An FPT Algorithm for Minimum Additive Spanner Problem

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Abstract

For a positive integer t and a graph G, an additive t-spanner of G is a spanning subgraph in which the distance between every pair of vertices is at most the original distance plus t. The Minimum Additive t-Spanner Problem is to find an additive t-spanner with the minimum number of edges in a given graph, which is known to be NP-hard. Since we need to care about global properties of graphs when we deal with additive t-spanners, the Minimum Additive t-Spanner Problem is hard to handle and hence only few results are known for it. In this paper, we study the Minimum Additive t-Spanner Problem from the viewpoint of parameterized complexity. We formulate a parameterized version of the problem in which the number of removed edges is regarded as a parameter, and give a fixed-parameter algorithm for it. We also extend our result to the case with both a multiplicative approximation factor α and an additive approximation parameter β, which we call (α, β)-spanners.

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1 Introduction

1.1 Spanners

A *spanner* of a graph G is a spanning subgraph of G that approximately preserves the distance between every pair of vertices in G. Spanners were introduced in [4, 40, 41] in the context of synchronization in networks. Since then, spanners have been studied with applications to several areas such as space efficient routing tables [19, 42], computation of approximate shortest paths [17, 18, 26], distance oracles [6, 45], and so on.

A main topic on spanners is trade-offs between the sparsity (i.e., the number of edges) of a spanner and its quality of approximation of the distance, and there are several ways to measure the approximation quality. In the early studies, the approximation quality of spanners was measured by a multiplicative factor, i.e., the ratio between the distance in the spanner and the original distance. Formally, for a positive integer t and a graph G, a spanning subgraph H of G is said to be a *multiplicative t-spanner* if $\mathrm{dist}_H(u,v) \leq t \cdot \mathrm{dist}_G(u,v)$ holds for any pair of vertices u and v. Here, $\mathrm{dist}_G(u,v)$ (resp. $\mathrm{dist}_H(u,v)$) denotes the distance between u and v in G (resp. in H). Note that we deal with only graphs with unit length edges, and hence the distance is defined as the number of edges on a shortest path. A well-known trade-off between the sparsity and the multiplicative factor is as follows: for any positive integer t and any graph G, there exists a $(2t-1)$-spanner with $O(n^{1+1/t})$ edges [3], where n denotes the number of vertices in G. This bound is conjectured to be tight based on the popular Girth Conjecture of Erdős [30].

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Another natural measure of the approximation quality is the difference between the distance in the spanner and the original distance. For a positive integer t and a graph G, a spanning subgraph H of G is said to be an *additive t-spanner* if $\mathrm{dist}_H(u, v) \leq \mathrm{dist}_G(u, v) + t$ holds for any pair of vertices u and v. Since an additive spanner was introduced in [36, 37], trade-offs between the sparsity and the additive term have been actively studied. It is shown in [2, 25] that every graph has an additive 2-spanner with $O(n^{3/2})$ edges. In addition, every graph has an additive 4-spanner with $O(n^{7/5}\mathrm{poly}(\log n))$ edges [14], and every graph has an additive 6-spanner with $O(n^{4/3})$ edges [7]. On the negative side, it is shown in [1] that these bounds cannot be improved to $O(n^{4/3-\epsilon})$ for any $\epsilon > 0$.

As a common generalization of these two concepts, (α, β)-spanners have also been studied in the literature. For $\alpha \geq 1$, $\beta \geq 0$, and a graph G, a spanning subgraph H of G is said to be an *(α, β)-spanner* if $\mathrm{dist}_H(u, v) \leq \alpha \cdot \mathrm{dist}_G(u, v) + \beta$ holds for any pair of vertices u and v. See [9, 27, 32, 43, 44, 46, 48, 49] for other results on trade-offs between the sparsity of a spanner and its approximation quality.

In this paper, we consider a classical but natural and important problem that finds a spanner of minimum size. In particular, we focus on additive t-spanners and consider the following problem for a positive integer t.

Minimum Additive t-Spanner Problem
Instance. A graph $G = (V, E)$.
Question. Find an additive t-spanner $H = (V, E_H)$ of G that minimizes $|E_H|$.

The Minimum Multiplicative t-Spanner Problem and the Minimum (α, β)-Spanner Problem are defined in the same way. Such a problem is sometimes called the Sparsest Spanner Problem.

Although additive t-spanners have attracted attention as described above, there are only few results on the Minimum Additive t-Spanner Problem. For any positive integer t, the Minimum Additive t-Spanner Problem is shown to be NP-hard in [37]. It is shown by Chlamtáč et al. [16] that there is no polynomial-time $2^{(\log^{1-\epsilon} n)/t^3}$-approximation algorithm for any $\epsilon > 0$ under a standard complexity assumption. In [16], an $O(n^{3/5+\epsilon})$-approximation algorithm is proposed for any $\epsilon > 0$ for a more general problem. We can obtain algorithms for some special cases as consequences of known results. Every connected interval graph has an additive 2-spanner that is a spanning tree [35], which implies that the Minimum Additive t-Spanner Problem in interval graphs can be solved in polynomial time for $t \geq 2$. The same result holds for AT-free graphs [35]. It is shown in [15] that every chordal graph has an additive 4-spanner with at most $2n - 2$ edges, which implies that there exists a 2-approximation algorithm for the Minimum Additive 4-Spanner Problem in chordal graphs. To the best of our knowledge, no other results exist for the Minimum Additive t-Spanner Problem, which is in contrast to the fact that the Minimum Multiplicative t-Spanner Problem has been actively studied from the viewpoints of graph classes and approximation algorithms (see Section 1.3).

We make a remark on a difference between multiplicative t-spanners and additive t-spanners. As in [13, 38, 33], multiplicative t-spanners can be characterized as follows: a subgraph $H = (V, E_H)$ of $G = (V, E)$ is a multiplicative t-spanner if and only if $\mathrm{dist}_H(u, v) \leq t$ holds for any $uv \in E \setminus E_H$. This characterization means that if two edges in $E \setminus E_H$ are far from each other, then they do not interfere with each other. Thus, we only need to care about local properties of graphs when we deal with multiplicative t-spanners. In contrast, for additive t-spanners, no such characterization exists, and hence we have to consider global properties of graphs. In this sense, handling the Minimum Additive t-Spanner Problem is much harder than the Minimum Multiplicative t-Spanner Problem, which might be a reason why only few results exist for the Minimum Additive t-Spanner Problem.

