Relational Width of First-Order Expansions of Homogeneous Graphs with Bounded Strict Width

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Abstract

Solving the algebraic dichotomy conjecture for constraint satisfaction problems over structures first-order definable in countably infinite finitely bounded homogeneous structures requires understanding the applicability of local-consistency methods in this setting. We study the amount of consistency (measured by relational width) needed to solve $\mathrm{CSP}(\mathbb{A})$ for first-order expansions $\mathbb{A}$ of countably infinite homogeneous graphs $\mathcal{H} := (A; E)$, which happen all to be finitely bounded. We study our problem for structures $\mathbb{A}$ that additionally have bounded strict width, i.e., for which establishing local consistency of an instance of $\mathrm{CSP}(\mathbb{A})$ not only decides if there is a solution but also ensures that every solution may be obtained from a locally consistent instance by greedily assigning values to variables, without backtracking.

Our main result is that the structures $\mathbb{A}$ under consideration have relational width exactly $(2, \mathbb{L}_{\mathcal{H}})$ where $\mathbb{L}_{\mathcal{H}}$ is the maximal size of a forbidden subgraph of $\mathcal{H}$, but not smaller than 3. It beats the upper bound: $(2m, 3m)$ where $m = \max(\mathrm{arity}(\mathbb{A}) + 1, \mathbb{L}, 3)$ and $\mathrm{arity}(\mathbb{A})$ is the largest arity of a relation in $\mathbb{A}$, which follows from a sufficient condition implying bounded relational width given in [10]. Since $\mathbb{L}_{\mathcal{H}}$ may be arbitrarily large, our result contrasts the collapse of the relational bounded width hierarchy for finite structures $\mathbb{A}$, whose relational width, if finite, is always at most $(2, 3)$.

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1 Introduction

The constraint satisfaction problem (CSP) is one of the most important problems in theoretical and applied computer science and at the same time it is a general framework in which many other computational problems may be formalized. Given a number of constraints imposed on variables one asks if there is a global solution, i.e., a function assigning values to variables so that all the constraints are simultaneously satisfied. Boolean satisfiability and graph colouring are among the most prominent examples of NP-hard problems that can be formalized as CSPs and hence the CSP is NP-hard in general. Thus, one considers the problem $\mathrm{CSP}(\mathbb{A})$ parametrized by a relational structure (called also a constraint language, a language or a template) $\mathbb{A}$. (In this paper, $\mathbb{A}$ is always over a finite signature). A longstanding open problem in this area was to verify the Feder-Vardi [20] conjecture which states that for every finite $\mathbb{A}$ the problem $\mathrm{CSP}(\mathbb{A})$ is either in P or it is NP-complete. After over thirty years of work and a number of important partial results this so-called Dichotomy Conjecture was confirmed independently in [26] and [16]. In both cases the proof was carried out in the

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so-called universal-algebraic approach to the complexity of CSPs [22, 17]. The approach not only provided appropriate tools but also suggested the delineation. This so-called algebraic dichotomy conjecture [17] saying that CSP($\mathbb{A}$) is hard under the condition that the algebra corresponding to $\mathbb{A}$ lacks interesting operations also has been confirmed in both proofs.

The universal-algebraic approach to finite-domain constraint satisfaction problems has been generalized to capture the computational complexity in many other similar settings. In particular, the complexity of CSP($\mathbb{A}$) depends on the algebra corresponding to $\mathbb{A}$ when $\mathbb{A}$ is ω-categorical [11], i.e., all countable models of the first-order theory of $\mathbb{A}$ are isomorphic. In particular, all structures first-order definable in (reducts of) (countably infinite) homogeneous structures over finite signatures are ω-categorical structure. (A structure is homogeneous if every isomorphism between its finite substructures may be extended to an automorphism of a structure.) Considering these infinite structures significantly broadens the class of problems that may be captured within the CSP framework. In particular, the order over rational numbers $(\mathbb{Q}, <)$, which is homogeneous, gives rise to CSP($\mathbb{Q}; <$) that can be seen as the digraph acyclicity problem. The latter cannot be expressed as the CSP over a finite template. Furthermore a number of problems of interest in qualitative reasoning may be captured by CSP($\mathbb{A}$) where $\mathbb{A}$ is a reduct of a homogeneous structure $\mathbb{B}$. It concerns constraint satisfaction problems in formalisms such as Allen's interval algebra or RCC-5, see [6] for a survey. Many of the homogeneous structures $\mathbb{B}$ of interest are finitely bounded, i.e., there exists a finite unique minimal set $\mathcal{F}_{\mathbb{B}}$ of finite structures over the signature of $\mathbb{B}$ such that a finite structure Δ embeds into $\mathbb{B}$ if and only if none of the structures in $\mathcal{F}_{\mathbb{B}}$ embeds into Δ. A dichotomy for algebras corresponding to reducts of countably infinite finitely bounded homogeneous structures was proved in [3]. As in the finite case, it suggests the delineation between polynomial-time solvable and NP-hard CSPs. Although the complexity dichotomy is still far from being obtained, the algebraic dichotomy conjecture for reducts of finitely bounded homogeneous structures is known to hold in the number of cases including the reducts of $(\mathbb{N}, =)$ [7], $(\mathbb{Q}, <)$ [8], the random partial order [23] or a countably infinite homogeneous graph [13, 14, 9].

Theoretical research on CSPs is focused not only on providing classifications of computational complexity but also on settling the limits of applicability of widely known algorithms or algorithmic techniques such as establishing local consistency. This method is used not only for finite CSP but is also considered to be the most important (if not the only) algorithmic technique for qualitative CSPs [25]. The algebraic characterization of finite structures $\mathbb{A}$ with bounded width [2], i.e., for which CSP($\mathbb{A}$) can be solved by establishing local consistency, is considered to be an important step towards solving the Feder-Vardi conjecture. Thus, in order to understand the complexity of CSPs for reducts $\mathbb{A}$ of finitely bounded homogeneous structures, we need to characterize $\mathbb{A}$ with bounded width and to understand how different notions of consistency relate to each other for templates under consideration. The focus of this paper is on the latter.

The amount of consistency needed to solve CSP($\mathbb{A}$) for $\mathbb{A}$ with bounded width is measured here [1] and here [15] by relational width. The relational width of $\mathbb{A}$ is a pair of numbers (k, l) with $k \leq l$ (for the exact definition we refer the reader to Section 4). The following question was of interest for finite structures.

▶ **Question 1.** *What is the exact relational width of* $\mathbb{A}$ *with bounded width?*

Question 1 for finite $\mathbb{A}$ was completely answered in [1] where it was proved that $\mathbb{A}$ with bounded width has always either relational width $(1, 1)$ or $(2, 3)$, see [15] for another proof. Both proofs rely, however, on the algebraic characterization of structures $\mathbb{A}$ with bounded width. Although the notion of bounded width has been generalized to ω-categorical structures [5], according to our knowledge, no algebraic characterization of bounded width for such structures is within sight. Nevertheless the algebraic characterization of strict bounded width has been quite easily lifted from finite [20] to infinite domains [5]. (Again, for a detailed definition we refer the reader to Section 4.) A reduct of a finitely bounded homogeneous structure has bounded strict width if and only if it is preserved by so-called oligopotent quasi near-unanimity operation. This algebraic characterization gives us a hope to answer the following question analogous to Question 1.

▶ **Question 2.** *What is the exact relational width of reducts $\mathbb{A}$ of finitely bounded homogeneous structures with bounded strict width?*

In this paper we answer Question 2 for first-order expansions of countably infinite homogeneous graphs. We believe that our method may be used to provide the general answer in the near future. We note that the answer to Question 2 would be not only a nice theoretical result but should be also of particular interest for structures that give rise to constraint satisfaction problems in qualitative reasoning. In this context strict width is called *local-to-global consistency* and has been widely studied see, e.g., [19].

