_1} E_{j_1} E^{\nu_1}) \times (E_{\mu_2} E^{i_2} P_{\psi_2} E_{j_2} E^{\nu_2}). \quad (\text{A1})$$

In order to compute the average in this expression, we will consider that the initial states $P_{\psi_1} \otimes P_{\psi_2}$ are sampled from the set of states X_{\otimes} , i.e. the tensor product of 2-designs.

First, let us consider the average of the survival probability over the set X_{\otimes} . Using Eq. (6) this average is given by:

$$\bar{F}_{\otimes}(\mathcal{E}_{j_1 j_2}^{i_1 i_2}) = \sum_{\substack{\mu_1 \nu_1 \\ \mu_2 \nu_2}} \chi_{\nu_1 \nu_2}^{\mu_1 \mu_2} \left(\frac{D_1^2 \delta_{\mu_1}^{i_1} \delta_{j_1}^{\nu_1} + \text{Tr}[E_{\mu_1} E^{i_1} E_{j_1} E^{\nu_1}]}{D_1(D_1 + 1)} \right) \times \left(\frac{D_2^2 \delta_{\mu_2}^{i_2} \delta_{j_2}^{\nu_2} + \text{Tr}[E_{\mu_2} E^{i_2} E_{j_2} E^{\nu_2}]}{D_2(D_2 + 1)} \right). \quad (\text{A2})$$

Now, let us consider the survival probability of system 1 independent of the result of the measurement over system 2. This mean value can be thought of as a reduced

mean survival probability over system 1, and it can be evaluated just by tracing out system 2 in the circuit described in Sec. III B. Thus, using this idea and Eq. (6) over system 1:

$$\bar{F}_1(\mathcal{E}_{j_1 j_2}^{i_1 i_2}) = \sum_{\substack{\mu_1 \nu_1 \\ \mu_2 \nu_2}} \chi_{\nu_1 \nu_2}^{\mu_1 \mu_2} \left(\frac{D_1^2 \delta_{\mu_1}^{i_1} \delta_{j_1}^{\nu_1} + \text{Tr}[E_{\mu_1} E^{i_1} E_{j_1} E^{\nu_1}]}{D_1(D_1 + 1)} \right) \times \left(\sum_{\psi_2 \in X_2} \frac{\text{Tr}[E_{\mu_2} E^{i_2} P_{\psi_2} E_{j_2} E^{\nu_2}]}{D_2(D_2 + 1)} \right).$$

Then, using that $\sum_{\psi_2 \in X_2} P_{\psi_2} = (1 + D_2) \mathbb{1}_2$ and the trace preserving condition $\sum_{\mu_2 \nu_2} \chi_{\mu_2 \nu_2}^{\mu_1 \mu_2} E^{\nu_1 \nu_2} E^{\mu_1 \mu_2} = \mathbb{1}$ we arrive at:

$$\bar{F}_1(\mathcal{E}_{j_1 j_2}^{i_1 i_2}) = \sum_{\mu_2 \nu_2} \chi_{j_1 \nu_2}^{i_1 \mu_2} \text{Tr}[E_{\mu_2} E^{i_2} E_{j_2} E^{\nu_2}] \frac{D_1}{D_2(D_1 + 1)} + \frac{\delta_{j_1}^{i_1} \delta_{j_2}^{i_2}}{D_1 + 1}. \quad (\text{A3})$$

Similarly, we can obtain an analogous expression if we consider the reduced mean survival probability over system 2. Finally, we can relate the mean fidelities by:

$$\bar{F}_{\otimes}(\mathcal{E}_{j_1 j_2}^{i_1 i_2}) = \frac{d \chi_{j_1 j_2}^{i_1 i_2} - \delta_{j_1}^{i_1} \delta_{j_2}^{i_2}}{(D_1 + 1)(D_2 + 1)} + \frac{\bar{F}_1(\mathcal{E}_{j_1 j_2}^{i_1 i_2})}{D_2 + 1} + \frac{\bar{F}_2(\mathcal{E}_{j_1 j_2}^{i_1 i_2})}{D_1 + 1} \quad (\text{A4})$$

We can also see explicitly the correction to the average over the Haar given by the mean fidelity:

$$\bar{F}(\mathcal{E}_{j_1 j_2}^{i_1 i_2}) = \bar{F}_{\otimes}(\mathcal{E}_{j_1 j_2}^{i_1 i_2}) \frac{(D_1 + 1)(D_2 + 1)}{d + 1} + \delta_{j_1}^{i_1} \delta_{j_2}^{i_2} \frac{2}{d + 1} - \bar{F}_1(\mathcal{E}_{j_1 j_2}^{i_1 i_2}) \frac{(D_1 + 1)}{d + 1} - \bar{F}_2(\mathcal{E}_{j_1 j_2}^{i_1 i_2}) \frac{(D_2 + 1)}{d + 1}.$$

Appendix B: Efficiency

In the bipartite case, the circuit of Fig. 2 can give eight different results: they are of the form $\pm \vec{r}$ where the first symbol (\pm) refers to the polarization obtained for the auxiliary qubit, and \vec{r} is a binary string of length two such that $r_i = 1$ if the projective measurement on subsystem \mathcal{H}_i gives survival of the initial state and $r_i = 0$ if not. If we perform M runs of the experiment (where each run corresponds to choosing the initial states randomly from the 2-designs X_1 and X_2), we define $M_{\pm \vec{r}}$ as the number of times we obtain result $\pm \vec{r}$. So, for example, M_{-01} is the number of times we measure -1 for the polarization of the auxiliary qubit, non-survival of the initial state on \mathcal{H}_1 and survival of the initial state on \mathcal{H}_2 . Having said that, if we perform M runs of the experiment, estimations