1.2 Our Results

In this paper, we consider the MINIMUM ADDITIVE t-SPANNER PROBLEM from the viewpoint of fixed-parameter tractability and give the first fixed-parameter algorithm parameterized by the number of deleted edges for it. A parameterized version of the MINIMUM MULTIPLICATIVE t-SPANNER PROBLEM is studied in [33]. Since an additive (or multiplicative) t-spanner of a connected graph contains $\Omega(|V|)$ edges, the number of edges of a minimum additive (or multiplicative) t-spanner is not an appropriate parameter. Therefore, as in [33], a parameter is defined as the number of edges that are removed to obtain an additive (or multiplicative) t-spanner. This parameterization has a meaning when we remove a small number of edges, and such a situation might appear in a subroutine of other algorithms, e.g., in order to obtain a small additive spanner, we can consider a heuristic algorithm that removes a small number of edges repeatedly. Furthermore, the number of removed edges is a solution size in a certain sense. For these reasons, it will be the most natural parameter when we deal with spanners. Note that the same parameterization is also adopted in [5] for another network design problem. Our problem is formulated as follows.

Parameterized Minimum Additive t-Spanner Problem

Instance. A graph $G = (V, E)$.

Parameter. A positive integer k.

Question. Find an edge set $E' \subseteq E$ with $|E'| \geq k$ such that $H = (V, E \setminus E')$ is an additive t-spanner of G or conclude that such E' does not exist.

Note that if there exists a solution of size at least k, then its subset of size k is also a solution, which means that we can replace the condition $|E'| \geq k$ with $|E'| = k$ in the problem. In this paper, we show that there exists a fixed-parameter algorithm for this problem, where an algorithm is called a *fixed-parameter algorithm* (or an *FPT algorithm*) if its running time is bounded by $f(k)(|V| + |E|)^{O(1)}$ for some computable function f. See [20, 31, 39] for more details.

Formally, our result is stated as follows.

▶ **Theorem 1.** *For a positive integer t, there exists a fixed-parameter algorithm for the PARAMETERIZED MINIMUM ADDITIVE t-SPANNER PROBLEM that runs in $(t+1)^{O(k^2+tk)}|V||E|$ time. In particular, the running time is $2^{O(k^2)}|V||E|$ if t is fixed.*

This result implies that there exists a fixed-parameter algorithm for the problem even when $t + k$ is the parameter. As described in Section 1.1, the MINIMUM ADDITIVE t-SPANNER PROBLEM is a really hard problem and only few results were previously known for it. Therefore, this result may be a starting point for further research on the problem. A technical key ingredient of our algorithm is Lemma 5 that constructs a sequence of edge-disjoint cycles satisfying a certain condition, which is of independent interest.

By using almost the same argument, we can show that a parameterized version of the MINIMUM (α, β)-SPANNER PROBLEM is also fixed-parameter tractable. We define the PARAMETERIZED MINIMUM (α, β)-SPANNER PROBLEM in the same way as the PARAMETERIZED MINIMUM ADDITIVE t-SPANNER PROBLEM.

▶ **Theorem 2.** *For real numbers $\alpha \geq 1$ and $\beta \geq 0$, there exists a fixed-parameter algorithm for the PARAMETERIZED MINIMUM (α, β)-SPANNER PROBLEM that runs in $(\alpha + \beta)^{O(k^2+(\alpha+\beta)k)}|V||E|$ time.*

1.3 Related Work: Minimum Multiplicative Spanner Problem

As mentioned in Section 1.1, there are a lot of results on the Minimum Multiplicative t-Spanner Problem, whereas only few results are known for the Minimum Additive t-Spanner Problem.

The Minimum Multiplicative t-Spanner Problem is NP-hard for any $t \geq 2$ in general graphs [11, 40], and there are several results on the problem for some graph classes. It is NP-hard even when the input graph is restricted to be planar [10, 33]. Cai and Keil [13] showed that the Minimum Multiplicative 2-Spanner Problem can be solved in linear time if the maximum degree of the input graph is at most 4, whereas this problem is NP-hard even if the maximum degree is at most 9. Venkatesan et al. [47] revealed the complexity of the Minimum Multiplicative t-Spanner Problem for several graph classes such as chordal graphs, convex bipartite graphs, and split graphs. For the weighted version of the problem in which each edge has a positive integer length, Cai and Corneil [12] showed the NP-hardness of the Minimum Multiplicative t-Spanner Problem for $t > 1$.

Another direction of research is to design approximation algorithms for the Minimum Multiplicative t-Spanner Problem. Kortsarz [34] gave an $O(\log n)$-approximation algorithm for $t = 2$, and Elkin and Peleg [28] gave an $O(n^{2/(t+1)})$-approximation algorithm for $t > 2$. The approximation ratio was improved to $O(n^{1/3} \log n)$ for $t = 3$ by Berman et al. [8] and to $O(n^{1/3}\text{poly}(\log n))$ for $t = 4$ by Dinitz and Zhang [22]. On the negative side, for any $t \geq 2$, it is shown in [29] that no $o(\log n)$-approximation algorithm exists unless $P = NP$. This lower bound was improved to $2^{(\log^{1-\epsilon} n)/k}$ for any $\epsilon > 0$ in [21] under a standard complexity assumption. Dragan et al. [23] gave an EPTAS for the problem in planar graphs. When the input graph is a 4-connected planar triangulation, a PTAS is proposed for the Minimum Multiplicative 2-Spanner Problem in [24].

A parameterized version of the Minimum Multiplicative t-Spanner Problem is introduced in [33], where the parameter is the number of deleted edges, and a fixed-parameter algorithm for it is presented in the same paper.

1.4 Organization

The remainder of this paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we give an FPT algorithm for the Parameterized Minimum Additive t-Spanner Problem and prove Theorem 1. In Section 4, we extend the argument in Section 3 to the Parameterized Minimum (α, β)-Spanner Problem and prove Theorem 2. Finally, in Section 5, we conclude the paper with a summary.

2 Preliminaries

In this paper, we deal with only undirected graphs with unit length edges. Since we can remove all the parallel edges and self-loops when we consider spanners, we assume that all the graphs in this paper are simple. Let $G = (V, E)$ be a graph. For $u, v \in V$, an edge connecting u and v is denoted by uv. For a subgraph H of G, the set of vertices and the set of edges in H are denoted by $V(H)$ and $E(H)$, respectively. For an edge $e \in E$, let $G - e$ denote the subgraph $G' = (V, E \setminus \{e\})$. We say that an edge set $F \subseteq E$ *contains* a path P if $E(P) \subseteq F$. For a path P and for two vertices $u, v \in V(P)$, let $P[u, v]$ denote the subpath of P between u and v. For $u, v \in V$, let $\text{dist}_G(u, v)$ denote the length of the shortest path between u and v in G. Note that the length of a path is the number of edges in it. If G is clear from the context, $\text{dist}_G(u, v)$ is simply denoted by $\text{dist}(u, v)$. For a positive integer t, a subgraph

$H = (V, E_H)$ of $G = (V, E)$ is said to be an *additive t-spanner* if $\mathrm{dist}_H(u, v) \le \mathrm{dist}_G(u, v) + t$ or $\mathrm{dist}_G(u, v) = +\infty$ holds for any $u, v \in V$. For real numbers $\alpha \ge 1$ and $\beta \ge 0$, a subgraph $H = (V, E_H)$ of $G = (V, E)$ is said to be an *(α, β)-spanner* if $\mathrm{dist}_H(u, v) \le \alpha \cdot \mathrm{dist}_G(u, v) + \beta$ or $\mathrm{dist}_G(u, v) = +\infty$ holds for any $u, v \in V$. In what follows, we may assume that the input graph $G = (V, E)$ is connected and $\mathrm{dist}_G(u, v)$ is finite for any $u, v \in V$, since we can deal with each connected component separately. For a positive integer p, let $[p] := \{1, \dots, p\}$.

