Published for SISSA by Springer

Received: *August 14, 2018*
Revised: *December 4, 2018*
Accepted: *January 10, 2019*
Published: *January 14, 2019*

JHEP01(2019)117

L_∞ algebras and tensor hierarchies in Exceptional Field Theory and Gauged Supergravity

Yago Cagnacci, Tomas Codina and Diego Marques

Instituto de Astronomía y Física del Espacio (CONICET-UBA), Buenos Aires, Argentina

E-mail: yago@iafe.uba.ar, tcodina@iafe.uba.ar, diegomarques@iafe.uba.ar

Abstract: We show how the gauge and field structure of the tensor hierarchies in Double and $E_{7(7)}$ Exceptional Field Theory fits into L_∞ algebras. Special attention is paid to redefinitions, the role of covariantly constrained fields and intertwiners. The results are connected to Gauged Supergravities through generalized Scherk-Schwarz reductions. We find that certain gauging-dependent parameters generate trivial gauge transformations, giving rise to novel symmetries for symmetries that are absent in their ungauged counterparts.

Keywords: Extended Supersymmetry, String Duality

ArXiv ePrint: 1807.06028

https://doi.org/10.1007/JHEP01(2019)117

Contents

1 Introduction

While the homotopy associative A_∞ algebra [1, 2] is the mathematical structure underlying the classical open string field theory sector, closed string field theory is organized by a homotopy Lie algebra, an L_∞ algebra whose axioms and identities were given in [3]–[4]. These algebras feature in a vast range of field theories [5], including consistent truncations of closed string field theory [6], higher-spin theories [7] and gauge theories [8–11]. Given this context it was recently suggested in [12] that L_∞ algebras provide a classification of perturbative gauge invariant classical field theories. This expectation was supported in that

JHEP01(2019)117

paper by a number additional examples of field theories fitting into L_∞ algebras: Chern-Simons theories, Einstein gravity, etc. Later on the scope was even expanded [13–16]. More aligned with our plan is the fact that Double Field Theory (DFT) [17, 18]–[19] also enjoys an underlying L_∞ structure: Courant algebroids can be cast in this language [20] as well as their duality covariant counterparts [21]–[22, 23], but more generally the full interacting theory exhibits this structure [12].

The explicit relation between the elements appearing in an L_∞ algebra and those in field theories has been systematized in [12]. The fields, gauge transformations and equations of motion belong to distinct graded subspaces in the algebra. Certain products can be read from the gauge transformations of the fields, the equations of motion, the closure identities, etc. Other products are then chosen to satisfy the L_∞ identities, which eventually force the inclusion of additional graded subspaces.

This paper is devoted to discuss how tensor hierarchy [24] algebras fit into L_∞ algebras.[1] We consider a number of representative examples in four space-time dimensions, such as the tensor hierarchy entering the Kaluza-Klein formulation of DFT (KK-DFT) [39], the one in $E_{7(7)}$ Exceptional Field Theory (EFT) [40, 41], and those of gauged supergravities (in particular half-maximal [42], maximal [43, 44] and we also give a general discussion based on [45]). For general reviews see [46–50].
A number of caveats are in order:

- We will only discuss the tensor hierarchy sector, namely p-form fields, and ignore other fields such as the graviton and scalars. The reason for this is that the metric must be invertible, and in duality covariant theories the scalars are grouped into a group-valued tensor, so they necessarily involve a background field expansion making the analysis cumbersome. Considering the tensor hierarchy fields only involves finite expansions and so can be dealt with exactly and non-iteratively. It is important to emphasize that the products and identities involving field perturbations of the metric and scalars will be ignored, so those considered here are only a subset of the full story.

- Tensor hierarchies are saturated by space-time dimensionality, namely, in n space-time dimensions they can only include up to n-forms. However, the hierarchy can be projected to end at any given level $\leq n$, forming a subalgebra that closes exactly. In some cases the tower of p-forms ends before space-time saturation, this is the case of KK-DFT, where the tower has a single unprojected two-form and the tensor hierarchy algebra closes exactly. Other cases, like in EFT and some gauged supergravities the hierarchy is space-time saturated, but only a projection of it (via intertwiners) is dynamical and enters the action. Democratic formulations including all p-forms in the action are possible [51]. All these situations will be discussed here.

- We will mostly focus on L_∞^{gauge} or $L_\infty^{\text{gauge+fields}}$, subalgebras of the full L_∞^{full} algebra that includes interactions and dynamics. There will be exceptions such as when analyzing the KK-DFT tensor hierarchy, where we discuss the full L_∞^{full} algebra including the

[1] Interesting papers with similar goals are [25–29]. Tensor hierarchies are also discussed in [30–35] and in EFT in [36–38].

JHEP01(2019)117

equations of motion. Crucial to our analysis will be a theorem stating sufficient criteria for an algebra to fit into L_∞^{gauge} [52], and a small extension introduced here to the $L_\infty^{\text{gauge+fields}}$ case.

Let us briefly anticipate some points to be discussed in the paper:

- It is well known that Double and Exceptional Field Theories are restricted by a section condition, and as a consequence the gauge transformations admit non-vanishing trivial parameters. We show that something similar happens in gauged supergravities: the quadratic constraints imply that the gauge transformations admit non-vanishing trivial parameters that depend on the gaugings and are then absent in the ungauged theories. We will show how these two facts are connected through generalized Scherk-Schwarz compactifications.

- It is a common saying that in Double and Exceptional Field Theories the gauge algebra closes with respect to a bracket whose Jacobiator is a trivial parameter. This is only true for a specific choice of variables. While field redefinitions leave the gauge algebra intact, parameter redefinitions change the brackets and their field dependence in such a way that the Jacobiator is no longer a trivial parameter. Any two sets of variables are of course equivalent, and must correspond to an isomorphism of the L_∞ algebra, so the fact that the gauge algebra is an L_∞ algebra remains intact, but the way in which the products are defined changes. In Double and Exceptional Field Theories, and in gauged supergravities there are two interesting set of parameters. One in which the gauge transformations of the fields can be cast in a manifestly covariant form (with respect to generalized diffeomorphisms and gauge transformations respectively): this is the usual set chosen in ExFT and gauged supergravities. For this set the brackets and trivial parameters are field dependent, and the Jacobiator in not a trivial parameter. There is a different set of parameters, related to the previous one through field dependent redefinitions, for which the brackets and trivial parameters do not depend on the fields, and the Jacobiator is a trivial parameter.

- When dynamics is taken into account — so as to obtain the L_∞^{full} — the tensor hierarchy couples to the other fields through the equations of motion. It is then impossible to disentangle the tensor hierarchy from the rest of the theory and, even if the perturbations of the metric and scalars are ignored, the L_∞ products and identities must depend on their background values. We will show this explicitly with some examples.

- Some ExFT feature the presence of "covariantly constrained fields" satisfying section-type conditions. These are required by closure of the algebra and covariance of the field strengths, and so play a crucial role in the L_∞ analysis. They are also crucial to establish how to extend the tensor hierarchy to higher forms, which also require new covariantly constrained fields.

JHEP01(2019)117

The paper is organized as follows. Section 2 is a self-contained review of L_∞ algebras, closely following [12]. We review sufficient criteria [52] for a theory to fit into L_∞^{gauge}, and slightly generalized the theorem to the case $L_\infty^{\text{gauge+fields}}$. In section 3 we review the KK-formulation of DFT and show how the tensor hierarchy sector embeds into L_∞^{full}. The gauge algebra of the tensor hierarchy in $E_{7(7)}$ EFT is discussed in section 4, together with its embedding in $L_\infty^{\text{gauge+fields}}$ algebras. Finally in section 5 we show how tensor hierarchies in gauged supergravities fit into L_∞ algebras. The appendices contain complementary computations.

2 L_∞ algebra and its field theory

In this section we give a brief self-contained review of L_∞ algebras in the ℓ-picture and their relation to field theories, their gauge sector and dynamics, as presented in [12]. We refer to that paper and references therein for more details.

To have an L_∞ algebra we first define a vector graded space X which is the direct sum of vector spaces X_n, each of which has degree n

$$X = \bigoplus_n X_n\,, \quad n \in \mathbb{Z}\,. \tag{2.1}$$

We will denote by x an element of X with definite degree, *i.e*, $x \in X_p$ for some fixed p. Next we introduce multilinear products ℓ_k

$$\ell_k : X^{\otimes k} \to X\,, \tag{2.2}$$

with intrinsic degree $k-2$, meaning that when acting on elements x_i we obtain

$$\deg(\ell_k(x_1, x_2, \ldots, x_k)) = k - 2 + \sum_{i=1}^{k} \deg(x_i)\,. \tag{2.3}$$

It is useful to note that ℓ_1 acts as a mapping as follows $\ell_1 : X_p \to X_{p-1}$. The products are graded commutative and obey the L_∞ relations, which constitute a deformation of Jacobi identities and will be introduced soon. The graded commutative property means that a sign might appear when exchanging the arguments. For a permutation σ of k labels we have

$$\ell_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = (-1)^\sigma \epsilon(\sigma; x)\, \ell_k(x_1, \ldots, x_k)\,. \tag{2.4}$$

The $(-1)^\sigma$ factor gives a plus or minus sign if the permutation is even or odd, respectively. The $\epsilon(\sigma; x)$ factor is the Koszul sign. We first take a graded commutative algebra $\Lambda(x_1, x_2, \cdots)$ with

$$x_i \wedge x_j = (-1)^{\deg(x_i)\deg(x_j)}\, x_j \wedge x_i\,, \quad \forall i, j\,, \tag{2.5}$$

and then define the Koszul sign for a general permutation as

$$x_1 \wedge \ldots \wedge x_k = \epsilon(\sigma; x)\; x_{\sigma(1)} \wedge \ldots \wedge\; x_{\sigma(k)}\,. \tag{2.6}$$

For example, for ℓ_2 we get

$$\ell_2(x_1, x_2) = (-1)^{1 + x_1 x_2}\, \ell_2(x_2, x_1)\,, \tag{2.7}$$

JHEP01(2019)117

where we introduced the notation

$$(-1)^{\deg(x_i)\deg(x_j)} \equiv (-1)^{x_i x_j}\,, \tag{2.8}$$

that is, within exponents the x_i always refer to its degree.

The L_∞ relations are labeled by a positive integer n given by the number of inputs. Explicitly they take the general form

$$\sum_{i+j=n+1} (-1)^{i(j-1)} \sum_{\sigma} (-1)^{\sigma} \epsilon(\sigma; x)\, \ell_j \left(\ell_i(x_{\sigma(1)}\,, \dots\,, x_{\sigma(i)})\,, x_{\sigma(i+1)}, \dots\, x_{\sigma(n)} \right) = 0\,. \tag{2.9}$$

The sum over σ is a sum over "unshuffles", it includes only the terms which satisfy

$$\sigma(1) < \cdots < \sigma(i)\,, \qquad \sigma(i+1) < \cdots < \sigma(n)\,. \tag{2.10}$$

As a shorthand notation, it is common to write these relations as

$$\sum_{i+j=n+1} (-1)^{i(j-1)} \ell_j\, \ell_i = 0\,, \tag{2.11}$$

such that

$$n = 1 \qquad 0 = \ell_1 \ell_1 \tag{2.12}$$

$$n = 2 \qquad 0 = \ell_1 \ell_2 - \ell_2 \ell_1 \tag{2.13}$$

$$n = 3 \qquad 0 = \ell_1 \ell_3 + \ell_2 \ell_2 + \ell_3 \ell_1 \tag{2.14}$$

$$n = 4 \qquad 0 = \ell_1 \ell_4 - \ell_2 \ell_3 + \ell_3 \ell_2 - \ell_4 \ell_1\,, \ \dots \tag{2.15}$$

For $n = 1$ we have

$$\ell_1(\ell_1(x)) = 0\,. \tag{2.16}$$

This means that ℓ_1 is a nilpotent operator, sometimes called Q as the BRST operator.

The $n = 2$ identity is

$$\ell_1(\ell_2(x_1, x_2)) = \ell_2(\ell_1(x_1), x_2) + (-1)^{x_1} \ell_2(x_1, \ell_1(x_2))\,. \tag{2.17}$$

It implies that ℓ_1 acts as a (graded) distributive operator on the arguments of ℓ_2.

For $n = 3$ we have

$$\begin{aligned} 0 = {} & \ell_2(\ell_2(x_1, x_2), x_3) + (-1)^{(x_1+x_2)x_3} \ell_2(\ell_2(x_3, x_1), x_2) + (-1)^{(x_2+x_3)x_1} \ell_2(\ell_2(x_2, x_3), x_1) \\ & + \ell_1(\ell_3(x_1, x_2, x_3)) + \ell_3(\ell_1(x_1), x_2, x_3) + (-1)^{x_1} \ell_3(x_1, \ell_1(x_2), x_3) \\ & + (-1)^{x_1+x_2} \ell_3(x_1, x_2, \ell_1(x_3))\,. \end{aligned} \tag{2.18}$$

In the following we will see that when the arguments of ℓ_2 are the gauge parameters of some field theory, it will be related to the bracket of the gauge algebra. As such, the first line above will become the Jacobiator. For this reason, the last line characterizes a *deformation* of a strict Lie algebra. Since the Jacobi identity holds modulo a BRST exact term $(\ell_1 \ell_3 + \ell_3 \ell_1)$, in the language of homological algebra ℓ_3 is a chain homotopy, so the

JHEP01(2019)117

X_0	X_{-1}	X_{-2}
Gauge Parameters ζ	Fields Ψ	E ($\ni EOM\mathcal{F}$)

Table 1. Graded subspaces and the elements they contain.

saying is that ℓ_2 satisfies the Jacobi identity "up to homotopy", or that this is an homotopy Lie algebra [4].

We also display the $n = 4$ identity, as will be needed later

$$\begin{aligned}
0 = {} & \ell_1(\,\ell_4(x_1,x_2,x_3,x_4)) \\
& - \ell_2(\,\ell_3(x_1,x_2,x_3),x_4) + (-1)^{x_3x_4}\,\ell_2(\,\ell_3(x_1,x_2,x_4),x_3) \\
& + (-1)^{(1+x_1)x_2}\ell_2(x_2,\ell_3(x_1,x_3,x_4)) - (-1)^{x_1}\ell_2(x_1,\ell_3(x_2,x_3,x_4)) \\
& + \ell_3(\ell_2(x_1,x_2),x_3,x_4) + (-1)^{1+x_2x_3}\,\ell_3(\ell_2(x_1,x_3),x_2,x_4) \\
& + (-1)^{x_4(x_2+x_3)}\ell_3(\ell_2(x_1,x_4),x_2,x_3) - \ell_3(x_1,\ell_2(x_2,x_3),x_4) \\
& + (-1)^{x_3x_4}\ell_3(x_1,\ell_2(x_2,x_4),x_3) + \ell_3(x_1,x_2,\ell_2(x_3,x_4)) \\
& - \ell_4(\ell_1(x_1),x_2,x_3,x_4) - (-1)^{x_1}\ell_4(x_1,\ell_1(x_2),x_3,x_4) \\
& - (-1)^{x_1+x_2}\ell_4(x_1,x_2,\ell_1(x_3),x_4) - (-1)^{x_1+x_2+x_4}\ell_4(x_1,x_2,x_3,\ell_1(x_4))\,.
\end{aligned} \tag{2.19}$$

Notice that, in the same sense that for $n = 3$ the "Jacobiator" $\ell_2\ell_2$ vanishes "up to homotopy", the same is true in $n = 4$ for $\ell_2\ell_3 - \ell_3\ell_2$.

After this brief self-contained introduction to L_∞ algebras, we now show how to relate these results with field theories [12]. In the first place, we must assign a given degree p to gauge parameters, fields, EOM, etc. and so specify to what vector subspace X_p they belong. The general rule is to take the gauge parameters ζ as vectors of degree $p = 0$, the dynamical fields Ψ as vectors of degree $p = -1$ and the EOM $\mathcal{F}$ as vectors of degree $p = -2$. The EOM's will form a subset of X_{-2} and, when needed, we will refer to a general element of X_{-2} as "E". If the field theory exhibits symmetries for symmetries, this picture is incomplete and requires an extra graded subspace X_1 with elements parameterizing such ambiguity, and possibly further graded subspaces X_2, X_3, ... Here, ζ, Ψ and $\mathcal{F}$ stand for direct sums in case there are more than one of each. After these assignments, one can readily read some brackets from certain equations in the field theory. Lets see some examples.

- The gauge transformations *define* the brackets $\ell_{n+1}(\zeta,\Psi^n)$ as follows

$$\delta_\zeta\Psi = \sum_{n\geq 0}\frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\,\ell_{n+1}(\zeta,\Psi^n) = \ell_1(\zeta) + \ell_2(\zeta,\Psi) - \frac{1}{2}\ell_3(\zeta,\Psi^2) - \dots\,, \tag{2.20}$$

  where we use the exponential notation for short

$$\Psi^k = \underbrace{\Psi,\dots,\Psi}_{k\text{ times}}\,. \tag{2.21}$$

  It can be checked that $\delta_\zeta\Psi$ so defined consistently belongs to the same vector subspace than the fields, namely X_{-1}.

JHEP01(2019)117

- The EOM instead *define* the $\ell_n(\Psi^n)$ brackets as follows[2]

$$\mathcal{F}(\Psi) = \sum_{n=1}^{\infty} \frac{(-1)^{\frac{n(n-1)}{2}}}{n!} \ell_n(\Psi^n) \;=\; \ell_1(\Psi) - \frac{1}{2}\ell_2(\Psi^2) - \frac{1}{3!}\ell_3(\Psi^3) + \frac{1}{4!}\ell_4(\Psi^4) + \cdots . \tag{2.24}$$

- The study of the gauge algebra leads to interesting features. Taking the commutator of two gauge transformations and using the L_∞ identities one finds

$$[\delta_{\zeta_1}, \delta_{\zeta_2}]\Psi = \delta_{-\mathbf{C}(\zeta_1,\zeta_2,\mathbf{\Psi})}\Psi + \delta^T_{\zeta_1,\zeta_2}\,, \Psi. \tag{2.25}$$

with

$$\mathbf{C}(\zeta_1, \zeta_2, \mathbf{\Psi}) \equiv \sum_{n\geq 0} \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\; \ell_{n+2}(\zeta_1, \zeta_2, \Psi^n)\,, \tag{2.26}$$

and

$$\delta^T_{\zeta_1,\zeta_2}\Psi \equiv \sum_{n\geq 0} \frac{1}{n!}(-1)^{\frac{(n-2)(n-3)}{2}}\; \ell_{n+3}(\zeta_1, \zeta_2, \mathcal{F}, \Psi^n)\,. \tag{2.27}$$

The supralabel in $\delta^T_{\zeta_1,\zeta_2}$ stands for "trivial" as it is a term that vanishes on-shell. Suppose we want to rewrite in the L_∞ framework some gauge algebra. If it closes off-shell, then $\delta^T_{\zeta_1,\zeta_2}\Psi = 0$ and we obtain a gauge algebra that closes under the bracket given by $\mathbf{C}(\zeta_1, \zeta_2, \mathbf{\Psi})$, which might be field dependent. If furthermore it turns out to be field independent, then we get

$$[\delta_{\zeta_1}, \delta_{\zeta_2}]\Psi = \delta_{-\ell_2(\zeta_1,\zeta_2)}\Psi\,, \qquad \ell_{n+2}(\zeta_1,\, \zeta_2,\, \Psi^n) = 0\,, \tag{2.28}$$

and the closure bracket is simply given by ℓ_2, as anticipated under (2.18).

- With respect to the gauge algebra, one can also consider the *gauge Jacobiator* $\mathcal{J}$, given by

$$\mathcal{J}(\zeta_1, \zeta_2, \zeta_3) \;\equiv\; \sum_{\text{cyc}} \big[\delta_{\zeta_3}\,,\, [\delta_{\zeta_2}\,,\, \delta_{\zeta_1}]\big] = 0\,. \tag{2.29}$$

This vanishes by definition, as can be seen by expanding the terms and acting on a probe field from right to left, i.e. $\delta_{\zeta_1}\delta_{\zeta_2}\delta_{\zeta_3}\Psi = \delta_{\zeta_1}\left(\delta_{\zeta_2}\left(\delta_{\zeta_3}\Psi\right)\right)$. Using the L_∞ relations

[2] As a side remark we point out that it might also be possible to go a step further and define an action from which the EOM are obtained, study the gauge algebra, the covariance of the EOM under gauge transformations, etc. For this one should be able to define an inner product $\langle x_1, x_2\rangle$ with the properties

$$\begin{aligned} \langle x_1, x_2\rangle &= (-1)^{x_1 x_2}\langle x_2, x_1\rangle\,, \\ \langle x,\, \ell_n(x_1, \ldots x_n)\rangle &= (-1)^{x x_1 + 1}\langle x_1,\, \ell_n(x, x_2 \ldots x_n)\rangle\,. \end{aligned} \tag{2.22}$$

The first one accounts for its graded symmetry, and the second one implies that it is a multilinear graded-commutative function of all the arguments. With this inner product the action is given by

$$S \;=\; \sum_{n=1}^{\infty} \frac{(-1)^{\frac{n(n-1)}{2}}}{(n+1)!}\,\langle \Psi, \ell_n(\Psi^n)\rangle\,, \tag{2.23}$$

and it can be checked that the EOM are obtained from it.

JHEP01(2019)117

Product	From	Equation
$\ell_n(\Psi^n)$	Equations of motion	(2.24)
$\ell_{n+1}(\zeta,\,\Psi)$	Field gauge transformations	(2.20)
$\ell_{n+2}(\zeta^2,\,\Psi)$	Closure	(2.26)
$\ell_{n+3}(\zeta^3,\,\Psi)$	Jacobiator	(2.31)
$\ell_{n+2}(\zeta,\,\mathcal{F},\,\Psi^n)$	EOM gauge transformations	(2.34)
$\ell_{n+3}(\zeta^2,\,\mathcal{F},\,\Psi^n)$	On-shell closure	(2.27)

Table 2. Products that can be read from kinematic and dynamical equations in a field theory.

over (2.29) one finds

$$\sum_{\text{cyc}}\big[\delta_{\zeta_3}\,,\,[\delta_{\zeta_2}\,,\,\delta_{\zeta_1}]\big]\,\Psi\,=\,-\delta_{Q'\chi}\,\Psi\,-\,\delta^T_{\chi}\Psi\,, \tag{2.30}$$

with $\chi\in X_1$ given by

$$\chi=\sum_{n\geq 0}\frac{1}{n!}(-1)^{\frac{(n-2)(n-3)}{2}}\;\ell_{n+3}(\zeta_1,\zeta_2,\zeta_3,\Psi^n)\,, \tag{2.31}$$

and

$$Q'\chi=\sum_{n\geq 0}\frac{1}{n!}(-1)^{\frac{n(n+1)}{2}}\;\ell_{n+1}(\chi,\Psi^n) \tag{2.32}$$

$$\delta^T_{\chi}\Psi=\sum_{n\geq 0}\frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\;\ell_{n+2}(\mathcal{F},\chi,\Psi^n)\,. \tag{2.33}$$

It can be shown that the r.h.s. of (2.30) vanishes identically when using the L_∞ relations, as it must be.

- It will be useful in what follows to consider the gauge transformation of the equations of motion, which gives

$$\delta_\zeta\mathcal{F}=\sum_{n\geq 0}\frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\;\ell_{n+2}(\zeta_1,\mathcal{F},\mathbf{\Psi})=\ell_2(\zeta,\mathcal{F})+\ell_3(\zeta,\mathcal{F},\Psi)-\frac{1}{2}\ell_4(\zeta,\mathcal{F},\Psi^2)+\ldots \tag{2.34}$$

We collect in a table 2 some of the products that can be directly read from standard expressions in a field theory. To see if a particular theory can be written in this framework one has to determine all ℓ_n products acting over all vectors (fields, gauge parameters and equations of motion) and then check all L_∞ relations. It could also be possible that the vector subspaces defined so far are not enough (namely that table 1 is incomplete) and one has to consider additional ones, as it happens for example in DFT where one has to add a vector subspace X_1. The fundamental formulas are (2.20) and (2.24). Knowing the specific form of the gauge transformations of the particular theory we are interested in, we can immediately read off the products $\ell_{n+1}(\zeta,\Psi^n)$ and $\ell_n(\Psi^n)$ respectively. In principle with

JHEP01(2019)117

these products and the L_∞ relations we can determine all products. However, this can be a tedious work. At this point formulas like (2.25), (2.30) and (2.34) come to our rescue. They are a consequence of the previous ones plus the L_∞ relations but they are also important as they allow to read off certain products immediately. We insist however that whatever the route to identify products is, in the end all L_∞ identities must be checked explicitly.

Take for example the case of DFT [21]. The gauge algebra closes under the C-bracket, which does not depend on fields

$$[\delta_{\zeta_1}, \delta_{\zeta_2}]\,\Psi = \delta_{-[\zeta_1,\zeta_2]_{(C)}}\Psi\,. \tag{2.35}$$

Comparing this expression with (2.28) we can readily make the identification

$$\ell_2(\zeta_1,\zeta_2) = [\zeta_1,\zeta_2]_{(C)}\ . \tag{2.36}$$

Now we could evaluate the identity for three gauge parameters (2.18). We have pointed out that the last line of (2.18) in this particular case gives the Jacobiator, which in the case of DFT is famously given by

$$J(\zeta_1,\zeta_2,\zeta_3) \equiv 3\big[\,[\zeta_{[1},\zeta_2]\,,\zeta_{3]}\big]\ = \partial\, N(\zeta_1,\zeta_2,\zeta_3)\,, \tag{2.37}$$

with the so-called Nijenhuis scalar defined by

$$N(\zeta_1,\zeta_2,\zeta_3) \equiv \frac{1}{2}\,[\zeta_{[1},\zeta_2]_{(C)}\cdot\zeta_{3]}\,. \tag{2.38}$$

After setting $\ell_3(\ell_1(\zeta_i),\zeta_j,\zeta_k)\equiv 0$ we can identify

$$\begin{aligned}\ell_3(\zeta_1,\zeta_2,\zeta_3) &= -N(\zeta_1,\zeta_2,\zeta_3)\,,\\ \ell_1(N(\zeta_1,\zeta_2,\zeta_3)) &= \partial N(\zeta_1,\zeta_2,\zeta_3)\,.\end{aligned} \tag{2.39}$$

These two last identifications done after evaluating a particular L_∞ identity could also be obtained by directly comparing the DFT gauge Jacobiator with (2.30)

$$\sum_{\rm cyc}\big[\delta_{\Lambda_3}\,,\,[\delta_{\Lambda_2}\,,\,\delta_{\Lambda_1}]\big]\,\Psi = \delta_{\partial N(\zeta_1,\zeta_2,\zeta_3)}\Psi\,, \tag{2.40}$$

which, of course, is zero when evaluating the gauge variation of the field after using the strong constraint.

A clarification is in order. We have said that we need to determine all products acting over all vectors and that some equations make this possible, for example (2.24) leads to the knowledge of $\ell_n(\Psi^n)$. However, we would like to know not only the diagonal part of ℓ_n acting on fields, but $\ell_n(\Psi_1,\Psi_2,\ldots,\Psi_n)$ with non diagonal entries. This can be achieved with the help of polarization identities. For the simplest case, ℓ_2, we get

$$2\ell_2(\Psi_1,\Psi_2) = \ell_2(\Psi_1+\Psi_2,\Psi_1+\Psi_2) - \ell_2(\Psi_1,\Psi_1) - \ell_2(\Psi_2,\Psi_2)\,, \tag{2.41}$$

and this expression can be generalized to any ℓ_n.

After we have identified the products, we must check the L_∞ identities. However, there is a large subset of identities which are automatically fulfilled due to the general

JHEP01(2019)117

construction of an L_∞ field theory and it is not necessary to do an explicit check. As shown in [12], the identities acting over a list of fields $(\Psi_1, \ldots, \Psi_n)$ hold true provided we define

$$\ell_{n+1}(E, \Psi_1, \ldots, \Psi_n) = 0\,, \quad n \geq 0, \quad E \in X_2\,. \tag{2.42}$$

The identities acting over a list $(\zeta_1, \zeta_2, \Psi_1, \ldots, \Psi_n)$, and $(\zeta, \Psi_1, \ldots, \Psi_n)$ with $n \geq 1$ also hold true. These identities come from closure of the gauge algebra over fields and the gauge transformations of the equations of motion, respectively.

2.1 Sufficient criteria for the gauge sector to fit into L_∞

In [52] a theorem was presented, stating sufficient conditions for a gauge algebra to lie within an L_∞^{gauge} structure. Since we will refer to it often, we dedicate a separate subsection to review it briefly. The theorem only refers to the gauge algebra, not the complete field theory. Given an algebra $(V, [\,\cdot,\cdot\,])$ with bilinear antisymmetric 2-bracket and a vector space U with a linear map $\mathcal{D} : U \to V$ satisfying

$$[\mathrm{Im}(\mathcal{D}), V] \;\subset\; \mathrm{Im}(\mathcal{D})\,, \tag{2.43}$$

and

$$\forall v_1, v_2, v_3 \in V : \;\; \mathrm{Jac}(v_1, v_2, v_3) \;\in\; \mathrm{Im}(\mathcal{D})\,, \tag{2.44}$$

where $\mathrm{Im}(\mathcal{D})$ denotes the image of $\mathcal{D}$, there exists a 3-term L_∞ structure with $\ell_2(v, w) = [v, w]$ on the graded vector space with

$$X_2 \xrightarrow{\ell_1 = \iota} X_1 \xrightarrow{\ell_1 = \mathcal{D}} X_0\,, \tag{2.45}$$

where $X_0 = V$, $X_1 = U$, $X_2 = \mathrm{Ker}(\mathcal{D})$, $\mathrm{Ker}(\mathcal{D})$ denotes the kernel of $\mathcal{D}$ and ι denotes the inclusion of $\mathrm{Ker}(\mathcal{D})$ into U.

In a more physical language, the hypothesis of this theorem are roughly

- The field transformations admit trivial parameters. The gauge algebra closes with respect to a given bracket. The bracket between a trivial parameter and a generic one is itself a trivial parameter.
- The Jacobiator computed from the bracket is a trivial parameter.

When these hypothesis are met, the theorem states that the gauge algebra fits into an L_∞ structure, with a few non-vanishing products written in equation (4.26) in [52].

We will show that all the cases considered in this paper satisfy these hypothesis. However, this will prove not to be always a simple task: the form of the brackets and the trivial parameters is highly sensitive to redefinitions, and the criteria outlined above is only useful when a specific set of parameters is considered. Meeting the criteria involves finding an appropriate set of variables, and we will comment on what happens when other sets are considered.

In the next subsection we point out that the consequences of these hypothesis are in fact a little broader: when the hypothesis are met the algebra fits into $L_\infty^{\text{gauge+fields}}$.

JHEP01(2019)117

2.2 Sufficient criteria for the gauge + field sectors to fit into L_∞

The theorem of the previous subsection 2.1 can be extended so as to consider in addition the graded space of fields, in what was called in [12] $L_\infty^{\text{gauge+fields}}$. To be more precise, we will take the fields to belong to the graded subspace X_{-1} and consider their gauge transformations given by (2.20). We will not consider their EOM, so all the products in (2.24), namely $\ell_n(\Psi^n)$, are taken to be null and thus there is no need for a graded subspace X_{-2}. Under the same conditions, we show that the algebra is in fact that of $L_\infty^{\text{gauge+fields}}$ and the only additional non-vanish products (apart from those in equation (4.26) in [52]) are those obtained from the gauge transformations of the fields (2.20), namely

$$\delta_\zeta \Psi = \sum_{n\geq 0} \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\, \ell_{n+1}(\zeta, \Psi^n) = \ell_1(\zeta) + \ell_2(\zeta, \Psi) - \frac{1}{2}\ell_3(\zeta, \Psi^2) - \dots . \tag{2.46}$$

We have to deal now with products involving fields $\Psi \in X_{-1}$ and other elements from other subspaces. We can consider the L_∞ relations acting over the following lists:

- Two $\zeta' s \in X_0$ and any number $n \geq 0$ of $\Psi' s \in X_{-1}$: $(\zeta_1, \zeta_2, \Psi_1, \Psi_2, \dots)$.
- One $\zeta \in X_0$ and any number $n > 0$ of $\Psi' s \in X_{-1}$: $(\zeta, \Psi_1, \Psi_2, \dots)$.
- Only one $\zeta \in X_0$. There is no $\Psi \in X_{-1}$ in this list, but we have to consider it because the product ℓ_1 maps to the subspace of fields: $\ell_1(\zeta) \in X_{-1}$.
- At least one $c \in X_2$ (and any other vector on the list, including at least one $\Psi \in X_{-1}$): $(c, \dots)$.
- At least one $\chi \in X_1$ (and any other vector on the list, including at least one $\Psi \in X_{-1}$): $(\chi, \dots)$.
- Three or more $\zeta' s \in X_0$ and any number $n > 0$ of $\Psi' s \in X_{-1}$: $(\zeta_1, \zeta_2, \zeta_3 \dots, \zeta_n, \Psi_1, \Psi_2, \dots)$.

As shown in [12], the products given by the first case satisfy the L_∞ relations as a consequence of closure of the gauge algebra over fields. The second item is trivially fulfilled once we make the choice $\ell_1(x) = 0$, with $x \in X_{-1}$. The third one gives rise to the identity $\ell_1(\ell_1(\zeta)) = 0$ which is verified trivially for the same reason.

For the other cases, we will construct all possible terms of the identities, which are of the form $\ell_i(\dots \ell_j(\dots))$ and show that they are null or compensate each other. We will make use of the nontrivial products defined in equation (4.26) in [52].

JHEP01(2019)117

At least one $c \in X_2$ (and any other vector on the list, including at least one $\Psi \in X_{-1}$). There are only two nontrivial products involving a c in X_2: $\ell_1(c)$, $\ell_2(c,\zeta)$. First consider that we have at least two $c's$. We would get the products

$$\ell_i(c_1,\Psi,\dots\underbrace{\ell_2(c_2,\zeta)}_{\equiv\tilde{c}\in X_2}) \Rightarrow \ell_i(c_1,\Psi,\dots\tilde{c}) = 0, \tag{2.47}$$

$$\ell_i(c_1,\Psi,\dots\underbrace{\ell_1(c_2)}_{\equiv\chi\in X_1}) \Rightarrow \ell_i(c_1,\Psi,\dots\chi) = 0, \tag{2.48}$$

$$\ell_i(c_1,c_2\dots\ell_j(\dots)) = 0\,. \tag{2.49}$$

For only one c we get

$$\ell_i(\Psi,\dots\underbrace{\ell_1(c)}_{\equiv\chi\in X_1}) \Rightarrow \ell_i(\Psi,\dots\chi) = 0, \tag{2.50}$$

$$\ell_i(\Psi,\dots\underbrace{\ell_2(c,\zeta)}_{\equiv\tilde{c}\in X_2}) \Rightarrow \ell_i(\Psi,\dots\tilde{c}) = 0, \tag{2.51}$$

$$\ell_i(c,\ \underbrace{\ell_j(\Psi\dots)}_{\text{must be in } X_0.}\ )\,, \tag{2.52}$$

but it is not possible to form a nontrivial $\ell_j(\Psi\dots)\in X_0$, thus the last product is also zero.