1.1 Our results

In contrast to all homogeneous structures, all countably infinite homogeneous graphs are well understood and have been classified in [24]. It happens that every such graph $\mathcal{H}$ is also finitely bounded, i.e., in each case there exists a finite unique minimal set of finite graphs $\mathcal{F}_{\mathcal{H}}$ such that a finite graph G embeds into $\mathcal{H}$ if and only if none of the graphs in $\mathcal{F}_{\mathcal{H}}$ embeds into G. We will write $\mathbb{L}_{\mathcal{H}}$ for the maximum of the number 3 and the size of the largest finite structure in $\mathbb{L}_{\mathcal{H}}$. Perhaps the best known example of a homogeneous graph is the random graph that is determined up to isomorphism by the two properties of being homogeneous and universal (i.e., it contains all countable graphs as induced subgraphs). Equivalently, the random graph is a unique countably infinite graph which has this extension property: for all disjoint finite subsets U, U' of the domain there exists an element v such that v is adjacent to all members of U and to none in U'. In this case the finite set of bounds consists of a single directed edge and a loop, and hence $\mathbb{L}_G$ for the random graph G is 3. Furthermore, the family of homogeneous graphs contains universal countable k-clique free graphs H_k with $k \geq 3$, called also Henson graphs, in which case $\mathcal{F}_{H_k}$ contains also a k-clique, and hence $\mathbb{L}_{H_k}$ is k or the graphs C_n^s that are disjoint sums of n cliques of size s where $1 \leq n, s \leq \omega$ and either n or s equals ω. Observe that $\mathcal{F}_{C_n^s}$ contains a graph on three vertices with two edges and one non-edge as well as a null graph over $n+1$ vertices in case n is finite or a $(s+1)$-clique in the case where s is finite. Thus, $\mathbb{L}_{C_n^s}$ is either $3, n+1$ or $s+1$. All remaining homogeneous graphs are the complements of graphs H_k or C_n^s. In this paper we prove the following.

▶ **Main Result.** *Let $\mathbb{A}$ be a first-order expansion of a countably infinite homogeneous graph $\mathcal{H}$ such that $\mathbb{A}$ has bounded strict width. Then $\mathbb{A}$ has relational width $(2, \mathbb{L}_{\mathcal{H}})$.*

In fact, we obtain a more general result. Some sufficient conditions implying that a first-order expansion of a homogeneous graph $\mathcal{H}$ has relational width $(2, \mathbb{L}_{\mathcal{H}})$ are given in Section 5. In particular, the conditions cover all languages under consideration preserved by binary canonical operations considered in [13, 14, 9] where an analysis of algebras corresponding to reducts of homogeneous graphs and the computational dichotomy is provided. Our result: relational width $(2, \mathbb{L}_{\mathcal{H}})$ beats the upper bound $(2m, 3m)$, where $m = \max(\text{arity}(\mathbb{A}) + 1, \mathbb{L}, 3)$ and $\text{arity}(\mathbb{A})$ is the largest arity of a relation in $\mathbb{A}$, that can be easily obtained from the proof of Theorem 4.10 in [10].

We believe that measuring relational width of structures with bounded width is interesting in its own rights. Nevertheless, our research has complexity consequences. As in the finite case, it was proved in [5] that CSP($\mathbb{A}$) for an ω-categorical $\mathbb{A}$ with strict width k may be solved by establishing $(k, k+1)$-consistency and hence in time $O(n^{k+1})$ where n is the number of variables in an instance. Our main result implies that such CSP($\mathbb{A}$) for a first-order expansion $\mathbb{A}$ of a homogeneous graph $\mathcal{H}$ may be solved by establishing $(2, \mathbb{L}_{\mathcal{H}})$-minimality, and hence in time $O(n^m)$ where $m = \max(\mathbb{L}_{\mathcal{H}}, \mathrm{arity}(\mathbb{A}))$.

1.2 Outline of the paper

We start with general preliminaries in Section 2. Then we review canonical operations providing tractability for reducts of homogeneous graphs, Section 3. Bounded (relational) width, strict width and other notions related to local consistency are provided in Section 4. There we also give a number of examples explaining the applicability of our main result. The proof of the main result is divided into Section 5 and Section 6. In the former one, we give a number of sufficient conditions implying relational width $(2, \mathbb{L}_{\mathcal{H}})$, while in the latter one we show that the sufficient conditions are satisfied whenever a first-order expansion of $\mathcal{H}$ has bounded strict width. In Section 5 we additionally show that the sufficient condition are also satisfied by first-order expansions of homogeneous graphs preserved by the studied binary canonical operations. As a consequence, we obtain that all tractable (whose CSP is solvable in polynomial time) reducts of $\mathcal{H}$ where $\mathcal{H}$ is $C_1^\omega, C_\omega^1, C_\omega^\omega$ or H_k with $k \geq 3$ have bounded relational width $(2, \mathbb{L}_{\mathcal{H}})$ and hence can be solved by establishing $(2, \mathbb{L}_{\mathcal{H}})$-minimality.

2 Preliminaries

We write $t = (t[1], ..., t[n])$ for a tuple of elements and $[n]$ to denote the set $\{1, \dots, n\}$.

2.1 Relations, languages and formulas

In this paper we consider first-order expansions $\mathbb{A} := (A; E, R_1, \dots, R_k)$ over a finite signature τ of homogeneous graphs, called also (constraint) languages or templates, where all $R_1, \dots, R_k$ have a first-order definition in $(A; E)$. We assume that $\mathbb{A}$ constains $=$ and N whenever N is pp-definable in $\mathbb{A}$. Relations E and N refer always to a homogeneous graph $\mathcal{H}$ known from the context. For the sake of presentation we usually do not distinguish between a relation symbol R in the signature of $\mathbb{A}$ and the relation $R^{\mathbb{A}}$ and use the former symbol for both. We often write $O, O_1, O_2, \dots$ for elements of $\{E, N, =\}$ and $\underline{\underline{E}}, \underline{\underline{N}}, \underline{\underline{O}}, \underline{\underline{O_1}}, \underline{\underline{O_2}}$ to denote relations $(E \cup =), (N \cup =), (O \cup =), (O_1 \cup =), (O_2 \cup =)$, respectively.

For a structure $\mathbb{A}$ over domain A and a tuple $t \in A^k$, the orbit of t in $\mathbb{A}$ is the relation $\{(\alpha(t[1]), \dots, \alpha(t[k])) \mid \alpha \in \mathrm{Aut}(\mathbb{A})\}$ where $\mathrm{Aut}(\mathbb{A})$ is the set of automorphisms of $\mathbb{A}$. In particular, E, N and $=$ are orbits of pairs, called also orbitals. We would like to note that all structures considered in this paper are ω-categorical. By a theorem proved independently by Ryll-Nardzewski, Engeler and Svenonius, a structure $\mathbb{A}$ is ω-categorical if and only if its automorphism group is oligomorphic, i.e., for every n the number of orbits of n-tuples is finite. See [21] for a textbook on model theory.

A primitive-positive (pp-)formula is a first-order formula built exclusively out of existential quantifiers $\exists$, conjunction $\wedge$ and atomic formulas $R(x_1, \dots, x_k)$ where R is a k-ary relation symbol and $x_1, \dots, x_k$ are variables, not necessarily pairwise different.

2.2 The universal-algebraic approach

We say that an operation $f : A^n \to A$ is a polymorphism of an m-ary relation R iff for any m-tuples $t_1, \ldots, t_n \in R$, it holds that the tuple $(f(t_1[1], \ldots, t_n[1]), \ldots, f(t_1[m], \ldots, t_n[m]))$ is also in R. We write $f(t_1, \ldots, t_n)$ as a shorthand for $(f(t_1[1], \ldots, t_n[1]), \ldots, f(t_1[m], \ldots, t_n[m]))$. An operation f is a polymorphism of $\mathbb{A}$ if it is a polymorphism of every relation in $\mathbb{A}$. If $f : A^n \to A$ is a polymorphism of $\mathbb{A}, R$, we say that f preserves $\mathbb{A}, R$, otherwise that f violates $\mathbb{A}, R$. A set of polymorphisms of an ω-categorical structure $\mathbb{A}$ forms an algebraic object called an oligomorphic locally closed clone [4], which in particular contains an oligomorphic permutation group [18].