3 Proof of Theorem 1

3.1 Outline

In this subsection, we show an outline of our proof of Theorem 1.

Define $F \subseteq E$ as the set of all edges contained in cycles of length at most $t + 2$. In other words, an edge $e = uv \in E$ is in F if and only if $G - e$ contains a u-v path of length at most $t + 1$. By definition, if $H = (V, E \setminus E')$ is an additive t-spanner of G, then $\mathrm{dist}_{G-e}(u, v) \le \mathrm{dist}_H(u, v) \le \mathrm{dist}_G(u, v) + t = t + 1$ holds for each $e = uv \in E'$, which implies that $E' \subseteq F$. Thus, if $|F|$ is small, then we can solve the Parameterized Minimum Additive t-Spanner Problem by checking whether $H = (V, E \setminus E')$ is an additive t-spanner of G or not for every subset E' of F with $|E'| = k$.

If $|F|$ is sufficiently large (as a function of t and k), then there exist many cycles of length at most $t+2$. In what follows, we show that if G has many cycles of length at most $t+2$, then there always exists $E' \subseteq E$ with $|E'| = k$ such that $H = (V, E \setminus E')$ is an additive t-spanner of G. To this end, we prove the following statements in Sections 3.2–3.4, respectively.

- If there are many cycles of length at most $t+2$, then we can find either many edge-disjoint cycles of length at most $t + 2$ or a desired set $E' \subseteq E$ (Section 3.2).
- If there are many edge-disjoint cycles of length at most $t + 2$, then we can construct a sequence $(C_1, \dots, C_p)$ of edge-disjoint cycles with a certain condition (Section 3.3). Roughly speaking, this condition means that if $h < i < j$, then removing edges in $E(C_j)$ does not affect the distance between C_h and C_i.
- If we have a sequence of edge-disjoint cycles with the above condition, then we can construct a desired set $E' \subseteq E$ (Section 3.4).

Finally, in Section 3.5, we put them together and describe our entire algorithm.

3.2 Finding Edge-disjoint Cycles

The objective of this subsection is to show that if there are many cycles of length at most $t + 2$, then we can find either many edge-disjoint cycles of length at most $t + 2$ or a desired set $E' \subseteq E$. We first show the following lemma.

▶ **Lemma 3.** *For positive integers r and ℓ, there exists an integer $f_1(r, \ell) = (r\ell)^{O(\ell)}$ satisfying the following condition. For any pair of distinct vertices $u, v \in V$ in a graph $G = (V, E)$, if there exists a set $\mathcal{P}$ of u-v paths of length at most ℓ with $|\mathcal{P}| \ge f_1(r, \ell)$, then G contains two distinct vertices $u', v' \in V$ and r edge-disjoint u'-v' paths of length at most $\ell - \mathrm{dist}(u, u') - \mathrm{dist}(v, v')$.*

Proof. We show that $f_1(r, \ell) := 2(r\ell^3)^{\ell-1}$ satisfies the condition by induction on ℓ. The claim is obvious when $\ell = 1$, because $|\mathcal{P}| \le 1$ holds as G is simple and $f_1(r, 1) = 2$. Thus, it suffices to consider the case of $\ell \ge 2$. Let $\mathcal{P}$ be a set of u-v paths of length at most ℓ with $|\mathcal{P}| \ge f_1(r, \ell)$. We consider the following two cases separately.

We first consider the case when $|\{P \in \mathcal{P} \mid e \in E(P)\}| < \frac{f_1(r,\ell)}{r\ell}$ holds for every $e \in E$. In this case, $|\{Q \in \mathcal{P} \mid E(P) \cap E(Q) \neq \emptyset\}| < \frac{f_1(r,\ell)}{r}$ for every $P \in \mathcal{P}$. This shows that we can take r edge-disjoint u-v paths in $\mathcal{P}$ by a greedy algorithm (i.e., repeatedly taking a u-v path P in $\mathcal{P}$ and removing all the paths sharing an edge with P). They form a desired set of paths in which $u' = u$ and $v' = v$.

We next consider the case when there exists an edge $e = xy \in E$ such that $|\{P \in \mathcal{P} \mid e \in E(P)\}| \geq \frac{f_1(r,\ell)}{r\ell} = 2\ell^2(r\ell^3)^{\ell-2}$. Since $\{x, y\} \neq \{u, v\}$, by changing the roles of x and y if necessary, we may assume that $x \notin \{u, v\}$. For $i = 1, \dots, \ell - 1$, let $\mathcal{P}^i_{ux}$ be the set of all u-x paths of length i and $\mathcal{P}^i_{xv}$ be the set of all x-v paths of length i. Then, since each path $P \in \mathcal{P}$ containing e can be divided into a u-x path and an x-v path, we obtain

$$\sum_{i+j\leq \ell} |\mathcal{P}^i_{ux}| \cdot |\mathcal{P}^j_{xv}| \geq |\{P \in \mathcal{P} \mid e \in E(P)\}| \geq 2\ell^2(r\ell^3)^{\ell-2}.$$

Since the number of pairs (i, j) with $i + j \leq \ell$ is at most $\frac{\ell(\ell-1)}{2} < \frac{\ell^2}{2}$, there exist $i, j \in [\ell - 1]$ with $i + j \leq \ell$ such that

$$|\mathcal{P}^i_{ux}| \cdot |\mathcal{P}^j_{xv}| \geq 2\ell^2(r\ell^3)^{\ell-2} \cdot \frac{2}{\ell^2} \geq 2(r\ell^3)^{i-1} \cdot 2(r\ell^3)^{j-1} \geq f_1(r, i) \cdot f_1(r, j).$$