At least one $\chi \in X_1$ (and any other vector on the list, including at least one $\Psi \in X_{-1}$). There are only four nontrivial products involving χ: $\ell_1(\chi)$, $\ell_2(\chi,\zeta)$, $\ell_2(\chi_1,\chi_2)$, $\ell_3(\chi,\zeta_1,\zeta_2)$. This means that we can have at most two χ's.

If we have more than two χ's we would necessary have to group at least one with a Ψ inside a product (as we cannot have more than two $\chi's$ together in the same product), which is zero. Take for example the case of three χ's:

$$\ell_i(\chi_1,\chi_2,\dots\underbrace{\ell_j(\chi_3,\Psi\dots)}_{=0}) \qquad \text{or} \qquad \ell_i(\chi_1,\dots\underbrace{\ell_j(\chi_2,\chi_3,\Psi\dots)}_{=0})\,. \tag{2.53}$$

Now take the list $(\chi_1,\chi_2,\dots)$. The possibilities are:

$$\ell_i(\chi_1\dots\underbrace{\ell_j(\chi_2,\Psi\dots)}_{=0}) = 0 \tag{2.54}$$

$$\underbrace{\ell_i(\chi_1,\Psi\dots\ell_j(\chi_2,\dots))}_{=0} = 0 \tag{2.55}$$

$$\ell_i(\Psi\dots\underbrace{\ell_2(\chi_1,\chi_2)}_{\equiv c\in X_2}) = \ell_i(\Psi,\dots c) = 0\,, \tag{2.56}$$

where we have used in the last line that there is no product mixing a $c\in X_2$ and a $\Psi\in X_{-1}$.

Now take the list with only one χ and consider it as an argument of the second product. As we cannot take together a $\Psi\in X_{-1}$ and a $\chi\in X_1$ inside the same product,

JHEP01(2019)117

the possibilities are

$$\ell_i(\Psi\dots\underbrace{\ell_3(\chi,\zeta_1,\zeta_2)}_{\equiv c\in X_2}) = \ell_i(\Psi\dots c) = 0 \tag{2.57}$$

$$\ell_i(\Psi\dots\underbrace{\ell_2(\chi,\zeta)}_{\equiv\tilde{\chi}\in X_1}) = \ell_i(\Psi\dots\tilde{\chi}) = 0 \tag{2.58}$$

$$\ell_i(\Psi\dots\underbrace{\ell_1(\chi)}_{\equiv\zeta_{triv.}\in X_0}) \Rightarrow \ell_{n+1}(\Psi^n,\zeta_{triv.}) = 0\,. \tag{2.59}$$

The last line gives zero, as these products come from the gauge transformations of the fields with a trivial parameter.

Considering that χ is in the outermost product, we get

$$\ell_i(\chi,\ell_j(\Psi\dots)) \Rightarrow \begin{cases}\ell_j(\Psi\dots)\in X_0, & \text{not possible,}\\ \ell_j(\Psi\dots)\in X_1, & \text{not possible}\,.\end{cases} \tag{2.60}$$

$$\ell_i(\chi,\zeta,\ell_j(\Psi\dots)) \Rightarrow \ell_j(\Psi\dots)\in X_0, \quad \text{not possible}\,. \tag{2.61}$$

Three or more $\zeta's \in X_0$ and any number $n > 0$ of $\Psi's \in X_{-1}$. We start by considering three gauge parameters. We first notice that we cannot use the products $\ell_{n+1}(\zeta,\Psi^n)$, which give an element $\tilde{\Psi}\in X_{-1}$. If they were in the second product ℓ_j we would have

$$\ell_i(\zeta_1,\zeta_2,\ell_{n+1}(\zeta,\Psi^n)) = \ell_3(\zeta_1,\zeta_2,\tilde{\Psi}) = 0\,. \tag{2.62}$$

Trying to use them in the first product does not work either. For $\ell_1(\zeta)$ we would need the second product to give an element of X_0:

$$\ell_1(\underbrace{\ell_{n+3}(\zeta_1,\zeta_2,\zeta_3,\Psi^n)}_{\notin X_0})\,. \tag{2.63}$$

For $\ell_2(\zeta,\Psi)$ we would need the second product to give an element of X_0 or X_{-1}:

$$\ell_2(\underbrace{\ell_{n+2}(\zeta_1,\zeta_2,\zeta_3,\Psi^{n-1})}_{\notin X_0},\Psi), \qquad \ell_2(\zeta_1,\underbrace{\ell_{n+2}(\zeta_2,\zeta_3,\Psi^n)}_{\notin X_{-1}})\,. \tag{2.64}$$

For $\ell_3(\zeta,\Psi,\Psi)$ we would need the second product to give an element of X_0 or X_{-1}:

$$\ell_3(\underbrace{\ell_{n+1}(\zeta_1,\zeta_2,\zeta_3,\Psi^{n-2})}_{\notin X_0},\Psi,\Psi), \qquad \ell_3(\zeta_1,\underbrace{\ell_{n+1}(\zeta_2,\zeta_3,\Psi^{n-1})}_{\notin X_{-1}},\Psi)\,. \tag{2.65}$$

Thus, one of the products must be $\ell_3(\zeta_1,\zeta_2,\zeta_3)\equiv\chi\in X_1$. If it was in the second product ℓ_j we would have

$$\ell_i(\dots\ell_j(\dots)) = \ell_{n+1}(\Psi^n,\ell_3(\zeta_1,\zeta_2,\zeta_3)) = \ell_{n+1}(\Psi^n,\chi) = 0\,. \tag{2.66}$$

If it was in the first product, we would need the second one to give a X_0 element to get a nontrivial term, but this is not the case

$$\ell_3(\zeta_1,\zeta_2,\underbrace{\ell_{n+1}(\zeta_3,\Psi^n)}_{\notin X_0})\,. \tag{2.67}$$

Considering more than three gauge parameters and repeating the arguments as before one also finds that the L_∞ identities are fulfilled trivially.

JHEP01(2019)117

3 L_∞ and Double Field Theory

The embedding of DFT into L_∞^{full} was performed in [12], and so there will be no surprises in this section. Here we consider the KK-formulation of DFT [39] with the goal of learning how its tensor hierarchy fits into L_∞. The advantage of this formulation is that it is similar in structure with EFT and gauge supergravities, and so will serve as a platform to extract lessons for future discussions.

3.1 Review of the Kaluza-Klein formulation of Double Field Theory

In this section we review the KK-formulation of DFT [39]. We present in the appendix the relation of this formulation with the generalized metric approach [53]. The fields and gauge parameters depend on external and internal coordinates $(x^\mu\,,\,X^M)$, with μ an external GL(n) index, and M an internal fundamental $O(d,d)$ index. The duality invariant metric η_{MN} and its inverse raise and lower internal indices, and the internal coordinate dependence is restricted by the strong constraint that states that fields and parameters and their products are annihilated by the duality invariant Laplacian

$$\partial_M \otimes \partial^M \cdots = 0\,. \tag{3.1}$$

The fields are

$$g_{\mu\nu}\,,\quad B_{\mu\nu}\,,\quad \phi\,,\quad A_\mu{}^M\,,\quad \mathcal{M}_{MN}\,, \tag{3.2}$$

the scalar matrix $\mathcal{M}_{MN}$ being a duality group-valued constrained field (i.e. $\mathcal{M}_M{}^P\mathcal{M}_P{}^N = \delta_M^N$). The gauge fields $A_\mu{}^M$ and $B_{\mu\nu}$ will be referred to as the fields of the *tensor hierarchy*. The symmetries of the theory are:

- A global $O(d,d|\mathbb{R})$ symmetry.
- A local $O(1,n-1|\mathbb{R})\times O(d|\mathbb{R})\times O(d|\mathbb{R})$ which is trivial in this formulation.
- External diffeomorphisms, parameterized by a vector ξ^μ.
- Gauge transformations of the two-form, parameterized by a one-form Ξ_μ.
- Internal generalized diffeomorphisms, parameterized by an $O(d,d)$ vector Λ^M.

We will deal with two different sets of parameters, related by field-redefinitions. Those noted with a hat $(\widehat{\Lambda}\,,\,\widehat{\Xi})$ are such that the gauge transformations can be written in a covariant form with respect to internal generalized diffeomorphisms. An alternative set of parameters $(\Lambda\,,\,\Xi)$ are more convenient in order to make contact with the usual formulation of DFT (see appendix) and to explore how the tensor hierarchy gauge algebra is that of L_∞. Both sets of parameters are related as follows

$$\begin{aligned}\widehat{\Lambda}^M &= \Lambda^M + \xi^\mu A_\mu{}^M\,,\\ \widehat{\Xi}_\mu &= \Xi_\mu - \left(B_{\mu\nu} - \frac{1}{2}A_\mu{}^M A_{\nu M}\right)\xi^\nu + A_\mu{}^M\Lambda_M\,.\end{aligned} \tag{3.3}$$

JHEP01(2019)117

The covariant form of the local symmetries is the following

$$\delta g_{\mu\nu} = \left(\mathcal{L}^{\mathcal{D}}_{\xi} + \widehat{\mathcal{L}}_{\widehat{\Lambda}}\right) g_{\mu\nu} = \xi^{\rho}\mathcal{D}_{\rho}g_{\mu\nu} + 2\,\mathcal{D}_{(\mu}\xi^{\rho}g_{\nu)\rho} + \widehat{\Lambda}^{M}\partial_{M}g_{\mu\nu}\,, \tag{3.4}$$

$$\delta B_{\mu\nu} = A_{[\mu}{}^{M}\,\delta A_{\nu]M} + \xi^{\rho}\,\mathcal{H}_{\rho\mu\nu} + 2\mathcal{D}_{[\mu}\widehat{\Xi}_{\nu]} - \mathcal{F}_{\mu\nu}{}^{M}\,\widehat{\Lambda}_{M}\,, \tag{3.5}$$

$$\delta e^{-2\phi} = \xi^{\rho}\mathcal{D}_{\rho}e^{-2\phi} + \partial_{M}\left(\widehat{\Lambda}^{M}e^{-2\phi}\right)\,, \qquad \delta\phi \;=\; \xi^{\rho}\mathcal{D}_{\rho}\phi + \widehat{\Lambda}^{M}\partial_{M}\phi - \frac{1}{2}\partial_{M}\widehat{\Lambda}^{M}\,, \tag{3.6}$$

$$\delta A_{\mu}{}^{M} = \xi^{\rho}\,\mathcal{F}_{\rho\mu}{}^{M} + g_{\mu\rho}\,\mathcal{M}^{MN}\partial_{N}\xi^{\rho} + \mathcal{D}_{\mu}\widehat{\Lambda}^{M} + \partial^{M}\widehat{\Xi}_{\mu}\,, \tag{3.7}$$

$$\delta\mathcal{M}_{MN} = \xi^{\rho}\mathcal{D}_{\rho}\mathcal{M}_{MN} + \widehat{\mathcal{L}}_{\widehat{\Lambda}}\mathcal{M}_{MN}\,. \tag{3.8}$$

Before defining the quantities that appear in these transformations, let us briefly go through a review of generalized diffeomorphisms. First we define the D-bracket

$$[\Lambda\,,\,V]^{M}_{(D)} = \Lambda^{P}\partial_{P}V^{M} + \left(\partial^{M}\Lambda_{P} - \partial_{P}\Lambda^{M}\right)V^{P}\,, \tag{3.9}$$

which is not antisymmetric but does satisfy the Jacobi identity. In terms of it we can define the generalized Lie derivative

$$\widehat{\mathcal{L}}_{\Lambda}V^{M} = [\Lambda\,,\,V]^{M}_{(D)} + \lambda(V)\,\partial_{P}\Lambda^{P}V^{M}\,. \tag{3.10}$$

Here $\lambda(V)$ is called a weight: all tensors in this section have vanishing weight except for the generalized dilaton. Extending the action of the generalized Lie derivative to tensors of higher rank is straightforward. Crucially, it admits trivial gauge parameters of the form

$$\Lambda^{M}_{\text{trivial}} = \partial^{M}\chi\,, \qquad \widehat{\mathcal{L}}_{\Lambda_{\text{trivial}}}\cdots = 0\,, \tag{3.11}$$

parameterized by functions $\chi(x, X)$. Then, we have a situation of symmetries for symmetries in DFT: two apparently different gauge parameters generate the same transformation if they are related by a trivial parameter. There is even a symmetry for symmetries for symmetries situation, given by constant shifts in the space of functions that leave the trivial parameters invariant.

While the symmetric part of the D-bracket is in fact a trivial parameter

$$\left[\Lambda_{(1}\,,\,\Lambda_{2)}\right]^{M}_{(D)} = \partial^{M}\left(\frac{1}{2}\Lambda_{1}^{P}\Lambda_{2P}\right)\,, \tag{3.12}$$

the antisymmetric part is called the C-bracket

$$[\Lambda_{1}\,,\,\Lambda_{2}]^{M}_{(C)} = \left[\Lambda_{[1}\,,\,\Lambda_{2]}\right]^{M}_{(D)} = 2\Lambda^{P}_{[1}\partial_{P}\Lambda^{M}_{2]} - \Lambda^{P}_{[1}\partial^{M}\Lambda_{2]P}\,, \tag{3.13}$$

and doesn't satisfy the Jacobi identity.

External derivatives ∂_{μ} of objects that transform tensorially with respect to these transformations are not covariant, so making these symmetries manifest requires the introduction of new derivatives

$$\mathcal{D}_{\mu} = \partial_{\mu} - \widehat{\mathcal{L}}_{A_{\mu}} \qquad \rightarrowtail \qquad \delta_{\Lambda}\,D_{\mu}\cdots = \widehat{\mathcal{L}}_{\Lambda}\,D_{\mu}\ldots\,, \tag{3.14}$$

JHEP01(2019)117

which are covariant because the connection transforms as follows with respect to internal generalized diffeomorphisms (3.7)[3]

$$\delta_\Lambda A_\mu{}^M = \partial_\mu \Lambda^M + [\Lambda\,,\, A_\mu]^{\,M}_{(D)}\,. \tag{3.15}$$

The dilaton $e^{-2\phi}$ transforms as a density (it is in fact the only tensor in KK-DFT with non-vanishing weight $\lambda\left(e^{-2\phi}\right)=1$)

$$\delta_\Lambda e^{-2\phi} = \partial_M\left(\Lambda^M e^{-2\phi}\right)\,, \tag{3.16}$$

and so its covariant derivative reads explicitly[4]

$$\begin{aligned}\mathcal{D}_\mu e^{-2\phi} &= \partial_\mu e^{-2\phi} - \partial_M\left(A_\mu{}^M\, e^{-2\phi}\right)\,,\\ \mathcal{D}_\mu\phi &= \partial_\mu\phi - A_\mu{}^M\partial_M\phi + \frac{1}{2}\partial_M A_\mu^M\,,\end{aligned} \tag{3.17}$$

such that $\mathcal{D}_\mu\phi$ transforms as a scalar with vanishing weight

$$\delta_\Lambda \mathcal{D}_\mu e^{-2\phi} = \partial_M\left(\Lambda^M\,\mathcal{D}_\mu e^{-2\phi}\right)\,, \tag{3.18}$$

$$\delta_\Lambda \mathcal{D}_\mu\phi = \Lambda^M\,\partial_M\mathcal{D}_\mu\phi\,. \tag{3.19}$$

The covariant curvatures of the gauge fields are given by

$$\begin{aligned}\mathcal{F}_{\mu\nu}{}^M &= 2\partial_{[\mu}A_{\nu]}{}^M - [A_\mu\,,\,A_\nu]^M_{(C)} - \partial^M B_{\mu\nu}\,,\\ \mathcal{H}_{\mu\nu\rho} &= 3\left(\mathcal{D}_{[\mu}B_{\nu\rho]} + A_{[\mu}{}^M\partial_\nu A_{\rho]M} - \frac{1}{3}A_{[\mu}{}^M\left[A_\nu\,,\,A_{\rho]}\right]_{(C)M}\right)\,,\end{aligned} \tag{3.20}$$

transforming under local symmetries as follows[5]

$$\delta\mathcal{F}_{\mu\nu}{}^M = \left(\mathcal{L}^{\mathcal{D}}_\xi + \widehat{\mathcal{L}}_{\widehat{\Lambda}}\right)\mathcal{F}_{\mu\nu}{}^M - \mathcal{H}_{\mu\nu\rho}\,\partial^M\xi^\rho + 2\mathcal{D}_{[\mu}\left(g_{\nu]\rho}\mathcal{M}^{MN}\partial_N\xi^\rho\right)\,, \tag{3.23}$$

$$\delta\mathcal{H}_{\mu\nu\rho} = \left(\mathcal{L}^{\mathcal{D}}_\xi + \widehat{\mathcal{L}}_{\widehat{\Lambda}}\right)\mathcal{H}_{\mu\nu\rho} + 3\mathcal{F}_{[\mu\nu}{}^M g_{\rho]\sigma}\mathcal{M}_{MN}\partial^N\xi^\sigma\,, \tag{3.24}$$

[3]Since the gauge vector A_μ enters the covariant derivative as a generator of the generalized Lie derivative, its gauge transformation is only determined up to trivial parameters of the form (3.11). However, all possible "orbits" are related through redefinitions of the parameters Ξ, as can be seen from (3.7).

[4]We note in passing that this covariant derivative can be integrated by parts in the presence of the measure $e^{-2\phi}$

$$\int d^nx d^{2d}X\, e^{-2\phi}\, V^{\mu M_1\ldots M_k}\,\mathcal{D}_\mu U_{M_1\ldots M_k} = -\int d^nx d^{2d}X\,\mathcal{D}_\mu\left(e^{-2\phi}V^{\mu M_1\ldots M_k}\right)U_{M_1\ldots M_k} + \text{T.D.}$$

[5]A useful intermediate step is

$$\delta\mathcal{F}_{\mu\nu}{}^M = 2\,\mathcal{D}_\mu\delta A_{\nu]}{}^M - \partial^M\Delta B_{\mu\nu}\,, \tag{3.21}$$

$$\delta\mathcal{H}_{\mu\nu\rho} = 3\,\mathcal{D}_{[\mu}\Delta B_{\nu\rho]} + 3\,\delta A_{[\mu}{}^M\,\mathcal{F}_{\nu\rho]M}\,, \tag{3.22}$$

with $\Delta B_{\mu\nu} = \delta B_{\mu\nu} - A^M_{[\mu}\,\delta A_{\nu]M}$.

JHEP01(2019)117

and they are related through Bianchi identities

$$3\,\mathcal{D}_{[\mu}\mathcal{F}_{\nu\rho]}{}^{M} + \partial^{M}\mathcal{H}_{\mu\nu\rho} = 0\,, \tag{3.25}$$
$$D_{[\mu}\mathcal{H}_{\nu\rho\sigma]} - \frac{3}{4}\mathcal{F}_{[\mu\nu}{}^{M}\mathcal{F}_{\rho\sigma]M} = 0\,.$$

Now that we have defined all the ingredients appearing in the gauge transformations, we can verify closure. The gauge transformations obviously close

$$[\delta_1\,,\,\delta_2] = -\delta_{\xi_{12}} - \delta_{\Lambda_{12}} - \delta_{\Xi_{12}}\,, \tag{3.26}$$

but only for the particular set of un-hatted parameters the brackets are field-independent

$$\xi^{\mu}_{12} = [\xi_1\,,\,\xi_2]^{\mu} + \Lambda^{P}_{[1}\partial_P\xi^{\mu}_{2]}\,, \tag{3.27}$$
$$\Lambda^{M}_{12} = [\Lambda_1\,,\,\Lambda_2]^{M}_{(C)} + 2\xi^{\rho}_{[1}\partial_\rho\Lambda^{M}_{2]} + \xi^{\rho}_{[1}\partial^{M}\Xi_{2]\rho} + \Xi_{[1\rho}\partial^{M}\xi^{\rho}_{2]}\,, \tag{3.28}$$
$$\Xi_{12\,\mu} = 2\Lambda^{P}_{[1}\partial_P\Xi_{2]\mu} + \Lambda^{P}_{[1}\partial_\mu\Lambda_{2]P} + \xi^{\rho}_{[1}\partial_\rho\Xi_{2]\mu} - \xi^{\rho}_{[1}\partial_\mu\Xi_{2]\rho} - \Xi_{[1\rho}\partial_\mu\xi^{\rho}_{2]}\,. \tag{3.29}$$

This fact makes this set of parameter particularly convenient to explore how the DFT tensor hierarchy fits into an L_∞ algebra.

We can now turn to dynamics. The action of KK-DFT is given by

$$S = \int d^{n}x\, d^{2d}X\, e^{-2d}\mathcal{L}\,, \tag{3.30}$$

where d depends as usual on $g = \det[g_{\mu\nu}]$ and ϕ

$$e^{-2d} = \sqrt{-g}\, e^{-2\phi}\,, \tag{3.31}$$

and the Lagrangian is

$$\mathcal{L} = \widehat{\mathcal{R}} - 4g^{\mu\nu}\mathcal{D}_\mu\phi\mathcal{D}_\nu\phi + 4\nabla_\mu\left(g^{\mu\nu}\mathcal{D}_\nu\phi\right) - \frac{1}{12}\mathcal{H}_{\mu\nu\rho}\mathcal{H}^{\mu\nu\rho}$$
$$+\ \frac{1}{8}g^{\mu\nu}\mathcal{D}_\mu\mathcal{M}_{MN}\,\mathcal{D}_\nu\mathcal{M}^{MN} - \frac{1}{4}\mathcal{M}_{MN}\mathcal{F}_{\mu\nu}{}^{M}\mathcal{F}^{\mu\nu N} - V\,. \tag{3.32}$$

Up to a single term, the "scalar potential" V is defined as minus the generalized Ricci scalar in DFT, but with the generalized metric replaced by the scalar matrix $\mathcal{M}$

$$-V = 4\,\mathcal{M}^{MN}\partial_{MN}d - \partial_{MN}\mathcal{M}^{MN} - 4\,\mathcal{M}^{MN}\partial_M d\partial_N d + 4\,\partial_M\mathcal{M}^{MN}\partial_N d \tag{3.33}$$
$$+\ \frac{1}{8}\mathcal{M}^{MN}\partial_M\mathcal{M}^{PQ}\partial_N\mathcal{M}_{PQ} + \frac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{PQ}\partial_P\mathcal{M}_{QN} + \frac{1}{4}\mathcal{M}^{MN}\partial_M g^{\mu\nu}\partial_N g_{\mu\nu}\,.$$

Given that the metric transforms as a scalar w.r.t. internal diffeomorphisms, within the Ricci scalar $\widehat{\mathcal{R}}$ all derivatives ∂_μ must be replaced by A-covariantized derivatives $\mathcal{D}_\mu$ in the following way

$$\widehat{\mathcal{R}} = g^{\mu\nu}\,\widehat{\mathcal{R}}^{\rho}{}_{\mu\rho\nu}\,, \tag{3.34}$$
$$\widehat{\mathcal{R}}^{\rho}{}_{\sigma\mu\nu} = \mathcal{D}_\mu\widehat{\Gamma}^{\rho}_{\nu\sigma} - \mathcal{D}_\nu\widehat{\Gamma}^{\rho}_{\mu\sigma} + \widehat{\Gamma}^{\rho}_{\mu\delta}\widehat{\Gamma}^{\delta}_{\nu\sigma} - \widehat{\Gamma}^{\rho}_{\nu\delta}\widehat{\Gamma}^{\delta}_{\mu\sigma}\,, \tag{3.35}$$
$$\widehat{\Gamma}^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\mathcal{D}_\mu g_{\nu\sigma} + \mathcal{D}_\nu g_{\mu\sigma} - \mathcal{D}_\sigma g_{\mu\nu}\right)\,, \tag{3.36}$$

JHEP01(2019)117

and we also defined a new covariant derivative $\nabla_\mu = \mathcal{D}_\mu + \widehat{\Gamma}_\mu$ that covariantizes the part of the external diffeomorphisms that involves external derivatives only.

Written like this, each term in the Lagrangian is independently and manifestly a Λ-scalar, Ξ-invariant and duality invariant (the scalar potential is the only exception, as the invariance with respect to internal diffeomorphisms is far from being manifest). The relative coefficients between the terms are fixed by external diffeomorphism invariance, and as a result

$$\delta\mathcal{L} = \xi^\mu \mathcal{D}_\mu \mathcal{L} + \widehat{\Lambda}^M \partial_M \mathcal{L} = \xi^\mu \partial_\mu \mathcal{L} + \Lambda^M \partial_M \mathcal{L} \,. \tag{3.37}$$

On the other hand, while $\sqrt{-g}$ transforms as a density under external diffeos and as a scalar under internal ones, $e^{-2\phi}$ plays the opposite role, and hence e^{-2d} acts as a density w.r.t. both

$$\delta e^{-2d} = \mathcal{D}_\mu \left(\xi^\mu e^{-2d}\right) + \partial_M \left(\widehat{\Lambda}^M e^{-2d}\right) = \partial_\mu \left(\xi^\mu e^{-2d}\right) + \partial_M \left(\Lambda^M e^{-2d}\right) , \tag{3.38}$$

where $\mathcal{D}_\mu$ acts on d the same way as on ϕ in (3.17). As a consequence, the action is invariant w.r.t. to all the local symmetries.

Regarding equations of motion, we find that under variations of the action w.r.t. the fields one has

$$\delta S = \int d^n x \, d^{2d} X \, \left(\delta g^{\mu\nu} \Delta g_{\mu\nu} + \delta B_{\mu\nu} \Delta B^{\mu\nu} + \dots\right) , \tag{3.39}$$

with

$$\begin{aligned}
e^{2d}\Delta g_{\mu\nu} = {} & \frac{1}{4} g_{\mu\nu}\, \Delta\phi + \widehat{\mathcal{R}}_{\mu\nu} + 2\nabla_{(\mu}\mathcal{D}_{\nu)}\phi - \frac{1}{4}\mathcal{H}_{\mu\rho\sigma}\mathcal{H}_\nu{}^{\rho\sigma} - \frac{1}{2}\mathcal{F}_{\mu\rho}{}^M \mathcal{F}_\nu{}^{\rho N}\mathcal{M}_{MN} \\
& + \frac{1}{8}\mathcal{D}_\mu \mathcal{M}_{MN} \mathcal{D}_\nu \mathcal{M}^{MN} - \frac{1}{2}\partial_M \left(\mathcal{M}^{MN}\partial_N g_{\mu\nu}\right) , \qquad (3.40) \\
\Delta B^{\mu\nu} = {} & \frac{1}{2}\mathcal{D}_\rho \left(e^{-2d}\mathcal{H}^{\rho\mu\nu}\right) - \frac{1}{2}\partial^M \left(e^{-2d}\mathcal{M}_{MN}\mathcal{F}^{\mu\nu N}\right) , \qquad (3.41) \\
e^{2d}\Delta\phi = {} & -2\mathcal{L} , \qquad (3.42) \\
\Delta A^\mu{}_M = {} & \Delta B^{\mu\nu} A_{\nu M} + \mathcal{D}_\rho \left(e^{-2d}\mathcal{M}_{MN}\mathcal{F}^{\rho\mu N}\right) - \frac{1}{2}e^{-2d}\mathcal{H}^{\mu\nu\rho}\mathcal{F}_{\nu\rho M} \qquad (3.43) \\
& - \partial_P \left(e^{-2d}\mathcal{M}^{PN} g^{\mu\nu}\mathcal{D}_\nu \mathcal{M}_{NM}\right) + \mathcal{D}_\nu \left(e^{-2d}\partial_M g^{\mu\nu}\right) \\
& + e^{-2d}\left(-4 g^{\mu\nu}\partial_M \mathcal{D}_\nu d - \frac{1}{4} g^{\mu\nu}\mathcal{D}_\nu \mathcal{M}^{PQ}\partial_M \mathcal{M}_{PQ} \right. \\
& \qquad \left. - \frac{1}{2} g^{\mu\nu}\mathcal{D}_\nu g^{\rho\sigma}\partial_M g_{\rho\sigma} + \mathcal{D}_\nu g^{\mu\rho}\partial_M g_{\rho\sigma} g^{\sigma\nu}\right) \\
e^{2d}\Delta\mathcal{M}_{MN} = {} & -\frac{1}{2}\mathcal{D}_\mu \left(e^{-2d} g^{\mu\nu}\mathcal{D}_\nu \mathcal{M}\right)_{(\overline{M}\underline{N})} - \frac{1}{2}\mathcal{F}_{\mu\nu(\overline{M}}\mathcal{F}^{\mu\nu}{}_{\underline{N})} \\
& + \mathcal{R}_{(\overline{M}\underline{N})}[\mathcal{M}, d] + \frac{1}{4}\partial_{(\overline{M}} g^{\mu\nu}\partial_{\underline{N})} g_{\mu\nu} \,. \qquad (3.44)
\end{aligned}$$

In the last line we used the standard notation for left-right projections,[6] and wrote the variation of the scalar potential in the same way as the generalized Ricci tensor because the former is equal to minus the generalized Ricci scalar.

[6]Defining $P_{MN} = \frac{1}{2}(\eta_{MN} - \mathcal{M}_{MN})$ and $\bar{P}_{MN} = \frac{1}{2}(\eta_{MN} + \mathcal{M}_{MN})$, the notation is such that $V_{\underline{M}} = P_M{}^N V_N$ and $V_{\overline{M}} = \bar{P}_M{}^N V_N$.

JHEP01(2019)117

3.2 The tensor hierarchy

Here we restrict attention to the degrees of freedom involved in the tensor hierarchy, namely $A_\mu{}^M$ and $B_{\mu\nu}$, and so set the gravitational and scalar degrees of freedom to background values (which we write with an overline). The symmetries relevant in the discussion of the tensor hierarchy are internal generalized diffeos Λ^M, and gauge transformations of the two-form Ξ_μ, so we will also ignore external diffeomorphisms from now on $\xi^\mu = 0$. This in particular implies that $\widehat{\Lambda} = \Lambda$ in (3.3), and

$$\widehat{\Xi}_\mu = \Xi_\mu + A_\mu{}^M \Lambda_M \,. \tag{3.45}$$

When the transformations (3.5) and (3.7)

$$\begin{aligned} \delta A_\mu^M &= \mathcal{D}_\mu \Lambda^M + \partial^M \widehat{\Xi}_\mu \,, \\ \delta B_{\mu\nu} &= A_{[\mu}{}^M \delta A_{\nu]M} + 2\mathcal{D}_{[\mu} \widehat{\Xi}_{\nu]} - \Lambda_M \mathcal{F}_{\mu\nu}{}^M \,, \end{aligned} \tag{3.46}$$

are written in terms of the un-hatted parameters the gauge covariance is no longer manifest

$$\begin{aligned} \delta A_\mu{}^M &= \partial_\mu \Lambda^M + \widehat{\mathcal{L}}_\Lambda A_\mu{}^M + \partial^M \Xi_\mu \,, \\ \delta B_{\mu\nu} &= \Lambda^M \partial_M B_{\mu\nu} + 2\partial_{[\mu} \Xi_{\nu]} + \partial_M \Xi_{[\mu} A_{\nu]}{}^M - A_{[\mu}{}^M \partial_{\nu]} \Lambda_M \,. \end{aligned} \tag{3.47}$$

These gauge transformations are annihilated by the following trivial parameters

$$\Lambda^M = \partial^M \chi \,, \qquad \Xi_\mu = -\partial_\mu \chi \,, \qquad \widehat{\Xi}_\mu = -\mathcal{D}_\mu \chi \,, \tag{3.48}$$

where we define the arbitrary function χ to carry vanishing weight.

Let us now discuss the gauge algebra. We saw in (3.28)–(3.29) that closure with respect to the parameters Λ^M and Ξ_μ was achieved through the brackets

$$\begin{aligned} \Lambda_{12}^M &= [\Lambda_1 \,,\, \Lambda_2]_{(C)}^M \,, \\ \Xi_{12\,\mu} &= 2\Lambda_{[1}^P \partial_P \Xi_{2]\mu} + \Lambda_{[1}^P \partial_\mu \Lambda_{2]P} \,. \end{aligned} \tag{3.49}$$

We can now collectively define the gauge parameters

$$\zeta = \left(\Lambda^M \,, \Xi_\mu\right) \,, \tag{3.50}$$

and note the brackets as

$$\Lambda_{12}^M = \zeta_{12}^M = [\zeta_1 \,,\, \zeta_2]^M \,, \tag{3.51}$$

$$\Xi_{12\,\mu} = \zeta_{12\,\mu} = [\zeta_1 \,,\, \zeta_2]_\mu \,. \tag{3.52}$$

This notation is useful to compute the Jacobiator

$$J(\zeta_1, \zeta_2, \zeta_3) = [[\zeta_1, \zeta_2] \,, \zeta_3] + \text{cyc.} = 3\, \big[\big[\zeta_{[1}, \zeta_2\big] \,, \zeta_{3]}\big] \,, \tag{3.53}$$

the components of which are known to be trivial parameters

$$J^M = \partial^M N \,, \qquad J_\mu = -\partial_\mu N \,, \tag{3.54}$$

JHEP01(2019)117

where we introduced the Nijenhuis scalar

$$N(\zeta_1,\,\zeta_2,\,\zeta_3) = \frac{1}{6}\,[\Lambda_1\,,\Lambda_2]^M_{(C)}\,\Lambda_{3M} + \text{cyc.} = \frac{1}{2}\,\big[\Lambda_{[1}\,,\Lambda_2\big]^M_{(C)}\,\Lambda_{3]M}\,. \tag{3.55}$$

If instead we choose to work with the redefined parameters

$$\widehat{\zeta} = \left(\Lambda^M\,,\,\widehat{\Xi}_\mu\right)\,, \tag{3.56}$$

we then find

$$\widehat{\zeta}^M_{12} = \left[\widehat{\zeta}_1\,,\,\widehat{\zeta}_2\right]^M = [\Lambda_1\,,\,\Lambda_2]^M_{(C)}\,, \tag{3.57}$$

$$\widehat{\zeta}_{12\,\mu} = \left[\widehat{\zeta}_1\,,\,\widehat{\zeta}_2\right]_\mu = \mathcal{D}_\mu\Lambda^M_{[1}\,\Lambda_{2]M}\,, \tag{3.58}$$

and now

$$\widehat{J}^M = \partial^M N\,, \qquad \widehat{J}_\mu = -\mathcal{D}_\mu N + 2\,\mathcal{D}_\mu\,\big[\Lambda_{[1}\,,\,\Lambda_2\big]^M_{(C)}\,\Lambda_{3]M} - \big[\Lambda_{[1}\,,\,\Lambda_2\big]^M_{(C)}\,\mathcal{D}_\mu\Lambda_{3]M}\,, \tag{3.59}$$

and so we see that the Jacobiator is no longer a trivial parameter. We will discuss this further and its relevance for EFT later.

