Shortest Reconfiguration of Colorings Under Kempe Changes

Marthe Bonamy
CNRS, LaBRI, Université de Bordeaux, Talence, France
marthe.bonamy@u-bordeaux.fr

Marc Heinrich
Université Lyon 1, LIRIS, UMR5205, France
marc.heinrich@univ-lyon1.fr

Takehiro Ito
Graduate School of Information Sciences, Tohoku University, Japan
takehiro@ecei.tohoku.ac.jp

Yusuke Kobayashi
Research Institute for Mathematical Sciences, Kyoto University, Japan
yusuke@kurims.kyoto-u.ac.jp

Haruka Mizuta
Graduate School of Information Sciences, Tohoku University, Japan
haruka.mizuta.s4@dc.tohoku.ac.jp

Moritz Mühlenthaler
Laboratoire G-SCOP, Grenoble INP, Université Grenoble Alpes, France
moritz.muhlenthaler@grenoble-inp.fr

Akira Suzuki
Graduate School of Information Sciences, Tohoku University, Japan
a.suzuki@ecei.tohoku.ac.jp

Kunihiro Wasa
National Institute of Informatics, Tokyo, Japan
wasa@nii.ac.jp

Abstract

A k-coloring of a graph maps each vertex of the graph to a color in $\{1, 2, \ldots, k\}$, such that no two adjacent vertices receive the same color. Given a k-coloring of a graph, a Kempe change produces a new k-coloring by swapping the colors in a bicolored connected component. We investigate the complexity of finding the smallest number of Kempe changes needed to transform a given k-coloring into another given k-coloring. We show that this problem admits a polynomial-time dynamic programming algorithm on path graphs, which turns out to be highly non-trivial. Furthermore, the problem is NP-hard even on star graphs and we show that on such graphs it admits a constant-factor approximation algorithm and is fixed-parameter tractable when parameterized by the number k of colors. The hardness result as well as the algorithmic results are based on the notion of a canonical transformation.

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1 Introduction

Reconfiguration problems ask for the existence of a transformation between two solutions of an instance of a combinatorial problem, such as the satifiability problem, the stable set problem, and so forth. Our reference problem is the k-coloring problem. Recall that a *k-coloring* of a graph partitions the vertex set into at most k stable sets, called *color classes*. A classical technique to obtain a new k-coloring from a given one is the Kempe change: Given a k-coloring, a *Kempe change* swaps the colors in a connected component of a subgraph induced by two color classes. We say that two k-colorings α and β of a graph admit a *Kempe-sequence of length ℓ* if there is a sequence $\gamma_0, \gamma_1, \ldots, \gamma_\ell$ of k-colorings, such that $\gamma_0 = \alpha$, $\gamma_\ell = \beta$, and for $0 \leq i < \ell$, the coloring γ_{i+1} can be obtained from γ_i by performing a single Kempe change. Two k-colorings that admit a Kempe-sequence (of any length) are called *Kempe-equivalent*. Kempe-equivalence has been of great interest in graph theory [15, 17, 19], sampling of colorings [8, 25], and statistical physics [2, 20, 21]. We consider a natural optimization variant of Kempe-equivalence: Given a number ℓ and two k-colorings of a graph, do the two colorings admit a Kempe-sequence of length at most ℓ?

One of the first algorithms that exhibits Kempe-sequences is due to Las Vergnas and Meyniel [15]. From their analysis it follows that for a d-degenerate graph and any $k > d$, any two k-colorings of the graph are Kempe-equivalent. More recently, several results of a similar flavor were obtained [2, 3, 10, 19]. Interestingly, no non-trivial subexponential upper bounds on the length of the Kempe-sequences computed by the algorithm of Las Vergnas and Meyniel are known, even for small values of k. Using different techniques, Bousquet and Heinrich showed recently that for $k \geq d + 2$, any two k-colorings of a d-degenerate graph admit a Kempe-sequence of polynomial length in d [6]. Note that none of the algorithms in the references mentioned above is known to exhibit Kempe-sequences of minimal length. Furthermore, to the best of our knowledge, the complexity of determining the minimal length of a Kempe-sequence between two k-colorings is open. The aim of this paper is to settle the complexity of this task and to provide efficient exact and approximate algorithms for it.

We show that a Kempe-sequence of minimal length for two k-colorings of a path graph can be found in polynomial time by a non-trivial dynamic programming algorithm. On the other hand, it is unlikely that this positive result can be generalized significantly, since we also show that the same problem on star graphs is NP-hard. Note that since star graphs and path graphs are bipartite, a Kempe-sequence of linear length in the input size can be found efficiently [3, 19].

In order to illustrate why finding Kempe-sequences of minimal length is non-trivial even on path graphs, let us consider the example shown in Figure 1. It shows two Kempe-sequences $\alpha, \gamma_1, \gamma_2, \beta$ and $\alpha, \gamma'_1, \gamma'_2, \gamma'_3, \beta$ between two 3-colorings α and β of a path on seven vertices. The latter Kempe-sequence changes only the colors of vertices that receive differents colors in α and β. On the other hand, the Kempe-sequence $\alpha, \gamma_1, \gamma_2, \beta$, which is optimal, changes also twice the color of the middle vertex v, which receives the same color in α and β. The purpose of changing the color of v is to build a large connected component consisting of the vertices with colors 1 and 2. As a result, with a single Kempe change, all vertices can be recolored to their target colors. We conclude from the example that a key difficulty is to build up suitable bicolored components in order to obtain a short Kempe-sequence.