▶ **Theorem 1** ([11]). *Let $\mathbb{A}$ be a countable ω-categorical structure. Then R is preserved by the polymorphisms of $\mathbb{A}$ if and only if it has a primitive-positive definition in $\mathbb{A}$, i.e., a definition via a primitive-positive formula.*

We say that a set of operations F generates a set of operations G if every $g \in G$ is in the smallest locally-closed clone containing F. We wite $\overline{\mathrm{Aut}(\mathbb{A})}$ to denote the clone generated by the automorphisms of the structure $\mathbb{A}$. An operation f of an oligomorphic clone F is called oligopotent if $\{g\}$ where $g(x) := f(x, \ldots, x)$ is generated by the permutations in F. We say that a k-ary operation f is a weak near-unanimity operation if $f(y, x, \ldots, x) = f(x, y, x, \ldots, x) = \cdots = f(x, \ldots, x, y)$ for all $x, y \in A$ and that f is a quasi near-unanimity operation (short, qnu-operation) if it is a weak near-unanimity and it additionally satisfies $f(x, \ldots, x) = f(x, \ldots, x, y)$ for all $x, y \in A$. We say that a k-ary operation f is a weak near-unanimity operation modulo $\overline{\mathrm{Aut}(\mathbb{A})}$ if there exist $e_1, \ldots, e_k \in \overline{\mathrm{Aut}(\mathbb{A})}$ such that: $e_1(f(y, x, \ldots, x)) = e_2(f(x, y, x, \ldots, x)) = \cdots = e_k(f(x, \ldots, x, y))$ for all $x, y \in A$.

2.3 The constraint satisfaction problem

We define the CSP to be a computational problem whose instance $\mathcal{I}$ is a triple $(\mathcal{V}, \mathcal{C}, A)$ where $\mathcal{V} = \{v_1, \ldots, v_n\}$ is a set of variables, $\mathcal{C}$ is a set of constraints each of which is of the form $((v_{i_1}, \ldots, v_{i_k}), R)$ where $\{v_{i_1}, \ldots, v_{i_k}\} \subseteq \mathcal{V}$ is *the scope* of the constraint and $R \subseteq A^k$. The question is whether there is a solution $\mathbf{s} : \mathcal{V} \to A$ to $\mathcal{I}$ satisfying $(\mathbf{s}(v_{i_1}), \ldots, \mathbf{s}(v_{i_k})) \in R$ for all $((v_{i_1}, \ldots, v_{i_k}), R) \subseteq \mathcal{C}$. Further, we define $\mathrm{CSP}(\mathbb{A})$ for a constraint language $\mathbb{A}$ to be the CSP restricted to instances where all relations come from $\mathbb{A}$. [1]

We define the projection of $((v_{i_1}, \ldots, v_{i_k}), R)$ to the set $\{w_1, \ldots, w_l\} \subseteq \{v_{i_1}, \ldots, v_{i_k}\}$ to be the constraint $(\{w_1, \ldots, w_l\}, R')$ where the relation R' is given by $(R'(w_1, \ldots, w_l) \equiv \exists x_1 \ldots \exists x_m\ R(v_{i_1}, \ldots, v_{i_k}))$ and $\{x_1, \ldots, x_m\} = \{v_{i_1}, \ldots, v_{i_k}\} \setminus \{w_1, \ldots, w_m\}$. Let $W \subseteq \mathcal{V}$. An assignment $\mathbf{a} : W \to A$ is a partial solution to $\mathcal{I}$ if $\mathbf{a}$ satisfies all projections of constraints in $\mathcal{I}$ to variables in W.

It is very well known that adding pp-definable relations to the template does not change the complexity of the problem.

▶ **Proposition 2.** *Let $\mathbb{A} = (A; R_1, \ldots, R_l)$ be a relational structure, and let R be a relation that has a primitive-positive definition in $\mathbb{A}$. Then $\mathrm{CSP}(\mathbb{A})$ and $\mathrm{CSP}(A, R, R_1, \ldots, R_l)$ are log-space equivalent.*

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[1] Equivalently, one defines an instance of $\mathrm{CSP}(\mathbb{A})$ as a conjunction φ of atomic formulae over the signature of $\mathbb{A}$. Then the question is whether φ is satisfiable in $\mathbb{A}$.

B_1	=	E	N
=	=	E	N
E	E	E	E
N	N	E	N

B_2	=	E	N
=	=	E	E
E	E	E	E
N	E	E	E

B_3	=	E	N
=	=	N	N
E	N	E	N
N	N	N	N

Figure 1 Three examplary binary behaviours: B_1, B_2, and B_3.

2.4 Efficient entailment

We say that a formula φ_1 entails a formula φ_2 both over free variables $x_1, \ldots, x_n$ if $(\forall x_1 \cdots \forall x_n\ (\varphi_1(x_1, \ldots, x_n) \implies \varphi_2(x_1, \ldots, x_n)))$ is a valid sentence. Furthermore, we say that an n-ary relation R entails φ over free variables $x_1, \ldots, x_n$ if $R(x_1, \ldots, x_n)$ entails φ. These definitions are quite standard but for the purposes of this paper we need a stronger notion of entailment.

▶ **Definition 3.** *We say that a quaternary relation R efficiently entails $\psi := (S_1(x_1, x_2) \implies S_2(x_3, x_4))$ where S_1, S_2 are binary relations if R entails ψ and R contains*

1. *a tuple t_1 such that $(t_1[1], t_1[2]) \in S_1$ and $(t_1[3], t_1[4]) \in S_2$, and*
2. *a tuple t_2 such that $(t_2[1], t_2[2]) \notin S_1$ and $(t_2[3], t_2[4]) \notin S_2$.*

We say that a quaternary relation R is a $[(S_1(x_1, x_2) \implies S_2(x_3, x_4)), (\varphi)]$-relation if R efficiently entails $(S_1(x_1, x_2) \implies S_2(x_3, x_4))$ and entails φ or that a quaternary relation R is a $[(S_1(x_1, x_2) \implies S_2(x_3, x_4))]$-relation if R efficiently entails $(S_1(x_1, x_2) \implies S_2(x_3, x_4))$.

3 Canonical Operations over Reducts of Homogeneous Graphs

The polymorphisms that appear in the complexity classifications of CSPs of reducts of homogeneous graphs display some regularities in the sense defined below.

Let $f : A^k \to A$, and let G be a permutation group on A. We say that f is canonical with respect to G if for all $m \in N, \alpha_1, \ldots, \alpha_k \in G$ and m-tuples $a_1, \ldots, a_k$, there exists $\beta \in G$ such that $\beta f(\alpha_1(a_1), \ldots, \alpha_k(a_k)) = f(a_1, \ldots, a_k)$. Equivalently, this means that f induces an operation $\xi^{\text{typ}}(f)$, called a k-ary behaviour, on orbits of m-tuples under G, by defining $\xi^{\text{typ}}(f)(O_1, \ldots, O_k)$ as the orbit of $f(a_1, \ldots, a_k)$ where a_i is any m-tuple in O_i. In what follows we are mainly interested in operations that are canonical with respect to $\text{Aut}(\mathcal{H})$ where $\mathcal{H}$ is a homogeneous graph. Therefore we usually say simply canonical. See [12] for a survey on canonical operations. Three simple binary behaviors are presented in Figure 1. According to Definition 6, a binary injection f such that $\xi^{\text{typ}}(f)$ is

- B_1 is said to be of behavior max and balanced,
- B_2 is said to be E-constant,
- B_3 is said to be of type min and N-dominated.

We introduce the following notation. Let $R_1, \ldots, R_k \subseteq A^2$ be binary relations. We write $R_1 \cdots R_k$ for the binary relation on A^k defined so that: $R_1 \cdots R_k(a_1, a_2)$ holds for k-tuples $a_1, a_2 \in A^k$ if and only if $R_i(a_1[i], a_2[i])$ holds for all $i \in [k]$. Here, we can find the list of all binary behaviours of interest.