Then, we have either (i) $|\mathcal{P}^i_{ux}| \geq f_1(r, i)$ and $|\mathcal{P}^j_{xv}| \geq 1$, or (ii) $|\mathcal{P}^j_{xv}| \geq f_1(r, j)$ and $|\mathcal{P}^i_{ux}| \geq 1$. Suppose that (i) holds. By induction hypothesis, $|\mathcal{P}^i_{ux}| \geq f_1(r, i)$ implies that there exist $u', v' \in V$ and r edge-disjoint u'-v' paths of length at most

$$\begin{aligned}
& i - \mathrm{dist}(u, u') - \mathrm{dist}(x, v') \\
& \quad \leq \ell - j - \mathrm{dist}(u, u') - \mathrm{dist}(x, v') && \text{(by } i + j \leq \ell\text{)} \\
& \quad \leq \ell - \mathrm{dist}(x, v) - \mathrm{dist}(u, u') - \mathrm{dist}(x, v') && \text{(by } |\mathcal{P}^j_{xv}| \geq 1\text{)} \\
& \quad \leq \ell - \mathrm{dist}(u, u') - \mathrm{dist}(v, v'). && \text{(by the triangle inequality)}
\end{aligned}$$

Thus, they form a desired set of paths. The same argument can be applied when (ii) holds. ◀

By using this lemma, we obtain the following proposition.

▶ **Proposition 4.** *Let $G = (V, E)$ be a graph and $\mathcal{C}$ be a set of cycles of length at most $t+2$. Let N be a positive integer and f_1 be a function as in Lemma 3. If $|\mathcal{C}| \geq N(t+2)f_1(k+t+1, t+1)$, then we have one of the following.*
- *There exist N edge-disjoint cycles in $\mathcal{C}$.*
- *There exists $E' \subseteq \bigcup_{C\in\mathcal{C}} E(C)$ with $|E'| = k$ such that $H = (V, E \setminus E')$ is an additive t-spanner of G.*

Proof. For each edge $e \in E$, let $\mathcal{C}_e := \{C \in \mathcal{C} \mid e \in E(C)\}$. We first consider the case when $|\mathcal{C}_e| < f_1(k + t + 1, t + 1)$ holds for every $e \in E$. In this case, $|\{C' \in \mathcal{C} \mid E(C) \cap E(C') \neq \emptyset\}| < (t + 2)f_1(k + t + 1, t + 1)$ for every $C \in \mathcal{C}$. This shows that we can take N edge-disjoint cycles in $\mathcal{C}$ by a greedy algorithm (i.e., repeatedly taking a cycle C in $\mathcal{C}$ and removing all the cycles sharing an edge with C), because $|\mathcal{C}| \geq N(t + 2)f_1(k + t + 1, t + 1)$.

We next consider the case when there exists an edge $e = uv \in E$ such that $|\mathcal{C}_e| \geq f_1(k + t + 1, t + 1)$. Since $\mathcal{P} := \{C - e \mid C \in \mathcal{C}_e\}$ consists of u-v paths of length at most $t + 1$, by Lemma 3, G contains two vertices $u', v' \in V$ and a set $\mathcal{P}'$ of $k + t + 1$ edge-disjoint u'-v' paths of length at most $t' := t + 1 - \mathrm{dist}_G(u, u') - \mathrm{dist}_G(v, v')$. Let Q_u and Q_v be a shortest u-u' path and a shortest v-v' path, respectively. Since $|E(Q_u)|+|E(Q_v)|+1 = t+2-t' \leq t+1$, there exists $\mathcal{P}'' \subseteq \mathcal{P}'$ with $|\mathcal{P}''| = |\mathcal{P}'|-(t+1) = k$ such that each path in $\mathcal{P}''$ does not contain

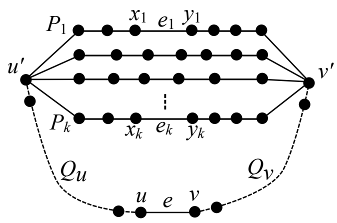

Figure 1 Definition of $e_1, \dots, e_k$ in Proposition 4.

edges in $E(Q_u) \cup E(Q_v) \cup \{e\}$. Let $P_1, \dots, P_k$ denote the paths in $\mathcal{P}''$. For $i = 1, \dots, k$, let e_i be the middle edge of P_i (see Fig. 1). Formally, we take $e_i = x_i y_i$ so that $P_i[u', x_i]$ contains $\lfloor \frac{|E(P_i)|-1}{2} \rfloor \le \lfloor \frac{t'-1}{2} \rfloor$ edges and $P_i[y_i, v']$ contains $\lceil \frac{|E(P_i)|-1}{2} \rceil \le \lceil \frac{t'-1}{2} \rceil$ edges. Define $E' := \{e_1, \dots, e_k\}$ and consider the graph $H = (V, E \setminus E')$. Then, for any $i, j \in [k]$ we can see that

$$\mathrm{dist}_H(x_i, x_j) \le |E(P_i[u', x_i])| + |E(P_j[u', x_j])| \le t' \le t + 1, \tag{1}$$

$$\mathrm{dist}_H(y_i, y_j) \le |E(P_i[y_i, v'])| + |E(P_j[y_j, v'])| \le t' \le t + 1, \tag{2}$$

$$\begin{aligned} \mathrm{dist}_H(x_i, y_j) &\le |E(P_i[u', x_i])| + |E(Q_u) \cup E(Q_v) \cup \{e\}| + |E(P_j[y_j, v'])| \\ &\le \left\lfloor \frac{t'-1}{2} \right\rfloor + (t + 2 - t') + \left\lceil \frac{t'-1}{2} \right\rceil \le t + 1. \end{aligned} \tag{3}$$

We now show that H is an additive t-spanner of G. Let x and y be distinct vertices in V and let P be a shortest x-y path in G. If $E(P) \cap E' = \emptyset$, then it is obvious that $\mathrm{dist}_H(x, y) = \mathrm{dist}_G(x, y)$. If $E(P) \cap E' \ne \emptyset$, then let $P[z, z']$ be the unique minimal subpath of P that contains all edges in $E(P) \cap E'$, where x, z, z', and y appear in this order along P. Since $z, z' \in \{x_1, y_1, \dots, x_k, y_k\}$, we have $\mathrm{dist}_H(z, z') \le t + 1$ by (1)–(3). Therefore,

$$\begin{aligned} \mathrm{dist}_H(x, y) &\le \mathrm{dist}_H(x, z) + \mathrm{dist}_H(z, z') + \mathrm{dist}_H(z', y) && \text{(by the triangle inequality)} \\ &\le |E(P[x, z])| + t + 1 + |E(P[z', y])| && \text{(by (1)–(3))} \\ &= |E(P)| - |E(P[z, z'])| + t + 1 \\ &\le \mathrm{dist}_G(x, y) + t, && \text{(by } |E(P)| = \mathrm{dist}_G(x, y)\text{)} \end{aligned}$$

which shows that H is an additive t-spanner of G. ◀

3.3 Finding a Good Sequence of Cycles

In this subsection, we construct a sequence of edge-disjoint cycles with a certain condition when we are given many edge-disjoint cycles.