3.3 The L_∞ structure of the tensor hierarchy

Here we discuss how the algebraic structure of the tensor hierarchy in DFT fits into L_∞. We begin by discussing the kinematics, and later move to consider the dynamics. The first step is to define the graded vector spaces, which divide here into the following subspaces

$$\begin{aligned}
X_1: &\quad \text{functions:} && \chi(x^\mu,\,X^M)\\
X_0: &\quad \text{gauge parameters (generically noted } \zeta\text{):} && \Lambda^M,\,\Xi_\mu\\
X_{-1}: &\quad \text{fields (generically noted } \Psi\text{):} && A_\mu{}^M,\,B_{\mu\nu}\\
X_{-2}: &\quad \text{field equations (generically noted } \mathcal{F}\text{):} && \Delta A^\mu{}_M,\,\Delta B^{\mu\nu}\,.
\end{aligned} \tag{3.60}$$

The space of functions might come as a surprise at the moment, but it turns out to be a general feature in theories that exhibit symmetries for symmetries. We will sometimes add indices to indicate which particular subspace we refer to within a given level. For example: for $\Psi \in X_{-1}$, $\Psi_\mu{}^M$ refers to a field $A_\mu{}^M$ and $\Psi_{\mu\nu}$ to a field $B_{\mu\nu}$. The same notation will be used over products, for instance $\chi \in X_1$, $\ell_1(\chi)_\mu$ indicates a parameter Ξ_μ as ℓ_1 maps from X_1 to X_0, etc.

We will first describe thoroughly how to construct the pure gauge part and then consider the dynamics of the fields. The pure gauge structure of DFT is given by the $O(D,D)$ covariantization of the Courant algebroid [21]. The relation between this structure and L_∞ algebras was done in [20] and later extended to the $O(D,D)$ case in [22, 23] and [12]. As discussed in subsection 2.1, a recent paper [52] established sufficient conditions for a gauge algebra to have an L_∞ structure. In this section we review these results, closely following the line of reasoning in [52] for a general bracket and then specializing to our case of interest.

JHEP01(2019)117

X_1	X_0	X_{-1}	X_{-2}
Functions χ	Gauge Parameters ζ	Fields Ψ	E ($\ni EOM\mathcal{F}$)

Table 3. New subspace X_1 required by the failure of Jacobi.

For the sake of simplicity we begin by focussing on the subspace X_0 with elements $\zeta = \Lambda^M + \Xi_\mu$, ignoring for the moment to the maximum extent the spaces X_{-1} and lower. We will proceed in an order that might seem capricious, but is only intended to reach the relevant results quickly. The first identity we consider is $n = 3$

$$\begin{aligned} -\ell_1(\ell_3(\zeta_1,\zeta_2,\zeta_3)) = \; & \ell_2(\ell_2(\zeta_1,\zeta_2),\zeta_3) + \ell_2(\ell_2(\zeta_3,\zeta_1),\zeta_2) + \ell_2(\ell_2(\zeta_2,\zeta_3),\zeta_1) \\ & + \ell_3(\ell_1(\zeta_1),\zeta_2,\zeta_3) + \ell_3(\zeta_1,\ell_1(\zeta_2),\zeta_3) + \ell_3(\zeta_1,\zeta_2,\ell_1(\zeta_3))\,. \end{aligned} \tag{3.61}$$

We have already all the required ingredients to evaluate the r.h.s.. As explained before, $\ell_2(\zeta_1,\,\zeta_2)$ must be read from the crossing of (2.28) and (3.49), yielding

$$\ell_2(\zeta_1,\,\zeta_2) = [\Lambda_1\,,\,\Lambda_2]^M_{(C)} + 2\Lambda^P_{[1}\partial_P\Xi_{2]\mu} + \Lambda^P_{[1}\partial_\mu\Lambda_{2]P}\,. \tag{3.62}$$

The first line in the r.h.s. of (3.61) then becomes the Jacobiator (3.53)–(3.54)

$$J(\zeta_1,\zeta_2,\zeta_3) = \partial^M N(\zeta_1,\zeta_2,\zeta_3) - \partial_\mu N(\zeta_1,\zeta_2,\zeta_3) \equiv \mathcal{D}N(\zeta_1,\zeta_2,\zeta_3)\,, \tag{3.63}$$

where N is the Nijenhuis scalar (3.55) and we have defined the operator

$$\mathcal{D} \equiv \partial^M - \partial_\mu\,. \tag{3.64}$$

The second line in (3.61) is schematically of the form $\ell_3(\zeta_i,\,\zeta_j,\,\ell_1(\zeta_k))$. Given that ℓ_1 maps the last argument into X_{-1} these terms are of the form $\ell_3(\zeta_1,\zeta_2,\Psi)$ and vanish in light of the discussion in (2.28). Then, the $n = 3$ equation (3.61) can be written as

$$-\,\ell_1(\ell_3(\zeta_1,\zeta_2,\zeta_3)) = \mathcal{D}\,N(\zeta_1,\zeta_2,\zeta_3) \in X_0\,. \tag{3.65}$$

Now note that the r.h.s. of (3.65) is not zero, and so $\ell_3(\zeta_1,\,\zeta_2,\,\zeta_3) \in X_1$ must also be non vanishing, implying as anticipated that a new graded subspace X_1 of functions $\chi(x, X)$ is required (see table 3).

From here it is trivial to see that (3.61) is satisfied for the following choice of products

$$\begin{aligned} \ell_3(\zeta_1,\zeta_2,\zeta_3) &= -N(\Lambda_1,\Lambda_2,\Lambda_3) \;\in X_1 \\ \ell_1(\chi) &= \mathcal{D}\chi,\; \chi \in X_1\,. \end{aligned} \tag{3.66}$$

Having introduced a new space of functions we now have to define new products and verify identities, that include such elements as arguments. We begin the verification of the first identity for $\chi \in X_1$

$$n = 1: \quad \ell_1(\ell_1(\chi)) \;= \ell_1(\mathcal{D}\chi) \stackrel{?}{=} 0\,. \tag{3.67}$$

JHEP01(2019)117

Since $\mathcal{D}\chi$ belongs to the space of parameters, and is in fact a trivial one (3.48) one then finds from (2.20) that

$$\delta_{\mathcal{D}\chi}\Psi = \ell_1(\mathcal{D}\chi) + \ell_2(\mathcal{D}\chi,\Psi) - \frac{1}{2}\ell_3(\mathcal{D}\chi,\Psi^2) - \ldots = 0\,. \tag{3.68}$$

Since each term contains different powers in field perturbations, each one must vanish separately, in particular the first one which implies that $\ell_1(\mathcal{D}\chi) = 0$, which is what we wanted to verify.

Now we consider the next identity for $n = 2$ (2.17), with $x_1 = \zeta$ and $x_2 = \chi$

$$\ell_1(\ell_2(\zeta,\chi)) = \ell_2(\ell_1(\zeta),\chi) + \ell_2(\zeta,\ell_1(\chi))\,. \tag{3.69}$$

An extra hypothesis was considered in [52]: for any $\zeta \in X_0$ we will need $[\mathrm{Im}(\mathcal{D}),\zeta] \subset \mathrm{Im}(\mathcal{D})$ i.e., $\mathrm{Im}(\mathcal{D})$ is an ideal of X_0 or, stated in other way, the commutator of a trivial gauge parameter and any gauge parameter gives back a trivial gauge parameter. A short computation shows that this is exactly what happens in the tensor hierarchy of DFT

$$[\mathcal{D}\chi,\,\zeta] = \mathcal{D}\left(-\frac{1}{2}\Lambda^P\partial_P\chi\right)\,. \tag{3.70}$$

Comparing this with the following rewriting of the $n = 2$ identity

$$\mathcal{D}\ell_2(\zeta,\,\chi) = \ell_2(\ell_1(\zeta),\chi) - [\mathcal{D}\chi,\,\zeta]\,, \tag{3.71}$$

we conclude that $\ell_2(\Psi,\chi) = 0$ and

$$\ell_2(\zeta,\,\chi) = \frac{1}{2}\Lambda^P\partial_P\chi\,. \tag{3.72}$$

So far we obtained the following non-vanishing products with entries belonging to the graded subspaces X_1 and X_0

$$\begin{aligned}
\ell_1(\chi) &= \partial^M\chi - \partial_\mu\chi\,,\\
\ell_1(\zeta) &= \partial_\mu\Lambda^M + \partial^M\Xi_\mu + 2\partial_{[\mu}\Xi_{\nu]}\,,\\
\ell_2(\zeta_1,\zeta_2) &= [\zeta_1,\zeta_2] = [\Lambda_1\,,\,\Lambda_2]^M_{(C)} + 2\Lambda^P_{[1}\partial_P\Xi_{2]\mu} + \Lambda^P_{[1}\partial_\mu\Lambda_{2]P}\,,\\
\ell_2(\zeta,\chi) &= \Lambda(\chi) \equiv \frac{1}{2}\Lambda^P\partial_P\chi\,,\\
\ell_3(\zeta_1,\zeta_2,\zeta_3) &= -N(\Lambda_1,\Lambda_2,\Lambda_3) = -\frac{1}{2}\left[\Lambda_{[1}\,,\,\Lambda_2\right]^P_{(C)}\Lambda_{3]P}\,.
\end{aligned} \tag{3.73}$$

These are the only non trivial products coming from the pure gauge sector and are enough to show that this sector fits into an L_∞ structure (with the help of the observation made in subsection 2.1). The remaining identities that have not been checked (involving two $\chi's \in X_1$ and for $\zeta's \in X_0$) are presented in the appendix. The outcome of the forthcoming analysis is that the products obtained so far will not be modified when considering entries with lower grade. In particular, the absence of a 4-product ℓ_4 indicates that $\ell_2\ell_3 - \ell_3\ell_2$ vanishes exactly as opposed to "up to homotopy", meaning that the gauge sector of the hierarchy tensor in DFT is governed by an L_3 algebra.

JHEP01(2019)117

We now take into account the fields, their gauge transformations and their dynamics. Some products that are trivial to identify are the following:

- From the gauge transformations of DFT (3.47) and the identifications (2.20) we rapidly find

$$\begin{aligned} \ell_2(\zeta,\,\Psi) &= [\Lambda, A_\mu]^M_{(D)} + \partial_P \Xi_{[\mu} A_{\nu]}{}^P - A_{[\mu}{}^P \partial_{\nu]} \Lambda_P + \Lambda^P \partial_P B_{\mu\nu}\,, \\ \ell_{n+1}(\zeta, \Psi^n) &= 0, \quad n > 1\,. \end{aligned} \tag{3.74}$$

- From the equations of motion of the gauge fields (3.43) and two-form (3.41) and crossing with (2.24) we read off the products involving only fields $\ell_n(\Psi^n)$. As mentioned previously, we will only consider the fields of the tensor hierarchy and ignore the fluctuations around the background for the metric $g_{\mu\nu} = \overline{g}_{\mu\nu}$, the dilaton $\phi = \overline{\phi}$ and $\mathcal{M}_{MN} = \overline{\mathcal{M}}_{MN}$. Note that these backgrounds are by no means assumed to be constant nor covariantly constant: they are only forced to satisfy the equations of motion. In this case, the non-trivial products are of the form $\ell_n(\Psi^n)$ with $n \leq 3$ and $\Psi = A_\mu{}^M + B_{\mu\nu}$. Here we only write explicitly the single product as will be needed later. Expanding the EOM (3.43) and (3.41) in powers of $A_\mu{}^M$ and $B_{\mu\nu}$ and keeping only the leading order gives

$$\begin{aligned} \ell_1(\Psi) = &-\partial^M \left(e^{-2\bar{d}} \overline{\mathcal{M}}_{MN} \overline{g}^{\mu\rho} \overline{g}^{\nu\sigma} \partial_{[\rho} A_{\sigma]}{}^N\right) + \frac{3}{2} \partial_\rho \left(e^{-2\bar{d}} \overline{g}^{\rho\sigma} \overline{g}^{\mu\delta} \overline{g}^{\nu\epsilon} \partial_{[\sigma} B_{\delta\epsilon]}\right) \\ &+ \frac{1}{2} \partial^M \left(e^{-2\bar{d}} \overline{\mathcal{M}}_{MN} \overline{g}^{\mu\rho} \overline{g}^{\nu\sigma} \partial^N B_{\rho\sigma}\right) + 2\partial_\rho \left(e^{-2\bar{d}} \overline{\mathcal{M}}_{MN} \overline{g}^{\rho\sigma} \overline{g}^{\mu\delta} \partial_{[\sigma} A_{\delta]}{}^N\right) \\ &+ \partial_P \left(e^{-2\bar{d}} \overline{\mathcal{M}}^{PN} \overline{g}^{\mu\nu} \widehat{\mathcal{L}}_{A_\nu} \overline{\mathcal{M}}_{NM}\right) - \widehat{\mathcal{L}}_{A_\nu} \left(e^{-2\bar{d}} \partial_M \overline{g}^{\mu\nu}\right) \\ &- \partial_\rho \left(e^{-2\bar{d}} \overline{\mathcal{M}}_{MN} \overline{g}^{\rho\sigma} \overline{g}^{\mu\delta} \partial^N B_{\sigma\delta}\right) - e^{-2\bar{d}} \Big(-4 \overline{g}^{\mu\nu} \partial_M \widehat{\mathcal{L}}_{A_\nu} \bar{d} \\ &- \frac{1}{4} \overline{g}^{\mu\nu} \widehat{\mathcal{L}}_{A_\nu} \overline{\mathcal{M}}^{PQ} \partial_M \overline{\mathcal{M}}_{PQ} - \frac{1}{2} \overline{g}^{\mu\nu} \widehat{\mathcal{L}}_{A_\nu} \overline{g}^{\rho\sigma} \partial_M \overline{g}_{\rho\sigma} + \widehat{\mathcal{L}}_{A_\nu} \overline{g}^{\mu\rho} \partial_M \overline{g}_{\rho\sigma} \overline{g}^{\sigma\nu} \Big)\,. \end{aligned} \tag{3.75}$$

  Higher products of fields are obtained in the same way, we don't show the expressions here in order to lighten the presentation, and also because the identities in which they appear are fulfilled by construction as explained at the end of section 2).

- We can use the gauge transformation of the EOM (2.34) to determine products of the form $\ell_{n+2}(\zeta, \mathcal{F}, \Psi^n)$, $n \geq 0$. First let us point out that the EOM for $B_{\mu\nu}$ (3.41) is covariant, namely it transforms as a tensorial density

$$\delta \Delta B_{\mu\nu} = \widehat{\mathcal{L}}_\Lambda \Delta B^{\mu\nu} = \partial_M \left(\Lambda^M \Delta B^{\mu\nu}\right)\,, \quad \lambda(\Delta B^{\mu\nu}) = 1\,, \tag{3.76}$$

  and the EOM for $A^\mu{}_M$ includes a covariant part also of weight one (which we denote Δ_c, *c.f* ((3.43)))

$$\Delta A^\mu{}_M = \Delta B^{\mu\nu} A_{\nu M} + \Delta_c A^\mu{}_M\,, \quad \delta \Delta_c A^\mu{}_M = \widehat{\mathcal{L}}_\Lambda \Delta_c A^\mu{}_M\,, \quad \lambda(\Delta_c A^\mu{}_M) = 1\,. \tag{3.77}$$

JHEP01(2019)117

Then, from (2.34) we read

$$\begin{aligned}\ell_2(\zeta,\mathcal{F}) &= \widehat{\mathcal{L}}_\Lambda \Delta B^{\mu\nu} + \Delta B^{\mu\nu}\left(\partial_\nu \Lambda_M + \partial_M \Xi_\nu\right) + \widehat{\mathcal{L}}_\Lambda \Delta_c A^\mu{}_M\,,\\ \ell_3(\zeta,\mathcal{F},\Psi) &= \widehat{\mathcal{L}}_\Lambda \Delta B^{\mu\nu}\, A_{\nu M} + \Delta B^{\mu\nu}\, \widehat{\mathcal{L}}_\Lambda A_{\nu M}\,,\\ \ell_{n+2}(\zeta,\mathcal{F},\Psi^n) &= 0\,, \qquad \text{for } n \geq 2\,.\end{aligned} \tag{3.78}$$

The products presented so far are enough to show that the theory lies within an L_∞ structure (modulo a subtlety to be discussed in what follows). The direct way to verify this is to write down all possible non-trivial identities with their corresponding arguments and to check that there is no need for new non-zero products. This is done in detail in the appendix. The strategy we follow consists in writing all possible lists of vectors with definite degree and then apply the identities over them. Such possibilities are given by the following lists:

- Any number of $\Psi's \in X_{-1}$: $(\Psi_1, \Psi_2, \ldots)$.
- One $\zeta \in X_0$ and any number $n > 0$ of $\Psi's \in X_{-1}$: $(\zeta, \Psi_1, \Psi_2, \ldots)$.
- Two $\zeta's \in X_0$ and any number $n \geq 0$ of $\Psi's \in X_{-1}$: $(\zeta_1, \zeta_2, \Psi_1, \Psi_2, \ldots)$.
- Three or more $\zeta's \in X_0$ and any number $n \geq 0$ of $\Psi's \in X_{-1}$: $(\zeta_1, \zeta_2, \zeta_3 \ldots, \zeta_n, \Psi_1, \Psi_2, \ldots)$.
- At least one $\chi \in X_1$ (and any other vector on the list): $(\chi, \ldots)$.
- At least one $E \in X_{-2}$ (and any other vector on the list): $(E, \ldots)$.
- Only one $\zeta \in X_0$.

The first three lists fulfill the L_∞ identities by construction as pointed out at the end of section 2. The first one is associated to the non-existence of a graded subspace X_{-3}, the second one to gauge transformations of the equations of motion, and the third one to closure of the gauge algebra over fields. The fourth, fifth and sixth ones fulfill the identities straightforwardly (see appendix).

The last one has an interesting implication: the dynamics of the tensor hierarchy can not fit alone within an L_∞ algebraic structure, it requires the extra field perturbations and their background values. So far we considered mostly the gauge sector and decoupled the degrees of freedom of the fields not belonging to the tensor hierarchy. In particular, we found that if the graded subspace X_{-2} is assumed to be absent (namely, the dynamics is ignored) then from (3.73) one finds that the following choice for ℓ_1 over gauge parameters is consistent with the L_∞ structure

$$\ell_1(\zeta) = \partial_\mu \Lambda^M + \partial^M \Xi_\mu + 2\partial_{[\mu}\Xi_{\nu]}\,. \tag{3.79}$$

Since this product maps into the space of fields, in order to verify the $\ell_1(\ell_1(\zeta)) = 0$ identity we must insert the above into (3.75). We rapidly get to the conclusion that it does not work. What we are missing are the contributions to X_{-1} coming from the gauge transformations

JHEP01(2019)117

of the other fields. This is in fact to be expected because on the one hand the EOM mix all the fields, and on the other because there are non-vanishing $\ell_1(\zeta)$ components along the fields that do not belong to the tensor hierarchy. Consider as an example the gauge transformation of the scalar fields

$$\delta_\Lambda \mathcal{M}_{PQ} = \widehat{\mathcal{L}}_\Lambda \mathcal{M}_{PQ} = \underbrace{\widehat{\mathcal{L}}_\Lambda \overline{\mathcal{M}}_{PQ}}_{\ell_1(\Lambda)_{PQ}} + \underbrace{\widehat{\mathcal{L}}_\Lambda m_{PQ}}_{\ell_2(\Lambda,m)_{PQ}} + \dots \tag{3.80}$$

where $m_{PQ} = m_{\overline{P}\underline{Q}}$ is the scalar first order perturbation. Only when these components of $\ell_1(\Lambda)$ are considered, namely

$$\ell_1(\zeta) = \partial_\mu \Lambda^M + \partial^M \Xi_\mu + 2\partial_{[\mu}\Xi_{\nu]} + \widehat{\mathcal{L}}_\Lambda \overline{\mathcal{M}}_{PQ} + \widehat{\mathcal{L}}_\Lambda \overline{d} + \widehat{\mathcal{L}}_\Lambda \overline{g}_{\mu\nu}\,, \tag{3.81}$$

the following identity is obtained

$$\ell_1(\ell_1(\Lambda))^\mu{}_M = \widehat{\mathcal{L}}_\Lambda \left(\Delta A^\mu{}_M\right)_{bckg} = 0, \tag{3.82}$$

where $\left(\Delta A^\mu{}_M\right)_{bckg}$ stands for the equations of motion evaluated on background fields, and we used that background fields are by definition constrained to satisfy the EOM.

This concludes the proof on how the tensor hierarchy in KK-DFT fits into an L_∞ algebra. We remark that it is not enough to restrict the analysis to the tensor hierarchy components of the different graded subspaces, but one should also consider other components that might contribute through dependence on background fields, even if their perturbations are ignored. The same situation arises when one considers the list $(\zeta, \Psi_1, \Psi_2, \dots)$: the corresponding L_∞ identity can be seen as coming from the gauge transformation of the equations of motion, and we have made use of the gauge transformations over all fields (including $\mathcal{M}_{PQ}$, d and $g_{\mu\nu}$) in order to get the products in (3.78).

3.4 Redefinitions

When discussing KK-DFT we considered two sets of parameters. Ones with a hat $(\widehat{\Lambda}^M, \widehat{\Xi}_\mu)$ that are convenient to write transformations covariantly with respect to internal generalized diffeomorphisms. This set is interesting as it is the one usually considered in Exceptional Field Theories and gauged supergravities. A property of this set of parameters is that, as shown before, the brackets with respect to which the gauge transformations close are field dependent, and the Jacobiator is not a trivial gauge parameter, and so identifying symmetries for symmetries in this set is a rather non-trivial task. We circumvented these issues by performing the redefinitions (3.3) that connect the hatted parameter with the un-hatted ones, which have the disadvantage of spoiling the manifest generalized diffeomorphism covariance of the gauge transformations, but are the specific set in which the brackets are field independent and the Jacobiator is a trivial parameter, making them easier to analyze in the context of L_∞ algebras.

Given that the redefinition (3.3) is field dependent, the graded subspace to which the hatted parameters belong must involve mixing between the graded subspaces of the other set of parameters. Both sets define equivalent algebras and so there must exist an

JHEP01(2019)117

isomorphism that connects them. Here we discuss how things fit into L_∞ in the hatted "frame". We start by considering the gauge transformations in the form of interest, given by

$$\begin{aligned}\delta_\zeta A_\mu{}^M &= \mathcal{D}_\mu \Lambda^M + \partial^M \widehat{\Xi}_\mu \\ \delta_\zeta B_{\mu\nu} &= 2\mathcal{D}_{[\mu}\widehat{\Xi}_{\nu]} - \Lambda_M \mathcal{F}_{\mu\nu}{}^M + A_{[\mu}{}^M \delta A_{\nu]M}\end{aligned} \tag{3.83}$$

from which we can identify the non-trivial products $\ell_1(\zeta)$, $\ell_2(\zeta,\Psi)$, $\ell_3(\zeta,\Psi_1,\Psi_2)$, $\ell_4(\zeta,\Psi_1,\Psi_2,\Psi_3)$.

We saw that closure

$$[\delta_1,\delta_2] = -\delta_{\Lambda_{12}} - \delta_{\widehat{\Xi}_{12}}\,, \tag{3.84}$$

holds with respect to field dependent brackets

$$\begin{aligned}\Lambda_{12}^M &= [\Lambda_1,\Lambda_2]_{(C)}^M \\ \widehat{\Xi}_{12\mu} &= \mathcal{D}_\mu \Lambda_{[1}^M \Lambda_{2]M}\,.\end{aligned}$$

Comparing with (2.25) we now get the identifications

$$\begin{aligned}\ell_2(\zeta_1,\zeta_2) &= [\Lambda_1,\Lambda_2]_{(C)}^M + \partial_\mu \Lambda_{[1}^M \Lambda_{2]M} \\ \ell_3(\zeta_1,\zeta_2,A) &= -\widehat{\mathcal{L}}_{A_\mu} \Lambda_{[1}^M \Lambda_{2]M}\,,\end{aligned} \tag{3.85}$$

which differ from the previous ones, the novelty being that we have a non-vanishing $\ell_3(\zeta_1,\zeta_2,\Psi)$, as expected for a field dependent bracket. There are however certain products that still take the same value. For example, consider the identity for $n=3$

$$\begin{aligned}-\ell_1(\ell_3(\zeta_1,\zeta_2,\zeta_3)) = \ell_2(\ell_2(\zeta_1,\zeta_2),\zeta_3) + \ell_2(\ell_2(\zeta_3,\zeta_1),\zeta_2) + \ell_2(\ell_2(\zeta_2,\zeta_3),\zeta_1) \\ + \ell_3(\ell_1(\zeta_1),\zeta_2,\zeta_3) + \ell_3(\zeta_1,\ell_1(\zeta_2),\zeta_3) + \ell_3(\zeta_1,\zeta_2,\ell_1(\zeta_3))\,.\end{aligned} \tag{3.86}$$

This time, the $\ell_2\ell_2$ terms on the r.h.s. of the first line are not anymore the Jacobiator of the bracket, because the bracket now depends on fields, and now the second line does not vanish. It would be convenient to separate this expression according to its index structure. For the M component, we get

$$[\ell_1(\ell_3(\zeta_1,\zeta_2,\zeta_3))]^M = -\mathrm{Jac}(\Lambda_1,\Lambda_2,\Lambda_3) = -\partial^M N(\Lambda_1,\Lambda_2,\Lambda_3)\,, \tag{3.87}$$

suggesting

$$\begin{aligned}\ell_3(\zeta_1,\zeta_2,\zeta_3) &= -N(\Lambda_1,\Lambda_2,\Lambda_3), && \in X_1 \\ [\ell_1(\chi)]^M &= \partial^M \chi, && \chi \in X_1\,.\end{aligned} \tag{3.88}$$

For the μ part we get

$$\begin{aligned}[\ell_1(\ell_3(\zeta_1,\zeta_2,\zeta_3))]_\mu = -\frac{3}{2}\partial_\mu \left[\Lambda_{[1},\Lambda_2\right]_{(C)}^P \Lambda_{3]P} + \frac{3}{2}\partial_\mu \Lambda_{[3\,P}\left[\Lambda_1,\Lambda_{2]}\right]_{(C)}^P \\ + 3\partial_\mu \Lambda_{[1}^P \partial_P \Lambda_2^N \Lambda_{3]N} + 6\Lambda_{[3}^P \partial_P \partial_\mu \Lambda_{1N} \Lambda_{2]}^N\,,\end{aligned} \tag{3.89}$$

JHEP01(2019)117

which can be rewritten as[7]

$$[\ell_1(\ell_3(\zeta_1,\zeta_2,\zeta_3))]_\mu = \partial_\mu N(\Lambda_1,\Lambda_2,\Lambda_3)\,. \tag{3.91}$$

Thus we see that this identity gives the same identifications we had before

$$\begin{aligned} \ell_3(\zeta_1,\zeta_2,\zeta_3) &= -N(\Lambda_1,\Lambda_2,\Lambda_3), \quad \in X_1 \\ \ell_1(\chi) &= \mathcal{D}\chi, \quad \chi \in X_1\,. \end{aligned} \tag{3.92}$$

Even though now we needed a non-trivial $\ell_3\ell_1$ to complement $\ell_2\ell_2$ to obtain the above identifications, we emphasize that now $\ell_1(\chi)$ is *not* a trivial gauge parameter. Instead, the product form of the trivial parameter must be read from (2.32).

4 L_∞ and $E_{7(7)}$ Exceptional Field Theory

In this section we present a self-contained review of $E_{7(7)}$ EFT [40, 41] and discuss how it fits into L_∞. The fields and gauge parameters depend on external and internal coordinates $(x^\mu\,, X^M)$, with μ an external GL(n) index, and M an internal fundamental $E_{7(7)}$ index. The invariant tensors are the generators $(t_\alpha)_M{}^N$ and the symplectic metric Ω_{MN}, which is antisymmetric and raises and lowers internal indices through the north-west south-east convention[8]

$$V^M = \Omega^{MN}V_N\,, \quad V_M = V^N\Omega_{NM}\,, \quad \Omega^{MK}\Omega_{KN} = -\delta^M_N\,. \tag{4.3}$$

Adjoint indices are instead raised and lowered with the *Cartan-Killing* form $\kappa_{\alpha\beta} = (t_\alpha)_M{}^N(t_\beta)_N{}^M$, and the $E_{7(7)}$ structure constants can be read from $[t_\alpha, t_\beta] = f_{\alpha\beta}{}^\gamma t_\gamma$. The adjoint representation is symmetric for $E_{7(7)}$, i.e. $(t_\alpha)_{[MN]} = 0$.

The internal coordinate dependence of fields and gauge parameters is restricted by the "section condition", which states the vanishing of the following duality covariant Laplacians

$$\Omega^{MN}\,\partial_M\otimes\partial_N\cdots = 0\,, \quad (t_\alpha)^{MN}\,\partial_M\otimes\partial_N\cdots = 0\,. \tag{4.4}$$

4.1 Exceptional generalized diffeomorphisms

Just so the paper is self-contained, here we discuss very briefly the structure of $E_{7(7)}$ covariant generalized diffeomorphisms [40, 41, 64, 65]. The generalized Lie derivative acting on a contravariant vector is given by

$$\widehat{\mathcal{L}}_\Lambda V^M = \Lambda^K\partial_K V^M - 12(t^\alpha)^M{}_N(t_\alpha)^K{}_L\,\partial_K\Lambda^L\,V^N + \lambda(V)\,\partial_K\Lambda^K\,V^M\,, \tag{4.5}$$

[7] The following identity is useful for this computation

$$\begin{aligned} \alpha\partial_\mu\left[\Lambda_{[1},\Lambda_2\right]^P_{(C)}\Lambda_{3]P} + \beta\partial_\mu\Lambda_{[3\,P}\left[\Lambda_1,\Lambda_{2]}\right]^P_{(C)} &= (2\alpha+\beta)\partial_\mu\Lambda^Q_{[1}\partial_Q\Lambda^P_2\Lambda_{3]P} \\ &\quad + 3\alpha\Lambda^Q_{[1}\partial_\mu\partial_Q\Lambda^P_2\Lambda_{3]P} - (\alpha+2\beta)\partial_\mu\Lambda^Q_{[1}\partial^P\Lambda_{2Q}\Lambda_{3]P}\,. \end{aligned} \tag{3.90}$$

[8] The only two $E_{7(7)}$ identities that are needed to reproduce the computations in this section are

$$(t_\alpha)_M{}^K(t^\alpha)_N{}^L = \frac{1}{24}\delta^K_M\delta^L_N + \frac{1}{12}\delta^L_M\delta^K_N + (t_\alpha)_{MN}(t^\alpha)^{KL} - \frac{1}{24}\Omega_{MN}\Omega^{KL}\,, \tag{4.1}$$

$$\begin{aligned} 0 = (t_\alpha)_{NL}(t^\alpha)^{M(K}(t_\beta)^{Q)L} - \frac{1}{24}(t_\beta)_N{}^{(K}\Omega^{Q)M} + \frac{1}{12}(t_\beta)^{M(K}\delta^{Q)}_N \\ - \frac{1}{2}(t^\alpha)^{ML}(t_\alpha)^{KQ}(t_\beta)_{NL} + \frac{1}{2}(t_\beta)^{ML}(t^\alpha)^{KQ}(t_\alpha)_{NL} + \frac{1}{24}(t_\beta)^{KQ}\delta^M_N\,. \end{aligned} \tag{4.2}$$

JHEP01(2019)117

with a natural generalization to higher rank tensors. The way this transformation acts on covariant tensors follows from the observation that the symplectic metric Ω_{MN} is invariant under generalized diffeomorphisms. The same is true for the generators

$$\widehat{\mathcal{L}}_\Lambda \Omega_{MN} = 0\,, \qquad \widehat{\mathcal{L}}_\Lambda (t^\alpha)_{MN} = 0\,. \tag{4.6}$$

There are two types of trivial parameters

$$\Lambda^M_{\text{trivial}} = (t^\alpha)^{MN}\partial_N T_\alpha\,, \qquad \Lambda^M_{\text{trivial}} = \Omega^{MN} T_N\,, \tag{4.7}$$

with T_M a covariantly constrained vector in the sense of the section condition

$$(t^\alpha)^{MN}\, T_M\, T_N = (t^\alpha)^{MN}\, T_M\, \partial_N \cdots = \Omega^{MN} T_M \otimes T_N = \Omega^{MN} T_M\, \partial_N \cdots = 0\,. \tag{4.8}$$

The symmetric part of the generalized Lie derivative with weight $\lambda = \frac{1}{2}$ turns out to be a trivial parameter

$$\{\Lambda_1,\, \Lambda_2\} = \widehat{\mathcal{L}}^{\lambda=\frac{1}{2}}_{\Lambda_{(1}}\, \Lambda_{2)} = 12(t^\alpha)^{MN}\partial_N \gamma_\alpha + \frac{1}{2}\Omega^{MN}\,\gamma_N\,, \tag{4.9}$$

with

$$\gamma_\alpha = -\frac{1}{2}(t_\alpha)_{KL}\Lambda_1^K \Lambda_2^L\,, \qquad \gamma_M = \Omega_{KL}\,\Lambda^K_{(1}\partial_M \Lambda^L_{2)}\,. \tag{4.10}$$

The antisymmetric part is instead the E-bracket with respect to which the generalized Lie derivative algebra closes

$$[\Lambda_1, \Lambda_2]^M_{(E)} = \widehat{\mathcal{L}}^{\lambda=\frac{1}{2}}_{\Lambda_{[1}}\Lambda_{2]} = 2\Lambda^K_{[1}\partial_K \Lambda^M_{2]} - 12(t^\alpha)^{MN}(t_\alpha)_{KL}\Lambda^K_{[2}\partial_N \Lambda^L_{1]} - \frac{1}{4}\Omega^{MN}\Omega_{KL}\partial_N\left(\Lambda_1^K \Lambda_2^L\right). \tag{4.11}$$

In this sense, $\widehat{\mathcal{L}}^{\lambda=\frac{1}{2}}_{\Lambda_1}\Lambda_2$ plays the role of an exceptional D-bracket.