Related Work

Finding shortest transformations between configurations has recently received much attention in different domains, ranging from the triangulations of point sets and polygons [16, 1] to satisfying assignments of Boolean formulas [22] to puzzles [9, 12, 24]; please see [13, 23]

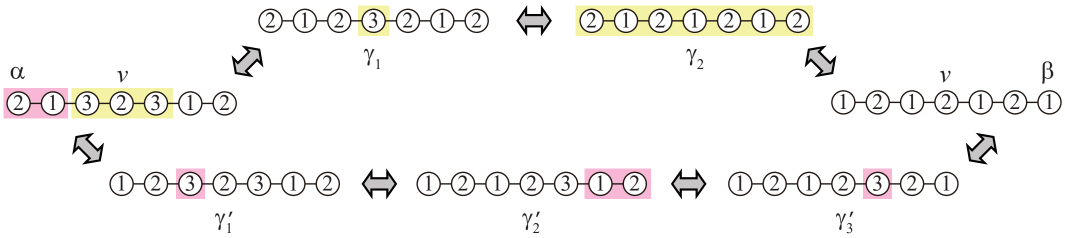

Figure 1 Two Kempe-sequences between 3-colorings α and β of a path on seven vertices. The upper sequence $\langle \alpha, \gamma_1, \gamma_2, \beta \rangle$ is shortest even though it recolors the middle vertex v twice; notice that v receives the same color 2 in α and β. On the other hand, the lower sequence $\langle \alpha, \gamma_1', \gamma_2', \gamma_3', \beta \rangle$ is not shortest although it recolors only the vertices receiving different colors in α and β.

for recent suveys. Such problems are often NP-hard, even if the existence of a suitable transformation between two configurations can be decided in polynomial time. A recent example is the NP-hardness of solving the n-Rubik's cube using a minimal number of steps [9].

We say that two k-colorings are *equivalent under elementary recolorings* if they admit a Kempe-sequence, such that any two consecutive k-colorings of the sequence differ with respect to the color of a single vertex. In contrast to Kempe-equivalence, the complexity of deciding equivalence under elementary recolorings is quite well understood [5, 7]. Furthermore, Johnson et al. showed that deciding the existence of a transformation of length ℓ between two k-colorings in this setting admits a polynomial-time algorithm for $k = 3$ and is FPT when parameterized by $k + \ell$ [14]. It may seem quite surprising that there is a polynomial-time algorithm for $k = 3$, since deciding if a graph admits a 3-coloring is NP-complete. This polynomial-time algorithm is based on a potential argument: Each vertex is assigned a non-negative potential value and in each step of a shortest transformation, the potential is decreased by a positive constant value. When the target coloring is reached the potential becomes zero. Due to long-range effects of general Kempe changes it is unlikely that a similar potential argument can be used in our setting; hence our techniques are very different.

We can think of a k-coloring in terms of placing labeled tokens the vertices of a graph. Transformations between labelled token configurations on graphs have been considered in a slightly different setting called *token swapping*: from a given token configuration a new configuration can be obtained by swapping two tokens on adjacent vertices. The goal is again to decide whether there is a transformation of length at most ℓ between two given token configurations. It is not required that adjacent vertices have tokens with different labels. The problem is related to the design of efficient sorting networks. It is NP-complete on graphs of treewidth two and constant diameter [4] and admits a polynomial-time 4-factor approximation algorithm [18]. Several variations of the problems have been studied, for example colored token swapping [26, 27], where tokens of the same color are indistinguishable.

Our Contribution

Let $\mathcal{G}$ be a class of graphs. We consider the following problem.

Kempe Distance on $\mathcal{G}$
input: Graph $G \in \mathcal{G}$, numbers $k, \ell \in \mathbb{N}$, two k-colorings α, β of G
question: Do α and β admit a Kempe-sequence of length at most ℓ?

We show that Kempe Distance on paths admits a $O(kn)$-time dynamic programming algorithm, where n is the number of vertices of the input graph. The algorithm can easily be modified to output a shortest Kempe-sequence. The analysis of the algorithm is highly non-trivial. Roughly speaking, given a path on vertices $v_1, v_2, \ldots, v_n$ and two k-colorings α and β of the input graph, the algorithm computes for $1 \leq i \leq n$ two kinds of quantities on the sub-path $v_1, v_2, \ldots, v_i$ according to eight rules (please refer to Section 2 for an overview). It computes i) the length of a shortest Kempe-sequence between the two colorings restricted to the current subpath and ii) for each color a that is different from the source and target color of v_i, the algorithm checks whether there is a shortest Kempe-sequence on the subpath such that v_i has color a at some point. To establish the correctness of the algorithm, we prove several interesting properties of Kempe-sequences between k-colorings of a path graph. For instance, we show that in a shortest Kempe-sequence, the color of each vertex changes at most twice.

We complement the above result by showing that the problem Kempe Distance on stars is NP-complete by a reduction from the problem Hamiltonian Cycle. In contrast, we show that a Kempe-sequence of minimal length that certifies the equivalence of two k-colorings of a star graph under elementary recolorings can be found efficiently. On the positive side, we show that there is a polynomial-time algorithm that computes a Kempe-sequence of length at most $4\,\mathrm{OPT}(I) + 1$, where $\mathrm{OPT}(I)$ denotes the length of a shortest Kempe-sequence for an instance I of Kempe Distance on stars, and give an almost matching lower bound. Furthermore, we show that Kempe Distance on stars parameterized by the number k of colors is fixed-parameter tractable. The algorithmic results as well as the hardness result are based on the notion of a *canonical transformation*. We would like to remark that many algorithms for reconfiguration problems make use of a *canonical configuration*, that is, the existence of a transformation is established by showing that any configuration can be transformed into a certain canonical one. Here, we show that for any Kempe-sequence between two k-colorings of a star there is a canonical Kempe-sequence of at most the same length. Hence, it is sufficient to consider canonical shortest Kempe-sequences.