Let $\mathcal{C}$ be a set of edge-disjoint cycles of length at most $t + 2$. For each cycle $C \in \mathcal{C}$, we apply the breadth-first search from $V(C)$ and obtain a shortest path $P(v, C)$ between $V(C)$ and each vertex $v \in V$. That is, $|E(P(v, C))| = \min\{|E(P)| \mid u \in V(C),\ P \text{ is a } u\text{-}v \text{ path}\}$. Then, $\bigcup_{v \in V} E(P(v, C))$ forms a forest for each cycle $C \in \mathcal{C}$. The objective of this subsection is to find a sequence $(C_1, \dots, C_p)$ of distinct p cycles $C_1, \dots, C_p \in \mathcal{C}$ satisfying the following condition:

($\star$) For any $h, i, j \in [p]$ with $h < i < j$ and for any vertex $v \in V(C_h)$, it holds that $E(P(v, C_i)) \cap E(C_j) = \emptyset$.

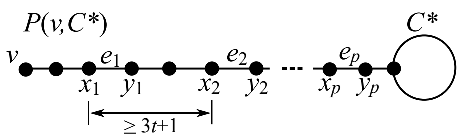

Figure 2 Definition of $e_1, \ldots, e_p$.

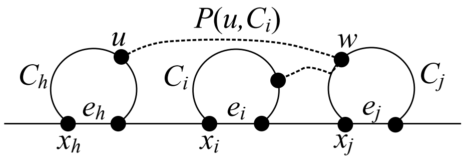

Figure 3 Definition of w.

Roughly speaking, this condition means that if $h < i < j$, then removing edges in $E(C_j)$ does not affect the distance between C_h and C_i.

▶ **Lemma 5.** *For any positive integers t and p, there exists an integer $f_2(t,p) = O(t^2p^4)$ satisfying the following condition. If $\mathcal{C}$ is a set of $f_2(t,p)$ edge-disjoint cycles of length at most $t+2$, then there exists a sequence $(C_1, \ldots, C_p)$ of distinct p cycles $C_1, \ldots, C_p \in \mathcal{C}$ satisfying the condition $(\star)$.*

Proof. We show that $f_2(t,p) := 27(t+2)(3t+1)p^4$ satisfies the condition in the lemma. Let $\mathcal{C}$ be a set of $f_2(t,p)$ edge-disjoint cycles of length at most $t+2$. We consider the following two cases separately.

Case 1. Suppose that there exist a vertex $v \in V$ and a cycle $C^* \in \mathcal{C}$ such that $|E(P(v,C^*)) \cap \bigcup_{C \in \mathcal{C}} E(C)| \geq (3t+1)p$. In this case, we can take edges $e_1, \ldots, e_p$ in $E(P(v,C^*)) \cap \bigcup_{C \in \mathcal{C}} E(C)$ such that $e_1 = x_1y_1, e_2 = x_2y_2, \ldots, e_p = x_py_p$ appear in this order along $P(v,C^*)$ and the subpath of $P(v,C^*)$ between x_i and x_{i+1} contains at least $3t+1$ edges for $i = 1, \ldots, p-1$ (see Fig. 2). For $i = 1, \ldots, p$, let $C_i \in \mathcal{C}$ be the cycle containing e_i. Note that C_i and C_j are distinct if $i \neq j$, since $\mathrm{dist}_G(x_i, x_j) \geq 3t+1 > |E(C_i)|$.

We now show that $(C_1, \ldots, C_p)$ satisfies the condition $(\star)$. Assume to the contrary that there exist indices $h, i, j \in [p]$ with $h < i < j$ and a vertex $u \in V(C_h)$ such that $E(P(u,C_i)) \cap E(C_j) \neq \emptyset$. Let w be the first vertex in $V(C_j)$ when we traverse $P(u,C_i)$ from u to $V(C_i)$ (see Fig. 3). Then, by using

$$\mathrm{dist}(u, x_h) \leq \left\lfloor \frac{|E(C_h)|}{2} \right\rfloor \leq t \quad \text{and} \quad \mathrm{dist}(x_j, w) \leq \left\lfloor \frac{|E(C_j)|}{2} \right\rfloor \leq t, \tag{4}$$

we obtain

$$\begin{aligned}
&\mathrm{dist}(x_h, x_i) + t \\
&\quad \geq \mathrm{dist}(x_h, x_i) + \mathrm{dist}(u, x_h) && \text{(by (4))} \\
&\quad \geq \mathrm{dist}(u, x_i) \geq |E(P(u, C_i))| \geq \mathrm{dist}(u, w) \\
&\quad \geq \mathrm{dist}(x_h, x_j) - \mathrm{dist}(x_h, u) - \mathrm{dist}(w, x_j) && \text{(by the triangle inequality)} \\
&\quad \geq |E(P[x_h, x_j])| - 2t && \text{(by (4) and } \mathrm{dist}(x_h, x_j) = |E(P[x_h, x_j])|) \\
&\quad \geq (|E(P[x_h, x_i])| + 3t + 1) - 2t && \text{(by } |E(P[x_i, x_j])| \geq 3t+1) \\
&\quad = \mathrm{dist}(x_h, x_i) + t + 1, && \text{(by } \mathrm{dist}(x_h, x_i) = |E(P[x_h, x_i])|)
\end{aligned}$$

which is a contradiction. Therefore, $(C_1, \ldots, C_p)$ satisfies the condition $(\star)$.