It was shown in [40, 41] that from the definitions above, it follows that a tensor in the adjoint representation T_α transforms as

$$\delta_\Lambda T_\alpha = \Lambda^K \partial_K T_\alpha + 12 f_{\alpha\beta}{}^\gamma (t^\beta)_L{}^K \partial_K \Lambda^L\, T_\gamma + \lambda'(T)\partial_K \Lambda^K T_\alpha\,. \tag{4.12}$$

Another useful result is that a trivial vector

$$\widetilde{T}^M = 12(t^\alpha)^{MN}\partial_N T_\alpha + \frac{1}{2}\Omega^{MN} T_N\,, \tag{4.13}$$

transforms as a vector of weight $\lambda(\widetilde{T}) = \frac{1}{2}$, provided $\lambda'(T_\alpha) = 1$ and the covariantly constrained field T_M transforms as follows

$$\delta_\Lambda T_M = \Lambda^K \partial_K T_M + 12(t^\alpha)^N{}_M (t_\alpha)^K{}_L\, \partial_K \Lambda^L\, T_N + \frac{1}{2}\partial_K \Lambda^K\, T_M - 24(t^\alpha)_L{}^K\, T_\alpha\, \partial_M \partial_K \Lambda^L\,. \tag{4.14}$$

Note that the transformation of T_M depends on T_α so both fields must always be considered jointly, forming a covariant pair $(T_M,\, T_\alpha)$.

We emphasize that every vector with a tilde in this paper indicates that it is of the trivial form (4.13), and as such must be considered constrained and with weight $\lambda = \frac{1}{2}$. Let

JHEP01(2019)117

us also point out that if the vectors Λ_1 and Λ_2 in (4.10) were transformed with respect the generalized Lie derivative with weight $\lambda = \frac{1}{2}$, then γ_α and γ_M would transform as (4.12) and (4.14) respectively, as expected.

At this point, we would like to make an observation that will be crucial in the next sections. Given the following adjoint and covariantly constrained tensors

$$\begin{aligned} \eta[\Lambda, T]_\alpha &= \delta_\Lambda T_\alpha + (t_\alpha)_{KL} \widetilde{T}^K \Lambda^L \,, \\ \eta[\Lambda, T]_M &= \delta_\Lambda T_M + \Omega_{KL} \left(\partial_M \widetilde{T}^K \Lambda^L - \widetilde{T}^K \partial_M \Lambda^L \right) \,, \end{aligned} \tag{4.15}$$

where again Λ^M is a vector, T_α is an adjoint tensor of weight $\lambda'(T) = 1$ and T_M is a covariantly constrained tensor, the following trivial parameter vanishes

$$\widetilde{\eta}^M = 12 (t^\alpha)^{MN} \partial_N \eta_\alpha + \frac{1}{2} \Omega^{MN} \eta_N = 0 \,, \tag{4.16}$$

as can easily be seen from the following rewriting

$$\widetilde{\eta} = \widehat{\mathcal{L}}_\Lambda \widetilde{T} - 2 \left\{ \widetilde{T}, \Lambda \right\} = - \widehat{\mathcal{L}}_{\widetilde{T}} \Lambda = 0 \,. \tag{4.17}$$

Let us now state what the Jacobiator looks like in terms of the above defined quantities

$$J(\Lambda_1, \Lambda_2, \Lambda_3) = 3 \left[[\Lambda_{[1}, \Lambda_2]_{(E)}, \Lambda_{3]} \right]_{(E)} = \widetilde{N} \,, \tag{4.18}$$

with $\widetilde{N}$ given by

$$\widetilde{N} = \left\{ \Lambda_{[1}, \left[\Lambda_2, \Lambda_{3]} \right]_{(E)} \right\} \,, \tag{4.19}$$

whose "Nijenhuis" components are

$$\begin{aligned} N_\alpha &= - \frac{1}{2} (t_\alpha)_{KL} \Lambda^K_{[1} \left[\Lambda_2, \Lambda_{3]} \right]^L_{(E)} \\ N_M &= \frac{1}{2} \Omega_{KL} \left(\Lambda^K_{[1} \partial_M \left[\Lambda_2, \Lambda_{3]} \right]^L_{(E)} - \partial_M \Lambda^K_{[1} \left[\Lambda_2, \Lambda_{3]} \right]^L_{(E)} \right) \,. \end{aligned} \tag{4.20}$$

Of course, these components cannot in general be disentangled due to the ambiguity discussed in (4.15), namely that they can be shifted in such a way to leave $\widetilde{N}$ invariant. However, in this particular case we will now argue that there is no redundancy. Taking a close look into (4.15) we see that it depends on three objects: a vector Λ, an adjoint tensor T_α and a covariantly constrained tensor T_M. In order to shift the components in (4.20) such that $\widetilde{N}$ in (4.19) remains invariant, we must find expressions for Λ^M, T_α and T_M depending only on Λ_1, Λ_2 and Λ_3. The only possibility that preserves the way in which the tensors transform is

$$\Lambda^M = \Lambda^M_i \,, \quad T_\alpha = - \frac{1}{2} (t_\alpha)_{KL} \Lambda^K_j \Lambda^L_k \,, \quad T_M = \Omega_{KL} \Lambda^K_{(j} \partial_M \Lambda^L_{k)} \,, \quad i,j,k = 1,2,3 \,, \tag{4.21}$$

but the antisymmetry in [123] eliminates it, leaving (4.20) as the only possible choice.

We end this brief subsection by noting that given a trivial vector $\widetilde{T}$, one has

$$\widehat{\mathcal{L}}_{\widetilde{T}} \Lambda = 0 = \left\{ \widetilde{T}, \Lambda \right\} + [\widetilde{T}, \Lambda]_{(E)} \,, \tag{4.22}$$

so the E-bracket between a trivial and a generic vector is itself a trivial vector. This observation will be useful later.

JHEP01(2019)117

4.2 The theory and its projected tensor hierarchy

The fields in $E_{7(7)}$ EFT [40, 41] are

$$e_\mu{}^a\,, \quad B_{\mu\nu\alpha}\,, \quad B_{\mu\nu M}\,, \quad A_\mu{}^M\,, \quad \mathcal{M}_{MN}\,, \tag{4.23}$$

where the scalar matrix $\mathcal{M}_{MN}$ is again a duality group-valued constrained field. Note that now the two-form field is non-Abelian, and takes values in the adjoint representation. In addition, this EFT requires extra two-forms $B_{\mu\nu M}$, which are covariantly constrained as in (4.8).

A parent background independent action based on a larger duality group (the analog of $O(10,10)$ for DFT) remains unknown in the case of exceptional duality (though see [66]). Constructions for the internal scalar sector were originally considered in [54–63]. One must then build the action as envisioned by O. Hohm and H. Samtleben [40, 41], by demanding that "internal" generalized diffeomorphisms behave as gauge symmetries in four dimensions and extending the KK-DFT framework to be compatible with exceptional duality symmetries and maximal supergravity (for ExFTs in other dimensions see [67–77]). In total, the bosonic action enjoys the following symmetries:

- A global continuous $E_{7(7)}$ symmetry.
- A local $\mathrm{SO}(1,3|\mathbb{R}) \times \mathrm{SU}(8|\mathbb{R})$ which is trivial in the bosonic sector.
- External improved diffeomorphisms, infinitesimally parameterized by ξ^μ.
- Internal generalized diffeomorphisms, inf. parameterized by Λ^M.
- Gauge transformations of the two-forms, parameterized by $\Xi_{\mu\alpha}$ and $\Xi_{\mu M}$.

As in DFT, there is a set of parameters that allows to write the gauge transformations in a covariant form, and another set that gives rise to field-independent brackets. As before, we note the former with a hat $\widehat{\Lambda}^M$, $\widehat{\Xi}_{\mu\alpha}$, $\widehat{\Xi}_{\mu M}$, and so on, and the later without hat. Also as before, and for the same reasons, we will restrict attention to the tensor hierarchy, and so will ignore external improved diffeos, and set the vielbein and scalars to background values, ignoring the perturbations around them. The way the transformations of the fields were obtained by Hohm and Samtleben are the following. The vector fields transform such that the derivative $\mathcal{D}_\mu = \partial_\mu - \widehat{\mathcal{L}}_{A_\mu}$ is covariant with respect to generalized diffeomorphisms. Given that it appears here as the argument of a generalized Lie derivative, this only determines the transformation of $A_\mu{}^M$ up to terms of the form (4.7) which we note with $\widehat{\Xi}_\mu$ bellow

$$\delta A_\mu{}^M = \mathcal{D}_\mu \Lambda^M + 12(t^\alpha)^{MN}\partial_N \widehat{\Xi}_{\mu\alpha} + \frac{1}{2}\Omega^{MN}\widehat{\Xi}_{\mu N}\,. \tag{4.24}$$

These of course will become the gauge parameters of the two-forms. We have written a hat on them to indicate that this is the set for which the gauge transformations are

JHEP01(2019)117

covariant with respect to internal diffeos, as before in DFT. Later one defines a covariant field strength as follows

$$\mathcal{F}_{\mu\nu}{}^{M} = 2\partial_{[\mu}A_{\nu]}{}^{M} - [A_\mu,\, A_\nu]^{M}_{(E)} - 12(t^\alpha)^{MN}\partial_N B_{\mu\nu\alpha} - \frac{1}{2}\Omega^{MN}B_{\mu\nu N}\,, \tag{4.25}$$

which transforms as

$$\delta\mathcal{F}_{\mu\nu}{}^{M} = 2\mathcal{D}_{[\mu}\delta A_{\nu]}{}^{M} - 12(t^\alpha)^{MN}\partial_N \Delta B_{\mu\nu\alpha} - \frac{1}{2}\Omega^{MN}\Delta B_{\mu\nu N}\,, \tag{4.26}$$

where $\lambda(\delta A) = \frac{1}{2}$ and we used the variation symbol Δ to isolate the covariant part of the transformation of the two-form

$$\Delta B_{\mu\nu\alpha} = \delta B_{\mu\nu\alpha} + (t_\alpha)_{KL}A_{[\mu}{}^{K}\delta A_{\nu]}{}^{L} \tag{4.27}$$

$$\Delta B_{\mu\nu M} = \delta B_{\mu\nu M} + \Omega_{KL}\left(A_{[\mu}{}^{L}\partial_M \delta A_{\nu]}{}^{K} - \partial_M A_{[\mu}{}^{L}\delta A_{\nu]}{}^{K}\right)\,. \tag{4.28}$$

By demanding covariance of the field strength one can obtain the gauge transformations of the two-forms from those of the gauge vector. There is a caveat though. Notice that the transformations of the two-forms enter the variation of the field strength (4.26) exactly in the same way as in (4.16). There, we made the observation that the combination $(t^\alpha)^{MN}\partial_M B_{\mu\nu\alpha}$ is not completely independent from $\Omega^{MN}B_{\mu\nu N}$. Then, it is not a priory possible to isolate the transformations $\Delta B_{\mu\nu\alpha}$ from those of $\Delta B_{\mu\nu M}$. At this point one should instead consider a combined two-form

$$\widetilde{B}_{\mu\nu M} = 12(t^\alpha)_M{}^{N}\partial_N B_{\mu\nu\alpha} + \frac{1}{2}B_{\mu\nu M}\,, \tag{4.29}$$

such that the ambiguity is eliminated, with its corresponding gauge parameter

$$\widetilde{\widehat{\Xi}}_{\mu M} = 12(t^\alpha)_M{}^{N}\partial_N \widehat{\Xi}_{\mu\alpha} + \frac{1}{2}\widehat{\Xi}_{\mu M}\,, \tag{4.30}$$

and then demanding that $\delta\mathcal{F}_{\mu\nu}{}^{M} = \widehat{\mathcal{L}}_\Lambda \mathcal{F}_{\mu\nu}{}^{M}$ the unambiguous fields and their gauge transformations are

$$\delta A_\mu{}^{M} = \mathcal{D}_\mu \Lambda^M + \Omega^{MN}\widetilde{\widehat{\Xi}}_{\mu N} \tag{4.31}$$

$$\begin{aligned}\Delta\widetilde{B}_{\mu\nu M} &= 2\mathcal{D}_{[\mu}\widetilde{\widehat{\Xi}}_{\nu]M} - 2\left\{\Lambda\,,\mathcal{F}_{\mu\nu}\right\}_M \\ &= 2\mathcal{D}_{[\mu}\widetilde{\widehat{\Xi}}_{\nu]M} + 12(t^\alpha)_M{}^{N}\partial_N\left[(t_\alpha)_{KL}\Lambda^K\mathcal{F}_{\mu\nu}{}^{L}\right] + \frac{1}{2}\left[\Omega_{KL}\left(\Lambda^L\partial_M\mathcal{F}_{\mu\nu}{}^{K} - \partial_M\Lambda^L\mathcal{F}_{\mu\nu}{}^{K}\right)\right],\end{aligned}$$

where we used the symmetrization $\{\}$ in (4.9) to show that the transformation is constrained in the same way as the field, and defined the covariant transformation $\Delta\widetilde{B}_{\mu\nu M} = \delta\widetilde{B}_{\mu\nu M} - 2\left\{A_{[\mu},\,\delta A_{\nu]}\right\}_M$. The expressions between brackets cannot be directly assigned to the Λ-transformations of the components of $\widetilde{B}_{\mu\nu M}$, they must be considered always in this combination. The naive separation would lead to the failure of closure of the gauge transformations, the absence of trivial parameters, etc. We will discuss later how to properly disentangle the components and their transformations.

JHEP01(2019)117

The Bianchi identities

$$3\mathcal{D}_{[\mu}\mathcal{F}_{\nu\rho]}{}^{M} + \widetilde{\mathcal{H}}_{\mu\nu\rho M} = 0 \tag{4.32}$$

$$\mathcal{D}_{[\mu}\widetilde{\mathcal{H}}_{\nu\rho\sigma]M} - \frac{3}{2}\left\{\mathcal{F}_{[\mu\nu},\, \mathcal{F}_{\rho\sigma]}\right\}_{M} = 0\,, \tag{4.33}$$

define the three-form field strength

$$\widetilde{\mathcal{H}}_{\mu\nu\rho M} = 3\left(\mathcal{D}_{[\mu}\widetilde{B}_{\nu\rho]M} + 2\left\{A_{[\mu}\,,\, \partial_{\nu}A_{\rho]} - \frac{1}{3}[A_{\nu},\, A_{\rho}]_{(E)}\right\}_{M}\right)\,, \tag{4.34}$$

which also is constrained in the same way as the tilded tensors and transforms as a vector under generalized diffeomorphisms, as can be easily computed from

$$\delta\widetilde{H}_{\mu\nu\rho M} = 3\mathcal{D}_{[\mu}\Delta\widetilde{B}_{\nu\rho]M} + 6\left\{\delta A_{[\mu},\, \mathcal{F}_{\nu\rho]}\right\}_{M}\,. \tag{4.35}$$

The diffeomorphism covariant transformations (4.31) admit the following trivial parameters

$$\Lambda^{M} = 12(t^{\alpha})^{MN}\partial_{N}\chi_{\alpha} + \frac{1}{2}\Omega^{MN}\chi_{N} = \Omega^{MN}\widetilde{\chi}_{N} \tag{4.36}$$

$$\widetilde{\widetilde{\Xi}}_{\mu M} = -\mathcal{D}_{\mu}\widetilde{\chi}_{M}\,, \tag{4.37}$$

where χ_{α} is a generic function in the adjoint of $E_{7(7)}$, χ_{M} a covariantly constrained function, such that $\widetilde{\chi}_{M}$ carries weight $\lambda = \frac{1}{2}$. The situation is analogous to that in DFT, where there is a set of parameters (noted with a hat) with respect to which the gauge transformations can be written in a generalized diffeomorphism covariant form, and such that the trivial set of parameters is field dependent. For this set of parameters the brackets with respect to which the algebra closes are field dependent, in full analogy with DFT, so we find it convenient to turn to a redefined set of non-covariant parameters $(\Lambda^{M},\, \widetilde{\Xi}_{\mu M})$

$$\widehat{\widetilde{\Xi}}_{\mu M} = \widetilde{\Xi}_{\mu M} + 2\left\{\Lambda,\, A_{\mu}\right\}_{M}\,, \tag{4.38}$$

such that for this set the trivial parameters are field independent

$$\begin{aligned}\Lambda^{M} &= \Omega^{MN}\widetilde{\chi}_{N}\,,\\ \widetilde{\Xi}_{\mu M} &= -\partial_{\mu}\widetilde{\chi}_{M}\,.\end{aligned} \tag{4.39}$$

The gauge transformations are now given by

$$\begin{aligned}\delta A_{\mu}{}^{M} &= \partial_{\mu}\Lambda^{M} + \widehat{\mathcal{L}}_{\Lambda}A_{\mu}{}^{M} + \Omega^{MN}\widetilde{\Xi}_{\mu N}\,,\\ \delta\widetilde{B}_{\mu\nu M} &= 2\partial_{[\mu}\widetilde{\Xi}_{\nu]M} + \widehat{\mathcal{L}}_{\Lambda}\widetilde{B}_{\mu\nu M} + 2\left\{\Sigma_{[\mu}\,,\, A_{\nu]}\right\}_{M}\,,\end{aligned} \tag{4.40}$$

where we defined

$$\Sigma_{\mu}{}^{M} = \Omega^{MN}\widetilde{\Xi}_{\mu N} + \partial_{\mu}\Lambda^{M}\,. \tag{4.41}$$

These transformations close

$$[\delta_{1}\,,\, \delta_{2}] = -\delta_{\Lambda_{12}} - \delta_{\widetilde{\Xi}_{12}}\,, \tag{4.42}$$

JHEP01(2019)117

with respect to field independent brackets

$$\begin{aligned}\Lambda_{12}^{M} &= [\Lambda_1,\,\Lambda_2]_{(E)}^{M}\,,\\ \widetilde{\Xi}_{12\mu M} &= 2\widehat{\mathcal{L}}_{\Lambda_{[1}}\widetilde{\Xi}_{2]\mu M} - 2\left\{\partial_\mu\Lambda_{[1}\,,\,\Lambda_{2]}\right\}\,.\end{aligned} \tag{4.43}$$

Merging both parameters into a single one, and defining the brackets

$$\zeta = \left(\Lambda^M,\,\widetilde{\Xi}_{\mu M}\right)\,,\qquad [\zeta_1,\,\zeta_2] = \left(\Lambda_{12}^M,\,\widetilde{\Xi}_{12\mu M}\right)\,, \tag{4.44}$$

the Jacobiator is given by

$$J = 3\,\left[\left[\zeta_{[1},\,\zeta_2\right]\,,\,\zeta_{3]}\right] = \left(\widetilde{N}^M,\,-\partial_\mu\widetilde{N}_M\right)\,, \tag{4.45}$$

with $\widetilde{N}$ defined in (4.19).

4.3 The L_∞ structure of the $E_{7(7)}$ projected tensor hierarchy

We begin here discussing the simplest case of the E-bracket gauge algebra and its formulation in terms of the L_∞ structure. As we will show, we only need to define two graded subspaces, which are

$$\begin{aligned} X_1:&\quad \text{functions:}\quad \chi = \chi_\alpha + \chi_N\,,\\ X_0:&\quad \text{gauge parameters:}\quad \Lambda^M\,.\end{aligned} \tag{4.46}$$

An arbitrary element $\chi \in X_1$ splits into two different parts: χ_α belonging to the adjoint representation **133** of $E_{7(7)}$, and a covariantly constrained function χ_N. The non-vanishing products are

$$\begin{aligned}\ell_2(\Lambda_1,\Lambda_2) &= [\Lambda_1,\Lambda_2]_{(E)}\,, && \in X_0\\ \ell_3(\Lambda_1,\Lambda_2,\Lambda_3) &= -N_\alpha - N_M, && \in X_1\\ \ell_1(\chi) &= \;12(t^\alpha)^{MN}\partial_N\chi_\alpha + \frac{1}{2}\Omega^{MN}\chi_N, && \in X_0\\ \ell_2(\Lambda,\chi) &= \frac{1}{2}\delta_\Lambda\chi_\alpha + \frac{1}{2}\delta_\Lambda\chi_N && \in X_1\end{aligned} \tag{4.47}$$

where we defined the "Nijenhuis" tensors in (4.20).

The construction goes as follows. We start by defining the graded spaces. Initially, we only define X_0 as the space of gauge parameters Λ^M. Next we identify

$$\ell_2(\Lambda_1,\Lambda_2) = [\Lambda_1,\Lambda_2]_{(E)}\,, \tag{4.48}$$

and we set $\ell_1(\Lambda) = 0$, so that the identities $\ell_1^2 = 0$ and $\ell_1\ell_2 - \ell_2\ell_1 = 0$ acting over gauge parameters are trivially fullfilled. The first non trivial identity we must verify is

$$\begin{aligned}-\ell_1(\ell_3(\Lambda_1,\Lambda_2,\Lambda_3)) = {}&\ell_2(\ell_2(\Lambda_1,\Lambda_2),\Lambda_3)+\ell_2(\ell_2(\Lambda_3,\Lambda_1),\Lambda_2)+\ell_2(\ell_2(\Lambda_2,\Lambda_3),\Lambda_1)\\ &+\ell_3(\ell_1(\Lambda_1),\Lambda_2,\Lambda_3)+\ell_3(\Lambda_1,\ell_1(\Lambda_2),\Lambda_3)+\ell_3(\Lambda_1,\Lambda_2,\ell_1(\Lambda_3))\,.\end{aligned} \tag{4.49}$$

On the first line of the r.h.s. we recognize the Jacobiator, which for the E-bracket is given by (4.19)

$$J^M(\Lambda_1,\Lambda_2,\Lambda_3) \;\equiv\; 3\left[\left[\Lambda_{[1},\Lambda_2\right],\Lambda_{3]}\right]^M = 12(t^\alpha)^{MN}\partial_N N_\alpha + \frac{1}{2}\Omega^{MN}N_M\,. \tag{4.50}$$

JHEP01(2019)117

Comparing the expressions (4.50) and (4.49) we get the identifications

$$\begin{aligned}\ell_3(\Lambda_1,\Lambda_2,\Lambda_3) &= -N_\alpha - N_M,\\ \ell_1(\chi) &= 12(t^\alpha)^{MN}\partial_N\chi_\alpha + \frac{1}{2}\Omega^{MN}\chi_N \equiv \widetilde{\chi}^M\,.\end{aligned} \tag{4.51}$$

Here we continue using the notation that a tilde over a tensor means that it takes the form of a trivial parameter (4.13). In order to accomplish this we have defined a new space X_1 because there is a nontrivial $\ell_3(\Lambda_1,\Lambda_2,\Lambda_3)$ which maps into this graded subspace. The elements belonging to this subspace will be denoted as χ. As already pointed out, $\chi \in X_1$ splits into two different parts: χ_α, which belongs to the adjoint representation of $E_{7(7)}$, and χ_N, a covariantly constrained field.

Now that we have this new space we have to reconsider the identity $\ell_1\ell_2 - \ell_2\ell_1 = 0$ acting on elements belonging to this new subspace

$$\ell_1(\ell_2(\Lambda,\chi)) = \ell_2(\ell_1(\Lambda),\chi) + \ell_2(\Lambda,\ell_1(\chi))\,. \tag{4.52}$$

We take the first term in the r.h.s. to zero as it contains $\ell_1(\Lambda) \in X_{-1}$, the space of fields that we are ignoring for the moment. The second term in the r.h.s. takes the form of a trivial parameter by virtue of (4.22). This determines $\ell_2(\Lambda,\chi) \in X_1$, but only up to terms of the form (4.15), which belong to the kernel of ℓ_1

$$\begin{aligned}\ell_2(\Lambda,\chi)_\alpha &= \frac{1}{2}\delta_\Lambda\chi_\alpha + \dots\\ \ell_2(\Lambda,\chi)_M &= \frac{1}{2}\delta_\Lambda\chi_M + \dots\,.\end{aligned} \tag{4.53}$$

The dots represent the ambiguity discussed in (4.15), which is resolved by analyzing the $n = 4$ identity. We will discuss this soon and for the moment take (4.53) without the dots as our definition. We now show that the products defined so far are the only ones needed to guarantee the L_∞^{gauge} structure of the gauge algebra given by the E-bracket.

We now proceed with the $n = 2$ identity $\ell_1\ell_2 - \ell_2\ell_1 = 0$, but now over $\chi_1,\ \chi_2 \in X_1$:

$$\ell_1(\ell_2(\chi_1,\chi_2)) = \ell_2(\ell_1(\chi_1),\chi_2) + \ell_2(\ell_1(\chi_2),\chi_1)\,. \tag{4.54}$$

The r.h.s. of this identity is zero because $\ell_1(\chi)$ is trivial, and we already chose $\ell_2(\Lambda,\chi)$ to be identified with a generalized diffeomorphic transformation (4.53). Then we can safely take

$$\ell_2(\chi_1,\chi_2) = 0\,. \tag{4.55}$$

Next, we must check the $n = 4$ identity $\ell_1\ell_4 - \ell_2\ell_3 + \ell_3\ell_2 - \ell_4\ell_1 = 0$ in the case of four gauge parameters (this identity is trivially fulfilled when evaluated with arguments in X_1). First we note that $\ell_4\ell_1 = 0$ by crossing (2.32) with the fact that the Jacobiator is field independent (4.19). We now point out that $\ell_3\ell_2 - \ell_2\ell_3$ vanishes per se

$$\ell_3\ell_2 - \ell_2\ell_3 \ = 6\,\ell_3([\Lambda_{[1},\Lambda_2],\Lambda_3,\Lambda_{4]}) - 4\,\ell_2(\ell_3(\Lambda_{[1},\Lambda_2,\Lambda_3),\Lambda_{4]}) = 0\,, \tag{4.56}$$

a computation that can be checked with some effort. Then, there is no need to define a $\ell_4(\Lambda_1,\,\Lambda_2,\,\Lambda_3,\,\Lambda_4) = 0$. Let us however analyze what would have happened if we had chosen

JHEP01(2019)117

a different product in (4.53), by deforming our previous choice with arbitrary elements η_α and η_M in the kernel of ℓ_1 (see (4.15))

$$\ell_2(\Lambda,\chi)_\alpha = \frac{1}{2}\delta_\Lambda\chi_\alpha + \alpha\,\eta_\alpha \tag{4.57}$$

$$\ell_2(\Lambda,\chi)_M = \frac{1}{2}\delta_\Lambda\chi_M + \alpha\,\eta_N\,, \tag{4.58}$$

with

$$\begin{aligned}\eta[\Lambda,\,\chi]_\alpha &= \delta_\Lambda\,\chi_\alpha + (t_\alpha)_{KL}\,\widetilde{\chi}^K\,\Lambda^L\,,\\ \eta[\Lambda,\,\chi]_M &= \delta_\Lambda\chi_M + \Omega_{KL}\left(\partial_M\widetilde{\chi}^K\,\Lambda^L - \widetilde{\chi}^K\,\partial_M\Lambda^L\right)\,,\end{aligned} \tag{4.59}$$

the identity would read

$$\begin{aligned}\ell_1\ell_4 &= 6\,\ell_3([\Lambda_{[1},\Lambda_2],\Lambda_3,\Lambda_{4]}) - 4\,\ell_2(\ell_3(\Lambda_{[1},\Lambda_2,\Lambda_3),\Lambda_{4]})\\ &= 4\alpha\eta[\Lambda_{[1},\,N(\Lambda_2,\Lambda_3,\Lambda_{4]})]_\alpha + 4\alpha\eta[\Lambda_{[1},\,N(\Lambda_2,\Lambda_3,\Lambda_{4]})]_N\,,\end{aligned} \tag{4.60}$$

where we have used the vanishing of (4.56) when the old products (4.53) were considered, and the components of N were defined in (4.20). We see that in order to satisfy the identity, the $\ell_1\ell_4$ term must absorb the ambiguity. In consequence, one should then define a new graded subspace X_2 and take $\ell_4(\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4) = 4\alpha\eta_\alpha + 4\alpha\eta_N \in X_2$. Notice that so defined, $\ell_4(\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4)$ would live in the kernel of $\widetilde{\eta}(\eta_a,\eta_N) = 12(t^\alpha)^{MN}\partial_N\eta_\alpha + \frac{1}{2}\Omega^{MN}\eta_N$. One should then take X_2 as the space of functions that live in the kernel of $\widetilde{\eta}$ in order to fulfill the $n=1$ L_∞ identity, and also define the action of the ℓ_1 product over this subspace, which following [12] can be simply taken to be an inclusion map into the subspace X_1. In summary, the choice made for $\ell_2(\Lambda,\chi)$ in (4.53) is the simplest one in that requires a lower number of non-vanishing products and graded subspaces.

Including to this analysis the tensors coming from the tensor hierarchy is now an easy task. Following the same steps as before, we defined three graded subspaces

$$\begin{aligned} X_1: &\quad \text{functions:} &&\chi = \chi_\alpha + \chi_N\,,\\ X_0: &\quad \text{gauge parameters:} &&\zeta = \Lambda^M + \widetilde{\Xi}_{\mu M}\,.\\ X_{-1}: &\quad \text{fields:} &&\Psi = A_\mu{}^M + \widetilde{B}_{\mu\nu}\,.\end{aligned} \tag{4.61}$$

We have added a new graded subspace in order to incorporate the fields and their gauge transformations. The only nontrivial products are given by

$$\begin{aligned}
\ell_2(\zeta_1,\zeta_2) &= [\zeta_1,\zeta_2]\,, && \in X_0\\
\ell_3(\zeta_1,\zeta_2,\zeta_3) &= -N_\alpha - N_M, && \in X_1\\
\ell_1(\chi) &= 12(t^\alpha)^{MN}\partial_N\chi_\alpha + \frac{1}{2}\Omega^{MN}\chi_N - 12(t^\alpha)_M{}^N\partial_N\partial_\mu\chi_\alpha - \frac{1}{2}\partial_\mu\chi_M, && \in X_0\\
\ell_2(\zeta,\chi) &= \frac{1}{2}\delta_\Lambda\chi_\alpha + \frac{1}{2}\delta_\Lambda\chi_N, && \in X_1\\
\ell_1(\zeta) &= \partial_\mu\Lambda^M + \Omega^{MN}\widetilde{\Xi}_{\mu N} + 2\partial_{[\mu}\widetilde{\Xi}_{\nu]M}, && \in X_{-1}\\
\ell_2(\zeta,\Psi) &= \widehat{\mathcal{L}}_\Lambda A^M_\mu + \delta_\Lambda\widetilde{B}_{\mu\nu} - 12(t^\alpha)^N_M\partial_N\left((t_\alpha)_{LK}\left(\Omega^{KP}\widetilde{\Xi}_{[\mu P} + \partial_{[\mu}\Lambda^K\right)A^L_{\nu]}\right)\\
&\quad + \frac{1}{2}\Omega_{LK}\left(\partial_M A^L_{[\mu}\left(\Omega^{KP}\widetilde{\Xi}_{\nu]P} + \partial_{\nu]}\Lambda^K\right) - A^L_{[\mu}\partial_M\left(\Omega^{KP}\widetilde{\Xi}_{\nu]P} + \partial_{\nu]}\Lambda^K\right)\right) && \in X_{-1}
\end{aligned} \tag{4.62}$$

JHEP01(2019)117

We have simply added here the L_∞ products coming from the gauge transformations of the fields (4.40) and their trivial versions (4.39).

This completes the analysis on how the tensor hierarchy of $E_{7(7)}$ fits into $L_\infty^{\text{gauge+fields}}$. We do not consider here the full dynamical theory. Even if restricted to the tensor hierarchy degrees of freedom, we expect similar features as those arising in DFT to appear here. Namely, that the spaces and products defined so far must be extended to include components along the other fields and to depend on their background values, see for example (3.81).

4.4 The unprojected tensor hierarchy

So far the two-forms we considered (and the one-form gauge parameters) were a specific combination (or projection) of two different contributions. For a general discussion, let us consider generic projected tensors

$$\widetilde{T}^M = 12(t^\alpha)^{MN}\partial_N T_\alpha + \frac{1}{2}\Omega^{MN}T_N\,. \tag{4.63}$$

While T_α is a generic adjoint tensor, T_M is covariantly constrained (i.e. as if it were a derivative) and $\widetilde{T}^M$ is also constrained to be of the form above. We saw that when the two-forms $\widetilde{B}_M$ and their gauge parameters $\widetilde{\Xi}_\mu$ were grouped into this form, the hierarchy closes exactly. This projection is well known in the context of gauged supergravities, in which case the responsible is usually called the *intertwining tensor*. Here, the projector is given in terms of a differential operator, and so we will refer to it as the *intertwining operator*.

The particular projection discussed so far is the one entering the action of $E_{7(7)}$ EFT, a duality covariantization of maximal supergravity. Un-projecting points towards new M-theoretical degrees of freedom not present in the supergravity limit. A full hierarchy contains in principle a tower of p-forms in different representations $A \in R_1$, $B \in R_2$, $C \in R_3$, etc. Then, R_1 here is the **56** of $E_{7(7)}$. The intertwining operator noted with a tilde "$\widetilde{\ }$" maps between these spaces $\widetilde{\ } : R_{p+1} \to R_p$, such that $\widetilde{\widetilde{\ }} = 0$. So far we considered $\widetilde{B} \in R_1$, and here we are interested in $B \in R_2$ and $\widetilde{C} \in R_2$. One could in principle continue the hierarchy until the p-forms saturate the space-time dimensionality. Modulo covariantly constrained fields, the representations we are expecting here are $R_2 = \mathbf{133}$, $R_3 = \mathbf{912}$ and $R_4 = \mathbf{8645} \oplus \mathbf{133}$.

The intertwining operator has a non-vanishing kernel, and so it is not invertible. At this stage it might seem a little confusing, but we will call $(\widetilde{T}_\alpha, \widetilde{T}_M)$ the elements of the kernell of $\widetilde{T}^M$, namely

$$12(t^\alpha)^{MN}\partial_N\widetilde{T}_\alpha + \frac{1}{2}\Omega^{MN}\widetilde{T}_N = 0\,. \tag{4.64}$$

The reason of this notation will become clear soon. In the meantime it is important to keep in mind that $\widetilde{T}_M$ is not $\widetilde{T}^M$ with its index lowered, but simply a constrained tensor that belongs to the kernel of $\widetilde{T}^M$. In the previous sections, we identified two tensors that belong to the kernel of the intertwining operator

$$\begin{aligned}\widetilde{T}[\Lambda, V]_\alpha &= \delta_\Lambda V_\alpha + (t_\alpha)_{KL}\,\widetilde{V}^K\,\Lambda^L\,,\\ \widetilde{T}[\Lambda, V]_M &= \delta_\Lambda V_M + \Omega_{KL}\left(\partial_M\widetilde{V}^K\,\Lambda^L - \widetilde{V}^K\,\partial_M\Lambda^L\right)\,,\end{aligned} \tag{4.65}$$

namely, for this specific choice one has $\widetilde{T}^M = 0$.