Notation

A *star graph* on n vertices consists of a *center vertex* of degree $n-1$ and $n-1$ *leaf vertices*, each of degree one. For a k-coloring α of a graph, a maximal monochromatic set of vertices is called a *color class*. Let G be a graph and let α and β be two k-colorings of G. Suppose that α and β admit a Kempe-sequence $s := (\gamma_0, \gamma_1, \ldots, \gamma_l)$. We say that s is a Kempe-sequence *from α to β* and denote its length by $|s|$. For a Kempe sequence $s = (\gamma_0, \gamma_1, \ldots, \gamma_l)$ and for a vertex $v \in V$, the *color transition of v in s* is a sequence of colors $c_0 \to c_1 \to \cdots \to c_p$ with $c_i \neq c_{i-1}$ for each i that represents the transition of colors of v in s. In other words, $c_0, c_1, \ldots, c_p$ is obtained from $\gamma_0(v), \gamma_1(v), \ldots, \gamma_l(v)$ by removing duplicates if a color appears consecutively in the sequence. When the color transition of v is $c_0 \to c_1 \to \cdots \to c_p$, we say that v is *recolored p times* in the Kempe sequence.

Proofs marked with $(*)$ have been omitted due to space limitations.

2 Kempe Distance on Paths: a polynomial-time algorithm

In this section we present a polynomial-time algorithm for the problem Kempe Distance on paths. In the following, let P be a path graph and we denote its vertex set by $V = \{v_1, v_2, \ldots, v_n\}$ and its edge set by $E = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$. Let $C = \{1, 2, \ldots, k\}$ be the color set and let $\alpha : V \to C$ and $\beta : V \to C$ be the initial and target k-colorings of P, respectively. For $i = 1, 2, \ldots, n$, the colors $\alpha(v_i)$ and $\beta(v_i)$ are denoted by α_i and β_i, respectively.

For $i \in \{1,\dots,n\}$, let P_i be the subpath of P induced by $V_i := \{v_1,\dots,v_i\}$. The restriction of α (resp. β) to $\{v_1,\dots,v_i\}$ is also denoted by α (resp. β) for notational convenience. For $i \in \{1,\dots,n\}$, let $l(P_i,\alpha,\beta) \in \mathbb{Z}_+$ be the length of a shortest Kempe sequence for the instance (P_i,α,β). We simply denote $l_i := l(P_i,\alpha,\beta)$ if α and β are clear. For each $i \in \{1,\dots,n\}$ and for a color $a \in C$, we define $f(i,a)$ as follows: $f(i,a) = \mathsf{yes}$ if $a \notin \{\alpha_i,\beta_i\}$ and there exists a Kempe sequence s of length l_i such that the color transition of v_i is $\alpha_i \to a \to \beta_i$, and $f(i,a) = \mathsf{no}$ otherwise.

Our algorithm adopts a dynamic programming approach that computes l_i and $f(i,a)$ for each $i \in \{1,\dots,n\}$ and for each $a \in C$. For $i \geq 2$, the update formula is as follows.

(U1) If $\alpha_i = \beta_i$, then $l_i = l_{i-1}$ and $f(i,a) = \mathsf{no}$ for any $a \in C$.
(U2) If $\alpha_i = \beta_{i-1}$ and $\alpha_{i-1} = \beta_i$, then $l_i = l_{i-1}$ and $f(i,a) = \mathsf{no}$ for any $a \in C$.
(U3) If $\alpha_i = \beta_{i-1}$, $\alpha_{i-1} \neq \beta_i$, and $f(i-1,\beta_i) = \mathsf{yes}$, then $l_i = l_{i-1}$ and $f(i,a) = \mathsf{no}$ for any $a \in C$.
(U4) If $\alpha_i \neq \beta_{i-1}$, $\alpha_{i-1} = \beta_i$, and $f(i-1,\alpha_i) = \mathsf{yes}$, then $l_i = l_{i-1}$ and $f(i,a) = \mathsf{no}$ for any $a \in C$.
(U5) If $\alpha_i = \beta_{i-1}$, $\alpha_{i-1} \neq \beta_i$, and $f(i-1,\beta_i) = \mathsf{no}$, then $l_i = l_{i-1}+1$ and

$$f(i,a) = \begin{cases} \mathsf{yes} & \text{if } a = \alpha_{i-1} \text{ or } f(i-1,a) = \mathsf{yes}, \\ \mathsf{no} & \text{otherwise} \end{cases}$$

for $a \in C$.
(U6) If $\alpha_i \neq \beta_{i-1}$, $\alpha_{i-1} = \beta_i$, and $f(i-1,\alpha_i) = \mathsf{no}$, then $l_i = l_{i-1}+1$ and

$$f(i,a) = \begin{cases} \mathsf{yes} & \text{if } a = \beta_{i-1} \text{ or } f(i-1,a) = \mathsf{yes}, \\ \mathsf{no} & \text{otherwise} \end{cases}$$

for $a \in C$.
(U7) If $\alpha_i, \beta_i, \alpha_{i-1}$, and β_{i-1} are distinct, then $l_i = l_{i-1}+1$ and

$$f(i,a) = \begin{cases} \mathsf{yes} & \text{if } a = \alpha_{i-1} \text{ and } f(i-1,\alpha_i) = \mathsf{yes}, \\ \mathsf{yes} & \text{if } a = \beta_{i-1} \text{ and } f(i-1,\beta_i) = \mathsf{yes}, \\ \mathsf{no} & \text{otherwise} \end{cases}$$

for $a \in C$.
(U8) If $\alpha_i \neq \beta_i$ and $\alpha_{i-1} = \beta_{i-1}$, then $l_i = l_{i-1}+1$ and

$$f(i,a) = \begin{cases} \mathsf{yes} & \text{if } i \geq 3,\ a = \alpha_{i-1},\ \alpha_i = \alpha_{i-2},\ \beta_i = \beta_{i-2}, \text{ and } f(i-2,a) = \mathsf{yes}, \\ \mathsf{no} & \text{otherwise} \end{cases}$$

for $a \in C$.