▶ **Definition 4.** *Let (A, E) be a countably infinite homogeneous graph. We say that a binary injective operation $f : A^2 \to A$ is*

- balanced *if for all $a, b \in A^2$ we have that* $\text{E=}(a, b)$ *and* $\text{=E}(a, b)$ *implies $E(f(a), f(b))$ as well as* $\text{N=}(a, b)$ *and* $\text{=N}(a, b)$ *implies $N(f(a), f(b))$,*

- *E-dominated (N-dominated) if for all $a, b \in A^2$ with $\neq\!=(a, b)$ or $=\!\neq(a, b)$ we have that $E(f(a), f(b))$ ($N(f(a), f(b))$);*
- *of behaviour* min *if for all $a, b \in A^2$ with $\neq\neq(a, b)$ we have $E(f(a), f(b))$ iff* $\mathrm{EE}(a, b)$*;*
- *of behaviour* max *if for all $a, b \in A^2$ with $\neq\neq(a, b)$ we have $N(f(a), f(b))$ iff* $\mathrm{NN}(a, b)$*;*
- *of behaviour* projection *if there exists $i \in [2]$ such that for all $a, b \in A^2$ with $\neq\neq(a, b)$ we have $E(f(a), f(b))$ iff $E(a[i], b[i])$,*
- *of behaviour* xor *if for all $a, b \in A^2$ with $\neq\neq(a, b)$ the relation $E(f(a), f(b))$ holds iff* $\mathrm{EN}(a, b)$ *or* $\mathrm{NE}(a, b)$ *holds;*
- *of behaviour* xnor *if for all $a, b \in A^2$ with $\neq\neq(a, b)$ the relation $E(f(a), f(b))$ holds iff* $\mathrm{EE}(a, b)$ *or* $\mathrm{NN}(a, b)$ *holds;*
- E-constant *if the image of f is a clique,*
- N-constant *if the image of f is an independent set.*

We now turn to ternary behaviours of interest.

▶ **Definition 5.** *Let $(A; E)$ be a countably infinite homogeneous graph. We say that a ternary injective operation $f : A^3 \to A$ is of behaviour*

- majority *if for all $a, b \in D^3$ satisfying $\neq\neq\neq (a, b)$ we have that $E(f(a), f(b))$ if and only if $EEE(a, b), EEN(a, b), ENE(a, b)$, or $NEE(a, b)$,*
- minority *if for all $a, b \in D^3$ satisfying $\neq\neq\neq (a, b)$ we have that $N(f(a), f(b))$ if and only if $NNN(a, b), EEN(a, b), ENE(a, b)$, or $NEE(a, b)$.*

Furthermore, let B be a binary behavior. A ternary function is hyperplanely of behaviour B if the binary functions $(x, y) \to f(x, y, c), (x, z) \to f(x, c, z)$, and $(y, z) \to f(c, y, z)$ have behavior B for all $c \in D$.

4 Consistency and Minimality

This section is devoted to the formal introduction of consistency and width notions. The main algorithm we are interested in is based on establishing minimality.

▶ **Definition 6.** *Let $l \geq k > 0$ be natural numbers. An instance $\mathcal{I} = (\mathcal{V}, \mathcal{C}, A)$ of the CSP is (k, l)-minimal if:*

1. *Every at most l-element set of variables is contained in the scope of some constraint in $\mathcal{I}$.*
2. *For every set W with $|W| \leq k$ and every pair of constraints C_1 and C_2 in $\mathcal{C}$ whose scopes contain W, the projections of the constraints C_1 and C_2 to W are the same.*

We say that $\mathcal{I}$ is trivial if it contains a constraint with an empty relation. Otherwise, we say that $\mathcal{I}$ is non-trivial.

As in the finite case, one may transform an instance $\mathcal{I}$ into an equivalent instance, i.e. with the same set of solutions by simply introducing at most $O(|\mathcal{V}|^l)$ new constraints so that the first condition in Definition 6 was satisfied and then by repeatedly removing orbits of tuples from constraints until the second condition is satisfied. Similarly to the finite CSP we have the following.

▶ **Proposition 7.** *Let $\mathbb{A}$ be an ω-categorical relational structure. Then for every instance $\mathcal{I}$ of* $\mathrm{CSP}(\mathbb{A})$ *and $l \geq k > 0$ there exists an instance $\mathcal{I}'$ of the CSP with the same sets of solutions as $\mathcal{I}$ such that $\mathcal{I}'$ is (k, l)-minimal.*

For fixed (k, l) and $\mathbb{A}$, the process of establishing (k, l)-minimality, i.e., transfoming $\mathcal{I}$ into $\mathcal{I}'$ takes time $O(|\mathcal{V}|^m)$ where $m = \max(l, \mathit{arity}(\mathbb{A}))$ is the maximum of l and the greatest arity of a relation in $\mathbb{A}$. If $\mathcal{I}'$ is trivial, then both $\mathcal{I}$ and $\mathcal{I}'$ have no solutions.

We are now ready to define the relational width.

▶ **Definition 8.** *We say that $\mathbb{A}$ has relational width (k, l) if and only if $\mathcal{I}$ has a solution provided any (k, l)-minimal instance of the CSP equivalent to I is non-trivial. We say that $\mathbb{A}$ has (relational) bounded width* [2] *if there exist (k, l) such that $\mathbb{A}$ has relational width (k, l).*

Finite structures with bounded relational width admit an algebraic characterization [2]. It is known that a finite structure $\mathbb{A}$ has bounded (relational) width if and only if it has a four-ary polymorphism f and a ternary polymorphism g that are weak near-unanimity operations and such that $f(y, x, x, x) = g(y, x, x)$ for all $x, y \in A$. We have a similar sufficient condition for reducts of finitely bounded homogeneous structures.

▶ **Theorem 9** ([10]). *Let $\mathbb{A}$ be a finite-signature reduct of a finitely bounded homogeneous structure $\mathbb{B}$. Suppose that $\mathbb{A}$ has a four-ary polymorphism f and a ternary polymorphism g that are canonical with respect to $\mathrm{Aut}(\mathbb{B})$ and are weak near-unanimity operations modulo $\overline{\mathrm{Aut}(\mathbb{B})}$, and such that there are operations $e_1, e_2 \in \overline{\mathrm{Aut}(\mathbb{B})}$ with $e_1(f(y, x, x, x)) = e_2(g(y, x, x))$ for all $x, y \in A$. Then $\mathrm{CSP}(\mathbb{A})$ has bounded relational width.*

A slight change in the proof of the above theorem gives us the upper bound for relational width of infinite structures under consideration.

▶ **Corollary 10.** *Let $\mathbb{A}$ be a finite-signature reduct of a finitely bounded homogeneous structure $\mathbb{B}$. Suppose that $\mathbb{A}$ has a four-ary polymorphism f and a ternary polymorphism g that are canonical with respect to $\mathrm{Aut}(\mathbb{B})$, that are weak near-unanimity operations modulo $\overline{\mathrm{Aut}(\mathbb{A})}$, and such that there are operations $e_1, e_2 \in \overline{\mathrm{Aut}(\mathbb{B})}$ with $e_1(f(y, x, x, x)) = e_2(g(y, x, x))$ for all $x, y \in A$. Then $\mathbb{A}$ has relational width $(2m, 3m)$ where $m = \max(\mathit{arity}(\mathbb{A}) + 1, \mathit{arity}(\mathbb{B}) + 1, \mathbb{L}_{\mathbb{B}}, 3)$.*

We now use Corollary 10 to provide the upper bound of the relational width for reducts of homogeneous graphs preserved by binary canonical operations considered in [13, 14, 9].

▶ **Proposition 11.** *Let $\mathbb{A}$ be a reduct of a countably infinite homogeneous graph $\mathcal{H}$ preserved by a binary injection:*
1. *of behaviour max which is either balanced or E-dominated, or*
2. *of behaviour min which is either balanced or N-dominated, or*
3. *which is E-constant, or*
4. *which is N-constant.*

Then it has relational width $(2m, 3m)$ where $m = \max(\mathit{arity}(\mathbb{A}) + 1, \mathbb{L}_{\mathcal{H}}, 3)$.

In Section 6.1, we use our approach to show that the exact relational width of structures under consideration in Proposition 11 is $(2, \mathbb{L}_{\mathcal{H}})$. The same is proved for first-order expansions of homogeneous graphs with bounded strict width.

Strict width is defined as follows. A (k, l)-minimal instance $\mathcal{I}$ of the CSP is called globally consistent, if every partial solution of $\mathcal{I}$ can be extended to a total solution of $\mathcal{I}$.