JHEP01(2019)117

It turns out however that there is a more general way to parameterize the kernel of the intertwining operator. It is in terms of new tensors in different representations than the ones considered so far. On the one hand, a tensor in the **912** $T_\alpha{}^M$ (more on representations and projectors can be found in [30, 31, 43, 44])

$$T_\alpha{}^M = P_\alpha^{(912)\,M\,\beta}{}_N\, T_\beta{}^N\,, \quad P_\alpha^{(912)\,M\,\beta}{}_N = \frac{1}{7}\left(\delta_\alpha^\beta\delta_N^M - 12\,(t_\alpha t^\beta)_N{}^M + 4\,(t^\beta t_\alpha)_N{}^M\right)\,, \tag{4.66}$$

and a covariantly constrained tensor $T_M{}^N$ (with unconstrained upper index)

$$\Omega^{NM}\partial_N\otimes T_M{}^K = \Omega^{MN}\,T_M{}^K\,T_N{}^L = (t^\alpha)^{NM}\partial_N\otimes T_M{}^K = (t^\alpha)^{MN}\,T_M{}^K\,T_N{}^L = 0\,. \tag{4.67}$$

Taking into account that due to the section condition

$$(t^\alpha)_M{}^K\,P_\alpha^{(912)L\,\beta}{}_N\,\partial_{(K}\otimes\partial_{L)} = 0\,, \tag{4.68}$$

more generally the kernel of $\widetilde{T}^M$ is given by [40, 41]

$$\begin{aligned}\widetilde{T}_\alpha &= 2\,\partial_M T_\alpha{}^M + (t_\alpha)_N{}^M\,T_M{}^N\\ \widetilde{T}_M &= -2\,\partial_N T_M{}^N - \partial_M T_N{}^N\,,\end{aligned} \tag{4.69}$$

namely, for *any* $T_\alpha{}^M$ and $T_M{}^N$ one has $\widetilde{T}^M = 0$.

The way to recover the specific case (4.65) is by making the choice

$$\begin{aligned}T_\alpha{}^M &= \frac{7}{2}P_\alpha^{(912)\,M\,\beta}{}_N\,\Lambda^N\,V_\beta\\ T_M{}^N &= (t^\alpha)_K{}^N\left(8\,\partial_M\Lambda^K\,V_\beta - 4\,\Lambda^K\,\partial_M V_\beta\right) - \frac{1}{2}\Lambda^N\,V_M\,.\end{aligned} \tag{4.70}$$

It should be clear now why we chose the tilde notation again. The equation (4.69) is the analog to (4.63) for the next level in the hierarchy. An interesting point is that from the definition (4.69), it can be checked that $(\widetilde{T}_\alpha,\,\widetilde{T}_M)$ transform as in (4.12) with weight $\lambda' = 1$ and (4.14), provided

$$\begin{aligned}\delta T_\alpha{}^M &= \Lambda^K\partial_K T_\alpha{}^M - 12(t^\beta)^M{}_N(t_\beta)^K{}_L\partial_K\Lambda^L T_\alpha{}^N + 12 f_{\alpha\gamma}{}^\beta(t^\gamma)^K{}_L\partial_K\Lambda^L T_\beta{}^M + \frac{3}{2}\partial_K\Lambda^K T_\alpha{}^M\\ \delta T_M{}^N &= \Lambda^K\partial_K T_M{}^N - 12(t^\alpha)^N{}_P(t_\alpha)^K{}_L\partial_K\Lambda^L T_M{}^P + 12(t^\alpha)^P{}_M(t_\alpha)^K{}_L\partial_K\Lambda^L T_P{}^N\\ &\quad + \partial_K\Lambda^K T_M{}^N - 24(t^\alpha)^N{}_L\partial_M\partial_K\Lambda^L T_\alpha{}^K\,.\end{aligned} \tag{4.71}$$

Of course, these transformations preserve the constrained nature of both tensors.

Given that the intertwining operator is not invertible, it is not possible to read T_α and T_M from a given $\widetilde{T}^M$, except *up to* terms of the form (4.69). Let us apply this lesson to disentangle the components of the projected two-form of the previous sections. From the transformation of $\widetilde{B}$ in (4.31) we can write the covariant form of the transformations of its components

$$\begin{aligned}\Delta B_{\mu\nu\alpha} &= 2\mathcal{D}_{[\mu}\widehat{\Xi}_{\nu]\alpha} + (t_\alpha)_{KL}\Lambda^K\mathcal{F}_{\mu\nu}{}^L + \widehat{\widetilde{\Sigma}}_{\mu\nu\alpha}\\ \Delta B_{\mu\nu M} &= 2\mathcal{D}_{[\mu}\widehat{\Xi}_{\nu]M} - \Omega_{KL}\left(\Lambda^K\partial_M\mathcal{F}_{\mu\nu}{}^L + \mathcal{F}_{\mu\nu}{}^K\partial_M\Lambda^L\right) + \widehat{\widetilde{\Sigma}}_{\mu\nu M}\,,\end{aligned} \tag{4.72}$$

JHEP01(2019)117

which are related to the genuine transformation through the relation $\Delta\widetilde{B}_{\mu\nu M} = \delta\widetilde{B}_{\mu\nu} - 2\left\{A_{[\mu},\, \delta A_{\nu]}\right\}$

$$\begin{aligned}\Delta B_{\mu\nu\alpha} &= \delta B_{\mu\nu\alpha} + (t_\alpha)_{KL} A_{[\mu}{}^K \delta A_{\nu]}{}^L \\ \Delta B_{\mu\nu M} &= \delta B_{\mu\nu M} - \Omega_{KL}\left(A_{[\mu}{}^K \partial_M \delta A_{\nu]}{}^L + \delta A_{[\nu}{}^K \partial_M A_{\mu]}{}^L\right) .\end{aligned} \tag{4.73}$$

The last terms in both lines of (4.72) parameterize the ambiguity in removing the projection, but will obviously correspond to the transformation of the two-form with respect to the parameter of the next field in the hierarchy (a three-form). The hat on it is to continue with the notation that parameters with a hat are those that allow to write the gauge transformations covariantly with respect to generalized diffeomorphisms. The tilde indicates that the parameters are of the form (4.69), namely they belong to the kernel of $\Delta\widetilde{B}_{\mu\nu M}$.

We now remove the projection from the three-form field strength (4.34)

$$\begin{aligned}\mathcal{H}_{\mu\nu\rho\alpha} &= 3\mathcal{D}_{[\mu} B_{\nu\rho]\alpha} - 3(t_\alpha)_{KL} A_{[\mu}{}^K \Gamma_{\nu\rho]}{}^L - \widetilde{C}_{\mu\nu\rho\alpha} \\ \mathcal{H}_{\mu\nu\rho M} &= 3\mathcal{D}_{[\mu} B_{\nu\rho]M} + 3\Omega_{KL}\left(A_{[\mu}{}^K \partial_M \Gamma_{\nu\rho]}{}^L + \Gamma_{[\nu\rho}{}^K \partial_M A_{\mu]}{}^L\right) - \widetilde{C}_{\mu\nu\rho M} ,\end{aligned} \tag{4.74}$$

where we defined the following combination relevant for the Chern-Simons contributions

$$\Gamma_{\mu\nu}{}^M = \partial_{[\mu} A_{\nu]}{}^M - \frac{1}{3}\left[A_\mu,\, A_\nu\right]^M_{(E)} . \tag{4.75}$$

The transformation of the three-form components (which can also be derived from (4.35)) are

$$\begin{aligned}\delta\mathcal{H}_{\mu\nu\rho\alpha} &= 3\mathcal{D}_{[\mu}\Delta B_{\nu\rho]\alpha} - 3(t_\alpha)\delta A_{[\mu}{}^K \mathcal{F}_{\nu\rho]}{}^L - \Delta\widetilde{C}_{\mu\nu\rho\alpha} \\ \delta\mathcal{H}_{\mu\nu\rho M} &= 3\mathcal{D}_{[\mu}\Delta B_{\nu\rho]M} + 3\Omega_{KL}\left(\delta A_{[\mu}{}^K \partial_M \mathcal{F}_{\nu\rho]}{}^L + \mathcal{F}_{[\nu\rho}{}^K \partial_M A_{\mu]}{}^L\right) - \Delta\widetilde{C}_{\mu\nu\rho M} ,\end{aligned} \tag{4.76}$$

where

$$\begin{aligned}\Delta\widetilde{C}_{\mu\nu\rho\alpha} &= \delta\widetilde{C}_{\mu\nu\rho\alpha} + 3\widetilde{T}_\alpha[\delta A_{[\mu},\, B_{\nu\rho]}] + 2\widetilde{T}_\alpha[A_\mu\,,\, \{A_\nu, \delta A_{\rho]}\}] \\ \Delta\widetilde{C}_{\mu\nu\rho M} &= \delta\widetilde{C}_{\mu\nu\rho M} + 3\widetilde{T}_M[\delta A_{[\mu},\, B_{\nu\rho]}] + 2\widetilde{T}_M[A_\mu\,,\, \{A_\nu, \delta A_{\rho]}\}] ,\end{aligned} \tag{4.77}$$

and the requirement of these transforming covariantly determines the covariant transformations of the three-form fields

$$\Delta\widetilde{C}_{\mu\nu\rho\alpha} = 3\mathcal{D}_{[\mu}\widehat{\widetilde{\Sigma}}_{\nu\rho]\alpha} - \widetilde{T}_\alpha[\Lambda,\, \mathcal{H}_{\mu\nu\rho}] - 3\widetilde{T}_\alpha[\mathcal{F}_{[\mu\nu},\, \widehat{\Xi}_{\rho]}] \tag{4.78}$$

$$\Delta\widetilde{C}_{\mu\nu\rho M} = 3\mathcal{D}_{[\mu}\widehat{\widetilde{\Sigma}}_{\nu\rho]M} - \widetilde{T}_M[\Lambda,\, \mathcal{H}_{\mu\nu\rho}] - 3\widetilde{T}_M[\mathcal{F}_{[\mu\nu},\, \widehat{\Xi}_{\rho]}] , \tag{4.79}$$

where $\widetilde{T}$ was defined in (4.65).

The new three-form field and its parameter are now constrained to take the form (4.69)

$$\begin{aligned}\widetilde{C}_{\mu\nu\rho\alpha} &= 2\partial_M C_{\mu\nu\rho\alpha}{}^M + (t_\alpha)_N{}^M\, C_{\mu\nu\rho M}{}^N \\ \widetilde{C}_{\mu\nu\rho M} &= -2\partial_N C_{\mu\nu\rho M}{}^N - \partial_M C_{\mu\nu\rho N}{}^N ,\end{aligned} \tag{4.80}$$

JHEP01(2019)117

and

$$\begin{aligned}\widetilde{\widehat{\Sigma}}_{\mu\nu\alpha} &= 2\partial_M \widehat{\Sigma}_{\mu\nu\alpha}{}^M + (t_\alpha)_N{}^M\, \widehat{\Sigma}_{\mu\nu M}{}^N \\ \widetilde{\widehat{\Sigma}}_{\mu\nu M} &= -2\partial_N \widehat{\Sigma}_{\mu\nu M}{}^N - \partial_M \widehat{\Sigma}_{\mu\nu N}{}^N\,.\end{aligned} \tag{4.81}$$

This is the projection required to truncate the hierarchy to this level. This new intertwining operator has its own non-vanishing kernel,[9] and so the projection is non-removable unless higher tensors enter the game.

We have been able to move a step upwards in the hierarchy, so let us resume the main results. The closed hierarchy now contains fields $A_\mu{}^M$, $B_{\mu\nu\alpha}$, $B_{\mu\nu M}$, $\widetilde{C}_{\mu\nu\rho\alpha}$ and $\widetilde{C}_{\mu\nu\rho M}$, the last three being constrained fields. Their gauge parameters are Λ^M, $\widehat{\Xi}_{\mu\alpha}$, $\widehat{\Xi}_{\mu M}$, $\widetilde{\widehat{\Sigma}}_{\mu\nu\alpha}$ and $\widetilde{\widehat{\Sigma}}_{\mu\nu M}$, and again the last three are also constrained. The gauge transformations are

$$\begin{aligned}\delta A_\mu{}^M &= \mathcal{D}_\mu \Lambda^M + \Omega^{MN} \widetilde{\widehat{\Xi}}_{\mu N} \\ \Delta B_{\mu\nu\alpha} &= 2\mathcal{D}_{[\mu} \widehat{\Xi}_{\nu]\alpha} + (t_\alpha)_{KL} \Lambda^K \mathcal{F}_{\mu\nu}{}^L + \widetilde{\widehat{\Sigma}}_{\mu\nu\alpha} \\ \Delta B_{\mu\nu M} &= 2\mathcal{D}_{[\mu} \widehat{\Xi}_{\nu]M} - \Omega_{KL} \left(\Lambda^K \partial_M \mathcal{F}_{\mu\nu}{}^L + \mathcal{F}_{\mu\nu}{}^K \partial_M \Lambda^L\right) + \widetilde{\widehat{\Sigma}}_{\mu\nu M} \\ \Delta \widetilde{C}_{\mu\nu\rho\alpha} &= 3\mathcal{D}_{[\mu} \widetilde{\widehat{\Sigma}}_{\nu\rho]\alpha} - \widetilde{T}_\alpha[\Lambda,\, \mathcal{H}_{\mu\nu\rho}] - 3\widetilde{T}_\alpha[\mathcal{F}_{[\mu\nu},\, \widehat{\Xi}_{\rho]}] \\ \Delta \widetilde{C}_{\mu\nu\rho M} &= 3\mathcal{D}_{[\mu} \widetilde{\widehat{\Sigma}}_{\nu\rho]M} - \widetilde{T}_M[\Lambda,\, \mathcal{H}_{\mu\nu\rho}] - 3\widetilde{T}_M[\mathcal{F}_{[\mu\nu},\, \widehat{\Xi}_{\rho]}]\,.\end{aligned} \tag{4.84}$$

They admit the following field-dependent trivial parameters

$$\begin{aligned}\Lambda^M_{\text{trivial}} &= \widetilde{\chi}^M = 12(t^\alpha)^{MN} \partial_N \chi_\alpha + \frac{1}{2}\Omega^{MN} \chi_N \\ \widehat{\Xi}^{\text{trivial}}_{\mu\alpha} &= -\mathcal{D}_\mu \chi_\alpha + \widetilde{\widehat{\chi}}_{\mu\alpha} \\ \widehat{\Xi}^{\text{trivial}}_{\mu M} &= -\mathcal{D}_\mu \chi_M + \widetilde{\widehat{\chi}}_{\mu M} \\ \widetilde{\widehat{\Sigma}}^{\text{trivial}}_{\mu\nu\alpha} &= -2\mathcal{D}_{[\mu} \widetilde{\widehat{\chi}}_{\nu]\alpha} - \widetilde{T}_\alpha[\mathcal{F}_{\mu\nu},\, \chi] \\ \widetilde{\widehat{\Sigma}}^{\text{trivial}}_{\mu\nu M} &= -2\mathcal{D}_{[\mu} \widetilde{\widehat{\chi}}_{\nu]M} - \widetilde{T}_M[\mathcal{F}_{\mu\nu},\, \chi]\,,\end{aligned} \tag{4.85}$$

where

$$12(t^\alpha)^{MN} \partial_N \widetilde{\widehat{\chi}}_{\mu\alpha} + \frac{1}{2}\Omega^{MN} \widetilde{\widehat{\chi}}_{\mu N} = 0\,. \tag{4.86}$$

The covariant transformations close

$$[\delta_1,\, \delta_2] = -\delta_{\Lambda_{12}} - \delta_{\widehat{\Xi}_{12}} - \delta_{\widetilde{\widehat{\Sigma}}_{12}}\,, \tag{4.87}$$

[9] It is easy to anticipate some representations of the next and final level of the hierarchy. Consider for instance a tensor $\Theta_\alpha{}^{MN} \in \mathbf{8645} \oplus \mathbf{133}$, namely satisfying

$$\Theta_\alpha{}^{MN} = (t^\beta)^{MN} \Theta_{[\alpha\beta]} = P^{(912)M\,\beta}_{\alpha\;\;\;P}\, \Theta_\beta{}^{PN}\,. \tag{4.82}$$

Due to the second identity we can choose a candidate in the **912** that because of the first one belongs to the kernel of $\widetilde{T}_\alpha$ and $\widetilde{T}_M$

$$\widetilde{\Sigma}_\alpha{}^M = \partial_N \Theta_\alpha{}^{MN}\,, \quad \widetilde{\Sigma}_M{}^N = 0\,, \tag{4.83}$$

namely $\widetilde{T}_\alpha = 2\partial_M \widetilde{\Sigma}_\alpha{}^M = 0$, thanks to the section condition.

JHEP01(2019)117

with respect to field-dependent brackets

$$\begin{aligned}
\Lambda_{12}^{M} &= [\Lambda_1,\, \Lambda_2]_{(E)}^{M} \\
\widehat{\Xi}_{12\mu\alpha} &= (t_\alpha)_{KL}\Lambda_{[1}^{K}\mathcal{D}_\mu\Lambda_{2]}^{L} \\
\widehat{\Xi}_{12\mu M} &= \Omega_{KL}(\mathcal{D}_\mu\Lambda_{[1}^{K}\partial_M\Lambda_{2]}^{L} + \Lambda_{[2}^{K}\partial_M\mathcal{D}_\mu\Lambda_{1]}^{L}) \\
\widetilde{\widehat{\Sigma}}_{12\mu\nu\alpha} &= 4\widetilde{T}_\alpha[\mathcal{D}_{[\mu}\Lambda_{[1},\, \widehat{\Xi}_{2]\nu]}] + 2\widetilde{T}_\alpha[\widetilde{\widehat{\Xi}}_{[1[\mu},\, \widehat{\Xi}_{2]\nu]}] + 2\widetilde{T}_\alpha[\Lambda_{[1},\, \{\Lambda_{2]},\, \mathcal{F}_{\mu\nu}\}] \\
\widetilde{\widehat{\Sigma}}_{12\mu\nu M} &= 4\widetilde{T}_M[\mathcal{D}_{[\mu}\Lambda_{[1},\, \widehat{\Xi}_{2]\nu]}] + 2\widetilde{T}_M[\widetilde{\widehat{\Xi}}_{[1[\mu},\, \widehat{\Xi}_{2]\nu]}] + 2\widetilde{T}_M[\Lambda_{[1},\, \{\Lambda_{2]},\, \mathcal{F}_{\mu\nu}\}]\,.
\end{aligned} \tag{4.88}$$

If we wished to eliminate the field dependence of the brackets, we have to perform the following field-dependent redefinitions of the gauge parameters

$$\begin{aligned}
\Xi_{\mu\alpha} &= \widehat{\Xi}_{\mu\alpha} + (t_\alpha)_{KL}A_\mu{}^{K}\Lambda^{L} \\
\Xi_{\mu M} &= \widehat{\Xi}_{\mu M} + \Omega_{KL}\left(\partial_M\Lambda^{K}A_\mu{}^{L} - \Lambda^{K}\partial_M A_\mu{}^{L}\right) \\
\widetilde{\Sigma}_{\mu\nu\alpha} &= \widetilde{\widehat{\Sigma}}_{\mu\nu\alpha} - 2\widetilde{T}_\alpha[A_{[\mu},\, \Xi_{\nu]}] - \widetilde{T}_\alpha[\Lambda,\, B_{\mu\nu}] - 2\widetilde{T}_\alpha[A_{[\mu},\, \{A_{\nu]},\, \Lambda\}] \\
\widetilde{\Sigma}_{\mu\nu M} &= \widetilde{\widehat{\Sigma}}_{\mu\nu M} - 2\widetilde{T}_M[A_{[\mu},\, \Xi_{\nu]}] - \widetilde{T}_M[\Lambda,\, B_{\mu\nu}] - 2\widetilde{T}_M[A_{[\mu},\, \{A_{\nu]},\, \Lambda\}]\,.
\end{aligned} \tag{4.89}$$

The redefinition is non-covariant with respect to generalized diffeomorphisms, and the trivial parameters are now given by field-independent quantities

$$\begin{aligned}
\Lambda_{\text{trivial}}^{M} &= 12(t^\alpha)^{MN}\partial_N\chi_\alpha + \frac{1}{2}\Omega^{MN}\chi_N \\
\Xi_{\mu\alpha}^{\text{trivial}} &= -\partial_\mu\chi_\alpha + \widetilde{\chi}_{\mu\alpha} \\
\Xi_{\mu M}^{\text{trivial}} &= -\partial_\mu\chi_M + \widetilde{\chi}_{\mu M} \\
\widetilde{\Sigma}_{\mu\nu\alpha}^{\text{trivial}} &= -2\partial_{[\mu}\widetilde{\chi}_{\nu]\alpha} \\
\widetilde{\Sigma}_{\mu\nu M}^{\text{trivial}} &= -2\partial_{[\mu}\widetilde{\chi}_{\nu]M}\,,
\end{aligned} \tag{4.90}$$

where we have made the redefinitions

$$\begin{aligned}
\widetilde{\chi}_{\mu\alpha} &= \widetilde{\widehat{\chi}}_{\mu\alpha} + \widetilde{T}_\alpha[A_\mu,\, \chi] \\
\widetilde{\chi}_{\mu M} &= \widetilde{\widehat{\chi}}_{\mu M} + \widetilde{T}_M[A_\mu,\, \chi]\,,
\end{aligned} \tag{4.91}$$

that preserve the constraint (4.86).

The brackets for the redefined parameters are now independent of the fields

$$\begin{aligned}
\Lambda_{12}^{M} &= [\Lambda_1,\, \Lambda_2]_{(E)}^{M} \\
\Xi_{12\mu\alpha} &= 2\delta_{\Lambda_{[1}}\Xi_{2]\mu\alpha} + (t_\alpha)_{KL}\partial_\mu\Lambda_{[1}^{K}\Lambda_{2]}^{L} \\
\Xi_{12\mu M} &= 2\delta_{\Lambda_{[1}}\Xi_{2]\mu M} + \Omega_{KL}\left(\partial_\mu\Lambda_{[2}^{K}\partial_M\Lambda_{1]}^{L} - \partial_M\partial_\mu\Lambda_{[2}^{K}\Lambda_{1]}^{L}\right) \\
\widetilde{\Sigma}_{12\mu\nu\alpha} &= 2\widetilde{T}_\alpha[\Lambda_{[1},\, \widetilde{\Sigma}_{2]\mu\nu}] - 4\widetilde{T}_\alpha[\partial_{[\mu}\Lambda_{[1},\, \Xi_{2]\nu}] - 2\widetilde{T}_\alpha[\widetilde{\Xi}_{[1[\mu},\, \Xi_{2]\nu}] \\
\widetilde{\Sigma}_{12\mu\nu M} &= 2\widetilde{T}_M[\Lambda_{[1},\, \widetilde{\Sigma}_{2]\mu\nu}] - 4\widetilde{T}_M[\partial_{[\mu}\Lambda_{[1},\, \Xi_{2]\nu}] - 2\widetilde{T}_M[\widetilde{\Xi}_{[1[\mu},\, \Xi_{2]\nu}]\,.
\end{aligned} \tag{4.92}$$

JHEP01(2019)117

One can then compute the Jacobiator components as we did repeatedly before

$$\begin{aligned} J^M &= \widetilde{N}^M = 12(t^\alpha)^{MP}\partial_P N_\alpha + \frac{1}{2}\Omega^{MP}N_P \\ J_{\mu\alpha} &= -\partial_\mu N_\alpha + \widetilde{N}_{\mu\alpha} \\ J_{\mu M} &= -\partial_\mu N_M + \widetilde{N}_{\mu M} \\ \widetilde{J}_{\mu\nu\alpha} &= -2\partial_{[\mu}\widetilde{N}_{\nu]\alpha} \\ \widetilde{J}_{\mu\nu M} &= -2\partial_{[\mu}\widetilde{N}_{\nu]M}\,, \end{aligned} \tag{4.93}$$

with the Nijenhuis tensors N_α and N_M defined in (4.20), and

$$\begin{aligned} \widetilde{N}_{\mu\alpha} &= -2\widetilde{T}_\alpha[\Lambda_{[1},\,\{\Lambda_2,\,\partial_\mu\Lambda_{3]}\}] \\ \widetilde{N}_{\mu M} &= -2\widetilde{T}_M[\Lambda_{[1},\,\{\Lambda_2,\,\partial_\mu\Lambda_{3]}\}]\,. \end{aligned} \tag{4.94}$$

We can now compute the bracket between a trivial parameter ζ_t of the form (4.90) and a generic parameter, finding

$$\begin{aligned} [\zeta_t,\,\zeta]^M &= \widetilde{\chi}'^M \\ [\zeta_t,\,\zeta]_{\mu\alpha} &= -\partial_\mu\chi'_\alpha + \widetilde{\chi}'_{\mu\alpha} \\ [\zeta_t,\,\zeta]_{\mu M} &= -\partial_\mu\chi'_M + \widetilde{\chi}'_{\mu M} \\ [\zeta_t,\,\zeta]_{\mu\nu\alpha} &= -\partial_{[\mu}\widetilde{\chi}'_{\nu]\alpha} \\ [\zeta_t,\,\zeta]_{\mu\nu M} &= -\partial_{[\mu}\widetilde{\chi}'_{\nu]M}\,, \end{aligned} \tag{4.95}$$

where

$$\begin{aligned} \chi'_\alpha &= \frac{1}{2}(t_\alpha)_{KL}\Lambda^L\widetilde{\chi}^L \\ \chi'_M &= -\frac{1}{2}\Omega_{KL}\left(\Lambda^K\partial_M\widetilde{\chi}^L - \partial_M\Lambda^K\widetilde{\chi}^L\right) \\ \widetilde{\chi}'_{\mu\alpha} &= -\widetilde{T}_\alpha[\Lambda,\Xi_{t\,\mu}] \\ \widetilde{\chi}'_{\mu M} &= -\widetilde{T}_M[\Lambda,\Xi_{t\,\mu}]\,. \end{aligned} \tag{4.96}$$

We have then shown that the next level in the hierarchy admits a set of parameters for which the criteria required by the theorem discussed in sections 2.1 and 2.2 is met. Namely, that the theory admits trivial field independent parameters (4.39), that for this set of parameters the bracket is field independent (4.92), that the bracket between a trivial parameter and a generic one is trivial (4.95), and that the Jacobiator is a trivial parameter (4.93). The theorem then states that the gauge algebra again fits into an $L_\infty^{\text{gauge+fields}}$ structure. This procedure could be extended all the way up to space-time saturation, starting by the next level discussed in footnote 9.

JHEP01(2019)117

5 L_∞ and gauged supergravity

As discussed in the previous sections, the gauge algebra of DFT and EFT does not satisfy the Jacobi identity, requiring a deformation therein produced by a non-vanishing triple product (so-called the Nijenhuis tensor). This is a manifestation of the fact that duality covariant theories contain trivial parameters, or said in a more fancy way, symmetries for symmetries.

On the other hand, generalized compactifications of these theories (originally introduced in [78, 79] in the context of DFT, see also [80–88], and later considered in an EFT context [89–98]) give rise to gauged supergravities. For reviews see [46–50]. It is then natural to ask the question of whether the L_∞ algebraic structure of the parent double and exceptional field theories leaves some imprint after compactification. Of course, this question can be asked and answered independently, without any mention to parent theories and compactifications, so we will address these questions from both ends.

Here we show that the most salient aspects of the algebraic structure of DFT and EFT are indeed reproduced by gauged supergravities. The failure of the Jacobi identity to be satisfied is governed by the so-called intertwining tensor, a particular projection of the embedding tensor. We explicitly compute the Jacobiator, and find that gauged supergravities admit novel symmetries for symmetries in which the trivial parameters are deformed by gaugings. Our goal is to show that gauged supergravities admit (via field redefinitions) a set of parameters that satisfy sufficient criteria (see subsections 2.1 and 2.2) for the algebra to fit into an $L_\infty^{\text{gauge+fields}}$ structure.

The presentation proceeds as follows. First we perform a generalized Scherk-Schwarz compactification of the KK formulation of DFT, ending with a gauged supergravity corresponding to the electric sector of a theory with half-maximal supersymmetry. This is the simplest possible scenario to explore these issues, serving as a prototype to confirm our expectations. We then move to the general case of generic gauged supergravities, without any mention to compactifications, for which the structure of the gauge transformations of the tensor hierarchy is well know. In this context we perform convenient redefinitions in order to identify field-independent brackets, their Jacobiator, the Nijenhuis tensors, and the general form of trivial gauge parameters.

5.1 Generalized Scherk-Schwarz compactification of KK-DFT

Let us briefly discuss the generalized Scherk-Schwarz (SS) compactification of the KK-DFT action so that our presentation is self-contained. The generalized metric and frame formulations were compactified in [78, 79], and here we use the same technique for this formulation. The KK-formulation is particularly useful to perform KK tower expansions in tori compactifictations [99]. We start by recalling the KK-DFT action (3.30)

$$S = \int d^n x d^{2d} X \sqrt{-g} e^{-2\phi} \mathcal{L} \,, \tag{5.1}$$

$$\begin{aligned} \mathcal{L} = \widehat{\mathcal{R}} &- 4\, g^{\mu\nu} \mathcal{D}_\mu \phi \mathcal{D}_\nu \phi + 4\, \nabla_\mu \left(g^{\mu\nu} \mathcal{D}_\nu \phi \right) - \frac{1}{12} \mathcal{H}_{\mu\nu\rho} \mathcal{H}^{\mu\nu\rho} \\ &+ \frac{1}{8} g^{\mu\nu} \mathcal{D}_\mu \mathcal{M}_{MN} \mathcal{D}_\nu \mathcal{M}^{MN} - \frac{1}{4} \mathcal{M}_{MN} \mathcal{F}_{\mu\nu}{}^M \mathcal{F}^{\mu\nu N} - V \,. \end{aligned} \tag{5.2}$$

JHEP01(2019)117

We will only discuss the case of $n = 4$ space-time dimensions: some results will suffer modifications in other dimensions. We can distinguish three types of global transformations (to be complemented with $\partial_M \to U_M{}^P \partial_P$ and $\partial_\mu \to e^{\frac{\gamma}{2}} \partial_\mu$)

$$\begin{aligned} g_{\mu\nu} &\to e^{\gamma}\, g_{\mu\nu} \\ \mathcal{M}^{MN} &\to U^M{}_P\, U^N{}_Q\, \mathcal{M}^{PQ} \\ \phi &\to \phi + c + \gamma \\ A_\mu{}^M &\to e^{\frac{\gamma}{2}}\, U^M{}_P\, A_\mu{}^P \\ B_{\mu\nu} &\to e^{\gamma} B_{\mu\nu}\,, \end{aligned} \tag{5.3}$$

where $c, \gamma \in \mathbb{R}$ and $U \in O(d,d|\mathbb{R})$. When a field is shifted with a warp factor $e^{\omega\gamma}$ we call ω the "external weight", such that

$$\omega(g) = \omega(B) = 1\,, \quad \omega(A) = \frac{1}{2}\,, \quad \omega(e^{-2\phi}) = -2\,, \quad \omega(\mathcal{M}) = 0\,. \tag{5.4}$$

While U-rotations (standard duality symmetry) and γ and c-sifts leave the Lagrangian invariant, the c-transformations re-scale the measure

$$\sqrt{-g} e^{-2\phi} \to e^{-2c} \sqrt{-g} e^{-2\phi}\,, \tag{5.5}$$

and so are not a symmetry of the action, but of the equations of motion only. The Scherk-Schwarz compactification procedure consists in proposing a compactification ansatz based on the global transformations

$$\begin{aligned} g_{\mu\nu}(x,X) &= e^{\gamma(X)}\, g_{\mu\nu}(x) \\ \mathcal{M}^{MN}(x,X) &= U^M{}_A(X) U^N{}_B(X)\, \mathcal{M}^{AB}(x) \\ \phi(x,X) &= \phi(x) + c(X) + \gamma(X) \\ A_\mu{}^M(x,X) &= e^{\frac{\gamma(X)}{2}} U^M{}_A(X)\, A_\mu{}^A(x) \\ B_{\mu\nu}(x,X) &= e^{\gamma(X)}\, B_{\mu\nu}(x)\,, \end{aligned} \tag{5.6}$$

where in this context $U^M{}_A$, γ and c are know as "twists". We distinguish between indices $M, N, P, \ldots$ transforming under $O(d,d)$ transformations in the parent DFT action, and those $A, B, C, \ldots$ transforming under the $O(d,d)$ in the effective gauge supergravity. Equivalently, we propose an ansatz for the gauge parameters

$$\begin{aligned} \Lambda^M(x,X) &= e^{\frac{\gamma(X)}{2}} U^M{}_A(X)\, \Lambda^A(x) \\ \Xi_\mu(x,X) &= e^{\gamma(X)}\, \Xi_\mu(x)\,, \end{aligned} \tag{5.7}$$

by assigning them external weights $\omega(\Lambda) = \frac{1}{2}$ and $\omega(\Xi) = 1$.

In general, the SS ansatz for a tensor $T^M(x,X)$ with weight λ and external weight ω is

$$T^M(x,X) = e^{\omega\,\gamma(X) - 2\lambda\, c(X)}\, U^M{}_A(X)\, T^A(x)\,. \tag{5.8}$$

JHEP01(2019)117

The generalized diffeomorphisms determine the form of the effective gauge transformation of $T^A(x)$ through the relation

$$\widehat{\mathcal{L}}_\Lambda T^M(x,X) = e^{\omega\,\gamma(X)-2\lambda\,c(X)}\,U^M{}_A(X)\,\mathcal{L}_\Lambda T^A(x)\,, \tag{5.9}$$

yielding

$$\mathcal{L}_\Lambda T^A = -X_{BC}{}^A\,\Lambda^B\,T^C + \left(\lambda+\omega-\frac{1}{2}\right)\,f_B\Lambda^B\,T^A\,, \tag{5.10}$$

where

$$X_{BC}{}^A = -f_{BC}{}^A + f_{[C}\,\delta^A_{B]} - \frac{1}{2}\,f^A\,\eta_{BC}\,, \tag{5.11}$$

and we finally arrive to the gaugings or fluxes f^A and $f_{AB}{}^C$, defined as

$$\begin{aligned} f^A &\equiv e^{\frac{\gamma}{2}}\,U_M{}^A\partial^M\gamma = -2\,e^{\frac{\gamma}{2}}\,\left(2U_M{}^A\partial^M c - \partial^M U_M{}^A\right)\\ f_{ABC} &\equiv 3\,e^{\frac{\gamma}{2}}\,U_{M[A}\partial^M U^P{}_B U^Q{}_{C]}\eta_{PQ}\,. \end{aligned}$$

Despite of being built from X-dependent twists, when they appear in this particular combination we demand that they are constant. Demanding that the fluxes are constant, plus strong constrained twists, is known to imply the quadratic constraints

$$\begin{aligned} f^A f_A &= 0\\ f^A f_{AB}{}^C &= 0\\ f_{[AB}{}^E f_{C]DE} &= \frac{2}{3} f_{[ABC} f_{D]}\,. \end{aligned} \tag{5.12}$$

The reverse implication is not true: the quadratic constraints are weaker than the strong constraint, which in the context of generalized Scherk-Schwarz compactifications is know to be unnecessary. We will not discuss this issue here, see [78–82] for discussions on this point.