In order to show the validity of (U1)–(U8), we introduce the following properties (P1) and (P2), and show simultaneously that (P1), (P2), and (U1)–(U8) hold for any $i \in \{1,\dots,n\}$ by induction.

(P1) There exists a Kempe sequence of length l_i such that v_i is recolored at most once in it, that is, v_i is recolored directly from α_i to β_i if $\alpha_i \neq \beta_i$, and v_i is never recolored if $\alpha_i = \beta_i$.
(P2) For any Kempe sequence s for (P_i,α,β), v_i is recolored at most $2|s| - 2l_i + 2$ times in s. In particular, v_i is recolored at most twice in any shortest Kempe sequence.

▶ **Proposition 1** ($*$). *For any pair of colorings $\alpha : V \to C$ and $\beta : V \to C$, (U1)–(U8) hold for $i \geq 2$, and (P1) and (P2) hold for any $i \geq 1$.*

By using this proposition, we can obtain a polynomial-time algorithm for KEMPE DISTANCE on paths.

▶ **Theorem 2.** *KEMPE DISTANCE on paths can be solved in $O(nk)$ time.*

Proof. We can easily compute l_1 and $f(1, a)$ for each $a \in C$. For $i = 2, 3, \dots, n$ in this order, we compute l_i and $f(i, a)$ for each $a \in C$ by using (U1)–(U8). Since each value can be computed in constant time, we can compute l_n in $O(nk)$ time. ◀

The algorithm returns l_n, the length of a shortest sequence. We note that we can easily modify our algorithm so that it outputs a sequence of Kempe changes of length l_n.

3 Kempe Distance on Stars

In this section we show that KEMPE DISTANCE on stars admits a constant-factor approximation algorithm and the same problem parameterized by the number k of colors is FPT. Furthermore, we give a lower bound instance for the approximation algorithm and show that KEMPE DISTANCE on stars is NP-complete. The key concept common to all our results in this section is the notion of a *sorted Kempe-sequence*.

3.1 Sorted Kempe-Sequences

In the following, let $S = (V, E)$ be a star graph with center vertex $c \in V$ and leaves $L = V \setminus \{c\}$. Any Kempe change performed on a k-coloring of S either changes the color of the center vertex or it does not. This observation motivates the following notion of a *sorted Kempe-sequence*. We consider a Kempe-sequence to be sorted, if there is some intermediate coloring γ, such that the color of the center vertex is constant up to γ and after γ it changes in each step. Formally, a Kempe-sequence $\gamma_0, \gamma_1, \dots, \gamma_\ell$ is *sorted* if there is some index $j \in \{1, 2, \dots, \ell\}$ and a color $a \in \{1, 2, \dots, k\}$, such that for $1 \leq i \leq j$, we have $\gamma_i(c) = a$ and for $j \leq i < \ell$, we have $\gamma_i(c) \neq \gamma_{i+1}(c)$. In the following we let $\gamma := \gamma_j$ and call γ the *intermediate coloring*. We first show that we may restrict our attention to sorted Kempe-sequences. We then provide tight bounds for shortest Kempe-sequences from α to γ and γ to β, respectively, which will imply our algorithmic results and our hardness result.

According to the next lemma, for any Kempe-sequence between two k-colorings of S there is a sorted one of at most the same length.

▶ **Lemma 3.** *Let $s := \gamma_0, \gamma_1, \dots, \gamma_\ell$ be a Kempe-sequence of length ℓ of k-colorings of S. Then there is a sorted Kempe-sequence from γ_0 to γ_ℓ of length at most ℓ.*

Proof. We assume without loss of generality that no two consecutive colorings in s are identical, since if this is the case, we may remove one of the two from s and thus obtain a shorter Kempe-sequence. Suppose furthermore that s is not sorted, that is, there is some index $i \in \{1, 2, \dots, \ell - 2\}$, such that $\gamma_i(c) \neq \gamma_{i+1}(c)$ and $\gamma_{i+1}(c) = \gamma_{i+2}(c)$. To keep our notation concise, let $a := \gamma_i(c)$ and $a' := \gamma_{i+1}(c)$ be the color of c in γ_i and γ_{i+1}, respectively. Since $\gamma_{i+1} \neq \gamma_{i+2}$, there is a unique leaf $v \in L$, such that $\gamma_{i+1}(v) \neq \gamma_{i+2}(v)$. We let $b := \gamma_{i+1}(v)$ and $b' := \gamma_{i+2}(v)$ be the color of the leaf v in γ_{i+1} and γ_{i+2}, respectively. By assumption we have $a \neq a'$ and $b \neq b'$. Furthermore, since $\gamma_{i+1}(c) = \gamma_{i+2}(c)$, we have that $b' \neq a'$. Finally, since γ_{i+1} is a k-coloring, we have $a' \neq b$.

We show that there is a Kempe-sequence $\gamma_0, \gamma_1, \ldots, \gamma_i, \gamma', \gamma_{i+2}, \ldots, \gamma_\ell$, such that $\gamma_i(c) = \gamma'(c)$. By applying this argument iteratively, we obtain a sorted Kempe-sequence between γ_0 and γ_ℓ of length at most ℓ. We consider two main cases.

Case 1: $\gamma_i(v) = \gamma_{i+1}(v)$

It remains to consider the two subcases $a = b'$ and $a \neq b'$. In the first subcase we have $\gamma_i(c) = a$ and $\gamma_{i+1}(c) = \gamma_{i+2}(c) = a'$, as well as $\gamma_i(v) = \gamma_{i+1}(v) = b$ and $\gamma_{i+2}(v) = a = b'$. So in this case we may first recolor v to color a' and then c to b' using Kempe changes. That is, we let the k-coloring γ' be given by $\gamma'(u) := \gamma_i(u)$ for each vertex $u \in V \setminus \{v\}$ and $\gamma'(v) := a'$. On the other hand, if $a \neq b'$ then a, a', b, b' are distinct, so we may choose $\gamma'(u) := \gamma_i(u)$ for $u \in V \setminus \{v\}$ and $\gamma'(v) := b'$. That is, we can reach γ' from γ_i by changing the color of v to b' and γ_{i+2} from γ' by changing the color of the center vertex c from a to a'.