▶ **Definition 12.** *We say that $\mathbb{A}$ has strict width k if for some $l \geq k \geq 2$ all instances of $\mathrm{CSP}(\mathbb{A})$ that are (k, l)-minimal are globally consistent. We say that $\mathbb{A}$ has bounded strict width if it has strict width k for some k.* [3]

We have the following algebraic characterization of ω-categorical structures with bounded strict width.

[2] We note that the definition of bounded width provided in [5] is equivalent to ours.
[3] Our definition of strict width slightly varies from a definition in [5] but again both definitions are equivalent.

▶ **Theorem 13** ([5, 4])**.** *Let $\mathbb{A}$ be an ω-categorical language. Then the following are equivalent.*
1. *$\mathbb{A}$ has strict width k.*
2. *$\mathbb{A}$ has an oligopotent $(k+1)$-ary quasi near-unanimity operation as a polymorphism.*

For a $(2,k)$-minimal instance over variables $\mathcal{V} = \{v_1,\ldots,v_n\}$ we write $\mathcal{I}_{i,j}$ with $i,j \in [n]$ to denote a subset of $\{E,N,=\}$ such that the projection of all constraints having v_i, v_j in its scope to $\{v_i, v_j\}$ equals $\bigcup_{O \in \mathcal{I}_{i,j}} O$. We will say that an instance is *simple* if $|\mathcal{I}_{i,j}| = 1$ for all $i,j \in [n]$.

We will now show that a simple non-trivial $(2, \mathbb{L}_{\mathcal{H}})$-instance of $\mathrm{CSP}(\mathbb{A})$ for a first-order expansion $\mathbb{A}$ of a homogeneous graph $\mathcal{H}$ always has a solution and that this amount of consistency is necessary.

▶ **Observation 14.** *Let $\mathcal{I}$ be a simple non-trivial $(2,\mathbb{L}_{\mathcal{H}})$-minimal instance of the CSP equivalent to an instance of $\mathrm{CSP}(\mathcal{H}')$ where $\mathcal{H}'$ is the expansion of $\mathcal{H}$ containing all orbitals pp-definable in $\mathcal{H}$. Then $\mathcal{I}$ has a solution.*

On the other hand, for every homogeneous graph $\mathcal{H}$ there exists a simple non-trivial $(1,\mathbb{L}_{\mathcal{H}})$-minimal instance $\mathcal{I}_1$ equivalent to an instance of $\mathrm{CSP}(\mathcal{H}')$ and a simple non-trivial $(2,\mathbb{L}_{\mathcal{H}}-1)$-minimal instance $\mathcal{I}_2$ equivalent to an instance of $\mathrm{CSP}(\mathcal{H}')$ that have no solutions.

Proof. We start from proving the first part of the observation. Define Δ to be a finite structure over the domain consisting of variables $\{v_1,\ldots,v_n\}$ in $\mathcal{I}$ and the signature $\tau \subseteq \{E,N,=\}$ such that $(v_i,v_j) \in R^{\Delta}$ for $i,j \in [n]$ and $R \in \tau$ if φ_I contains a constraint $((v_i,v_j),R)$. Since $\mathcal{I}$ is $(2,3)$-minimal, we have the following.

▶ **Observation 15.** *The binary relation $\sim := \{(v_i,v_j) \mid \mathcal{I}_{i,j} \subseteq \{=\}\} \cup \bigcup_{i\in[n]} \{(v_i v_i)\}$ is an equivalence relation.*

We claim that there is an embedding from $\Delta/\sim$ to $\mathcal{H}'$. Assume the contrary. Since $\mathcal{H}$ is finitely bounded, there exists G over variables $\{w_1,\ldots,w_l\}$ in $\mathcal{F}_{\mathcal{H}'}$ such that G embeds into $\Delta/\sim$ and $l \leq \mathbb{L}_{\mathcal{H}}$. Since $\mathcal{I}$ is $(2,\mathbb{L}_{\mathcal{H}})$-minimal, there is a constraint C in $\mathcal{I}$ whose scope contains $\{w_1,\ldots,w_l\}$ and the corresponding relation is empty. It contradicts with the assumption that $\mathcal{I}$ is non-trivial. Thus, $\Delta/\sim$ embeds into $\mathcal{H}$, and in consequence $\mathcal{I}$ has a solution. It completes the proof of the first part of the observation.

For the second part of the observation, we select $\mathcal{I}_1$ to be $\{((v_1,v_2),E),((v_1,v_2),N)\}$. Indeed, every subset of variables of $\mathcal{I}_1$ is in the scope of some constraint. The projection of each constraint to $\{v_1\}$ or $\{v_2\}$ is the set of all vertices in $\mathcal{H}$. It follows that $\mathcal{I}_1$ is $(1,\mathbb{L}_{\mathcal{H}})$-consistent. Clearly $\mathcal{I}_1$ has no solutions.

We now turn to $\mathcal{I}_2$. If $\mathbb{L}_{\mathcal{H}} > 3$ and $G = ([n],E)$ is a forbidden subgraph of size $n = \mathbb{L}_{\mathcal{H}}$ consider an instance $\mathcal{I}_2'$ over variables $\{v_1,\ldots,v_n\}$ containing a constraint $((v_i,v_j),E)$ if $(i,j) \in E^G$ and a constraint $((v_i,v_j),N)$ if $(i,j) \notin E^G$. Let $\mathcal{I}_2$ be a $(2,\mathbb{L}_{\mathcal{H}}-1)$-minimal instance of the CSP equivalent to $\mathcal{I}_2'$. By the minimality of $\mathcal{F}_{\mathcal{H}}$, we have that no induced subgraph of G is in $\mathcal{F}_{\mathcal{H}}$. It follows that $\mathcal{I}_2$ is non-trivial but, clearly, $\mathcal{I}_2$ has no solutions. If $\mathbb{L}_{\mathcal{H}} = 3$, then we select $\mathcal{I}_2$ to be an instance such that $\mathcal{C} = \{((v_1,v_2),=),((v_2,v_3),=),((v_1,v_3),E)$. It is again straightforward to check that $\mathcal{I}_2$ is $(2,2)$-minimal. Yet, it has no solutions. It completes the proof of the observation. ◀

We complete this section by giving some examples of first-order expansions of homogeneous graphs with bounded strict width.

▶ **Proposition 16.** *Let $\mathbb{A}$ be a first-order expansion of the random graph $\mathcal{H} = (A;E)$ such that every relation in $\mathbb{A}$ is pp-definable as a conjunction of clauses of the form:*

$$(x_1 \neq y_1 \vee \cdots \vee x_k \neq y_1 \vee R(y_1,y_2) \vee y_2 \neq z_1 \vee \cdots \vee y_2 \neq z_l),$$

where $R \in \{E,N\}$. Then $\mathbb{A}$ has bounded strict-width.

And here comes another example.

▶ **Proposition 17.** *The constraint language $\mathbb{A} = (A; E, N, R)$ where $(A; E)$ is C_2^ω and $R(x_1, x_2, x_3) \equiv ((E(x_1, x_2) \wedge N(x_2, x_3)) \vee (N(x_1, x_2) \wedge E(x_2, x_3)))$ has bounded strict width.*

5 Conditions Sufficient for Low Relational Width

In order to show that a non-trivial $(2, \mathbb{L}_{\mathcal{H}})$-minimal instance $\mathcal{I}$ of CSP($\mathbb{A}$) for first-order expansions $\mathbb{A}$ of a homogeneous graph $\mathcal{H}$ has a solution, we always use one scheme. We take advantage of the fact that certain quaternary and ternary relations are not pp-definable in $\mathbb{A}$ and we carefully narrow down $\mathcal{I}_{i,j}$ for $i, j \in [n]$ so that we end up with a simple non-trivial instance $\mathcal{I}'$ which is a 'subinstance' of $\mathcal{I}$ in the following sense: for every $C = ((x_1, \ldots, x_r), R)$ in $\mathcal{I}$ we have $C' = ((x_1, \ldots, x_r), R') \in \mathcal{I}'$ such that $R' \subseteq R$. Since $\mathcal{I}'$ is simple, by Observation 14, it has a solution. This solution is clearly a solution to the orginal instance $\mathcal{I}$.