Case 2. Suppose that $|E(P(v,C^*)) \cap \bigcup_{C \in \mathcal{C}} E(C)| < (3t+1)p$ holds for every $v \in V$ and $C^* \in \mathcal{C}$, which implies that $|\{C \in \mathcal{C} \mid E(P(v,C^*)) \cap E(C) \neq \emptyset\}| < (3t+1)p$ as $\mathcal{C}$ is a set of edge-disjoint cycles. We define $\mathcal{F}_3 \subseteq \mathcal{C}^3$ by

$$\mathcal{F}_3 := \{(C_h, C_i, C_j) \mid C_h, C_i, C_j \in \mathcal{C},\ E(P(v, C_i)) \cap E(C_j) \neq \emptyset \text{ for some } v \in V(C_h)\}.$$

Then, it holds that

$$\begin{aligned}
|\mathcal{F}_3| &= \sum_{C_h \in \mathcal{C}} \sum_{C_i \in \mathcal{C}} |\{C_j \in \mathcal{C} \mid E(P(v, C_i)) \cap E(C_j) \neq \emptyset \text{ for some } v \in V(C_h)\}| \\
&\leq \sum_{C_h \in \mathcal{C}} \sum_{C_i \in \mathcal{C}} \sum_{v \in V(C_h)} |\{C_j \in \mathcal{C} \mid E(P(v, C_i)) \cap E(C_j) \neq \emptyset\}| \\
&< \sum_{C_h \in \mathcal{C}} \sum_{C_i \in \mathcal{C}} \sum_{v \in V(C_h)} (3t+1)p \\
&\leq (t+2)(3t+1)p|\mathcal{C}|^2. \qquad (5)
\end{aligned}$$

We note that $(C_1, \ldots, C_p)$ satisfies the condition $(\star)$ if and only if $(C_h, C_i, C_j) \notin \mathcal{F}_3$ holds for any $h, i, j \in [p]$ with $h < i < j$. That is, $\mathcal{F}_3$ represents the set of forbidden orderings of three cycles. We define $\mathcal{F}_2 \subseteq \mathcal{C}^2$ and $\mathcal{F}_1 \subseteq \mathcal{C}$ by

$$\begin{aligned}
\mathcal{F}_2 &:= \left\{ (C_h, C_i) \in \mathcal{C}^2 \;\middle|\; |\{C \in \mathcal{C} \mid (C_h, C_i, C) \in \mathcal{F}_3\}| \geq \frac{|\mathcal{C}|}{3p^2} \right\}, \\
\mathcal{F}_1 &:= \left\{ C_h \in \mathcal{C} \;\middle|\; |\{C \in \mathcal{C} \mid (C_h, C) \in \mathcal{F}_2\}| \geq \frac{|\mathcal{C}|}{3p} \right\}.
\end{aligned}$$

By (5), we have

$$\begin{aligned}
|\mathcal{F}_2| &\leq |\mathcal{F}_3| \cdot \frac{3p^2}{|\mathcal{C}|} < 3(t+2)(3t+1)p^3|\mathcal{C}|, \\
|\mathcal{F}_1| &\leq |\mathcal{F}_2| \cdot \frac{3p}{|\mathcal{C}|} < 9(t+2)(3t+1)p^4 \leq \frac{|\mathcal{C}|}{3}. \qquad (6)
\end{aligned}$$

In order to obtain $(C_1, \ldots, C_p)$ satisfying the condition $(\star)$, we construct a sequence of cycles satisfying additional conditions.

▷ Claim 6. For each $q \in [p]$, there exists a sequence $(C_1, \ldots, C_q)$ of q distinct cycles $C_1, \ldots, C_q \in \mathcal{C}$ satisfying the following conditions:

- $C_h \notin \mathcal{F}_1$ for any $h \in [q]$,
- $(C_h, C_i) \notin \mathcal{F}_2$ for any $h, i \in [q]$ with $h < i$, and
- $(C_h, C_i, C_j) \notin \mathcal{F}_3$ for any $h, i, j \in [q]$ with $h < i < j$.

Proof. We show the claim by induction on q. When $q = 1$, we can choose $C_1 \in \mathcal{C} \setminus \mathcal{F}_1$ arbitrarily. Suppose that we have $C_1, \ldots, C_q \in \mathcal{C}$ satisfying the conditions in the claim, where $q \leq p - 1$. We evaluate the number of cycles that cannot be chosen as C_{q+1}. By the definitions of $\mathcal{F}_1$ and $\mathcal{F}_2$, we have that

$$N_2 := |\{C \in \mathcal{C} \mid (C_h, C) \in \mathcal{F}_2 \text{ for some } h \in [q]\}| \leq q \cdot \frac{|\mathcal{C}|}{3p} \leq \left(1 - \frac{1}{p}\right) \cdot \frac{|\mathcal{C}|}{3} < \frac{|\mathcal{C}|}{3} - p, \qquad (7)$$

$$N_3 := |\{C \in \mathcal{C} \mid (C_h, C_i, C) \in \mathcal{F}_3 \text{ for some } h, i \in [q] \text{ with } h < i\}| \leq q^2 \cdot \frac{|\mathcal{C}|}{3p^2} < \frac{|\mathcal{C}|}{3}, \qquad (8)$$

where we use $|\mathcal{C}| > 3p^2$ to obtain (7). Since $|\mathcal{C}| - |\mathcal{F}_1| - N_2 - N_3 > p \geq q + 1$ by (6)–(8), there exists a cycle $C_{q+1} \in \mathcal{C}$ that is different from $C_1, \ldots, C_q$ such that $(C_1, \ldots, C_q, C_{q+1})$ satisfies the conditions in the claim. This shows the claim by induction on q. ◁

By this claim, there exists a sequence $(C_1, \ldots, C_p)$ of p distinct cycles $C_1, \ldots, C_p \in \mathcal{C}$ such that $(C_h, C_i, C_j) \notin \mathcal{F}_3$ for any $h, i, j \in [p]$ with $h < i < j$, which means that $(C_1, \ldots, C_p)$ satisfies the condition $(\star)$. ◀

3.4 Constructing an Additive t-Spanner

In this subsection, we show that we can construct an additive t-spanner of G by using a sequence of edge-disjoint cycles satisfying the condition ($\star$).

▶ **Lemma 7.** *For any positive integers t and k, there exists an integer $f_3(t,k) = (t+2)^{O(k)}$ satisfying the following condition. If there exists a sequence $(C_1, \ldots, C_p)$ of $p = f_3(t,k)$ edge-disjoint cycles of length at most $t+2$ satisfying the condition ($\star$), then there exists an edge set $E' \subseteq \bigcup_{i\in[p]} E(C_i)$ with $|E'| = k$ such that $H = (V, E \setminus E')$ is an additive t-spanner of G.*

Proof. We show that $p = f_3(t,k) := k(t+2)^{k-1}$ satisfies the condition. For each edge $e \in E$, define

$$I(e) := \{i \in [p] \mid e \notin \bigcup_{v\in V} E(P(v, C_i))\}.$$

Since $\bigcup_{v\in V} E(P(v, C_i))$ forms a forest for each $i \in [p]$, for any cycle C there exists an edge $e \in E(C)$ such that $i \in I(e)$. In other words, $\bigcup_{e\in E(C)} I(e) = [p]$ for any cycle C. We prove the lemma by showing that Algorithm 1 always finds an edge set $E' \subseteq \bigcup_{i\in[p]} E(C_i)$ with $|E'| = k$ such that $H = (V, E \setminus E')$ is an additive t-spanner of G.

Algorithm 1 Constructing an additive t-spanner from a sequence with ($\star$).