Case 2: $\gamma_i(v) \neq \gamma_{i+1}(v)$

Recall that c receives different colors in γ_i and γ_{i+1}. Therefore, if v receives different colors in γ_i and γ_{i+1}, then we must have $\gamma_i(v) = a'$ and $\gamma_{i+1}(v) = a = b$. Furthermore, if $a = b'$ then v receives the same color in γ_{i+1} and γ_{i+2}, which contradicts our assumptions. Hence, it remains to consider the case that $a \neq b'$. We let $\gamma'(u) := \gamma_i(u)$ for $u \in V \setminus \{v\}$ and $\gamma'(v) := b'$ and observe that γ' is a k-coloring of S. By construction, the k-coloring γ' can be obtained from γ_i by performing a Kempe change that alters the color of v to b'. Then, γ_{i+2} can be obtained from γ' by a Kempe change that alters the color of c from a to a'. ◀

Note that a Kempe change that alters the color of the center vertex c, say, from color a to color b, also changes the color of every leaf of color b to color a. Let us fix two k-colorings α and β of the star S and let γ be the intermediate coloring of a sorted Kempe-sequence from α to β. The next lemma establishes that the color classes of an intermediate coloring γ agree with those of β.

▶ **Lemma 4** (∗). *Let $u, v \in L$ be two leaves of S. Then $\beta(u) = \beta(v)$ if and only if $\gamma(u) = \gamma(v)$.*

We will show below that, given some coloring γ whose color classes agree with β, we can find efficiently a corresponding shortest sorted Kempe-sequence from α to β via γ. Hence, in the light of Lemma 3, the task of finding a shortest Kempe-sequence from α to β reduces to the task of finding among such colorings one whose corresponding sorted Kempe-sequence is shortest. For this purpose, from the two k-colorings α and β of S, we construct an edge-weighted bipartite auxiliary graph and show that any sorted Kempe-sequence from α to β corresponds to a one-sided perfect matching in this graph. Furthermore, we determine the bounds on the length of a sorted Kempe-sequence from α to β in terms of the weight of the corresponding matching and structure of the graph it induces on the set of colors. Let $G_{\alpha\beta}$ be a complete bipartite graph on the vertex sets $([k] \setminus \{\alpha(c)\}, \beta(L))$. The weight w_{ij} of an edge $(i, j) \in E(G_{\alpha\beta})$ is given by

$$w_{ij} := |\alpha^{-1}(i) \cap \beta^{-1}(j)| \ .$$

In the following, let γ be a k-coloring of S whose color classes agree with those of β. That is the color classes of γ and β induce the same partition of the vertex set of S, but the parts may receive different colors in β and γ. Let $M := \{(i, j) \in E(G_{\alpha\beta}) \mid \gamma^{-1}(i) = \beta^{-1}(j)\}$. By Lemma 4, the set M is a $\beta(L)$-perfect matching of $G_{\alpha\beta}$. Hence, any sorted Kempe-sequence from α to β gives a $\beta(L)$-perfect matching of $G_{\alpha\beta}$. Also, each $\beta(L)$-perfect matching of $G_{\alpha\beta}$ corresponds to an intermediate coloring of a sorted Kempe-sequence from α to β.

We now show that, given γ, we can compute in polynomial time a shortest Kempe-sequence from α to β via γ.

▶ **Proposition 5.** *Let γ be a k-coloring of S whose color classes agree with those of β. Then there is a polynomial-time algorithm that computes a shortest sorted Kempe-sequence from α to β via γ.*

The next two lemmas give tight bounds for i) the length of a Kempe-sequence from α to γ that does not alter the color of the center vertex and ii) the length of a Kempe-sequence from γ to β such that the color of the center vertex is altered in each step. Since the proofs of the two lemmas are constructive and lead to polynomial-time algorithms, they imply Proposition 5.

▶ **Lemma 6** ($*$). *The length of a shortest Kempe-sequence from α to γ such that the color of the center vertex is constant is $|L| - w(M)$.*

Proof (sketch). Observe that the number of leaves $u \in L$ such that $\alpha(u) \neq \gamma(u)$ is a lower bound on the length of a Kempe-sequence from α to γ that does not alter the color of the center vertex. By performing for each leaf u such that $\alpha(u) \neq \gamma(u)$ a Kempe change that assigns to u the color $\gamma(u)$, we obtain a shortest Kempe-sequence from α to γ of the claimed length that does not alter the color of the center vertex. ◀

It remains to bound the length of a shortest Kempe-sequence from γ to β such that the color of the center vertex changes in each step. Note that matching M in $G_{\alpha\beta}$ obtained from γ defines at most one *successor* $M(u) \in \beta(L)$ for each color $u \in [k] \setminus \{\gamma(c)\}$, where c is the center vertex of the star S. By the construction of $G_{\alpha\beta}$, this partial successor map gives rise to a set $\mathcal{C}$ of cycles and a set $\mathcal{P}$ paths on the set $\{1, 2, \ldots, k\}$ of colors. To obtain a desired Kempe-sequence from γ to β, for each item $Z \in \mathcal{P} \cup \mathcal{C}$, the color classes corresponding to the vertices $V(Z)$ need to be altered one-by-one by changing the color of the center vertex.