We shrink an instance of CSP($\mathbb{A}$) using one of three different sets of relations presented in the three lemmas below.

▶ **Lemma 18.** *Let $\{O_1, O_2\}$ be $\{E, N\}$ and $\mathbb{A}$ be a first-order expansion of a homogeneous graph $\mathcal{H}$ such that none of the following types of relations is pp-definable in $\mathbb{A}$:*

1. $[(O_1(x_1, x_2) \implies \underline{\underline{\mathrm{O}_2}}(x_3, x_4))]$*-relations,*
2. $[(O_1(x_1, x_2) \implies x_3 = x_4)]$*-relations, and*
3. $[(O_2(x_1, x_2) \implies x_3 = x_4), (\underline{\underline{\mathrm{O}_2}}(x_1, x_2) \wedge \underline{\underline{\mathrm{O}_2}}(x_3, x_4))]$*-relations.*

Then $\mathbb{A}$ has relational width $(2, \overline{\mathbb{L}_{\mathcal{H}}})$.

Before we discuss the 'shrinking' strategy that stands behind Lemma 18, consider a non-trivial instance $\mathcal{I}$ of some $\mathbb{A}$ under consideration in the lemma and a constraint $((x_1, \ldots, x_r), R)$ for which there are i_1, j_1, i_2, j_2 such that $v_{i_1}, v_{j_1}, v_{i_2}, v_{j_2} \in \{x_1, \ldots, x_r\}$ and $O_1 \in \mathcal{I}_{i_1, j_1}$, $O_1 \in \mathcal{I}_{i_2, j_2}$. Since $\mathcal{I}$ is non-trivial and $(2, 3)$-minimal the relation $(R'(x_1, \ldots, x_r) \equiv (R(x_1, \ldots, x_r) \wedge O_1(v_{i_1}, v_{j_1})))$ is non-empty. But also $(R''(x_1, \ldots, x_r) \equiv (R(x_1, \ldots, x_r) \wedge O_1(v_{i_1}, v_{j_1}) \wedge O_1(v_{i_2}, v_{j_2})))$ is non-empty. Indeed, otherwise since $O_1 \cup \underline{\underline{\mathrm{O}_2}} = A^2$, the structure $\mathbb{A}$ would define a relation from Item 1 or Item 2. Generalizing the argument, one can easily transform $\mathcal{I}$ to a non-trivial $\mathcal{I}'$ where every $\mathcal{I}'_{i,j} = \{O_1\}$ whenever $\mathcal{I}_{i,j}$ contains O_1. Using a similar reasoning and Item 3, and taking care of some details, we have to skip here, one can then transform $\mathcal{I}'$ to $\mathcal{I}''$ so that $\mathcal{I}''_{i,j} = O_2$ whenever $\mathcal{I}'_{i,j}$ contains O_2. Since $\mathcal{I}''$ is simple and non-trivial, we can use Observation 14 to argue that both $\mathcal{I}''$ and $\mathcal{I}$ has a solution.

The next lemma considers a specific situation where $\mathcal{H}$ is a disjoint sum of ω edges and languages under consideration are preserved by oligopotent qnu-operations.

▶ **Lemma 19.** *Let $\mathbb{A}$ be a first-order expansion of C_ω^2 preserved by an oligopotent qnu-operation and such that none of the following types of relations is pp-definable in $\mathbb{A}$:*

1. $[(N(x_1, x_2) \implies \underline{\underline{\mathrm{E}}}(x_3, x_4))]$*-relations,*
2. $[(N(x_1, x_2) \implies E(x_3, x_4)), (\underline{\underline{\mathrm{E}}}(x_3, x_4))]$*-relations,*
3. $[(N(x_1, x_2) \implies x_3 = x_4)]$*-relations,*
4. $[(O_1(x_1, x_2) \implies O_2(x_3, x_4)), (\underline{\underline{\mathrm{E}}}(x_1, x_2) \wedge N(x_2, x_3) \wedge \underline{\underline{\mathrm{E}}}(x_3, x_4))]$*-relations where the set $\{O_1, O_2\}$ equals $\{E, =\}$.*

Then $\mathbb{A}$ has relational width $(2, 3)$.

The shrinking strategy for Lemma 19 is as follows. We start with a non-trivial $(2, 3)$-minimal instance $\mathcal{I}$ and use Items 1–3 to transform it into a non-trivial $(2, 3)$-minimal $\mathcal{I}'$ such that $\mathcal{I}'_{i,j} = \{N\}$ whenever $\mathcal{I}_{i,j}$ contains N. Since $\underline{\underline{\mathrm{E}}}$ fo-definable in C_ω^2 is transitive and

$\mathcal{I}'$ is $(2,3)$-minimal, it is easy to show that the graph over variables $\{v_1,\ldots,v_n\}$ and edges $\mathcal{I}'_{i,j}$ with $i,j \in [n]$ is a disjoint union of components $K_1,\ldots,K_\kappa$ such that for all $k \in [\kappa]$ and all $v_i, v_j \in K_k$ it holds that $\mathcal{I}'_{i,j} \subseteq \{E,=\}$ and whenever v_i, v_j are in different components, then $\mathcal{I}'_{i,j} = \{N\}$. Now, any $\mathcal{I}'_{K_i}$ – the instance $\mathcal{I}'$ restricted to variables in K_i, which is in fact an instance of CSP(Δ) for Δ over two-elements (some edge in C^2_ω, different for every $i \in [\kappa]$) preserved by a near-unanimity operation, is shown to have a solution $\mathbf{s}_i$. It follows by the characterization of relational width for finite structure. In order to prove that solution $\mathbf{s} := \bigcup_{i\in[\kappa]} \mathbf{s}_i$ is the solution to $\mathcal{I}'$ and hence to $\mathcal{I}$ we use the fact that relations from Item 4 are not pp-definable in $\mathbb{A}$.

Finally, we turn to the case where $\mathcal{H}$ is a disjoint sum of two infinite cliques and the structures $\mathbb{A}$ have oligopotent qnu-operations as polymorphisms.

▶ **Lemma 20.** *Let $\mathbb{A}$ be a first-order expansion of C^ω_2 preserved by an oligopotent qnu-operation and such that $\mathbb{A}$ pp-defines neither $\underline{\underline{\mathrm{N}}}$ nor $[(O(x_1,x_2) \rightarrow x_3 = x_4)]$ for any $O \in \{E,N\}$. Then $\mathbb{A}$ has relational width $(2,3)$.*

Clearly any tuple over C^ω_2 takes some of its values from one equivalence class in C^ω_2 and the remaining values from the other class. In order to prove Lemma 20, we consider a non-trivial $(2,3)$-minimal instance $\mathcal{I}$ of CSP($\mathbb{A}$) but this time we also assume without loss of generality that there are no i,j with $\mathcal{I}_{i,j} = \{=\}$. Since $\mathbb{A}$ does not define $\underline{\underline{\mathrm{N}}}$ we have that for all $i,j \in [n]$ the set $\mathcal{I}_{i,j}$ contains E. Then we transform $\mathcal{I}$ to $\mathcal{I}_B$ of CSP(Δ) where Δ is over the domain $\{0,1\}$ by replacing any tuple t in any relation in any constraint in $\mathcal{I}$ by a tuple over $\{0,1\}$ so that all values in one equivalence class are replaced by 0 and all values in the other equivalence class are replaced by 1. Since Δ is preserved by a near-unanimity operation, and hence has bounded relational width we have that the $(2,3)$-minimal $\mathcal{I}_B$ has a solution $\mathbf{s}_B : \{v_1,\ldots,v_m\} \rightarrow \{0,1\}$. We use $\mathbf{s}_B$ to transform $\mathcal{I}$ to $\mathcal{I}'$ so that we set $\mathcal{I}'_{i,j}$ to $\{N\}$ whenever $\mathbf{s}_B(v_i) \neq \mathbf{s}_B(v_j)$. No $[(N(x_1,x_2) \rightarrow x_3 = x_4)]$-relations are pp-definable in $\mathbb{A}$, and hence we have that $\mathcal{I}'_{i,j}$ for any $i,j \in [n]$ contains E. Since $\mathbb{B}$ pp-defines no $[(E(x_1,x_2) \rightarrow x_3 = x_4)]$-relations we may transform $\mathcal{I}'$ into $\mathcal{I}''$ so that $\mathcal{I}''_{i,j}$ is E whenever $\mathcal{I}'_{i,j} \neq \{N\}$. Thus, $\mathcal{I}''$ is a simple non-trivial $(2,3)$-minimal instance. It follows by Observation 14 that both $\mathcal{I}''$ and $\mathcal{I}$ have a solution. Again, we skipped many details but our goal was rather to convey some intuitions that stand behind the proofs of the lemmas in this section.