The gauge transformations of the fields in the parent action (3.4)–(3.8) can also be *twisted* in this way to extract their analogs in the effective theory. From now on, all the quantities we display correspond to those in the effective gauged supergravity unless the opposite is explicitly mentioned

$$\begin{aligned} \delta g_{\mu\nu} &= \mathcal{L}_\Lambda g_{\mu\nu}\\ \delta\mathcal{M}^{AB} &= \mathcal{L}_\Lambda\mathcal{M}^{AB}\\ \delta e^{-2\phi} &= \mathcal{L}_\Lambda e^{-2\phi}\\ \delta A_\mu{}^A &= \partial_\mu\Lambda^A + \mathcal{L}_\Lambda A_\mu{}^A + f^A\,\Xi_\mu\\ \delta B_{\mu\nu} &= 2\partial_{[\mu}\Xi_{\nu]} + \mathcal{L}_\Lambda B_{\mu\nu} + \partial_{[\mu}\Lambda^A A_{\nu]A} - f_A\,A_{[\mu}{}^A\Xi_{\nu]}\,, \end{aligned} \tag{5.13}$$

where all fields have vanishing weight except for $\lambda(e^{-2\phi})=1$, and the external weights are listed in (5.4).

Let us point out that similarly to DFT (3.46), the gauge transformations can be taken to a covariant form

$$\delta A_\mu{}^A = \mathcal{D}_\mu\Lambda^A + f^A\widehat{\Xi}_\mu \tag{5.14}$$

$$\delta B_{\mu\nu} = 2\mathcal{D}_{[\mu}\widehat{\Xi}_{\nu]} - \mathcal{F}_{\mu\nu}{}^A\Lambda_A + A_{[\mu}{}^A\delta A_{\nu]A}\,, \tag{5.15}$$

JHEP01(2019)117

by redefining the one-form parameter as in (3.45)

$$\widehat{\Xi}_\mu = \Xi_\mu + A_\mu{}^A \Lambda_A \,. \tag{5.16}$$

The brackets for the parameters with a hat are expected to be field dependent, while those without a hat depend on parameters only. The non-covariant ones are then more convenient to deal with when analyzing the gauge algebra, which will be discussed later.

The covariant derivatives (3.14) can also be twisted, leading to their analogs in gauged supergravity, namely

$$\mathcal{D}_\mu = \partial_\mu - \mathcal{L}_{A_\mu} \,, \tag{5.17}$$

such that for a covariant tensor transforming as in (5.10) one has

$$\mathcal{L}_\Lambda \mathcal{D}_\mu T^A = -X_{BC}{}^A \, \Lambda^B \, \mathcal{D}_\mu T^C + \left(\lambda + \omega - \frac{1}{2}\right) f_B \Lambda^B \, \mathcal{D}_\mu T^A \,. \tag{5.18}$$

For the field strengths (3.20) we have the same story

$$\mathcal{F}_{\mu\nu}{}^A = 2\partial_{[\mu} A_{\nu]}{}^A - f_{BC}{}^A \, A_\mu{}^B A_\nu{}^C - f_B \, A_{[\mu}{}^B A_{\nu]}{}^A - f^A \, B_{\mu\nu} \tag{5.19}$$

$$\mathcal{H}_{\mu\nu\rho} = 3\left(\mathcal{D}_{[\mu} B_{\nu\rho]} + A_{[\mu}{}^A \partial_\nu A_{\rho]A} - \frac{1}{3} f_{ABC} A_\mu{}^A A_\nu{}^B A_\rho{}^C\right) , \tag{5.20}$$

and they satisfy the following Bianchi identities

$$3\mathcal{D}_{[\mu} \mathcal{F}_{\nu\rho]}{}^A + f^A \, \mathcal{H}_{\mu\nu\rho} = 0 \tag{5.21}$$

$$\mathcal{D}_{[\mu} \mathcal{H}_{\nu\rho\sigma]} - \frac{3}{4} \mathcal{F}_{[\mu\nu}{}^A \mathcal{F}_{\rho\sigma]A} = 0 \,. \tag{5.22}$$

Then, we automatically see that the compactified effective action takes exactly the same form as the parent one (5.2), where the covariant derivatives and the field strengths must be replaced by their lower dimensional gauged versions. The only subtlety is a local overall shift $\int dX \, e^{-2c-\gamma}$ which integrates in the internal space to modify the effective action's Planck constant. There is also the non-trivial task of compactifying the scalar potential, which gives

$$V = \frac{1}{12} \mathcal{M}^{AB} \mathcal{M}^{CD} \mathcal{M}_{EF} f_{AC}{}^E f_{BD}{}^F + \frac{1}{4} \mathcal{M}^{AB} \left(f_{AC}{}^D f_{BD}{}^C + 3 f_A f_B\right) + \frac{1}{6} f_{ABC} f^{ABC} \,. \tag{5.23}$$

The last term can be seen to vanish due to the strong constraint [78–82], so we put it in by hand. When the term is non-vanishing the theory cannot be uplifted to maximal supergravity [100, 101].

The gauge invariance of the effective action is achieved due to the fact that the measure transforms like

$$\delta\left(\sqrt{-g} e^{-2\phi}\right) = \Lambda^A f_A \sqrt{-g} e^{-2\phi} \,, \tag{5.24}$$

which is compensated by the variation of the Lagrangian

$$\delta\mathcal{L} = -\Lambda^A f_A \mathcal{L} \,. \tag{5.25}$$

JHEP01(2019)117

The brackets with respect to which the gauge transformations close can be either obtained by direct computation, or by twisting those in (3.49)

$$\Lambda_{12}^{M}(x,X) = e^{\frac{\gamma(X)}{2}} U^{M}{}_{A}(X)\, \Lambda_{12}^{A}(x) \tag{5.26}$$

$$\Xi_{\mu 12}(x,X) = e^{\gamma(X)}\, \Xi_{\mu 12}(x)\,, \tag{5.27}$$

giving rise to

$$\Lambda_{12}^{A} = f_{BC}{}^{A} \Lambda_{1}^{B} \Lambda_{2}^{C} + f_{B}\, \Lambda_{[1}^{B} \Lambda_{2]}^{A} \tag{5.28}$$

$$\Xi_{\mu 12} = 2 f_{A}\, \Lambda_{[1}^{A} \Xi_{2]\mu} + \Lambda_{[1}^{A} \partial_{\mu} \Lambda_{2]A}\,, \tag{5.29}$$

which satisfy

$$[\delta_1,\, \delta_2] = -\delta_{12}\,, \tag{5.30}$$

for the compactified gauge transformations (5.13).

As before, merging the two parameters into a single one and defining the bracket notation

$$\zeta = (\Lambda^{A},\, \Xi_{\mu})\,, \quad [\zeta_1,\, \zeta_2]^{A} = \Lambda_{12}^{A}\,, \quad [\zeta_1,\, \zeta_2]_{\mu} = \Xi_{\mu 12}\,, \tag{5.31}$$

one can readily compute the Jacobiator

$$J = 3\, \big[\big[\zeta_{[1},\, \zeta_2\big]\,,\, \zeta_{3]}\big] = (f^{A}\, N,\, -\partial_{\mu} N)\,, \tag{5.32}$$

with the Nijenhuis scalar defined by

$$N(\Lambda_1, \Lambda_2, \Lambda_3) = \frac{1}{2}\, f_{ABC}\, \Lambda_{1}^{A} \Lambda_{2}^{B} \Lambda_{3}^{C}\,. \tag{5.33}$$

The same result is obtained by compactifying the parent quantities (3.54)–(3.55) with external weights $\omega(J^{A}) = \frac{1}{2}$, $\omega(J_{\mu}) = 1$, $\omega(N) = 1$.

We now have a clear indication that the electric sector of half-maximal supergravity admits trivial parameters

$$\Lambda_{\text{trivial}}^{A} = f^{A}\, \chi\,, \qquad \Xi_{\mu,\text{trivial}} = -\partial_{\mu} \chi\,, \tag{5.34}$$

which indeed can be verified by direct inspections using the quadratic constraints (5.12). Moreover, it is easy to see that the commutator of a trivial parameter with a generic one, yields a new trivial parameters characterized by the function

$$\chi' = -\frac{1}{2} f_{A} \Lambda^{A}\, \chi\,. \tag{5.35}$$

We then see that the electric sector of half-maximal gauged supergravity inherits from its parent theory the sufficient conditions for its gauge algebra to fit into $L_{\infty}^{\text{gauge+fields}}$.

JHEP01(2019)117

5.2 The tensor hierarchy in gauged supergravity

The gauge structure of the tensor hierarchy in gauged supergravities has the nice advantage of being writable in a universal form, regardless of the dimension and the amount of supersymmetry. Following [45], and motivated by the fact that our discussion so far reached the three-form in this paper only, we will project the hierarchy to this level here, and discuss the full hierarchy relevant for $n = 4$ space-time dimensions in the appendix. The conclusion will be that gauged supergravities admit symmetries for symmetries whose trivial parameters are deformed by the embedding tensor. We will afterwards show in the next subsection that the gauge algebra of the tensor hierarchy meets the criteria of subsections 2.1 and 2.2 implying that it has an $L_\infty^{\text{gauge+fields}}$ structure.

Starting from a generic ungauged supergravity with global symmetry group G_0, we can turn some subgroup $G \in G_0$ into local using the embedding tensor formalism, which maintains at least formally the G_0 covariance by treating the embedding tensor as a spurionic object. In this procedure, the generators of the algebra of the local subgroup X_M are parameterized as a projection of the $\mathfrak{g}_0$ generators, t_α

$$X_M \equiv \Theta_M{}^\alpha t_\alpha \,, \tag{5.36}$$

with the embedding tensor $\Theta_M{}^\alpha$ a mixed index tensor with $M = 1, \ldots, dim(\mathfrak{g})$ in some $\overline{V}$ representation, and α in the adjoint of G_0. This embedding tensor has to satisfy linear and quadratic constraints. The former is a supersymmetric requirement which can be written as a projection that selects some particular irreducible representations of the product $\overline{V} \otimes \mathfrak{g}_0$ to which the embedding tensor belongs. The latter is needed to ensure the closure of the gauge algebra and can be obtained by demanding gauge invariance of $(X_M)_N{}^O = \Theta_M{}^\alpha \, (t_\alpha)_N{}^O$, which leads to

$$[X_M, X_N] = -X_{MN}{}^O X_O \,. \tag{5.37}$$

The gauge transformations act on vectors as follows

$$\begin{aligned} \delta_\Lambda T^M &= -X_{NP}{}^M \, \Lambda^N T^P \\ \delta_\Lambda T_M &= X_{NM}{}^P \, \Lambda^N T_P \,, \end{aligned} \tag{5.38}$$

with respect to local gauge parameters $\Lambda^M(x)$. In order to preserve gauge invariance it is necessary to replace derivatives ∂_μ by covariant derivatives $\mathcal{D}_\mu$ through the minimal coupling procedure

$$\mathcal{D}_\mu = \partial_\mu - A_\mu{}^M X_M \,, \tag{5.39}$$

where $A_\mu{}^M$ are the gauge vectors. The derivative is covariant provided the vectors transform as follows

$$\delta A_\mu{}^M = \mathcal{D}_\mu \Lambda^M + \ldots \,, \tag{5.40}$$

where the dots represent terms that vanish when contracted with the embedding tensor. The story continues by introducing a field strength

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = -\mathcal{F}_{\mu\nu}{}^M X_M = -\left(2\partial_{[\mu} A_{\nu]}{}^M + X_{NP}{}^M A_{[\mu}{}^N A_{\nu]}{}^P + \ldots\right) X_M \,, \tag{5.41}$$

JHEP01(2019)117

which again is defined up to a projection with the embedding tensor. It turns out that the dots are fixed by demanding covariance of the embedding tensor, namely that it transforms as a vector (5.38), for which one must introduce new degrees of freedom $B_{\mu\nu I}$

$$\mathcal{F}_{\mu\nu}{}^{M} = 2\partial_{[\mu}A_{\nu]}{}^{M} + X_{NP}{}^{M}A_{[\mu}{}^{N}A_{\nu]}{}^{P} + Z^{MI}B_{\mu\nu I}\,, \tag{5.42}$$

where Z^{MI}, the so-called intertwining tensor, is constrained to vanish when projected with X_M

$$Z^{MI}\,X_M = 0\,. \tag{5.43}$$

This object can then be used to fill in the dots in the variation of the vector fields

$$\delta A_{\mu}{}^{M} = \mathcal{D}_{\mu}\Lambda^{M} - Z^{MI}\widehat{\Xi}_{\mu I}\,. \tag{5.44}$$

The gauge transformation of the two form is then forced to compensate the failure of the first two terms in (5.42) to transform covariantly, but then of course is defined only up to terms that vanish when projected in this case with the intertwining tensor. So, covariance of $\mathcal{F}_{\mu\nu}{}^{M}$ only teaches how to transform a projection of the two-form with respect to a projection of its one-form parameter

$$\widetilde{B}_{\mu\nu}{}^{M} = Z^{MI}B_{\mu\nu I}\,, \quad \widetilde{\widehat{\Xi}}_{\mu}{}^{M} = Z^{MI}\widehat{\Xi}_{\mu I}\,, \tag{5.45}$$

giving

$$\delta A_{\mu}{}^{M} = \mathcal{D}_{\mu}\Lambda^{M} - \widetilde{\widehat{\Xi}}_{\mu}{}^{M} \tag{5.46}$$

$$\delta\widetilde{B}_{\mu\nu}{}^{M} = 2\mathcal{D}_{[\mu}\widetilde{\widehat{\Xi}}_{\nu]}{}^{M} - 2X_{(NP)}{}^{M}\left(\Lambda^{N}\,\mathcal{F}_{\mu\nu}{}^{P} - A_{[\mu}{}^{N}\delta A_{\nu]}{}^{P}\right)\,. \tag{5.47}$$

Here we have used the same notation as in the rest of the paper, namely that tensors with a tilde are projected by intertwiners, and tensors with a hat allow to write the gauge transformations in gauge covariant form and involve brackets that are field dependent. In fact a quick computation shows that this *projected* tensor hierarchy is self consistent, as it closes with respect to the brackets

$$\Lambda_{12}{}^{M} = -X_{NP}{}^{M}\Lambda^{N}_{[1}\Lambda^{P}_{2]} \tag{5.48}$$

$$\widetilde{\widehat{\Xi}}_{12\mu}{}^{M} = 2X_{(NP)}{}^{M}\mathcal{D}_{\mu}\Lambda^{N}_{[1}\Lambda^{P}_{2]}\,. \tag{5.49}$$

By redefining the parameters

$$\widetilde{\widehat{\Xi}}_{\mu}{}^{M} = \widetilde{\Xi}_{\mu}{}^{M} + 2X_{(NP)}{}^{M}A_{\mu}{}^{N}\Lambda^{P}\,, \tag{5.50}$$

the one-form bracket becomes field independent

$$\widetilde{\Xi}_{12\mu}{}^{M} = 2X_{(NP)}{}^{M}\left(\Lambda^{N}_{[1}\partial_{\mu}\Lambda^{P}_{2]} - 2Z^{NI}\Lambda^{P}_{[1}\,\Xi_{2]\mu I}\right)\,. \tag{5.51}$$

We conclude this brief discussion by showing the projected three-form field strength

$$\widetilde{\mathcal{H}}_{\mu\nu\rho}{}^{M} = 3\mathcal{D}_{[\mu}\widetilde{B}_{\nu\rho]}{}^{M} + 6X_{(NP)}{}^{M}A_{[\mu}{}^{N}\left(\partial_{\nu}A_{\rho]}{}^{P} + \frac{1}{3}X_{OQ}{}^{P}A_{\nu}{}^{O}A_{\rho]}{}^{Q}\right)\,, \tag{5.52}$$

JHEP01(2019)117

and the projected Bianchi identities

$$3\mathcal{D}_{[\mu}\mathcal{F}_{\nu\rho]}{}^{M} - \widetilde{\mathcal{H}}_{\mu\nu\rho}{}^{M} = 0\,, \quad \mathcal{D}_{[\mu}\widetilde{\mathcal{H}}_{\nu\rho\sigma]}{}^{M} - \frac{3}{2}X_{(NP)}{}^{M}\mathcal{F}_{[\mu\nu}{}^{N}\mathcal{F}_{\rho\sigma]}{}^{P} = 0\,. \tag{5.53}$$

Let us now move one step upwards in the hierarchy, by un-projecting the two-form. This requires including a three-form which will now be projected. One could keep going until the space-time dimension puts an upper limit to the hierarchy, but we will stop here and discuss the general case in the appendix. Besides $A_\mu{}^M$ and $B_{\mu\nu I}$ we now introduce a three-form $C_{\mu\nu\rho}{}^A$, which in this case is required by demanding covariance of the un-projected three-form field strength $\mathcal{H}_{\mu\nu\rho I}$. For the sake of briefly we will not pursue the whole discussion, but simply state the results and refer to the details in [45]. The gauge transformations are given by

$$\delta A_\mu{}^M = \mathcal{D}_\mu\Lambda^M - Z^{MI}\,\widehat{\Xi}_{\mu I} \tag{5.54}$$

$$\delta B_{\mu\nu I} = 2\mathcal{D}_{[\mu}\widehat{\Xi}_{\nu]I} - 2d_{IMN}\Lambda^M\mathcal{F}_{\mu\nu}{}^{N} + 2d_{IMN}A_{[\mu}{}^{M}\delta A_{\nu]}{}^{N} - Y_{IA}\,\widehat{\Sigma}_{\mu\nu}{}^{A} \tag{5.55}$$

$$\begin{aligned}\delta\left(Y_{IA}\,C_{\mu\nu\rho}{}^{A}\right) = Y_{IA}\Big(&3\mathcal{D}_{[\mu}\widehat{\Sigma}_{\nu\rho]}{}^{A} + 3g_M{}^{AI}\mathcal{F}_{[\mu\nu}{}^{M}\widehat{\Xi}_{\rho]I} + g_M{}^{AI}\Lambda^M\mathcal{H}_{\mu\nu\rho I}\\ &+3g_M{}^{AI}B_{I[\mu\nu}\delta A_{\rho]}{}^{M} + 2g_M{}^{AI}d_{INP}A_{[\mu}{}^{M}A_\nu{}^{N}\delta A_{\rho]}{}^{P}\Big)\,,\end{aligned} \tag{5.56}$$

where we have introduced the field strength tensors

$$\mathcal{F}_{\mu\nu}{}^{M} = 2\partial_{[\mu}A_{\nu]}{}^{M} + X_{NO}{}^{M}A_{[\mu}{}^{N}A_{\nu]}{}^{O} + Z^{MI}B_{\mu\nu I} \tag{5.57}$$

$$\mathcal{H}_{\mu\nu\rho I} = 3\mathcal{D}_{[\mu}B_{\nu\rho]I} + 6d_{IMN}A_{[\mu}{}^{M}\left(\partial_\nu A_{\rho]}{}^{N} + \frac{1}{3}X_{OP}{}^{N}A_\nu{}^{O}A_{\rho]}{}^{P}\right) + Y_{IA}C_{\mu\nu\rho}{}^{A}\,, \tag{5.58}$$

with the covariant derivative

$$\mathcal{D}_\mu B_{\nu\rho I} \equiv \partial_\mu B_{\nu\rho I} - A_\mu{}^{M}X_{MI}{}^{J}B_{\nu\rho J}\,. \tag{5.59}$$

The field strengths satisfy the generalized Bianchi identities

$$3\mathcal{D}_{[\mu}\mathcal{F}_{\nu\rho]}{}^{M} - Z^{MI}\mathcal{H}_{\mu\nu\rho I} = 0 \tag{5.60}$$

$$4\mathcal{D}_{[\mu}\mathcal{H}_{\nu\rho\sigma]I} - 6d_{IMN}\mathcal{F}_{[\mu\nu}{}^{M}\mathcal{F}_{\rho\sigma]}{}^{N} - Y_{IA}\mathcal{G}_{\mu\nu\rho\sigma}{}^{A} = 0\,, \tag{5.61}$$

with $\mathcal{G}_{\mu\nu\rho\lambda}{}^{A}$ the 3-form strength tensor, defined by this equation up to a projection with Y_{IA}.

Let us explain the notation. The tensors d_{IMN} and $g_M{}^{AI}$ are gauging independent and represent the G_0-invariants of the theory. Just to put an example, in $n = 4$ maximal supergravity, where the gauge group is $E_{7(7)}$, these would the related to Ω_{MN} and $(t_\alpha)^{MN}$. The tensors Z^{MI} and Y_{IA} on the other hand do depend on the gaugings, and so together with X_M are responsible for its deformation. The intertwining tensor Z^{MI} for example is related to the symmetric part of the embedding tensor $X_{(MN)}{}^{P}$. The fields are $A_\mu{}^M$, $B_{\mu\nu I}$ and $C_{\mu\nu\rho}{}^A$, their associated gauge parameters are Λ^M, $\widehat{\Xi}_{\mu I}$ and $\widehat{\Sigma}_{\mu\nu}{}^A$, and their field strengths are $\mathcal{F}_{\mu\nu}{}^M$, $\mathcal{H}_{\mu\nu\rho I}$ and $\mathcal{G}_{\mu\nu\rho\sigma}{}^A$. The indices M, I, A, denote the representations to which the tensors belong. Then, $X_{MI}{}^J$ corresponds to the embedding tensor in the representation of the two-form field, $X_{MA}{}^B$ of the three-form field, etc.

JHEP01(2019)117

Gauge covariance is achieved provided the following constraints are imposed

$$\begin{aligned} d_{I[MN]} &= 0 \\ d_{I(MN}g_{O)}{}^{AI} &= 0 \\ Z^{MI}d_{INO} &= X_{(NO)}{}^{M} \\ X_{MI}{}^{J} + 2Z^{NJ}d_{IMN} &= Y_{IA}g_{M}{}^{AJ} \\ Z^{MI}Y_{IA} &= 0 \\ Z^{MI}X_{M} &= 0 \\ Y_{IA}g_{M}{}^{AJ}Z^{MK} &= 2Z^{MK}Z^{NJ}d_{IMN}\,, \end{aligned} \tag{5.62}$$

where the last two follow from the others and the quadratic constraints.

The vector fields $A_\mu{}^M$ in (5.54) transform exactly as in (5.46). Instead, the transformation of the two-form in (5.47) corresponds to a projection of (5.55) with the intertwining tensor Z^{MI}. When removing the projection in (5.55) one has to include terms that vanish when projected, which in this case are represented by $Y_{IA}\widehat{\Sigma}_{\mu\nu}{}^{A}$. The bracket of this parameter is responsible of absorbing the failure of closure of the unprojected gauge transformation, and its associated field $Y_{IA}C_{\mu\nu\rho}{}^{A}$ is responsible for guaranteing the covariance of $\mathcal{H}_{\mu\nu\rho I}$, namely to enforce

$$\delta\mathcal{H}_{\mu\nu\rho I} = X_{MI}{}^{J}\Lambda^{M}\mathcal{H}_{\mu\nu\rho J}\,. \tag{5.63}$$

In fact, if the three-form fields $C_{\mu\nu\rho}{}^{A}$ were absent in (5.58), then $\mathcal{H}_{\mu\nu\rho I}$ would fail to transform covariantly and the failure would be proportional to $X_{MI}{}^{J} + 2Z^{NJ}d_{IMN}$, which in turn is proportional to Y_{IA}, as can be seen in (5.62). To follow the notation we have been using, we could put a tilde on the three-form and its gauge parameter to denote that both are projected

$$\widetilde{C}_{\mu\nu\rho I} = Y_{IA}C_{\mu\nu\rho}{}^{A}\,, \qquad \widetilde{\widehat{\Sigma}}_{\mu\nu I} = Y_{IA}\widehat{\Sigma}_{\mu\nu}{}^{A}\,, \tag{5.64}$$

and we have to remember that now the tensors with tilde satisfy

$$Z^{MI}\widetilde{C}_{\mu\nu\rho I} = 0\,, \qquad Z^{MI}\widetilde{\widehat{\Sigma}}_{\mu\nu I} = 0\,. \tag{5.65}$$

If we decided to move a step forward and removed the projection with Y_{IA}, then the strength tensor $\mathcal{G}_{\mu\nu\rho\lambda}{}^{A}$ would not be covariant, and its failure would be proportional to $X_{MB}{}^{A} + g_{M}{}^{AJ}Y_{JB}$, forcing the inclusion of a four-form and so on. The same mechanism is repeated over and over. Then, if we want to analyze the tensor hierarchy up to the three-form only, a truncation scheme must be considered, which is possible given that

$$Y_{IA}\left(X_{MB}{}^{A} + g_{M}{}^{AJ}Y_{JB}\right) = 0\,, \tag{5.66}$$

vanishes thanks to the G-invariance of Y_{IA} and (5.62). We then cut the p-form chain at $p = 3$ and only consider $\left\{A_\mu{}^M, B_{\mu\nu I}, \widetilde{C}_{\mu\nu\rho I}\right\}$, their parameters $\left\{\Lambda^M, \widehat{\Xi}_{\mu I}, \widetilde{\widehat{\Sigma}}_{\mu\nu I}\right\}$ and their field strengths $\left\{\mathcal{F}_{\mu\nu}{}^{M}, \mathcal{H}_{\mu\nu\rho I}, \widetilde{\mathcal{G}}_{\mu\nu\rho\sigma I}\right\}$.

JHEP01(2019)117

To summarize, the gauge transformations of the projected tensor hierarchy up to the three-form are

$$\delta A_\mu{}^M = \mathcal{D}_\mu \Lambda^M - Z^{MI}\,\widehat{\Xi}_{\mu I} \tag{5.67}$$

$$\delta B_{\mu\nu I} = 2\mathcal{D}_{[\mu}\widehat{\Xi}_{\nu]I} - 2d_{IMN}\Lambda^M \mathcal{F}_{\mu\nu}{}^N + 2d_{IMN}A_{[\mu}{}^M \delta A_{\nu]}{}^N - \widehat{\widetilde{\Sigma}}_{\mu\nu I} \tag{5.68}$$

$$\begin{aligned}\delta \widetilde{C}_{\mu\nu\rho I} = 3\,\mathcal{D}_{[\mu}\widehat{\widetilde{\Sigma}}_{\nu\rho]I} + Y_{IA}\,g_M{}^{AJ}\Big(&3\,\mathcal{F}_{[\mu\nu}{}^M\widehat{\Xi}_{\rho]J} + \Lambda^M \mathcal{H}_{\mu\nu\rho J} \\ &+3\,B_{[\mu\nu J}\delta A_{\rho]}{}^M + 2\,d_{JNP}A_{[\mu}{}^M A_\nu{}^N \delta A_{\rho]}{}^P\Big)\,,\end{aligned} \tag{5.69}$$

the field strengths are defined as

$$\begin{aligned}\mathcal{F}_{\mu\nu}{}^M &= 2\partial_{[\mu}A_{\nu]}{}^M + X_{NO}{}^M A_{[\mu}{}^N A_{\nu]}{}^O + Z^{MI}B_{\mu\nu I} \\ \mathcal{H}_{\mu\nu\rho I} &= 3\mathcal{D}_{[\mu}B_{\nu\rho]I} + 6d_{IMN}A_{[\mu}{}^M\left(\partial_\nu A_{\rho]}{}^N + \frac{1}{3}X_{OP}{}^N A_\nu{}^O A_{\rho]}{}^P\right) + \widetilde{C}_{\mu\nu\rho I} \\ \widetilde{\mathcal{G}}_{\mu\nu\rho\sigma I} &= 4\mathcal{D}_{[\mu}\widetilde{C}_{\nu\rho\sigma]I} - Y_{IA}\,g_M{}^{AJ}\big(6B_{[\mu\nu J}\mathcal{F}_{\rho\sigma]}{}^M - 3Z^{MK}B_{[\mu\nu J}B_{\rho\sigma]K} \\ &\quad +8d_{JNP}A_{[\mu}{}^M A_\nu{}^N \partial_\rho A_{\sigma]}{}^P + 2d_{JPN}X_{QR}{}^P A_{[\mu}{}^M A_\nu{}^N A_\rho{}^Q A_{\sigma]}{}^R\big)\,,\end{aligned} \tag{5.70}$$

and satisfy the following Bianchi identities

$$\begin{aligned}3\,\mathcal{D}_{[\mu}\mathcal{F}_{\nu\rho]}{}^M - Z^{MI}\mathcal{H}_{\mu\nu\rho I} &= 0 \\ 4\,\mathcal{D}_{[\mu}\mathcal{H}_{\nu\rho\sigma]I} - 6d_{IMN}\mathcal{F}_{[\mu\nu}{}^M\mathcal{F}_{\rho\sigma]}{}^N - \widetilde{\mathcal{G}}_{\mu\nu\rho\sigma I} &= 0 \\ 5\,\mathcal{D}_{[\mu}\widetilde{\mathcal{G}}_{\nu\rho\sigma\lambda]I} + 10Y_{IA}g_M{}^{AJ}\mathcal{F}_{[\mu\nu}{}^M\mathcal{H}_{\rho\sigma\lambda]J} &= 0\,.\end{aligned} \tag{5.71}$$

Moving to the closure of the gauge algebra, we can now find the unprojected one-form bracket $\widehat{\Xi}_{12\mu I}$, and the projected two-form bracket $\widehat{\widetilde{\Sigma}}_{12\mu\nu I}$

$$\begin{aligned}\Lambda_{12}^M &= -X_{NP}{}^M\Lambda_{[1}^N\Lambda_{2]}^P \\ \widehat{\Xi}_{12\mu I} &= 2\,d_{IMN}\,\mathcal{D}_\mu\Lambda_{[1}^M\Lambda_{2]}^N \\ \widehat{\widetilde{\Sigma}}_{12\mu\nu I} &= -2\,Y_{IA}\,g_M{}^{AJ}\left(d_{JNP}\,\Lambda_{[1}^M\Lambda_{2]}^N\,\mathcal{F}_{\mu\nu}{}^P + 2\,\widehat{\Xi}_{[1[\mu J}\,\mathcal{D}_{\nu]}\Lambda_{2]}^M - Z^{MK}\,\widehat{\Xi}_{[1[\mu J}\,\widehat{\Xi}_{2]\nu]K}\right).\end{aligned} \tag{5.72}$$

As expected, these brackets are those of the set of parameters for which the gauge transformations are written covariantly, and so are field dependent. We show in the following subsection that via redefinitions one can reach the hypothesis of the theorem presented in subsections 2.1 and 2.2, implying that the gauge algebra of the tensor hierarchy in gauged supergravities fits into an $L_\infty^{\text{gauge+fields}}$ algebra.

We have projected the hierarchy to the three-form field to match these results with those of the EFT section, but it can of course be continued more generally to higher forms. In the appendix we show how to extend the hierarchy exhaustively for the case we are interested in here: $n = 4$ space-time dimensions.

JHEP01(2019)117

5.3 L_∞ algebra and gauged supergravity

In order to obtain field independent brackets we propose the following gauge parameter redefinitions

$$\begin{aligned}\widehat{\Xi}_{\mu I} &= \Xi_{\mu I} + 2 d_{IMN} A_\mu{}^M \Lambda^N \\ \widetilde{\widehat{\Sigma}}_{\mu\nu I} &= \widetilde{\Sigma}_{\mu\nu I} - Y_{IA}\, g_M{}^{AJ} \left(2 A_{[\mu}{}^M \Xi_{\nu]J} + B_{\mu\nu J}\Lambda^M + 2 A_{[\mu}{}^M A_{\nu]}{}^N d_{JNP}\Lambda^P\right) ,\end{aligned} \tag{5.73}$$

now yielding the brackets

$$\begin{aligned}\Lambda_{12}^M &= -X_{NP}{}^M \Lambda_{[1}^N \Lambda_{2]}^P \\ \Xi_{12\mu I} &= 2\, d_{IMN} \left(\Lambda_{[1}^M \partial_\mu \Lambda_{2]}^N + 2\, Z^{MJ}\, \Xi_{[1\mu J}\Lambda_{2]}^N\right) \\ \widetilde{\Sigma}_{12\mu\nu I} &= -2\, Y_{IA}\, g_M{}^{AJ} \left(2\Lambda_{[1}^M\, \partial_{[\mu}\Xi_{2]\nu]J} + Z^{MK}\, \Xi_{[1[\mu J}\, \Xi_{2]\nu]K} + \widetilde{\Sigma}_{[1\mu\nu J}\Lambda_{2]}^M\right) .\end{aligned} \tag{5.74}$$

Defining a composed parameter with its corresponding bracket

$$\zeta = \left(\Lambda^M,\, \Xi_{\mu I},\, \widetilde{\Sigma}_{\mu\nu I}\right) , \quad [\zeta_1,\, \zeta_2] = \left(\Lambda_{12}^M,\, \Xi_{12\mu I},\, \widetilde{\Sigma}_{12\mu\nu I}\right) , \tag{5.75}$$

we construct the Jacobiator

$$J(\zeta_1, \zeta_2, \zeta_3) = 3\left[\left[\zeta_{[1}, \zeta_2\right], \zeta_3\right] , \tag{5.76}$$

with components

$$\begin{aligned} J^M &= Z^{MI} N_I \\ J_{\mu I} &= \partial_\mu N_I + \widetilde{N}_{\mu I} \\ \widetilde{J}_{\mu\nu I} &= 2\, \partial_{[\mu}\widetilde{N}_{\nu]I} , \end{aligned} \tag{5.77}$$

where we defined the Nijenhuis tensors

$$N_I = d_{IMN} X_{OP}{}^M \Lambda_{[1}{}^O \Lambda_2{}^P \Lambda_{3]}{}^N \tag{5.78}$$

$$\widetilde{N}_{\mu I} = 2\, Y_{IA}\, g_M{}^{AJ}\, d_{JNP}\, \Lambda_{[1}^M \left(2\Lambda_2^N \partial_\mu \Lambda_{3]}^P + 3 Z^{NK} \Xi_{2\mu K} \Lambda_{3]}^P\right) . \tag{5.79}$$

We then have a strong indication that the following are trivial parameters

$$\begin{aligned} \Lambda^M_{\text{trivial}} &= Z^{MI}\chi_I \\ \Xi_{\mu I\,\text{trivial}} &= \partial_\mu \chi_I + \widetilde{\chi}_{\mu I} \\ \widetilde{\Sigma}_{\mu\nu I\,\text{trivial}} &= 2\, \partial_{[\mu}\widetilde{\chi}_{\nu]I} , \end{aligned} \tag{5.80}$$

where now χ_I is arbitrary and $\widetilde{\chi}_{\mu I}$ is constrained to satisfy $Z^{MI}\widetilde{\chi}_{\mu I} = 0$. It can be checked that this is indeed the case. For the other set of parameters one has instead

$$\begin{aligned} \Lambda^M_{\text{trivial}} &= Z^{MI}\chi_I \\ \widehat{\Xi}_{\mu I\,\text{trivial}} &= \mathcal{D}_\mu \chi_I + \widetilde{\widehat{\chi}}_{\mu I} \\ \widetilde{\widehat{\Sigma}}_{\mu\nu I\,\text{trivial}} &= 2\, \mathcal{D}_{[\mu}\widetilde{\widehat{\chi}}_{\nu]I} - Y_{IA}\, g_M{}^{AJ}\, \mathcal{F}_{\mu\nu}{}^M\, \chi_J , \end{aligned} \tag{5.81}$$

for a redefined but still constrained $\widetilde{\widehat{\chi}}_{\mu I}$.