Input : A sequence $(C_1, \ldots, C_p)$ of edge-disjoint cycles of length at most $t+2$ with the condition ($\star$)
Output : An edge set $E' \subseteq \bigcup_{i\in[p]} E(C_i)$ with $|E'| = k$ such that $H = (V, E \setminus E')$ is an additive t-spanner

```
I_0 := [p]
for i = 1, ..., k do
    Let ind(i) be the minimum index in I_{i-1}
    C'_i := C_{ind(i)}
    Choose an edge e_i ∈ E(C'_i) that maximizes |(I_{i-1} \ {ind(i)}) ∩ I(e_i)|
    I_i := (I_{i-1} \ {ind(i)}) ∩ I(e_i)
end
Return E' := {e_1, ..., e_k}
```

We first show that the algorithm returns a set of k edges. For $i = 1, \ldots, k-1$, since $\bigcup_{e\in E(C'_i)} I(e) = [p]$ and $|E(C'_i)| \le t+2$, we have that $|I_i| \ge \frac{|I_{i-1}\setminus\{\mathrm{ind}(i)\}|}{|E(C'_i)|} \ge \frac{|I_{i-1}|-1}{t+2}$. By combining this with $|I_0| = k(t+2)^{k-1}$, we see that $|I_i| \ge (k-i)(t+2)^{k-i-1}$ for each i by induction, because

$$|I_i| \ge \frac{|I_{i-1}| - 1}{t+2} \ge \frac{(k-i+1)(t+2)^{k-i} - 1}{t+2} \ge (k-i)(t+2)^{k-i-1}.$$

In particular, $|I_{k-1}| \ge 1$ holds, and hence the algorithm returns a set $E' = \{e_1, \ldots, e_k\}$.

We next show that $H = (V, E \setminus E')$ is an additive t-spanner. Let x and y be distinct vertices in V and let P be a shortest x-y path in G. If $E(P) \cap E' = \emptyset$, then it is obvious that $\mathrm{dist}_H(x,y) = \mathrm{dist}_G(x,y)$. If $E(P) \cap E' = \{e_i\}$ for some $i \in \{1, \ldots, k\}$, then $(E(P) \setminus \{e_i\}) \cup (E(C'_i) \setminus \{e_i\})$ contains an x-y path, and hence we obtain $\mathrm{dist}_H(x,y) \le |(E(P) \setminus \{e_i\}) \cup (E(C'_i) \setminus \{e_i\})| \le \mathrm{dist}_G(x,y) + t$.

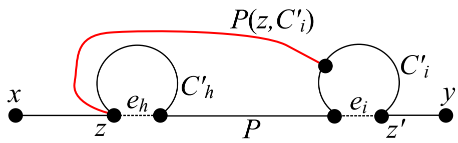

Figure 4 Proof of Lemma 7.

Thus, it suffices to consider the case when $|E(P) \cap E'| \geq 2$. Let $P[z, z']$ be the unique minimal subpath of P that contains all edges in $E(P) \cap E'$, where x, z, z', and y appear in this order along P. Suppose that z and z' are endpoints of edges e_h and e_i in $E(P) \cap E'$, respectively. We may assume that $h < i$ by changing the roles of x and y if necessarily. We now observe the following properties of $P(z, C'_i)$.

- Since $(C_1, \dots, C_p)$ satisfies $(\star)$, $(C'_1, \dots, C'_k)$ also satisfies $(\star)$. It follows that $P(z, C'_i)$ does not contain edges in $E(C'_j)$ for any $j > i$, because $z \in V(C'_h)$ and $h < i$. In particular, $P(z, C'_i)$ does not contain e_j for any $j > i$.
- Since $\mathrm{ind}(i) \in I_{i-1} \subseteq I(e_1) \cap I(e_2) \cap \cdots \cap I(e_{i-1})$ by the algorithm, $P(z, C'_i)$ does not contain e_j for any $j < i$.
- It is obvious that $P(z, C'_i)$ does not contain e_i by the definition of $P(z, C'_i)$.

By these observations, $P(z, C'_i)$ does not contain edges in E', which means that $P(z, C'_i)$ is a path in H (see Fig. 4). Since $C'_i - e_i$ contains a path connecting an endpoint of $P(z, C'_i)$ and z', $E(P(z, C'_i)) \cup (E(C'_i) \setminus \{e_i\})$ contains a path between z and z', and hence we have that

$$\mathrm{dist}_H(z, z') \leq |E(P(z, C'_i))| + |E(C'_i) \setminus \{e_i\}| \leq |E(P(z, C'_i))| + t + 1. \tag{9}$$

Since $P[z, z'] - e_i$ forms a path from z to C'_i, we obtain

$$|E(P(z, C'_i))| \leq |E(P[z, z']) \setminus \{e_i\}|. \tag{10}$$

By (9) and (10), we have that

$$\begin{aligned}
&\mathrm{dist}_H(x, y) \\
&\quad \leq \mathrm{dist}_H(x, z) + \mathrm{dist}_H(z, z') + \mathrm{dist}_H(z', y) && \text{(by the triangle inequality)} \\
&\quad \leq \mathrm{dist}_H(x, z) + |E(P(z, C'_i))| + t + 1 + \mathrm{dist}_H(z', y) && \text{(by (9))} \\
&\quad \leq \mathrm{dist}_H(x, z) + |E(P[z, z']) \setminus \{e_i\}| + t + 1 + \mathrm{dist}_H(z', y) && \text{(by (10))} \\
&\quad = |E(P[x, z])| + |E(P[z, z'])| + t + |E(P[z', y])| \\
&\quad = |E(P[x, y])| + t \\
&\quad = \mathrm{dist}_G(x, y) + t.
\end{aligned}$$

Therefore, H is an additive t-spanner of G. ◀

3.5 The Entire Algorithm

In this subsection, we describe our entire algorithm for the Parameterized Minimum Additive t-Spanner Problem and prove Theorem 1 by using Proposition 4 and Lemmas 5 and 7. Define

$$p := f_3(t, k), \qquad N := f_2(t, p), \qquad f_4(t, k) := N(t + 2)^2 f_1(k + t + 1, t + 1),$$

where f_1, f_2, and f_3 are as in Lemmas 3, 5, and 7, respectively. Then, $N = (t+2)^{O(k)}$ and $f_1(k+t+1, t+1) = (kt)^{O(t)}$, and hence

$$f_4(t,k) = (t+2)^{O(k)} \cdot (kt)^{O(t)}. \tag{11}$$

When $t \geq k$, (11) is bounded by $(t+2)^{O(t)} \cdot (t^2)^{O(t)} = (t+1)^{O(t)}$. When $t \leq k$, (11) is bounded by $(t+2)^{O(k)} \cdot (k^2)^{O(t)} = (t+1)^{O(k)}$. By combining them, we obtain $f_4(t,k) = (t+1)^{O(k+t)}$. Note that we can simply denote $f_4(t,k) = t^{O(k+t)}$ unless $t = 1$.