▶ **Lemma 7** ($*$). *The length of a shortest Kempe-sequence from γ to β such that the color of the center vertex changes in each step is*

$$\left(\sum_{Z \in \mathcal{C} \cup \mathcal{P}} |E(Z)| + 1\right) - |\{\gamma(c)\} \cap \beta(L)\}| + 1 - |\{\beta(c)\} \cap \gamma(L)\}| \ .$$

Proof (sketch). By Lemma 4 it remains to assign the correct colors (given by β) to the color classes of γ. We observe that for any cycle $C \in \mathcal{C}$, since the color of the center vertex is not in $V(C)$, at least $|E(C)| + 1$ Kempe changes altering the color of the center vertex are required in order to assign to each leaf $u \in L$ with $\gamma(u) \in V(C)$ the color $\beta(u)$. By a similar observation, each path $P \in \mathcal{P}$ requires at $|E(P)| + 1$ Kempe changes if the center vertex has $\gamma(c) \notin V(P)$ and $|E(P)|$ Kempe changes otherwise. ◀

3.2 Fixed-parameter and Approximation Algorithms

Based on the insights about sorted Kempe-sequences from Section 3.1 we show that Kempe Distance on stars is fixed-parameter tractable, where the parameter is the number k of available colors. The correspondence between sorted Kempe-sequences and matchings in the auxiliary graph $G_{\alpha\beta}$ defined in Section 3.1, together with Proposition 5, yields the following FPT result.

▶ **Theorem 8** ($*$). *Kempe Distance on stars can be decided in time $O(k! \cdot \mathsf{poly}(k) \cdot n)$, where k is the number of available colors and n is the size of the instance.*

Proof (sketch). Let $(S, k, \ell, \alpha, \beta)$ be an instance of Kempe Distance on stars. Let L be the set of leaves of the star graph S. Each intermediate coloring of a sorted Kempe-sequence corresponds to a $\beta(L)$-perfect matching in the graph $G_{\alpha\beta}$ from Section 3.1. This graph can be constructed in time $O(\mathsf{poly}(k) \cdot n)$ by iterating over the leaves of G and keeping track of the source and target colors. For each $\beta(L)$-perfect matching M of $G_{\alpha\beta}$ we obtain in time $O(n)$ the length of a shortest Kempe-sequences according to the proofs of lemmas 6 and 7. If the sum of the two lengths is at most ℓ for some matching output Yes, otherwise output No. ◀

Note that for a given instance of Kempe Distance, a shortest Kempe-sequence can be obtained in a straight-forward way by turning the constructive proofs of Lemmas 6 and 7 into an algorithm. For two k-colorings α and β of a graph, we let $\alpha \triangle \beta := \{i \in \{1, 2, \ldots, k\} \mid \alpha^{-1}(i) \neq \beta^{-1}(i)\}$ be the set of colors on which the color classes of α and β are different. Since changing the color of any leaf u that has the same source and target color results in at least two additional Kempe changes, any shortest Kempe-sequence from α to β only involves colors in $\alpha \triangle \beta$. This observation implies the following corollary.

▶ **Corollary 9.** *Kempe Distance on stars is FPT in the number of color classes on which the two colorings differ.*

For an instance I of Kempe Distance on stars, we denote by $\mathrm{OPT}(I)$ the smallest value t, such that α and β admit a Kempe-sequence of length at most t. Note that $\mathrm{OPT}(I)$ is at most the order of input graph. We show that for a maximum-weight matching M of $G_{\alpha\beta}$, the length of a sorted Kempe-sequence with intermediate coloring γ_M gives a constant-factor approximation.

▶ **Theorem 10.** *There is a polynomial-time algorithm that, given an instance I of Kempe Distance on stars with k-colorings α and β, computes a Kempe-sequence from α to β of length at most $4 \cdot \mathrm{OPT}(I) + 1$.*

Proof. Let $I = (S, \alpha, \beta, k, \ell)$ be an instance of Kempe Distance on stars and let $G_{\alpha\beta} = (A + B, E)$ be the bipartite graph obtained from the instance as in Section 3.1. We denote by L the set of leaves of the graph S. Let τ^* be a shortest Kempe-sequence from α to β of length $\mathrm{OPT}(I)$. By Lemma 3, we may assume that τ^* is sorted, so there is a matching M^* of $G_{\alpha\beta}$, such that τ^* is composed of a Kempe-sequence τ_1^* from α to an intermediate coloring γ_{M^*} and a Kempe-sequence τ_2^* from γ_{M^*} to β. Let M be a maximum-weight matching of $G_{\alpha\beta}$ and let τ_1 (resp., τ_2) be the Kempe-sequence from α to γ_M (resp., from γ_M to β) obtained according to Lemma 6 (resp., Lemma 7). We show that τ has length at most $4 \cdot \mathrm{OPT}(I) + 1$.

By lemmas 4 and 6, the τ_1 has minimal length among all Kempe-sequences from α to a coloring γ' such that for any two leaves $u, v \in L$, we have $\gamma'(u) = \gamma'(v)$ if and only if $\beta(u) = \beta(v)$. So the length of τ_1 is at most $\mathrm{OPT}(I)$. It will be convenient in the remainder of this proof to consider Kempe-sequences also to be sequences of Kempe changes.

It remains to bound the length of τ_2. Let $\mathcal{C}$ (resp., $\mathcal{P}$) be the set of cycles (resp., paths) on $\{1, 2, \ldots, k\}$ given by the successor map M. Let $j \in \beta(L)$ be a target color of some leaf of S and suppose that there is at least one Kempe change with target color j in τ_2. Then j is a vertex of an item in $\mathcal{C} \cup \mathcal{P}$ (relative to the matching M). We call a color $j \in \beta(L)$ *deficient* if $j \in V(Z)$ for some $Z \in \mathcal{C} \cup \mathcal{P}$ and τ^* contains no Kempe change with target color j.

▷ Claim 11. The only edge with positive weight incident to a deficient vertex j in $G_{\alpha\beta}$ is (j,j). Furthermore, $(j,j) \in M^*$.