6 Constraint Languages with Low Relational Width

In this section we employ lemmas from Section 5 to provide the exact characterization of relational width of first-order expansions of homogeneous graphs with bounded strict width and first-order expansions of homogeneous graphs preserved by binary canonical operations from Proposition 11. In order to prove the former, we also show which quaternary relations of interest are violated by ternary injections used in the complexity classification (see Subsection 6.2). To rule out some other relations, we have to use oligopotent qnu-operations directly (see Subsection 6.3).

6.1 Binary Injections and Low Relational Width

We start with first-order expansions $\mathbb{A}$ of homogeneous graphs $\mathcal{H}$ whose tractability has been shown in Proposition 8.22 in [13], Proposition 6.2 in [14] as well as Proposition 37 and Theorem 62 in [9].

▶ **Lemma 21.** *Let $\mathbb{A}$ be a first-order expansion of a countably infinite homogeneous graph $\mathcal{H}$ preserved by a binary injection:*
1. *of behaviour max which is either balanced or E-dominated, or*
2. *of behaviour min which is either balanced or N-dominated, or*
3. *which is E-constant, or*
4. *which is N-constant.*

Then $\mathbb{A}$ has relational width $(2, \mathbb{L}_{\mathcal{H}})$.

The above lemma gives an opportunity to reformulate the dichotomy results for reducts $\mathbb{A}$ of $C^1_\omega, C^\omega_1, C^\omega_\omega$ and H_k for any $k \geq 3$.

▶ **Corollary 22.** *Let $\mathbb{A}$ be a reduct of a homomorphism graph $\mathcal{H}$ which is $C^1_\omega, C^\omega_1, C^\omega_\omega$ or H_k for any $k \geq 3$. Then either* $\mathrm{CSP}(\mathbb{A})$ *is NP-complete or $\mathbb{A}$ has relational width $(2, \mathbb{L}_{\mathcal{H}})$.*

Proof. We have that any tractable first-order expansion of $(\mathbb{N}; =, \neq)$ is preserved by a binary injection [7]. It follows that every tractable reduct of C^ω_1 is either preserved by a constant operation or is a first-order expansion of C^ω_1 and preserved by a binary injection which is of behaviour max and E-dominated. A similar reasoning holds for reducts of C^1_ω with a difference that we replace E with N. Further, every reduct of C^ω_ω is either homomorphically equivalent to a reduct of $(\mathbb{N}; =)$ or pp-defines both E and N, see Theorem 4.5 [14]. In the former case we are done, while in the latter a tractable $\mathbb{A}$ is preserved by a binary injection of behaviour min and balanced, Corollary 7.5 in [14]. The corollary follows by Lemma 21. By Proposition 15 and Lemma 17 in [9], a tractable reduct of H_k with $k \geq 3$ is either homomorphically equivalent to a reduct of $(\mathbb{N}; =)$ or pp-defines both E and N. In the former case we are done while in the latter, we have that $\mathrm{CSP}(\mathbb{A})$ is in P when it is preserved by a binary injection of behaviour min and N-dominated (see Theorem 38 in [9]). Again, the corollary follows by Lemma 21. ◀

6.2 Types of Relations violated by Ternary Canonical Operations

Here we look at quaternary relations of interest violated by canonical ternary operations. We start with ternary injections of behaviour majority.

▶ **Lemma 23.** *Let $\mathbb{A}$ be a reduct of a countably infinite homogeneous graph preserved by a ternary injection of behaviour majority which additionally is:*
- *hyperplanely balanced and of behaviour projection, or*
- *hyperplanely E-constant, or hyperplanely N-constant, or*
- *hyperplanely of behaviour max and E-dominated, or*
- *hyperplanely of behaviour min and N-dominated.*

Then $\mathbb{A}$ pp-defines no $[(O(x_1, x_2) \implies x_3 = x_4)]$-relations with $O \in \{E, N\}$.

We continue with ternary injections of behaviour minority.

▶ **Lemma 24.** *Let $\mathbb{A}$ be a reduct of a countably infinite homogeneous graph preserved by a ternary injection of behaviour minority which additionally is:*
- *hyperplanely balanced and of behaviour projection,*
- *hyperplanely of behaviour projection and E-dominated, or*
- *hyperplanely of behaviour projection and N-dominated, or*
- *hyperplanely balanced of behaviour xnor, or*
- *hyperplanely balanced of behaviour xor.*

Then $\mathbb{A}$ does not pp-define $[(O(x_1, x_2) \implies x_3 = x_4)]$-relations with $O \in \{E, N\}$.

We are already in the position to prove that another large family of CSP($\mathbb{A}$) under consideration may be solved by establishing minimality.

▶ **Corollary 25.** *Let $\mathbb{A}$ be a first-order expansion of C_2^ω preserved by a canonical ternary injection of behaviour minority which is hyperplanely balanced of behaviour xnor and an oligopotent qnu-operation. Then $\mathbb{A}$ has relational width $(2, 3)$.*

Proof. By Lemma 24, the structure $\mathbb{A}$ does not pp-define $[(O(x_1, x_2) \implies x_3 = x_4)]$ with $O \in \{E, N\}$. Since a canonical ternary injection of behaviour minority which is hyperplanely balanced of behaviour xnor does not preserve $\underline{\underline{\mathrm{N}}}$, the result follows by appealing to Lemma 20. ◀

The third lemma of this subsection takes care of the third kind of ternary operations that occurrs in complexity classifications of CSPs for reducts of homogeneous graphs.

▶ **Lemma 26.** *Let $\mathbb{A}$ be a reduct of a countably infinite homogeneous graph preserved by a ternary canonical operation h with $h(N, \cdot, \cdot) = h(\cdot, N, \cdot) = h(\cdot, \cdot, N) = N$ and which behaves like a minority on $\{E, =\}$, i.e., h satisfies the behaviour B such that $B(E, E, E) = B(E, =, =) = B(=, E, =) = B(=, =, E) = E$ and $B(=, =, =) = B(=, E, E) = B(E, =, E) = B(E, E, =)$ equals $=$. Then $\mathbb{A}$ pp-defines none of the following types of relations:*

- $[(N(x_1, x_2) \implies \underline{\underline{\mathrm{E}}}(x_3, x_4))]$*-relations,*
- $[(N(x_1, x_2) \implies x_3 = x_4)]$*-relations,*
- $[(N(x_1, x_2) \implies E(x_3, x_4)), (\underline{\underline{\mathrm{E}}}(x_3, x_4))]$*-relations.*

6.3 Types of Relations violated by Oligopotent QNUs

Here we provide a list of quaternary relations of interest violated by ternary canonical operations and oligopotent qnu-operations. We start with the case where the considered homogeneous graph is the random graph.

▶ **Lemma 27.** *Let $\mathbb{A}$ be a first-order expansion of the random graph preserved by a ternary injection of behaviour majority which additionally satisfies one of the conditions in Lemma 23 or of behaviour minority which additionally satisfies one of the conditions in Lemma 24, and an oligopotent qnu-operation. Then $\mathbb{A}$ pp-defines at most one of the following:*

1. *either a* $[(E(x_1, x_2) \implies \underline{\underline{\mathrm{N}}}(x_3, x_4))]$*-relation or*
2. *a* $[(N(x_1, x_2) \implies \underline{\underline{\mathrm{E}}}(x_3, x_4))]$*-relation.*

Here comes the corollary.