JHEP01(2019)117

We conclude this section by pointing out that gauged supergravities share with DFT and EFT the interesting feature that the bracket between a generic parameter and a trivial one, gives another trivial one

$$[\zeta_{\text{trivial}}, \zeta] = \zeta'_{\text{trivial}} \,,$$

where ζ_{trivial} is defined in (5.80) and

$$\zeta'_{\text{trivial}} = \left(Z^{MI} \chi'_I,\, \partial_\mu \chi'_I + \widetilde{\chi}'_{\mu I},\, 2\,\partial_{[\mu} \widetilde{\chi}'_{\nu] I} \right) , \tag{5.82}$$

with

$$\chi'_I = Z^{MJ} d_{IMN} \Lambda^N \chi_J \tag{5.83}$$

$$\widetilde{\chi}'_{\mu I} = -Y_{IA}\, g_M{}^{AJ}\, Z^{MK}\, \Xi_{\mu J}\, \chi_K \,. \tag{5.84}$$

Once again we see that there is a set of parameters that meet the hypothesis of the theorem in subsections 2.1 and 2.2, implying that the tensor hierarchy in gauged supergravities lie within an $L_\infty^{\text{gauge+fields}}$ structure.

The concrete graded spaces and products are obtained following the procedure described in previous sections. For completion we give here some products in the L_∞^{gauge} sector, which are trivial to read from the results above. The only graded subspaces in this case are

$$X_2 \;:\; \eta = (\eta_I,\, \widetilde{\eta}_{\mu I}) = \left(Y_{IA} C^A \,, -Y_{IA} \partial_\mu C^A \right) \quad \forall C^A \,, \tag{5.85}$$

$$X_1 \;:\; \chi = (\chi_I,\, \widetilde{\chi}_{\mu I}) \,, \tag{5.86}$$

$$X_0 \;:\; \zeta = (\Lambda^M,\, \Xi_{\mu I},\, \widetilde{\Sigma}_{\mu\nu I}) \,. \tag{5.87}$$

The product ℓ_1 maps $X_2 \to X_1$ through inclusion. Some explicit products are

$$\ell_1(\eta) = \eta_I + \widetilde{\eta}_{\mu I} \,, \tag{5.88}$$

$$\ell_1(\chi) = \Lambda^M_{\text{trivial}} + \Xi_{\mu I\,\text{trivial}} + \widetilde{\Sigma}_{\mu\nu I\,\text{trivial}} \,, \tag{5.89}$$

$$\ell_2(\zeta_1,\, \zeta_2) = \Lambda^M_{12} + \Xi_{12\mu I} + \widetilde{\Sigma}_{12\mu\nu I} \,, \tag{5.90}$$

$$\ell_2(\zeta,\, \chi) = -\chi'_I - \widetilde{\chi}'_{\mu I} + \mathcal{K}_I + \widetilde{\mathcal{K}}_{\mu I} \,, \tag{5.91}$$

$$\ell_3(\zeta_1,\, \zeta_2,\, \zeta_3) = -N_I - \widetilde{N}_{\mu I} \,. \tag{5.92}$$

As done in a previous section, the product $\ell_2(\zeta,\, \chi)$ is defined here up to terms $\mathcal{K}_I$, $\widetilde{\mathcal{K}}_{\mu I}$ taking values in the kernel of ℓ_1, namely

$$0 = Z^{MI} \mathcal{K}_I + \partial_\mu \mathcal{K}_I + \widetilde{\mathcal{K}}_{\mu I} + 2\partial_{[\mu} \widetilde{\mathcal{K}}_{\nu] I} \,. \tag{5.93}$$

These can be chosen in such a way that[10]

$$\ell_2(\zeta,\, \chi) = \frac{1}{2} \delta_\Lambda \chi_I + Y_{IA} g_M{}^{AJ} \left(Z^{MK} \Xi_{\mu J} \chi_K - \frac{1}{2} \partial_\mu (\Lambda^M \chi_J) \right) , \tag{5.94}$$

[10]Note that the last terms proportional to the Y-tensor would vanish if the hierarchy were only projected to the two-form field, as expected from the results in section 4.3.

JHEP01(2019)117

	Maximal	**Half-maximal** $n = 6$
G_0	$E_{7(7)}$	$\mathrm{SL}(2) \times O(6,6)$
Embedding tensor	**912**	(**2**, **12**) + (**2**, **220**)
1-forms	**56**	(**2**, **12**)
2-forms	**133**	(**3**, **1**) + (**1**, **66**)
3-forms	**912**	Projected out in [42]
4-forms	**133** + **8645**	Projected out in [42]

Table 4. Duality groups and representations of the tensor hierarchy in maximal and half-maximal supergravities in four space-time dimensions.

which in turn implies

$$\ell_2(\chi_1, \chi_2) = 0\,, \tag{5.95}$$

$$\ell_3(\zeta_1, \zeta_2, \chi) = Y_{IA} C'^A - Y_{IA} \partial_\mu C'^A\,, \tag{5.96}$$

with

$$C'^A = \frac{1}{3} g_M{}^{AJ} \left(\Lambda^M_{12} \chi_J - \frac{1}{2} Y_{JB} g_N{}^{BK} \Lambda^M_{[1} \Lambda^N_{2]} \chi_K \right)\,, \tag{5.97}$$

among higher products defined from higher identities. As stated by the theorem discussed in section 2.1 no further graded subspaces are required beyond X_2. Instead, the effect of unprojecting the tensor hierarchy is to populate X_2, X_1 andX_0 with the new representations.

5.4 Maximal and half-maximal gauged supergravity

Here we discuss how the general framework introduced in section 5.2 reproduces the tensor hierarchy of maximal and half-maximal gauged supergravities in four space-time dimensions. In table 4 we write the representations for each theory.

The general discussion in section 5.2 perfectly matches the standard formulation of the maximal theory [43, 44], and so there is not much to say beyond the general discussion. The formulation of the half-maximal theory [42] requires instead a closer inspection. We can read from table 4 that the global symmetry of the ungauged maximal supergravity is a continuous $\mathrm{SL}(2) \times O(6,n)$. We will take the fundamental representation to have split indices (α, M), where α, β, γ are $\mathrm{SL}(2)$ indices and M, N, P are fundamental $O(6,n)$ indices. This slightly modifies the notation in section 5.2, there the whole fundamental index was noted by M, N, P. Hopefully this will not cause confusion. The invariants of the global group are the antisymmetric Levi-Civita tensor $\epsilon_{\alpha\beta}$ with the conventions $\epsilon_{+-} = \epsilon^{+-} = 1$ and $\epsilon_{\alpha\gamma}\epsilon^{\beta\gamma} = \delta^\beta_\alpha$. The $O(6,n)$ invariant tensor is η_{MN}. Both tensors are used like metrics to raise and lower indices. In terms of these, the generators are

$$\begin{aligned} (t_{MN})_P{}^Q &= \delta^Q_{[M} \eta_{N]P} \\ (t_{\alpha\beta})_\gamma{}^\delta &= \delta^\delta_{(\alpha} \epsilon_{\beta)\gamma}\,. \end{aligned} \tag{5.98}$$

In order to gauge the supergravity we promote a subgroup $G \subset G_0$ to local with the help of the embedding tensor as we did in section 5.2. The precise identifications are listed in table 5.

JHEP01(2019)117

General	**Half-Maximal**
M	αM
I	$(\alpha\beta) \quad [MN]$
Λ^M , $\widehat{\Xi}_{\mu I}$	$\Lambda^{\alpha M}$, $\widehat{\Xi}_{\mu,\alpha\beta}$, $\widehat{\Xi}_{\mu,MN}$
$A_\mu{}^M$, $B_{\mu\nu I}$	$A_\mu{}^{\alpha M}$, $B_{\mu\nu,\alpha\beta}$, $B_{\mu\nu,MN}$
d_{IMN}	$d_{\gamma\delta,\alpha M\beta N} = \eta_{MN}\epsilon_{\alpha(\gamma}\epsilon_{\delta)\beta}$, $\quad d_{OP,\alpha M\beta N} = \epsilon_{\alpha\beta}\eta_{M[O}\eta_{P]N}$
$\Theta_M{}^I$	$\Theta_{\alpha M}{}^{\beta\gamma} = f_{\delta M}\epsilon^{\delta(\beta}\delta_\alpha^{\gamma)}$, $\quad \Theta_{\alpha M}{}^{NP} = f_{\alpha M}{}^{NP} + \delta_M^{[N} f_\alpha{}^{P]}$
$X_{MN}{}^O$	$X_{\alpha M\beta N}{}^{\gamma O} = \Theta_{\alpha M}{}^{\delta\lambda}\,(t_{\delta\lambda})_\beta{}^\gamma\delta_N^O + \Theta_{\alpha M}{}^{PQ}\,(t_{PQ})_N{}^O\delta_\beta^\gamma$ $= -\delta_\beta^\gamma f_{\alpha MN}{}^O + \frac{1}{2}\left(\delta_M^O\delta_\beta^\gamma f_{\alpha N} - \delta_N^O\delta_\alpha^\gamma f_{\beta M} - \eta_{MN}\delta_\beta^\gamma f_\alpha{}^O + \epsilon_{\alpha\beta}\epsilon^{\delta\gamma}\delta_N^O f_{\delta M}\right)$
Z^{MI}	$Z^{\alpha M,\beta\gamma} = \frac{1}{2}\eta^{MN}\epsilon^{\alpha\delta}\Theta_{\delta N}{}^{\beta\gamma} = -\frac{1}{2}\epsilon^{\alpha(\beta}\epsilon^{\gamma)\delta} f_\delta{}^M$ $Z^{\alpha M,NO} = \frac{1}{2}\eta^{MP}\epsilon^{\alpha\beta}\Theta_{\beta P}{}^{NO} = \frac{1}{2}\epsilon^{\alpha\beta}\left(f_\beta{}^{MNO} + \eta^{M[N} f_\beta{}^{O]}\right)$

Table 5. Identification between tensors in the general structure of the tensor hierarchy (section 5.2) and the $n = 4$ half-maximal gauged supergravity.

The gaugings now take the form $f_{\alpha M}$ and $f_{\alpha MNP} = f_{\alpha[MNP]}$, in terms of which the quadratic constraints (5.37) take the form

$$\begin{aligned}
f_\alpha{}^M f_{\beta M} &= 0\\
f_{(\alpha}{}^P f_{\beta)PMN} &= 0\\
3f_{\alpha[MN}{}^P f_{\beta R]OP} - 2f_{(\alpha[MNR} f_{\beta)O]} &= 0\\
\epsilon^{\alpha\beta}\left(f_\alpha{}^P f_{\beta PMN} + f_{\alpha M} f_{\beta N}\right) &= 0\\
\epsilon^{\alpha\beta}\left(f_{\alpha MNR} f_{\beta PQ}{}^R - f_\alpha{}^R f_{\beta R[M}\eta_{P][Q}\eta_{N]]} - f_{\alpha[M} f_{N][PQ]\beta} + f_{\alpha[P} f_{Q][MN]\beta}\right) &= 0\,.
\end{aligned} \tag{5.99}$$

From the table 5 and the general expressions in section 5.2 we can compute the gauge transformations of the vector fields and two-forms

$$\begin{aligned}
\delta A_\mu{}^{\alpha M} &= \mathcal{D}_\mu\Lambda^{\alpha M} - \frac{1}{2}\epsilon^{\alpha\beta}\left[f_\gamma{}^M\epsilon^{\gamma\delta}\widehat{\Xi}_{\mu,\delta\beta} + \left(f_\beta{}^{MNP} + \eta^{MN} f_\beta{}^P\right)\widehat{\Xi}_{\mu,NP}\right]\\
\delta B_{\mu\nu,\alpha\beta} &= 2\mathcal{D}_{[\mu}\widehat{\Xi}_{\nu],\alpha\beta} - 2\eta_{MN}\epsilon_{\gamma(\alpha}\epsilon_{\beta)\delta}(\Lambda^{\gamma M}\mathcal{F}_{\mu\nu}{}^{\delta N} - A_{[\mu}{}^{\gamma M}\delta A_{\nu]}{}^{\delta N})\\
\delta B_{\mu\nu,MN} &= 2\mathcal{D}_{[\mu}\widehat{\Xi}_{\nu],MN} - 2\epsilon_{\alpha\beta}\eta_{O[M}\eta_{N]P}(\Lambda^{\alpha O}\mathcal{F}_{\mu\nu}{}^{\beta P} - A_{[\mu}{}^{\alpha O}\delta A_{\nu]}{}^{\beta P})\,,
\end{aligned} \tag{5.100}$$

their field strengths

$$\begin{aligned}
\mathcal{F}_{\mu\nu}{}^{\alpha M} &= 2\partial_{[\mu}A_{\nu]}{}^{\alpha M} + X_{\beta N\gamma P}{}^{\alpha M}A_{[\mu}{}^{\beta N}A_{\nu]}{}^{\gamma P}\\
&\quad + \frac{1}{2}\epsilon^{\alpha\beta}\left[f_\gamma{}^M\epsilon^{\gamma\delta}B_{\mu\nu,\delta\beta} + \left(f_\beta{}^{MNP} + \eta^{MN} f_\beta{}^P\right)B_{\mu\nu,NP}\right]\\
\mathcal{H}_{\mu\nu\rho,\alpha\beta} &= 3\mathcal{D}_{[\mu}B_{\nu\rho],\alpha\beta} + 6\eta_{MN}\epsilon_{\gamma(\alpha}\epsilon_{\beta)\delta}A_{[\mu}{}^{\gamma M}\left(\partial_\nu A_{\rho]}{}^{\delta N} + \frac{1}{3}X_{\lambda O\epsilon P}{}^{\delta N}A_\nu{}^{\lambda O}A_{\rho]}{}^{\epsilon P}\right)\\
\mathcal{H}_{\mu\nu\rho,MN} &= 3\mathcal{D}_{[\mu}B_{\nu\rho],MN} + 6\epsilon_{\alpha\beta}\eta_{O[M}\eta_{N]P}A_{[\mu}{}^{\alpha O}\left(\partial_\nu A_{\rho]}{}^{\beta P} + \frac{1}{3}X_{\gamma Q\delta R}{}^{\beta P}A_\nu{}^{\gamma Q}A_{\rho]}{}^{\delta R}\right),
\end{aligned} \tag{5.101}$$

JHEP01(2019)117

and the covariant derivatives

$$\mathcal{D}_\mu = \partial_\mu - A_\mu{}^{\alpha M}\Theta_{\alpha M}{}^{NO}t_{NO} - A_\mu{}^{\alpha M}\Theta_{\alpha M}{}^{\beta\gamma}t_{\beta\gamma}\,. \tag{5.102}$$

With these results we can also compute the Y_{IA} tensor using its definition in (5.62)

$$Y_{IA}g_M{}^{AJ} = X_{MI}{}^{J} + 2Z^{NJ}d_{IMN}\,, \tag{5.103}$$

with components

$$\begin{aligned}
Y_{\alpha\beta,A}\, g_{\gamma M}{}^{A,\delta\lambda} &= -f_{(\alpha M}\delta^{(\delta}_{\beta)}\delta^{\lambda)}_{\gamma} \\
Y_{\alpha\beta,A}\, g_{\gamma M}{}^{A,NP} &= \left(f_{(\alpha M}{}^{NP} + \delta^{[N}_{M}f_{(\alpha}{}^{P]}\right)\epsilon_{\beta)\gamma} \\
Y_{NP,A}\, g_{\gamma M}{}^{A,\delta\lambda} &= -\frac{1}{2}\delta^{(\delta}_{\gamma}\epsilon^{\lambda)\beta}f_{\beta[N}\eta_{P]M} \\
Y_{NP,A}\, g_{\gamma M}{}^{A,OQ} &= -2f_{\gamma M[N}{}^{[O}\delta^{Q]}_{P]} + f_{\gamma[N}{}^{OQ}\eta_{P]M} - f_{\gamma[N}\delta^{[O}_{P]}\delta^{Q]}_{M}\,.
\end{aligned} \tag{5.104}$$

The fact that these are non-vanishing implies that in half-maximal gauged supergravity the algebra closes because the two-forms are projected with the intertwiner Z^{MI}. In principle the full algebra can be obtained un-projecting and adding higher tensors through the standard procedure.

We can now as a final check see how to relate these expressions to those in the electric sector computed from DFT. The identifications for the gaugings are

$$\begin{aligned}
f_{-M} &= 0 \\
f_{-MNP} &= 0 \\
f_{+M} &= f_M \\
f_{+MNO} &= f_{MNO}\,.
\end{aligned} \tag{5.105}$$

Then, the last two equations of (5.99) vanish trivially and the remaining constraints reduce to

$$\begin{aligned}
f^M f_M &= 0 \\
f^P f_{PMN} &= 0 \\
3f_{[MN}{}^{P} f_{R]OP} - 2f_{[MNR} f_{O]} &= 0\,,
\end{aligned} \tag{5.106}$$

and match (5.12) exactly.

To obtain full identification the parameters must be related as

$$\Lambda^{+M} = \Lambda^M\,, \quad \widehat{\Xi}_{\mu\,--} = -2\widehat{\Xi}_\mu\,, \tag{5.107}$$

and the fields as follows

$$A_\mu{}^{+M} = A_\mu{}^{M}\,, \quad B_{\mu\nu\,--} = -2B_{\mu\nu}\,. \tag{5.108}$$

We call this selection the "electric" section of the theory. It turns out that by selecting the electric gaugings (5.105), the transformations of the electric components above do not

JHEP01(2019)117

depend on the magnetic ones (namely, the parameters Λ^{-M}, $\widehat{\Xi}_{\mu\,-+}$, $\widehat{\Xi}_{\mu\,++}$ and $\widehat{\Xi}_{\mu\,MN}$ and the fields $A_\mu{}^{-M}$, $B_{\mu\nu\,-+}$, $B_{\mu\nu\,++}$ and $B_{\mu\nu\,MN}$ disappear from the transformations of $A_\mu{}^{+M}$ and $B_{\mu\nu\,--}$). Then, they decouple forming a closed subalgebra. With these identifications the gauge transformations of the effective action of DFT are recovered (5.14).

We are now in the position of understanding why the small electric sector of gauged supergravity, consisting in vector fields and single two-form, closes exactly without projection with an intertwining tensor nor the inclusion of higher forms. The reason is that the electric section yields the following vanishing components of the intertwining tensor

$$Y_{--,A}\, g_{\gamma M}{}^{A,I} = 0\,, \tag{5.109}$$

and this particular combination is responsible for the closure of the gauge algebra of the two-form to fail.

We can also re-derive the Jacobiator starting with (5.77) for half-maximal

$$\begin{aligned} J^{\alpha M} &= Z^{\alpha M\beta\gamma} N_{\beta\gamma} + Z^{\alpha MNO} N_{NO} \\ J_{\mu,\alpha\beta} &= \partial_\mu N_{\alpha\beta} \\ J_{\mu,MN} &= \partial_\mu N_{MN}\,, \end{aligned} \tag{5.110}$$

with the Nijenhuis tensors given by

$$\begin{aligned} N_{\alpha\beta} &= \eta_{MN}\epsilon_{\gamma(\alpha}\epsilon_{\beta)\delta} X_{\lambda O\epsilon P}{}^{\gamma M} \Lambda_{[1}{}^{\lambda O} \Lambda_2{}^{\epsilon P} \Lambda_{3]}{}^{\delta N} \\ N_{MN} &= \epsilon_{\alpha\beta}\eta_{O[M}\eta_{N]P} X_{\gamma Q\delta R}{}^{\alpha O} \Lambda_{[1}{}^{\gamma Q} \Lambda_2{}^{\delta R} \Lambda_{3]}{}^{\beta P}\,. \end{aligned} \tag{5.111}$$

Then, imposing the electric section, it is easy to see that

$$N_{--} = f_{MNO} \Lambda_{[1}{}^{M} \Lambda_2{}^{N} \Lambda_{3]}{}^{O} = 2N\,, \tag{5.112}$$

where in the last equality we established the relation with (5.33). Finally, using that the only non-vanishing components of the intertwining tensor are

$$\begin{aligned} Z^{-M,OP} &= -\frac{1}{2}\left(f^{MOP} + \eta^{M[O} f^{P]}\right) \\ Z^{-M,+-} &= -\frac{1}{4} f^M \\ Z^{+M,--} &= \frac{1}{2} f^M\,, \end{aligned} \tag{5.113}$$

the following relevant components of the Jacobiator survive

$$J^{+M} = \frac{1}{2} f^M\, N_{--} = J^M \tag{5.114}$$

$$J_{\mu,--} = \partial_\mu N_{--} = -2J_\mu\,, \tag{5.115}$$

which are trivial parameters, as expected, and in the last equality we established the relation with (5.32). This completes the embedding of the tensor hierarchy for the electric and full half-maximal supergravity in the general structure discussed before.

JHEP01(2019)117

Acknowledgments

We thank S. Iguri and C. Nuñez for comments. Our work is supported by CONICET.

A Relating the generalized metric and the KK formulations of DFT

In this appendix we relate the degrees of freedom and parameters of the generalized metric formulation of DFT [53] ($\mathcal{H}_{\mathcal{MN}}$, d, $\xi^{\mathcal{M}}$) with those in the KK formulation [39] ($g_{\mu\nu}$, $B_{\mu\nu}$, $A_\mu{}^M$, $\mathcal{M}_{MN}$, ϕ, ξ^μ, Λ^M, Ξ_μ). The $O(D,D)$ indices $\mathcal{M},\mathcal{N} = 1,\dots,2D$ with $D = n+d$ are here distinguished from the $O(d,d)$ indices $M,N = 1,\dots,2d$. The relation is given by

$$\eta_{\mathcal{MN}} = \begin{pmatrix} 0 & \delta^\mu_\nu & 0 \\ \delta^\nu_\mu & 0 & 0 \\ 0 & 0 & \eta_{MN} \end{pmatrix}, \qquad \partial_{\mathcal{M}} = (0, \partial_\mu, \partial_M) , \tag{A.1}$$

such that the strong constraint in $O(D,D)$ $\partial_{\mathcal{M}}\partial^{\mathcal{M}} = 0$ becomes that in $O(d,d)$ $\partial_M\partial^M = 0$. The parameters are related by

$$\xi^{\mathcal{M}} = \begin{pmatrix} -\Xi_\mu \\ \xi^\mu \\ \Lambda^M \end{pmatrix} , \tag{A.2}$$

and the dynamical degrees of freedom by

$$\mathcal{H}_{\mathcal{MN}} = \begin{pmatrix} g^{\mu\nu} & -g^{\mu\rho}C_{\rho\nu} & -g^{\mu\rho}A_{\rho N} \\ -g^{\nu\rho}C_{\rho\mu} & g_{\mu\nu} + C_{\rho\mu}C_{\sigma\nu}g^{\rho\sigma} + A_\mu{}^P\mathcal{M}_{PQ}A_\nu{}^Q & C_{\rho\mu}g^{\rho\sigma}A_{\sigma N} + A_\mu{}^P\mathcal{M}_{PN} \\ -g^{\nu\rho}A_{\rho M} & C_{\rho\nu}g^{\rho\sigma}A_{\sigma M} + A_\nu{}^P\mathcal{M}_{PM} & \mathcal{M}_{MN} + g^{\rho\sigma}A_{\rho M}A_{\sigma N} \end{pmatrix} , \tag{A.3}$$

where $C_{\mu\nu} = -B_{\mu\nu} + \frac{1}{2}A_\mu{}^P A_{\nu P}$. From here it is straightforward to reproduce the KK-formulation from the $O(D,D)$ covariant expressions of the generalized Lie derivative and DFT action.

B Checking some L_∞ identities

Here we check some non-trivial identities for section 3.3. We first identify the terms that are non vanishing and construct the possible lists of arguments from them.

All identities that include at least one $E \in X_{-2}$ hold true. The argument is very similar to the one presented in [1701.08824], but there are some differences. Each term of an L_∞ relation is of the form $\ell_i(\dots \ell_j(\dots))$, where $\dots$ represents a list of arguments. The only non vanishing products involving an $E \in X_{-2}$ are $\ell_2(E,\zeta)$ and $\ell_3(E,\zeta,\Psi)$, see (3.78). If we consider that there is an E in the second list $\ell_i(\dots \ell_j(E,\dots))$ then the possibilities are:

$$\ell_i(\dots \underbrace{\ell_2(E,\zeta_1)}_{\equiv\tilde{E}\in X_{-2}}) \Rightarrow \ell_2(\zeta_2,\tilde{E}) \text{ or } \ell_3(\zeta_2,\Psi,\tilde{E}), \tag{B.1}$$

$$\ell_i(\dots \underbrace{\ell_3(E,\zeta_1,\Psi_1)}_{\equiv\hat{E}\in X_{-2}}) \Rightarrow \ell_2(\zeta_2,\hat{E}) \text{ or } \ell_3(\zeta_2,\Psi_2,\hat{E}). \tag{B.2}$$

JHEP01(2019)117

From this we learn that the possible lists are (ζ_1, E, ζ_2), $(\zeta_1, E, \zeta_2, \Psi)$, or $(E, \zeta_1, \Psi_1, \zeta_2, \Psi_2)$. If instead we consider that there is an E in the first list $\ell_i(E, \ldots \ell_j(\ldots))$, then the possibilities are:

$$\begin{aligned}
\ell_2(E, \underbrace{\ell_j(\ldots)}_{\in X_0}) &\Rightarrow \ell_2(E, \ell_1(\chi)) \text{ or } \ell_2(E, \ell_2(\zeta_1, \zeta_2)), \\
\ell_3(E, \zeta_1, \underbrace{\ell_j(\ldots)}_{\in X_{-1}}) &\Rightarrow \ell_3(E, \zeta_1, \ell_1(\zeta_2)) \text{ or } \ell_3(E, \zeta, \ell_2(\zeta_2, \Psi)), \\
\ell_3(E, \Psi, \underbrace{\ell_j(\ldots)}_{\in X_0}) &\Rightarrow \ell_3(E, \Psi, \ell_1(\chi)) \text{ or } \ell_3(E, \Psi, \ell_2(\zeta_1, \zeta_2)) \,.
\end{aligned} \tag{B.3}$$

There are lists of arguments that are the same to those of the previous case and we have the new ones (χ, E) and (χ, Ψ, E).

Now we consider the list (E, χ), for which the identity reads:

$$\ell_1(\ell_2(E, \chi)) = \ell_2(\ell_1(E), \chi) + \ell_2(\ell_1(\chi), E) \,, \tag{B.4}$$

which holds because $\ell_2(E, \chi) = 0$, $\ell_1(E) = 0$ and

$$\ell_2(\mathcal{D}\chi, E) = \begin{cases} \widehat{\mathcal{L}}_{\partial^M \chi} \Delta B^{\mu\nu} = 0 \,, \\ \Delta B^{\mu\nu} \left(\partial_\mu \partial^M \chi - \partial^M \partial_\mu \chi \right) + \widehat{\mathcal{L}}_{\partial^M \chi} \Delta_c A^\mu{}_M = 0 \,. \end{cases} \tag{B.5}$$

Now we consider the list (χ, Ψ, E) with its corresponding identity:

$$\begin{aligned}
0 \stackrel{?}{=}\, & \ell_1 \left(\ell_3(\chi, \Psi, E) \right) - \ell_2 \left(\ell_2(\chi, E), \Psi \right) + \ell_2 \left(\ell_2(\chi, \Psi), E \right) - \ell_2 \left(\ell_2(\Psi, E), \chi \right) \\
& + \ell_3 \left(\ell_1(E), \chi, \Psi \right) - \ell_3 \left(\ell_1(\chi), E, \Psi \right) - \ell_3 \left(\ell_1(\Psi), E, \chi \right) , \\
0 \stackrel{?}{=}\, & \ell_3 \left(\ell_1(\chi), E, \Psi \right) = \ell_3 \left(\mathcal{D}\chi, E, \Psi \right) = \widehat{\mathcal{L}}_{\partial^N \chi} \Delta B^{\mu\nu} A_{\nu M} + \Delta B^{\mu\nu} \widehat{\mathcal{L}}_{\partial^N \chi} A^\mu{}_M = 0 \,.
\end{aligned} \tag{B.6}$$

The remaining lists, (ζ_1, E, ζ_2), $(\zeta_1, E, \zeta_2, \Psi)$, or $(E, \zeta_1, \Psi_1, \zeta_2, \Psi_2)$, also satisfy the identities as can be seen from the closure of gauge transformations over field equations

$$[\delta_{\zeta_1}, \delta_{\zeta_2}] \, \mathcal{F} = -\delta_{[\zeta_1, \zeta_2]} \mathcal{F} \,. \tag{B.7}$$

Expanding in powers of Ψ we arrive at the L_∞ relations over the list of the form $(\zeta_1, \zeta_2, E, \Psi^n)$, with $n \geq 0$. To lowest order in Ψ we get

$$\ell_2(\zeta_2, \ell_2(\zeta_1, \mathcal{F})) + \ell_3(\zeta_2, \mathcal{F}, \ell_1(\zeta_1)) - \ell_2(\zeta_1, \ell_2(\zeta_2, \mathcal{F})) - \ell_3(\zeta_1, \mathcal{F}, \ell_1(\zeta_2)) = -\ell_2(\ell_2(\zeta_1, \zeta_2), \mathcal{F}) \,. \tag{B.8}$$

This is the L_∞ relation needed for the list (ζ_1, E, ζ_2), once one takes into account that the missing term $\ell_1(\ell_3(\zeta_1, \zeta_2, \mathcal{F}))$ expected in such identity does not appears as it was set equal to zero. The linear order yields

$$\begin{aligned}
-\ell_3(\ell_2(\zeta_1, \zeta_2), \mathcal{F}, \Psi) = & \ell_2(\zeta_2, \ell_3(\zeta_1, \mathcal{F}, \Psi)) + \ell_3(\zeta_2, \ell_2(\zeta_1, \mathcal{F}), \Psi) + \ell_3(\zeta_2, \mathcal{F}, \ell_2(\zeta_1, \Psi)) \\
& - \ell_4(\zeta_2, \mathcal{F}, \ell_1(\zeta_1), \Psi) - \ell_2(\zeta_1, \ell_3(\zeta_2, \mathcal{F}, \Psi)) - \ell_3(\zeta_1, \ell_2(\zeta_2, \mathcal{F}), \Psi) \\
& - \ell_3(\zeta_1, \mathcal{F}, \ell_2(\zeta_2, \Psi)) + \ell_4(\zeta_1, \mathcal{F}, \ell_1(\zeta_2), \Psi) \,,
\end{aligned} \tag{B.9}$$

JHEP01(2019)117

which again is an L_∞ relation once one considers the terms that are equal to zero. Finally the quadratic Ψ^2 order has an extra difficulty because from (B.7) one gets the identity evaluated on diagonal arguments $(\zeta_1, \zeta_2, E, \Psi, \Psi)$ instead of $(\zeta_1, \zeta_2, E, \Psi_1, \Psi_2)$. However, this is not a problem because an L_∞ relation which holds true on diagonal arguments also holds true on non-diagonal arguments, which can be proved using the polarization identities.

All identities that include at least one $\chi \in X_1$ hold true. We recall first that there are only two non trivial products involving χ: $\ell_1(\chi)$ and $\ell_2(\chi, \zeta)$. This means that we can have at most two χ's. If we had more than two χ's we would have had two or more χ's in the same product which is zero. Take for example the case of three χ's:

$$\underbrace{\ell_i(\chi_a, \chi_b, \ell_j(\chi_c \dots))}_{=0} \qquad \text{or} \qquad \ell_i(\chi_a, \underbrace{\ell_j(\chi_b, \chi_c \dots)}_{=0}) \,. \tag{B.10}$$

Now take the list $(\chi_1, \chi_2, \dots)$. A term in the identities would look like $\ell_i(\chi_1 \dots \ell_j(\chi_2, \dots))$. The possibilities are:

$$\ell_i(\chi_1 \dots \underbrace{\ell_2(\chi_2, \zeta)}_{\equiv\tilde{\chi}\in X_1}) \Rightarrow \ell_i(\chi_1, \dots \tilde{\chi}) = 0, \tag{B.11}$$

$$\ell_i(\chi_1 \dots \underbrace{\ell_1(\chi_2)}_{\equiv\tilde{\zeta}\in X_0}) \Rightarrow \ell_2(\chi_1, \tilde{\zeta}) \,. \tag{B.12}$$

The only list of arguments is then (χ_1, χ_2). The identity reads

$$\ell_1(\underbrace{\ell_2(\chi_1, \chi_2)}_{=0}) = \ell_2(\ell_1(\chi_1), \chi_2) + \ell_2(\ell_1(\chi_2), \chi_1) = \ell_2(\mathcal{D}\chi_1, \chi_2) + \ell_2(\mathcal{D}\chi_2, \chi_1) = 0 \,. \tag{B.13}$$

Now take the list with only one χ, $(\chi, \dots)$. If χ is in the first sublist, we have

$$\ell_i(\chi \dots \ell_j(\dots)) \Rightarrow \ell_2(\chi, \underbrace{\ell_j(\dots)}_{\in X_0}) \Rightarrow \ell_2(\chi, \ell_2(\zeta_1, \zeta_2)) \,. \tag{B.14}$$

One can then check that the identity for the list (χ, ζ_1, ζ_2) holds.