Proof of Claim. Each vertex $j \in \beta(L)$ of $G_{\alpha\beta}$ has at least one incident edge with positive weight; otherwise no vertex has target color j implying that $j \notin \beta(L)$. Since, by assumption, there is no Kempe change with target color j in τ_2^*, we have that $(j,j) \in M^*$. Furthermore, we have that each edge (i,j) with $i \neq j$ has weight zero in $G_{\alpha\beta}$; otherwise there is a Kempe change in τ_1^* with target color j. ◁

We may ignore the colors $j \in \beta(L)$ such that $(j,j) \in M$, since no Kempe change in τ_2 has target color j. So let us assume that $\beta(L)$ contains no such colors. Furthermore, let d be the number of deficient colors in $\beta(L)$. We distinguish two cases.

Case 1: $d \leq |\beta(L)|/2$.

For each non-deficient color in $\beta(L)$, the Kempe-sequence τ^* contains at least one Kempe change with target color j. Therefore, $\mathrm{OPT}(I) \geq |\beta(L)|/2$. For each deficient color $j \in \beta(L)$, the corresponding vertex j on other side of $G_{\alpha\beta}$ is covered by M, since otherwise M does not have maximum weight. Therefore, each path or cycle in $\mathcal{C} \cup \mathcal{P}$ contains at least two vertices. By Lemma 7, the Kempe-sequence τ_2 has length at most $(3|\beta(L)|/2) + 1 \leq 3\,\mathrm{OPT}(I) + 1$.

Case 2: $d > |\beta(L)|/2$.

Consider the graph with edges $H := (V(G_{\alpha\beta}), M \triangle M^*)$. Since M and M^* are $\beta(L)$-perfect, each component of H has an even number of edges. Therefore, the graph H is a disjoint union of even paths $\mathcal{P}'$ and cycles $\mathcal{C}'$. Let $Z \in \mathcal{P}' \cup \mathcal{C}'$ and consider any deficient vertex j of Z and the two edges $(j,j) \in M^*$ and $(i,j) \in M$ incident to it. By Claim 11, we have that $w_{jj} > 0$ and $w_{ij} = 0$. On the other hand, since M has maximum weight, we have $w(M \cap E(Z)) \geq w(M^* \cap E(Z))$. Therefore, the weight of the edges in $M \cap E(Z)$ exceeds that of the edges in $M^* \cap E(Z)$ by at least the number of deficient vertices of Z. Now observe that for each edge $(i,j) \in M \setminus M^*$, there are at least w_{ij} Kempe changes in τ_1^* due to Lemma 4. We conclude that τ_1^* has length at least $d \geq |\beta(L)|/2 + 1$. By again considering the worst-case length of τ_2 according to Lemma 7, we bound the length of the Kempe-sequence τ by

$$3\,\mathrm{OPT}(I) \geq 3|\tau_1^*| \geq 3|\beta(L)|/2 + 1 \geq |\tau_2|$$

By combining the bounds for τ_1 and τ_2, we obtain that the Kempe-sequence τ has length at most $4\,\mathrm{OPT}(I) + 1$. ◀

We conclude this subsection by showing that our analysis in the proof of Theorem 10 is almost tight. For this purpose, consider the instance of Kempe Distance on stars and the corresponding graph $G_{\alpha\beta}$ shown in Figure 2. Note that just one Kempe change is needed to transform the source coloring into the target coloring. Both perfect matchings of the shown graph $G_{\alpha\beta}$ have weight 1, so the approximation algorithm may select the one consisting of the two crossing edges. This yields a transformation that first recolors the vertex with source and target color 1 to color 2. It remains to permute the color classes to obtain the target coloring; the permutation of the colors is a cycle of length two. By Lemma 7, takes precisely three Kempe changes to reach the target coloring. Hence, the algorithm outputs a 4-approximate solution in the worst case.

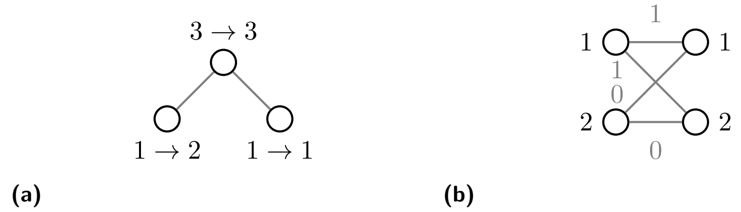

Figure 2 Instance of Kempe Distance on stars (a) and corresponding edge-weighted graph $G_{\alpha\beta}$ (b) for which the approximation algorithm gives a 4-approximate solution. A label $a \to b$ indicates source color a and target color b for the corresponding vertex.

3.3 NP-completeness of Kempe Distance

We complement our polynomial-time algorithms from Section 3.2 by the following result, which we will prove in the remainder of this section.

▶ **Theorem 12.** *Kempe Distance on stars is* NP*-complete.*

Recall that any two k-colorings of a star on n vertices admit a Kempe-sequence of length $O(n)$. Hence, Kempe Distance on stars is in NP. To show NP-hardness, we give a polynomial-time reduction from the problem Hamiltonian Cycle, which asks, whether a given graph contains a simple cycle visiting each vertex. It was shown by Garey, Johnson, and Tarjan that Hamiltonian Cycle remains NP-complete even on planar cubic graphs [11]. In order to show that Kempe Distance on stars is NP-hard, we first reduce Hamiltonian Cycle on cubic graphs to the following minimum-cost permutation problem.

Minimum-Cost Permutation
Input: non-negative weights $w \in \mathbb{Z}_{\geq 0}^{V \times V}$ and number $z \in \mathbb{Z}$
For a permutation π we denote by $c(\pi)$ the number of cycles of a cycle decomposition of π. The *cost* of a permutation π of V is given by $c(\pi) - \sum_{v \in V} w(v, \pi_v)$.
Question: Is there a permutation π of V, such that π has cost at most z?