▶ **Corollary 28.** *Let $\mathbb{A}$ be a first-order expansion of the random graph preserved by a ternary injection of behaviour majority which additionally satisfies one of the conditions in Lemma 23 or of behaviour minority which additionally satisfies one of the conditions in Lemma 24 and an oligopotent qnu-operation. Then $\mathbb{A}$ has relational width $(2, 3)$.*

Proof. By appeal to Lemma 27, it follows that there are $\{O_1, O_2\} = \{E, N\}$ such that $\mathbb{A}$ does not pp-define a $[(O_1(x_1, x_2) \implies \underline{\underline{O_2}}(x_3, x_4))]$-relation. By Lemmas 23 and 24, $\mathbb{A}$ pp-defines neither $[(O_1(x_1, x_2) \implies x_3 = x_4)]$-relations nor $[(O_2(x_1, x_2) \implies x_3 = x_4)]$-relations. Since $\mathbb{L}_{\mathcal{H}}$ in the case where $\mathcal{H}$ is the random graph equals 3, the result follows by Lemma 18. ◀

We now turn to the case where the considered homogeneous graph is the disjoint union of ω edges.

▶ **Lemma 29.** *Let $\mathbb{A}$ be a first-order expansion of C^2_ω preserved by a ternary canonical operation h with $h(N,\cdot,\cdot) = h(\cdot,N,\cdot) = h(\cdot,\cdot,N) = N$ and which behaves like a minority on $\{E,=\}$ and an oligopotent qnu-operation. Then $\mathbb{A}$ pp-defines neither*

- *a* $[(E(x_1,x_2) \implies (x_3 = x_4)), (\underline{\underline{\mathrm{E}}}(x_1,x_2) \wedge N(x_2,x_3) \wedge \underline{\underline{\mathrm{E}}}(x_3,x_4))]$*-relation nor*
- *a* $[((x_1 = x_2) \implies E(x_3,x_4)), (\underline{\underline{\mathrm{E}}}(x_1,x_2) \wedge N(x_2,x_3) \wedge \underline{\underline{\mathrm{E}}}(x_3,x_4))]$*-relation.*

Then we provide another similar lemma.

▶ **Lemma 30.** *Let $\mathbb{A}$ be a first-order expansion of C^2_ω preserved by a ternary canonical operation h with $h(N,\cdot,\cdot) = h(\cdot,N,\cdot) = h(\cdot,\cdot,N) = N$ and which behaves like a minority on $\{E,=\}$ and an oligopotent qnu-operation. Then $\mathbb{A}$ pp-defines neither*

- *a* $[(E(x_1,x_2) \implies E(x_3,x_4)), (\underline{\underline{\mathrm{E}}}(x_1,x_2) \wedge N(x_2,x_3) \wedge \underline{\underline{\mathrm{E}}}(x_3,x_4))]$*-relation nor*
- *a* $[((x_1 = x_2) \implies (x_3 = x_4)), (\underline{\underline{\mathrm{E}}}(x_1,x_2) \wedge N(x_2,x_3) \wedge \underline{\underline{\mathrm{E}}}(x_3,x_4))]$*-relation.*

Then comes the corollary.

▶ **Corollary 31.** *Let $\mathbb{A}$ be a first-order expansion of C^2_ω preserved by a ternary canonical operation h with $h(N,\cdot,\cdot) = h(\cdot,N,\cdot) = h(\cdot,\cdot,N) = N$ and which behaves like a minority on $\{E,=\}$ and an oligopotent qnu-operation. Then $\mathbb{A}$ has relational width $(2,3)$.*

Proof. By Lemmas 26, 29 and 30 none of the types of relations mentioned in Lemma 19 is pp-definable in $\mathbb{A}$. Appealing to Lemma 19 completes the proof of the corollary. ◀

6.4 The Main Result

Here we prove our main result.

▶ **Theorem 32.** *Let $\mathbb{A}$ be a first-order expansion of a countably infinite homogeneous graph $\mathcal{H}$ which has bounded strict width. Then $\mathbb{A}$ has relational width $(2,\mathbb{L}_{\mathcal{H}})$.*

Proof. By the classification of Lachlan and Woodrow [24], we have that $\mathcal{H}$ is either the random graph, a Henson graph H_k with a forbidden k-clique where $k \geq 3$, a disjoint set of n cliques of size s denoted by C^s_n or a complement of either C^s_n or H_k. The case where $\mathcal{H}$ is C^ω_1, C^1_ω, C^ω_ω or H_k with $k \geq 3$ follows by Corollary 22. If C^s_n is such that $3 \leq n < \omega$ or $3 \leq s < \omega$, then by Theorem 60 in [9], a first-order expansion $\mathbb{A}$ of C^s_n is either homomorphically equivalent to a reduct of $(\mathbb{N};=)$ or is not preserved by an oligopotent qnu-operation and we are done. If $\mathbb{A}$ is a first-order expansion of C^ω_2, then by Theorem 61 in [9] either it is homomorphically equivalent to a reduct of $(\mathbb{N};=)$ or is not preserved by an oligopotent qnu-operation or pp-defines both E and N and is preserved by a canonical ternary injection of behaviour minority which is hyperplanely balanced of behaviour xnor and then $\mathbb{A}$ has relational width $(2,3)$ by Corollary 25. If $\mathbb{A}$ is a first-order expansion of C^2_ω, then by Theorem 62 in [9], we have that either $\mathbb{A}$ is homomorphically equivalent to a reduct of $(\mathbb{N};=)$, or it is not preserved by an oligopotent qnu-operation or it pp-defines both E and N and is preserved by a canonical binary injection of behaviour min that is N-dominated or a ternary canonical operation h with $h(N,\cdot,\cdot) = h(\cdot,N,\cdot) = h(\cdot,\cdot,N) = N$ and which behaves like a minority on $\{E,=\}$. In the former case the language $\mathbb{A}$ has relational width $(2,3)$ by Lemma 21, in the latter by Corollary 31.

The remaining case is where $\mathbb{A}$ is a first-order expansion of the random graph G preserved by an oligopotent qnu-operation. By Theorem 6.1 in [13] we have that a first-order expansion of G is either homorphically equivalent to a reduct of $(\mathbb{N};=)$ and then we are done or pp-defines both E and N in which case, by Theorem 9.3 in [13], we have that:

- $\mathbb{A}$ is preserved by a binary injection of behaviour max which is either balanced or E-dominated, by a binary injection of behaviour min which is either balanced or N-dominated, by a binary injection which is E-constant, or a binary injection which is N-constant, and then the theorem follows by Lemma 21, or
- $\mathbb{A}$ is preserved by a ternary injection of behaviour majority which additionally satisfies one of the conditions in Lemma 23 or of behaviour minority which additionally satisfies one of the conditions in Lemma 24, and then the theorem holds by Corollary 28.

It completes the proof of the theorem. ◀

7 Summary and Future Work

In this paper we proved in particular that:

1. every first-order expansion of a homogeneous graph $\mathcal{H}$ preserved by a canonical binary operation considered in [13, 14, 9] and
2. every first-order expansion of a homogeneous graph $\mathcal{H}$ with bounded strict width has relational width exactly $(2, \mathbb{L}_{\mathcal{H}})$.

A nice consequence of the former result is that all tractable reducts of $C^1_\omega, C^\omega_1, C^\omega_\omega$ and H_k with $k \geq 3$ have relational width exactly $(2, \mathbb{L}_{\mathcal{H}})$, and thus all tractable CSP($\mathbb{A}$) may be solved by establishing $(2, \mathbb{L}_{\mathcal{H}})$-minimality. Nevertheless, we find the latter result to be the main result of this paper. It is for the following reason.

Our general strategy is that we show that constraint languages $\mathbb{A}$ under consideration do not express "too many implications", i.e., quaternary relations that efficiently entail formulas of the form $(R_1(x_1, x_2) \implies R_2(x_3, x_4))$, see definitions in Section 2.4 and lemmas in Section 5 and then use these facts in order to find a strategy of how to shrink a non-trivial $(2, \mathbb{L}_{\mathcal{H}})$-minimal instance of the CSP so that it became a simple instance. In this paper, in order to show that certain relations are not pp-definable in $\mathbb{A}$ we employ in particular some binary and ternary canonical operations. We believe that it is not in fact necessary and theorems analogous to Theorem 32 may be obtained for large families of constraint languages using only the fact that structures $\mathbb{A}$ under consideration are preserved by oligopotent qnu-operations. Thereby we believe that Question 2 may be answered in full generality.

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