of $\bar{F}_\otimes(\mathcal{E}_{j_1 j_2}^{i_1 i_2})$, $\bar{F}_1(\mathcal{E}_{j_1 j_2}^{i_1 i_2})$ and $\bar{F}_2(\mathcal{E}_{j_1 j_2}^{i_1 i_2})$ will be given by:

$$\begin{aligned}\bar{F}_\otimes(\mathcal{E}_{j_1 j_2}^{i_1 i_2}) &\simeq \frac{1}{M} (M_{+11} - M_{-11}) \\ \bar{F}_1(\mathcal{E}_{j_1 j_2}^{i_1 i_2}) &\simeq \frac{1}{M} (M_{+11} + M_{+10} - M_{-11} - M_{-10}) \quad (\text{B1}) \\ \bar{F}_2(\mathcal{E}_{j_1 j_2}^{i_1 i_2}) &\simeq \frac{1}{M} (M_{+11} + M_{+01} - M_{-11} - M_{-01}) .\end{aligned}$$

Using this in Eq. (13) we obtain an expression for the estimation of $\chi_{j_1 j_2}^{i_1 i_2}$:

$$\begin{aligned}\chi_{j_1 j_2}^{i_1 i_2} &\simeq \frac{1}{M} \left\{ \left(1 - \frac{1}{d}\right) [M_{+11} - M_{-11}] \right. \\ &\quad - \left(\frac{D_1 + 1}{d}\right) [M_{+10} - M_{-10}] \quad (\text{B2}) \\ &\quad \left. - \left(\frac{D_2 + 1}{d}\right) [M_{+01} - M_{-01}] \right\} ,\end{aligned}$$

In the general case of N subsystems, a similar reasoning gives

$$\chi_j^i \simeq \frac{1}{M} \sum_{\vec{r}} C_{\vec{r}} (M_{+\vec{r}} - M_{-\vec{r}}) , \quad (\text{B5})$$

where now \vec{r} is a binary string of length N indicating survival (or not) in each subsystem. The coefficients $C_{\vec{r}}$ are:

$$C_{\vec{r}} = \begin{cases} 0 & \text{if } \vec{r} = \vec{0} \\ \frac{1}{d} \left[\prod_{a=1}^N (D_a + 1) - \sum_{\vec{m} \notin \{\vec{0}, \vec{1}\}} \prod_{a/\vec{m}_a=1} (D_a + 1) \right] & \text{if } \vec{r} = \vec{1} \\ -\frac{1}{d} \sum_{\vec{m} \notin \{\vec{0}, \vec{1}\}} \prod_{a/\vec{m}_a=1} (D_a + 1) & \text{otherwise,} \end{cases} \quad (\text{B6})$$

where $\|\vec{w}\|_1$ is the sum of the elements of \vec{w} . From the last equation, it is easy to get a useful bound for those coefficients: $|C_{\vec{r}}| < 4^N$. Hence, in general, we can think of χ_j^i as the mean value of a random variable in $[-4^N, 4^N]$ and Hoeffding's inequality gives:

$$p \leq 2 \exp\left(\frac{-M\varepsilon^2}{2 \times 4^N}\right) , \quad (\text{B7})$$

where p is the failure probability for the estimation of χ_j^i with error ε using the results of M experiments. From there, it is easy to show that

$$M \geq \frac{2 \log(2/p) 4^N}{\varepsilon^2} \quad (\text{B8})$$

experiments are enough to achieve the desired precision. By definition, $N = \omega(d)$ (the number of distinct prime divisors of d), and the Hardy–Ramanujan theorem [17]

so we can think of $\chi_{j_1 j_2}^{i_1 i_2}$ as the average of a random variable which can take values $\pm(1 - \frac{1}{d})$, $\pm(\frac{D_1+1}{d})$, $\pm(\frac{D_2+1}{d})$ and 0 (corresponding to the experiments where none of the initial states survive). Hoeffding's inequality [16] tells us that if x is a random variable that takes values in the interval $[a, b]$ and $\bar{x} = \frac{1}{n} \sum_i^n x_i$ is the average of n realizations of this variable, then:

$$\text{Prob}(|\bar{x} - E[x]| \geq \epsilon) \leq 2 \exp\left(\frac{-2n\epsilon^2}{(b-a)^2}\right) , \quad (\text{B3})$$

where $E[x]$ is the mean value of x . In our case, $\chi_{j_1 j_2}^{i_1 i_2}$ is estimated as the mean value of M realizations of a random variable which takes values in $[-(1 - \frac{1}{d}), (1 - \frac{1}{d})]$, so the probability p of failure when estimating this coefficient with error ε is bounded as:

$$p \leq 2 \exp\left(\frac{-M\varepsilon^2}{2(1 - \frac{1}{d})^2}\right) , \quad (\text{B4})$$

which is the same as in Eq. (14).

states that:

$$|\omega(d) - \log(\log(d))| < (\log(\log(d)))^{\frac{1}{2} + \delta} \quad (\text{B9})$$

for almost all $d \in \mathbb{N}$ and for all $\delta > 0$. In particular, $\omega(d) < \log(\log(d)) + (\log(\log(d)))^{\frac{1}{2} + \delta}$. Picking $\delta = 1/2$, we get:

$$\omega(d) < 2 \log(\log(d)) \quad (\text{B10})$$

for almost all $d \in \mathbb{N}$. Using this in Eq. (B8), and recalling that $N = \omega(d)$, we obtain that

$$M \geq \frac{2 \log(2/p) \log(d)^4}{\varepsilon^2} \quad (\text{B11})$$

experiments are enough for almost all dimensions d . Thus, the efficiency of the protocol follows from the fact that the number of experiments grows polynomially with the logarithm of the dimension.