We then establish that the minimum cost of a permutation corresponds to the length of a shortest Kempe-sequence of a suitable instance of Kempe Distance. Let $G = (V, E)$ be a cubic graph. Consider the weights $w \in \mathbb{Z}^{V \times V}$ given by

$$w_{uv} = \begin{cases} K & \text{if } u \text{ and } v \text{ are adjacent in } G \\ 0 & \text{otherwise,} \end{cases}$$

where K is a suitably large number, say $K = |V|^2$. The following lemma implies that Minimum-Cost Permutation is NP-hard.

▶ **Lemma 13.** *G has a Hamiltonian cycle if and only if there is a permutation π of V, such that π has cost at most $1 - K \cdot |V|$.*

Proof. Let $C = v_1, v_2, \dots, v_t$ be a Hamiltonian cycle of G, where $t = |V|$. Furthermore, let π be given by $\pi(v_i) := v_{i+1}$ for $1 \leq i < t$ and $\pi(v_t) := v_1$ Since C is a Hamiltonian cycle, the cost π is $1 - K \cdot |V|$.

Now suppose that π is a permutation of V of cost at most $1 - K \cdot |V|$. Then for each $v \in V$ contributes at least $-K$ to the cost and π contains not fixpoints (due to the choice of K). Therefore, each vertex $v \in V$ is on some cycle of π. Since the cost π is at most $1 - K \cdot |V|$ we have that $c(\pi) \leq 1$. It follows from the construction of w that π corresponds Hamiltonian cycle of G. ◀

Note that since w is symmetric, it corresponds to a complete edge-weighted bipartite graph G' on the vertex set (V, V). We construct an instance (S, α, β) of Kempe Distance on stars, such that the corresponding edge-weighted auxiliary graph $G_{\alpha\beta}$ (see Section 3.1) is isomorphic to the weighted graph G'. The star graph S has $3K \cdot |V|$ leaves $\bigcup_{v \in V}\{\ell_1^v, \ell_2^v, \dots, \ell_{3K}^v\}$. We denote its center vertex by c. The source coloring α is given by $\alpha(\ell_i^v) := v$ for $1 \leq i \leq 3K$ and $v \in V$. For $1 \leq i \leq 3$, let $n_i(v)$ be the three neighbors of v in G in arbitrary order. For $1 \leq i \leq 3K$ and $v \in V$, let

$$\beta(\ell_i^v) := \begin{cases} n_1(v) & \text{if } 1 \leq i \leq K \\ n_2(v) & \text{if } K+1 \leq i \leq 2K \\ n_3(v) & \text{it } 2k+1 \leq i \leq 3K \end{cases}$$

Finally, let $\alpha(c) = \beta(c) = |V| + 1$. It is readily verified that the weighted graph G' is isomorphic to the graph $G_{\alpha\beta}$ obtained from S and the colorings α and β as in Section 3.1. Hence, we may invoke lemmas 6 and 7 and obtain the following relation between minimum-cost permutations and shortest Kempe-sequences from α to β.

► **Lemma 14.** *There is a permutation of V of cost at most $1 - K \cdot |V|$ if and only if α and β admit a Kempe-sequence of length at most $|V|(2K+1)+1$.*

Proof. Let π be a permutation of V of cost at most $1 - K \cdot |V|$. Since π corresponds to a perfect matching M of the weighted graph G' and G' is isomorphic to $G_{\alpha\beta}$, we may invoke lemmas 6 and 7 to obtain a sorted Kempe-sequence τ of length $|V|(2K+1)+1$ from α to β via γ_M.

Now let τ be a Kempe-sequence from α to β of length at most $|V|(2K+1)+1$. We show that there is a permutation π of V of cost at most $1 - K \cdot |V|$. We may assume that τ is sorted, so let γ be the intermediate coloring and let M_γ be the corresponding matching of $G_{\alpha\beta}$ (see Section 3.1). The successor map given by the matching M_γ gives rise to a permutation π on V, which in turn induces a set $\mathcal{C}$ of cycles on V; and potentially fixpoints. By lemmas 6 and 7 we have

$$|V|(2K+1)+1 = |\tau| \geq 3K|V| - w(M_\gamma) + c(\pi)$$

Rearranging gives $1 - K|V| + |V| \geq -w(M_\gamma) + c(\pi) = \sum_{v \in V} w(v, \pi_v)$. Since $K > 2|V|$, we have $w(v, \pi_v) = K$ for each $v \in V$. Therefore, by the construction of w, the permutation π has no fixpoints. Hence, by lemmas 6 and 7, we get the following sharper bound on the length of τ.

$$|V|(2K+1)+1 = |\tau| \geq 3K|V| - w(M_\gamma) + |V| + c(\pi)$$

Since $w(v, \pi_v) = K$ for each $v \in V$, we obtain from this inequality that $c(\pi) = 1$. Therefore, the permutation π has cost at most $1 - K \cdot |V|$. ◄

From Lemmas 13 and 14, the NP-hardness of Kempe Distance on stars is immediate. Observe that if we restrict ourselves to elementary recolorings then the problem is tractable.

► **Proposition 15** (∗). *There is a polynomial-time algorithm that, given two k-colorings α and β of a star graph, finds, if it exists, a Kempe-sequence of minimal length that certifies the equivalence of α and β under elementary recolorings.*

4 Conclusion

We showed that Kempe Distance on paths admits a polynomial-time algorithm and that the same problem on stars is NP-complete. Furthermore, we show that Kempe Distance on stars is FPT in the number k of colors and it admits a constant-factor approximation algorithm. There are some interesting open questions related to our results.

- Is it possible to generalize the dynamic programming algorithm for Kempe Distance on paths to trees with a bounded number of leaves?
- Does Kempe Distance on stars admit a polynomial-time approximation scheme?
- Does Kempe Distance on stars admit a polynomial kernel?

We conjecture that the answer to the last question is negative.

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