In our algorithm, we first compute the set $F \subseteq E$ of all edges contained in cycles of length at most $t+2$. Note that we can do it in $O(|V||E|)$ time by applying the breadth-first search from each vertex.

As described in Section 3.1, if $H = (V, E \setminus E')$ is an additive t-spanner of G for $E' \subseteq E$, then $E' \subseteq F$ holds. Thus, if $|F| \leq f_4(t,k)$, then we can solve the Parameterized Minimum Additive t-Spanner Problem in $O(f_4(t,k)^k |V||E|)$ time by checking whether $H = (V, E \setminus E')$ is an additive t-spanner of G or not for every subset E' of F with $|E'| = k$.

Otherwise, we have $|F| \geq f_4(t,k) = N(t+2)^2 f_1(k+t+1, t+1)$. Since there exist at least $\frac{|F|}{t+2} \geq N(t+2) f_1(k+t+1, t+1)$ cycles of length at most $t+2$ by the definition of F, we can take a set $\mathcal{C}$ of $N(t+2)f_1(k+t+1,t+1)$ cycles of length at most $t+2$ by a greedy algorithm. The procedure is formally described as follows: for $i = 1, 2, \dots, N(t+2)f_1(k+t+1, t+1)$, we pick up an edge $e_i \in F$, find a cycle C_i of length at most $t+2$ that contains e_i, and remove $E(C_i)$ from F. Then, define $\mathcal{C} := \{C_1, C_2, \dots, C_{N(t+2)f_1(k+t+1,t+1)}\}$.

By Proposition 4 and Lemmas 5 and 7, there always exists a set $E' \subseteq \bigcup_{C \in \mathcal{C}} E(C)$ with $|E'| = k$ such that $H = (V, E \setminus E')$ is an additive t-spanner of G. Furthermore, such E' can be found in $O(((t+2)|\mathcal{C}|)^k |V||E|) = O(f_4(t,k)^k |V||E|)$ time by checking all the edge sets of size k in $\bigcup_{C \in \mathcal{C}} E(C)$. Note that it will be possible to improve the running time of this part by following the proofs of Proposition 4 and Lemmas 5 and 7. However, we do not do it in this paper, because it does not improve the total running time.

Overall, our algorithm solves the Parameterized Minimum Additive t-Spanner Problem in $O(f_4(t,k)^k |V||E|) = (t+1)^{O(k^2+tk)} |V||E|$ time, and hence we obtain Theorem 1. The entire algorithm is shown in Algorithm 2.

4 Extension to (α, β)-Spanners

In this section, we extend the argument in the previous section to (α, β)-spanners and give a proof of Theorem 2.

Let $t := \lfloor \alpha + \beta \rfloor - 1$. We compute the set $F \subseteq E$ of all edges contained in cycles of length at most $t + 2 = \lfloor \alpha + \beta \rfloor + 1$. If $H = (V, E \setminus E')$ is an (α, β)-spanner of G for $E' \subseteq E$, then $\mathrm{dist}_H(u,v) \leq \alpha \cdot \mathrm{dist}_G(u,v) + \beta \leq \alpha + \beta$ for each $uv \in E'$. By integrality, $\mathrm{dist}_H(u,v) \leq \lfloor \alpha + \beta \rfloor$ for each $uv \in E'$, which shows that $E' \subseteq F$ holds. This implies that the problem is trivial if $t = 0$. Thus, we consider the case when $t \geq 1$ and define $f_4(t,k)$ as in Section 3.5. If $|F| \leq f_4(t,k)$, then we can solve the Parameterized Minimum (α, β)-Spanner Problem in $O(f_4(t,k)^k |V||E|)$ time by checking whether $H = (V, E \setminus E')$ is an (α, β)-spanner of G or not for every subset E' of F with $|E'| = k$.

Otherwise, by the argument in Section 3.5, in $O(f_4(t,k)^k |V||E|)$ time, we can find an edge set E' with $|E'| = k$ such that $H = (V, E \setminus E')$ is an additive t-spanner. Then, H is also an (α, β)-spanner, because

$$\begin{aligned}
\mathrm{dist}_H(u,v) &\leq \mathrm{dist}_G(u,v) + t \leq (\mathrm{dist}_G(u,v) - 1) + \alpha + \beta \\
&\leq \alpha \cdot (\mathrm{dist}_G(u,v) - 1) + \alpha + \beta = \alpha \cdot \mathrm{dist}_G(u,v) + \beta
\end{aligned}$$

for every pair of vertices u and v. Therefore, it suffices to return the obtained set E'. This completes the proof of Theorem 2.

Algorithm 2 Entire Algorithm.

Input : A graph $G = (V, E)$
Output : An edge set $E' \subseteq E$ with $|E'| = k$ such that $H = (V, E \setminus E')$ is an additive t-spanner (or conclude that such E' does not exist)

Compute $F := \{e \in E \mid e$ is contained in some cycle of length at most $t + 2\}$
if $|F| \leq f_4(t, k)$ **then**
  for *each* $E' \subseteq F$ *with* $|E'| = k$ **do**
    if $H = (V, E \setminus E')$ *is an additive* t*-spanner of* G **then**
      Return E'
    end
  end
  Conclude that such E' does not exist
end
else
  for $i = 1, 2, \ldots, N(t + 2)f_1(k + t + 1, t + 1)$ **do**
    Find a cycle C_i of length at most $t + 2$ that contains $e_i \in F$
    $F := F \setminus E(C_i)$
  end
  $\mathcal{C} := \{C_1, C_2, \ldots, C_{N(t+2)f_1(k+t+1,t+1)}\}$
  for *each* $E' \subseteq \bigcup_{C \in \mathcal{C}} E(C)$ *with* $|E'| = k$ **do**
    if $H = (V, E \setminus E')$ *is an additive* t*-spanner of* G **then**
      Return E'
    end
  end
end

5 Conclusion

In this paper, we studied the Minimum Additive t-Spanner Problem from the viewpoint of fixed-parameter tractability. We formulated a parameterized version of the Minimum Additive t-Spanner Problem in which the number of removed edges is regarded as a parameter, and gave a fixed-parameter algorithm for it. We also extended our result to the Minimum (α, β)-Spanner Problem.

As described in the last paragraph in Section 1.1, handling the Minimum Additive t-Spanner Problem is much harder than the Minimum Multiplicative t-Spanner Problem, because we have to care about global properties of graphs. Since only few results were previously known for the Minimum Additive t-Spanner Problem, this work may be a starting point for further research on the problem.

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