Returning to the confection of lists, if χ is in the second sublist, we have more possibilities. In the first place we could have

$$\ell_i(\dots, \underbrace{\ell_2(\chi, \zeta)}_{\tilde{\chi}\in X_1}) \Rightarrow \begin{cases} \ell_2(\tilde{\zeta}, \ell_2(\chi, \zeta)), & \text{same list as before,} \\ \ell_1(\ell_2(\chi, \zeta_2)) \,. \end{cases} \tag{B.15}$$

The last case was also checked when we discussed the pure gauge structure. In the second place

$$\ell_i(\dots, \underbrace{\ell_1(\chi)}_{\tilde{\zeta}\in X_0}) \Rightarrow \begin{cases} \ell_2(\zeta, \ell_1(\chi)), & \text{same list as before,} \\ \ell_1(\ell_1(\chi)), & \text{checked when we discussed pure gauge structure,} \\ \ell_2(\Psi, \ell_1(\chi)), & \text{with list } (\Psi, \chi). \end{cases} \tag{B.16}$$

This last case gives the following identity which is true

$$0 = \ell_1\left(\ell_2(\chi, \Psi)\right) - \ell_2\left(\ell_1(\chi), \Psi\right) - \ell_2\left(\ell_1(\Psi), \chi\right) \,. \tag{B.17}$$

JHEP01(2019)117

Finally, all identities which include at least 3 ζ's and any number of Ψ hold true. The list with only three gauge parameters was already checked when we analized the pure gauge structure (it was our starting point). Next we could consider four gauge parameters. The identity is $\ell_1\ell_4 - \ell_2\ell_3 + \ell_3\ell_2 - \ell_4\ell_1 = 0$ and without any ℓ_4, the following expression must vanish

$$\ell_3\ell_2 - \ell_2\ell_3 = \; 6\,\ell_3([\zeta_{[1},\zeta_2],\zeta_3,\zeta_{4]}) - 4\,\ell_2(\ell_3(\zeta_{[1},\zeta_2,\zeta_3),\zeta_{4]})\,. \tag{B.18}$$

Taking into account that ℓ_3 over three gauge parameters of the form $\zeta = \Lambda^M + \Xi_\mu$ discards the Ξ_μ part, we get for the first term on the r.h.s. the following expression

$$6\,\ell_3([\Lambda_{[1},\Lambda_2]_{(C)},\Lambda_3,\Lambda_{4]}) = 2\,\big[\,[\Lambda_{[1},\Lambda_2]_{(C)},\Lambda_3\,]^M_{(C)}\,\Lambda_{4]\,M} + [\Lambda_{[3},\Lambda_4]^M_{(C)}[\Lambda_1,\Lambda_{2]}]_{(C)\,M}\,. \tag{B.19}$$

On the other hand

$$-4\,\ell_2(\ell_3(\zeta_{[1},\zeta_2,\zeta_3),\zeta_{4]}) = \widehat{\mathcal{L}}_{\Lambda_{[4}}\left([\Lambda_1,\Lambda_2]^M_{(C)}\Lambda_{3]\,M}\right)\,, \tag{B.20}$$

so using the Leibnitz rule for the Generalized Lie derivative and the fact that the D-bracket can be replaced by the C-bracket because of the antisymmetrization the identity can be proved.

Now lets take into account gauge parameters and fields: $(\zeta_1,\zeta_2,\zeta_3,\Psi^n)$. We first notice that we cannot use the products $\ell_1(\zeta),\ell_2(\zeta,\Psi),\ell_3(\zeta,\Psi,\Psi)$, which give an element $\tilde{\Psi}\in X_{-1}$. If they were in the second product ℓ_j we would have

$$\ell_i(\ldots\ell_j(\ldots)) = \ell_3(\zeta_1,\zeta_2,\tilde{\Psi}) = 0\,. \tag{B.21}$$

Trying to use them in the first product does not work either. For $\ell_1(\zeta)$ we would need the second product to give an element of X_0:

$$\ell_1(\underbrace{\ell_{n+3}(\zeta_1,\zeta_2,\zeta_3,\Psi^n)}_{\notin X_0})\,. \tag{B.22}$$

For $\ell_2(\zeta,\Psi)$ we would need the second product to give an element of X_0 or X_{-1}:

$$\ell_2(\underbrace{\ell_{n+2}(\zeta_1,\zeta_2,\zeta_3,\Psi^{n-1})}_{\notin X_0},\Psi)\,,\qquad \ell_2(\zeta_1,\underbrace{\ell_{n+2}(\zeta_2,\zeta_3,\Psi^n)}_{\notin X_{-1}})\,. \tag{B.23}$$

For $\ell_3(\zeta,\Psi,\Psi)$ we would need the second product to give an element of X_0 or X_{-1}:

$$\ell_3(\underbrace{\ell_{n+1}(\zeta_1,\zeta_2,\zeta_3,\Psi^{n-2})}_{\notin X_0},\Psi,\Psi),\qquad \ell_3(\zeta_1,\underbrace{\ell_{n+1}(\zeta_2,\zeta_3,\Psi^{n-1})}_{\notin X_{-1}},\Psi)\,. \tag{B.24}$$

Thus, one of the products must be $\ell_3(\zeta_1,\zeta_2,\zeta_3)\equiv\chi\in X_1$. If it was in the second product ℓ_j we would have

$$\ell_i(\ldots\ell_j(\ldots)) = \ell_{n+1}(\Psi^n,\ell_3(\zeta_1,\zeta_2,\zeta_3)) = \ell_{n+1}(\Psi^n,\chi) = 0\,. \tag{B.25}$$

JHEP01(2019)117

If it was in the first product, we would need the second one to give a X_0 element to get a nontrivial term, but this is not the case

$$\ell_3(\zeta_1,\zeta_2,\underbrace{\ell_{n+1}(\zeta_3,\Psi^n)}_{\notin X_0})\,. \tag{B.26}$$

Considering more than three gauge parameters and repeating the arguments as before one also finds that the L_∞ identities are fulfilled trivially.

C Full hierarchy in four space-time dimensions

Having discussed in detail the three-form tensor hierarchy in generic gauged supergravities we can move a step upwards and introduce the four-form field which corresponds to the last field in the p-form chain for $n = 4$ space-time dimensions (some of these results can also be found in [51]). To do this we continue by un-projecting the three-form fields

$$\widetilde{C}_{\mu\nu\rho I} \equiv Y_{IA} C_{\mu\nu\rho}{}^{A}\,;\;\; \widetilde{\widehat{\Sigma}}_{\mu\nu I} \equiv Y_{IA}\widehat{\Sigma}_{\mu\nu}{}^{A}\,;\;\; \widetilde{\mathcal{G}}_{\mu\nu\rho\sigma I} \equiv Y_{IA}\mathcal{G}_{\mu\nu\rho\sigma}{}^{A}\,, \tag{C.1}$$

at the expense of introducing a new projected four-form field $\widetilde{D}_{\mu\nu\rho\sigma}{}^{A}$ and its projected shift parameter $\widetilde{\widehat{\Theta}}_{\mu\nu\rho}{}^{A}$, modifying the previous results

$$\begin{aligned}\delta C_{\mu\nu\rho}{}^{A} = {}& 3D_{[\mu}\widehat{\Sigma}_{\nu\rho]}{}^{A} + g_M{}^{AI}\Big(3\mathcal{F}_{[\mu\nu}{}^{M}\widehat{\Xi}_{\rho]I} + \Lambda^{M}\mathcal{H}_{\mu\nu\rho I} + 3B_{[\mu\nu I}\delta A_{\rho]}{}^{M} \\ & +2d_{IMN}A_{[\mu}{}^{M}A_{\nu}{}^{N}\delta A_{\rho]}{}^{P}\Big) - \widetilde{\widehat{\Theta}}_{\mu\nu\rho}{}^{A}\end{aligned} \tag{C.2}$$

$$\begin{aligned}\mathcal{G}_{\mu\nu\rho\sigma}{}^{A} = {}& 4\mathcal{D}_{[\mu}C_{\nu\rho\sigma]}{}^{A} - g_M{}^{AI}\big(6B_{[\mu\nu I}\mathcal{F}_{\rho\sigma]}{}^{M} - 3Z^{MJ}B_{[\mu\nu I}B_{\rho\sigma]J} + 8d_{INP}A_{[\mu}{}^{M}A_{\nu}{}^{N}\partial_\rho A_{\sigma]}{}^{P} \\ & +2d_{IPN}X_{QR}{}^{P}A_{[\mu}{}^{M}A_{\nu}{}^{N}A_{\rho}{}^{Q}A_{\sigma]}{}^{R}\big) + \widetilde{D}_{\mu\nu\rho\sigma}{}^{A}\,,\end{aligned} \tag{C.3}$$

and satisfying

$$Y_{IA}\widetilde{D}_{\mu\nu\rho\sigma}{}^{A} = Y_{IA}\widetilde{\widehat{\Theta}}_{\mu\nu\rho}{}^{A} = 0\,. \tag{C.4}$$

Then, imposing gauge covariance of the unprojected four-form field strength tensor

$$\delta\mathcal{G}_{\mu\nu\rho\sigma}{}^{A} = -X_{MB}{}^{A}\Lambda^{M}\mathcal{G}_{\mu\nu\rho\sigma}{}^{B}\,, \tag{C.5}$$

we find the transformation of the new field

$$\begin{aligned}&\delta\widetilde{D}_{\mu\nu\rho\sigma}{}^{A} \\ &= 4\mathcal{D}_{[\mu}\widetilde{\widehat{\Theta}}_{\nu\rho\sigma]}{}^{A} - W^{AX}h_{XBM}\Big(6\mathcal{F}_{[\mu\nu}{}^{M}\widehat{\Sigma}_{\rho\sigma]}{}^{B} + \Lambda^{M}\mathcal{G}_{\mu\nu\rho\sigma}{}^{B} - 4C_{[\mu\nu\rho}{}^{B}\delta A_{\sigma]}{}^{M} \\ &\quad +2g_N{}^{BI}d_{IPQ}A_{[\mu}{}^{M}A_{\nu}{}^{N}A_{\rho}{}^{P}\delta A_{\sigma]}{}^{Q}\Big) + 4V^{AIJ}\Big(\widehat{\Xi}_{[\mu I}\mathcal{H}_{\nu\rho\sigma]J} + 3d_{INP}B_{[\mu\nu J}A_{\rho}{}^{N}\delta A_{\sigma]}{}^{P}\Big)\,,\end{aligned}$$

where we introduced the new intertwining tensor W^{AX}, which has to satisfy

$$\begin{aligned}W^{AX}h_{XBM} &= X_{MB}{}^{A} + g_M{}^{AI}Y_{IB} \\ W^{AX}Y_{IA} &= 0\,,\end{aligned}$$

JHEP01(2019)117

and the G_0-invariant tensor, h_{XBM}. In addition we defined

$$V^{AIJ} \equiv 2g_M{}^{A[I}Z^{J]M}\,,$$

which also vanishes when contracted with Y_{KA} thanks to the constraints. Due to space-time saturation in four dimensions the field strength of the four-form field and its Bianchi identity vanish.

It can be shown that the gauge algebra of the new system $\left\{A_\mu{}^M, B_{\mu\nu I}, C_{\mu\nu\rho}{}^A, \widetilde{D}_{\mu\nu\rho\sigma}{}^A\right\}$ closes up to terms proportional to $h_{XB(M}g_{N)}{}^{BI}$ and V^{AIJ}, which must then be set to zero as new constraints. In fact these are related by the equation

$$W^{AX}\left(h_{XB(M}g_{N)}{}^{BI}\right) = V^{AIJ}d_{JMN}\,, \tag{C.6}$$

and the new constraints are

$$\begin{aligned} h_{XB(M}g_{N)}{}^{BI} &= 0 \\ V^{AIJ} = 2g_M{}^{A[I}Z^{J]M} &= 0\,. \end{aligned} \tag{C.7}$$

The brackets that arise from closure of the algebra are given by

$$\Lambda^M_{12} = -X_{NP}{}^M\Lambda^N_{[1}\Lambda^P_{2]} \tag{C.8}$$

$$\widehat{\Xi}_{12\mu I} = 2d_{IMN}\mathcal{D}_\mu\Lambda^M_{[1}\Lambda^N_{2]} \tag{C.9}$$

$$\widehat{\Sigma}_{12\mu\nu}{}^A = -2g_M{}^{AI}\left(d_{INP}\Lambda^M_{[1}\Lambda^N_{2]}\mathcal{F}_{\mu\nu}{}^P + 2\widehat{\Xi}_{I[\mu[1}\mathcal{D}_{\nu]}\Lambda^M_{2]} - Z^{MJ}\widehat{\Xi}_{I[\mu[1}\widehat{\Xi}_{\nu]2]J}\right) \tag{C.10}$$

$$\widetilde{\widehat{\Theta}}_{12\mu\nu\rho}{}^A = -W^{AX}h_{XBM}\left(g_N{}^{BI}\Lambda^M_{[1}\Lambda^N_{2]}\mathcal{H}_{\mu\nu\rho I} + 6\widehat{\Sigma}_{[\mu\nu[1}{}^B\mathcal{D}_{\rho]}\Lambda^M_{2]}\right)\,. \tag{C.11}$$

To finish this chain we can unproject the 4-form

$$\widetilde{D}_{\mu\nu\rho\sigma}{}^A = W^{AX}D_{\mu\nu\rho\sigma X}\,;\ \widetilde{\widehat{\Theta}}_{\mu\nu\rho}{}^A = W^{AX}\widehat{\Theta}_{\mu\nu\rho X}\,, \tag{C.12}$$

and introduce a new gauge parameter $\widehat{\Phi}_{\mu\nu\rho\sigma X}$ into its transformation

$$\begin{aligned} \delta D_{\mu\nu\rho\sigma X} = 4\mathcal{D}_{[\mu}\widehat{\Theta}_{\nu\rho\sigma]X} - h_{XBM}\Big(&6\mathcal{F}_{[\mu\nu}{}^M\widehat{\Sigma}_{\rho\sigma]}{}^B + \Lambda^M\mathcal{G}_{\mu\nu\rho\sigma}{}^B - 4C_{[\mu\nu\rho}{}^B\delta A_{\sigma]}{}^M \\ &+2g_N{}^{BI}d_{IPQ}A_{[\mu}{}^MA_\nu{}^NA_\rho{}^P\delta A_{\sigma]}{}^Q\Big) - \widehat{\Phi}_{\mu\nu\rho\sigma X}\,. \end{aligned}$$

In higher dimensions the new parameter would be identified with the gauge parameter of the five-form field. Consistency requires that it vanishes under the W^{AX} projection

$$W^{AX}\widehat{\Phi}_{\mu\nu\rho\sigma X} = 0\,.$$

JHEP01(2019)117

We can finally summarize all the results for the full tensor hierarchy in generic gauged supergravities in four space-time dimensions:

- Parameters, fields and curvatures in four-dimensions

Representation	**Parameter**	**Field**	**Curvature**
R_1	Λ^M	$A_\mu{}^M$	$\mathcal{F}_{\mu\nu}{}^M$
R_2	$\widehat{\Xi}_{\mu I}$	$B_{\mu\nu I}$	$\mathcal{H}_{\mu\nu\rho I}$
R_3	$\widehat{\Sigma}_{\mu\nu}{}^A$	$C_{\mu\nu\rho}{}^A$	$\mathcal{G}_{\mu\nu\rho\sigma}{}^A$
R_4	$\widehat{\Theta}_{\mu\nu\rho X}$	$D_{\mu\nu\rho\sigma X}$	0
R_4	$\widehat{\Phi}_{\mu\nu\rho\sigma X}$	0	0

- Gauge transformations

$$\begin{aligned}
\delta A_\mu{}^M &= \mathcal{D}_\mu \Lambda^M - Z^{MI}\widehat{\Xi}_{\mu I} \\
\delta B_{\mu\nu I} &= 2\mathcal{D}_{[\mu}\widehat{\Xi}_{\nu]I} - 2d_{IMN}\left(\Lambda^M \mathcal{F}_{\mu\nu}{}^N - A_{[\mu}{}^M \delta A_{\nu]}{}^N\right) - Y_{IA}\widehat{\Sigma}_{\mu\nu}{}^A \\
\delta C_{\mu\nu\rho}{}^A &= 3D_{[\mu}\widehat{\Sigma}_{\nu\rho]}{}^A + g_M{}^{AI}\Big(3\mathcal{F}_{[\mu\nu}{}^M\widehat{\Xi}_{\rho]I} + \Lambda^M \mathcal{H}_{\mu\nu\rho I} + 3B_{[\mu\nu I}\delta A_{\rho]}{}^M \\
&\quad + 2d_{IMN}A_{[\mu}{}^M A_\nu{}^N \delta A_{\rho]}{}^P\Big) - W^{AX}\widehat{\Theta}_{\mu\nu\rho X} \\
\delta D_{\mu\nu\rho\sigma X} &= 4\mathcal{D}_{[\mu}\widehat{\Theta}_{\nu\rho\sigma]X} - h_{XBM}\Big(6\mathcal{F}_{[\mu\nu}{}^M\widehat{\Sigma}_{\rho\sigma]}{}^B + \Lambda^M \mathcal{G}_{\mu\nu\rho\sigma}{}^B - 4C_{[\mu\nu\rho}{}^B\delta A_{\sigma]}{}^M \\
&\quad + 2g_N{}^{BI}d_{IPQ}A_{[\mu}{}^M A_\nu{}^N A_\rho{}^P \delta A_{\sigma]}{}^Q\Big) - \widehat{\Phi}_{\mu\nu\rho\sigma X}\,.
\end{aligned}$$

- Field Strengths

$$\begin{aligned}
\mathcal{F}_{\mu\nu}{}^M &= 2\partial_{[\mu}A_{\nu]}{}^M + X_{NO}{}^M A_{[\mu}{}^N A_{\nu]}{}^O + Z^{MI}B_{\mu\nu I} \\
\mathcal{H}_{\mu\nu\rho I} &= 3\mathcal{D}_{[\mu}B_{\nu\rho]I} + 6d_{IMN}A_{[\mu}{}^M\left(\partial_\nu A_{\rho]}{}^N + \frac{1}{3}X_{OP}{}^N A_\nu{}^O A_{\rho]}{}^P\right) + Y_{IA}C_{\mu\nu\rho}{}^A \qquad \text{(C.13)} \\
\mathcal{G}_{\mu\nu\rho\sigma}{}^A &= 4\mathcal{D}_{[\mu}C_{\nu\rho\sigma]}{}^A - g_M{}^{AI}\left(6B_{[\mu\nu I}\mathcal{F}_{\rho\sigma]}{}^M - 3Z^{MJ}B_{[\mu\nu I}B_{\rho\sigma]J} + 8d_{INP}A_{[\mu}{}^M A_\nu{}^N \partial_\rho A_{\sigma]}{}^P\right. \\
&\quad \left. + 2d_{IPN}X_{QR}{}^P A_{[\mu}{}^M A_\nu{}^N A_\rho{}^Q A_{\sigma]}{}^R\right) + W^{AX}D_{\mu\nu\rho\sigma X}\,.
\end{aligned}$$

- Bianchi identities

$$2\mathcal{D}_{[\mu}\mathcal{D}_{\nu]} + \mathcal{F}_{\mu\nu}{}^M X_M = 0 \qquad \text{(C.14)}$$

$$3\mathcal{D}_{[\mu}\mathcal{F}_{\nu\rho]}{}^M - Z^{MI}\mathcal{H}_{\mu\nu\rho I} = 0 \qquad \text{(C.15)}$$

$$4\mathcal{D}_{[\mu}\mathcal{H}_{\nu\rho\sigma]I} - 6d_{IMN}\mathcal{F}_{[\mu\nu}{}^M\mathcal{F}_{\rho\sigma]}{}^N - Y_{IA}\mathcal{G}_{\mu\nu\rho\sigma}{}^A = 0\,. \qquad \text{(C.16)}$$

JHEP01(2019)117

- Constraints

$$
\begin{aligned}
Z^{MI} d_{INP} &= X_{(NP)}{}^{M} \\
Z^{MI} X_M &= 0 \\
d_{I[MN]} &= 0 \\
Y_{IA} g_M{}^{AJ} &= X_{MI}{}^{J} + 2 Z^{NJ} d_{IMN} \\
Y_{IA} Z^{MI} &= 0 \\
d_{I(MN} g_{P)}{}^{AI} &= 0 \\
W^{AX} h_{XBM} &= X_{MB}{}^{A} + g_M{}^{AI} Y_{IB} \\
W^{AX} Y_{IA} &= 0 \\
h_{XB(M} g_{N)}{}^{BI} &= 0 \\
W^{AX} \widehat{\Phi}_{\mu\nu\rho\sigma X} &= 0
\end{aligned}
$$

- Brackets

$$
\begin{aligned}
\Lambda_{12}^{M} &= -X_{NP}{}^{M} \Lambda_{[1}^{N} \Lambda_{2]}^{P} \\
\widehat{\Xi}_{12\mu I} &= 2 d_{IMN} \mathcal{D}_\mu \Lambda_{[1}^{M} \Lambda_{2]}^{N} \\
\widehat{\Sigma}_{12\mu\nu}{}^{A} &= -2 g_M{}^{AI} \left(d_{INP} \Lambda_{[1}^{M} \Lambda_{2]}^{N} \mathcal{F}_{\mu\nu}{}^{P} + 2 \widehat{\Xi}_{I[\mu[1} \mathcal{D}_{\nu]} \Lambda_{2]}^{M} - Z^{MJ} \widehat{\Xi}_{I[\mu[1} \widehat{\Xi}_{\nu]2]J} \right) \\
\widehat{\Theta}_{12\mu\nu\rho X} &= -h_{XBM} \left(g_N{}^{BI} \Lambda_{[1}^{M} \Lambda_{2]}^{N} \mathcal{H}_{\mu\nu\rho I} + 6 \widehat{\Sigma}_{[\mu\nu[1}{}^{B} \mathcal{D}_{\rho]} \Lambda_{2]}^{M} \right) \\
\widehat{\Phi}_{12\mu\nu\rho\sigma X} &= -\left(X_{MX}{}^{Y} + W^{AY} h_{XAM} \right) \left(h_{YBN} \Lambda_{[1}^{M} \Lambda_{2]}^{N} \mathcal{G}_{\mu\nu\rho\sigma}{}^{B} + 8 \widehat{\Theta}_{Y[\mu\nu\rho[1} \mathcal{D}_{\sigma]} \Lambda_{2]}^{M} \right) \\
&\quad + 4 h_{XBM} Z^{MI} \Big[2 W^{BY} \widehat{\Theta}_{Y[\mu\nu\rho[1} \widehat{\Xi}_{2]\sigma]I} - 6 \widehat{\Xi}_{J[\mu[1} \mathcal{D}_\nu \widehat{\Sigma}_{\rho\sigma]2]}{}^{B} + 2 g_N{}^{BJ} \mathcal{H}_{J[\mu\nu\rho} \widehat{\Xi}_{\sigma]I[1} \Lambda_{2]}^{N} \\
&\qquad + 3 \mathcal{F}_{[\mu\nu}{}^{N} \left(2 d_{INP} \Lambda_{[1}^{P} \widehat{\Sigma}_{2]\rho\sigma]}{}^{B} - g_N{}^{BJ} \widehat{\Xi}_{J\rho[1} \widehat{\Xi}_{2]\sigma]I} \right) \Big]
\end{aligned}
$$

- Trivial parameters

$$
\begin{aligned}
\Lambda_{\text{trivial}}^{M} &= Z^{MJ} \chi_J \\
\widehat{\Xi}_{\mu I \text{ trivial}} &= \mathcal{D}_\mu \chi_I + Y_{IA} \chi_\mu{}^{A} \\
\widehat{\Sigma}_{\mu\nu}{}^{A}{}_{\text{ trivial}} &= 2 \mathcal{D}_{[\mu} \chi_{\nu]}{}^{A} - g_M{}^{AJ} \mathcal{F}_{\mu\nu}{}^{M} \chi_J + W^{AX} \chi_{\mu\nu X} \\
\widehat{\Theta}_{\mu\nu\rho X \text{ trivial}} &= 3 \mathcal{D}_{[\mu} \chi_{\nu\rho]X} + 3 h_{XMB} \mathcal{F}_{[\mu\nu}{}^{M} \chi_{\rho]}{}^{B} + \chi_{\mu\nu\rho X} \\
\widehat{\Phi}_{\mu\nu\rho\sigma X \text{ trivial}} &= 4 \mathcal{D}_{[\mu} \chi_{\nu\rho\sigma]X} - 6 \left(X_{MX}{}^{Y} + W^{AY} h_{XAM} \right) \mathcal{F}_{[\mu\nu}{}^{M} \chi_{\rho\sigma]Y} \\
&\quad + h_{XMB} Z^{MI} \left(4 \mathcal{H}_{[\mu\nu\rho I} \chi_{\sigma]}{}^{B} - \mathcal{G}_{\mu\nu\rho\sigma}{}^{B} \chi_I \right) .
\end{aligned}
$$

We then see that the symmetry for symmetries effect in gauged supergravities is a generic feature even for dimensionally saturated tensor hierarchies.

JHEP01(2019)117

References

[1] J. Stasheff, *Homotopy associativity of H-spaces. II Trans. Am. Math. Soc.* **108** (1963) 293.

[2] J. Stasheff, *H-spaces from a hotompy point of view*, Lecture Notes in Mathematics, vol. 161, Springer Verlag (1970).

[3] B. Zwiebach, *Closed string field theory: Quantum action and the B-V master equation*, *Nucl. Phys.* **B 390** (1993) 33 [hep-th/9206084] [INSPIRE].

[4] T. Lada and J. Stasheff, *Introduction to SH Lie algebras for physicists*, *Int. J. Theor. Phys.* **32** (1993) 1087 [hep-th/9209099] [INSPIRE].

[5] G. Barnich, R. Fulp, T. Lada and J. Stasheff, *The sh Lie structure of Poisson brackets in field theory*, *Commun. Math. Phys.* **191** (1998) 585 [hep-th/9702176] [INSPIRE].

[6] A. Sen, *Wilsonian Effective Action of Superstring Theory*, *JHEP* **01** (2017) 108 [arXiv:1609.00459] [INSPIRE].

[7] F.A. Berends, G.J.H. Burgers and H. van Dam, *On the Theoretical Problems in Constructing Interactions Involving Higher Spin Massless Particles*, *Nucl. Phys.* **B 260** (1985) 295 [INSPIRE].

[8] R. Fulp, T. Lada and J. Stasheff, *sh-Lie algebras induced by gauge transformations*, *Commun. Math. Phys.* **231** (2002) 25 [INSPIRE].

[9] A.M. Zeitlin, *Homotopy Lie Superalgebra in Yang-Mills Theory*, *JHEP* **09** (2007) 068 [arXiv:0708.1773] [INSPIRE].

[10] A.M. Zeitlin, *String field theory-inspired algebraic structures in gauge theories*, *J. Math. Phys.* **50** (2009) 063501 [arXiv:0711.3843] [INSPIRE].

[11] A.M. Zeitlin, *Conformal Field Theory and Algebraic Structure of Gauge Theory*, *JHEP* **03** (2010) 056 [arXiv:0812.1840] [INSPIRE].

[12] O. Hohm and B. Zwiebach, *L_∞ Algebras and Field Theory*, *Fortsch. Phys.* **65** (2017) 1700014 [arXiv:1701.08824] [INSPIRE].

[13] R. Blumenhagen, M. Fuchs and M. Traube, *$\mathcal{W}$ algebras are L_∞ algebras*, *JHEP* **07** (2017) 060 [arXiv:1705.00736] [INSPIRE].

[14] R. Blumenhagen, M. Fuchs and M. Traube, *On the Structure of Quantum L_∞ algebras*, *JHEP* **10** (2017) 163 [arXiv:1706.09034] [INSPIRE].

[15] R. Blumenhagen, I. Brunner, V. Kupriyanov and D. Lüst, *Bootstrapping non-commutative gauge theories from L_∞ algebras*, *JHEP* **05** (2018) 097 [arXiv:1803.00732] [INSPIRE].

[16] R. Blumenhagen, M. Brinkmann, V. Kupriyanov and M. Traube, *On the Uniqueness of L_∞ bootstrap: Quasi-isomorphisms are Seiberg-Witten Maps*, *J. Math. Phys.* **59** (2018) 123505 [arXiv:1806.10314] [INSPIRE].

[17] W. Siegel, *Superspace duality in low-energy superstrings*, *Phys. Rev.* **D 48** (1993) 2826 [hep-th/9305073] [INSPIRE].

[18] W. Siegel, *Two vierbein formalism for string inspired axionic gravity*, *Phys. Rev.* **D 47** (1993) 5453 [hep-th/9302036] [INSPIRE].

[19] C. Hull and B. Zwiebach, *Double Field Theory*, *JHEP* **09** (2009) 099 [arXiv:0904.4664] [INSPIRE].

JHEP01(2019)117

[20] D. Roytenberg and A. Weinstein, *Courant Algebroids and Strongly Homotopy Lie Algebras*, math/9802118.

[21] C. Hull and B. Zwiebach, *The Gauge algebra of double field theory and Courant brackets*, *JHEP* **09** (2009) 090 [arXiv:0908.1792] [INSPIRE].

[22] A. Deser and C. Sämann, *Extended Riemannian Geometry I: Local Double Field Theory*, arXiv:1611.02772 [INSPIRE].

[23] A. Deser, M.A. Heller and C. Sämann, *Extended Riemannian Geometry II: Local Heterotic Double Field Theory*, *JHEP* **04** (2018) 106 [arXiv:1711.03308] [INSPIRE].

[24] B. de Wit, H. Nicolai and H. Samtleben, *Gauged Supergravities, Tensor Hierarchies and M-theory*, *JHEP* **02** (2008) 044 [arXiv:0801.1294] [INSPIRE].

[25] D. Baraglia, *Leibniz algebroids, twistings and exceptional generalized geometry*, *J. Geom. Phys.* **62** (2012) 903 [arXiv:1101.0856] [INSPIRE].

[26] M. Cederwall and J. Palmkvist, *L_∞ algebras for extended geometry from Borcherds superalgebras*, arXiv:1804.04377 [INSPIRE].

[27] M. Cederwall, *Algebraic structures in exceptional geometry*, 2017, arXiv:1712.06995, DOI [INSPIRE].

[28] A.S. Arvanitakis, *Brane Wess-Zumino terms from AKSZ and exceptional generalised geometry as an L_∞-algebroid*, arXiv:1804.07303 [INSPIRE].

[29] O. Hohm and H. Samtleben, *Leibniz-Chern-Simons Theory and Phases of Exceptional Field Theory*, arXiv:1805.03220 [INSPIRE].

[30] F. Riccioni and P.C. West, *The E_{11} origin of all maximal supergravities*, *JHEP* **07** (2007) 063 [arXiv:0705.0752] [INSPIRE].

[31] F. Riccioni, D. Steele and P. West, *The E_{11} origin of all maximal supergravities: The Hierarchy of field-strengths*, *JHEP* **09** (2009) 095 [arXiv:0906.1177] [INSPIRE].

[32] J. Greitz, P. Howe and J. Palmkvist, *The tensor hierarchy simplified*, *Class. Quant. Grav.* **31** (2014) 087001 [arXiv:1308.4972] [INSPIRE].

[33] J. Palmkvist, *The tensor hierarchy algebra*, *J. Math. Phys.* **55** (2014) 011701 [arXiv:1305.0018] [INSPIRE].

[34] J. Palmkvist, *Tensor hierarchies, Borcherds algebras and E11*, *JHEP* **02** (2012) 066 [arXiv:1110.4892] [INSPIRE].

[35] S. Lavau, *Tensor hierarchies and Lie n-extensions of Leibniz algebras*, arXiv:1708.07068 [INSPIRE].

[36] G. Aldazabal, M. Graña, D. Marqués and J.A. Rosabal, *The gauge structure of Exceptional Field Theories and the tensor hierarchy*, *JHEP* **04** (2014) 049 [arXiv:1312.4549] [INSPIRE].

[37] O. Hohm and Y.-N. Wang, *Tensor hierarchy and generalized Cartan calculus in* SL(3) × SL(2) *exceptional field theory*, *JHEP* **04** (2015) 050 [arXiv:1501.01600] [INSPIRE].

[38] Y.-N. Wang, *Generalized Cartan Calculus in general dimension*, *JHEP* **07** (2015) 114 [arXiv:1504.04780] [INSPIRE].

[39] O. Hohm and H. Samtleben, *Gauge theory of Kaluza-Klein and winding modes*, *Phys. Rev.* **D 88** (2013) 085005 [arXiv:1307.0039] [INSPIRE].

JHEP01(2019)117

[40] O. Hohm and H. Samtleben, *Exceptional field theory. II. $E_{7(7)}$*, *Phys. Rev.* **D 89** (2014) 066017 [arXiv:1312.4542] [INSPIRE].

[41] H. Godazgar, M. Godazgar, O. Hohm, H. Nicolai and H. Samtleben, *Supersymmetric $E_{7(7)}$ Exceptional Field Theory*, *JHEP* **09** (2014) 044 [arXiv:1406.3235] [INSPIRE].

[42] J. Schon and M. Weidner, *Gauged $N = 4$ supergravities*, *JHEP* **05** (2006) 034 [hep-th/0602024] [INSPIRE].

[43] B. de Wit, H. Samtleben and M. Trigiante, *On Lagrangians and gaugings of maximal supergravities*, *Nucl. Phys.* **B 655** (2003) 93 [hep-th/0212239] [INSPIRE].

[44] B. de Wit, H. Samtleben and M. Trigiante, *The Maximal $D = 4$ supergravities*, *JHEP* **06** (2007) 049 [arXiv:0705.2101] [INSPIRE].

[45] M. Weidner, *Gauged supergravities in various spacetime dimensions*, *Fortsch. Phys.* **55** (2007) 843 [hep-th/0702084] [INSPIRE].

[46] G. Aldazabal, D. Marques and C. Núñez, *Double Field Theory: A Pedagogical Review*, *Class. Quant. Grav.* **30** (2013) 163001 [arXiv:1305.1907] [INSPIRE].

[47] D.S. Berman and D.C. Thompson, *Duality Symmetric String and M-theory*, *Phys. Rept.* **566** (2014) 1 [arXiv:1306.2643] [INSPIRE].

[48] O. Hohm, D. Lüst and B. Zwiebach, *The Spacetime of Double Field Theory: Review, Remarks and Outlook*, *Fortsch. Phys.* **61** (2013) 926 [arXiv:1309.2977] [INSPIRE].

[49] H. Samtleben, *Lectures on Gauged Supergravity and Flux Compactifications*, *Class. Quant. Grav.* **25** (2008) 214002 [arXiv:0808.4076] [INSPIRE].

[50] M. Trigiante, *Gauged Supergravities*, *Phys. Rept.* **680** (2017) 1 [arXiv:1609.09745] [INSPIRE].

[51] E.A. Bergshoeff, J. Hartong, O. Hohm, M. Huebscher and T. Ortín, *Gauge Theories, Duality Relations and the Tensor Hierarchy*, *JHEP* **04** (2009) 123 [arXiv:0901.2054] [INSPIRE].

[52] O. Hohm, V. Kupriyanov, D. Lüst and M. Traube, *Constructions of L_∞ Algebras and Their Field Theory Realizations*, *Adv. Math. Phys.* **2018** (2018) 9282905 [arXiv:1709.10004] [INSPIRE].

[53] O. Hohm, C. Hull and B. Zwiebach, *Generalized metric formulation of double field theory*, *JHEP* **08** (2010) 008 [arXiv:1006.4823] [INSPIRE].

[54] C.M. Hull, *Generalised Geometry for M-theory*, *JHEP* **07** (2007) 079 [hep-th/0701203] [INSPIRE].

[55] P. Pires Pacheco and D. Waldram, *M-theory, exceptional generalised geometry and superpotentials*, *JHEP* **09** (2008) 123 [arXiv:0804.1362] [INSPIRE].

[56] C. Hillmann, *Generalized $E_{7(7)}$ coset dynamics and $D = 11$ supergravity*, *JHEP* **03** (2009) 135 [arXiv:0901.1581] [INSPIRE].

[57] D.S. Berman and M.J. Perry, *Generalized Geometry and M-theory*, *JHEP* **06** (2011) 074 [arXiv:1008.1763] [INSPIRE].

[58] A. Coimbra, C. Strickland-Constable and D. Waldram, *Supergravity as Generalised Geometry II: $E_{d(d)} \times \mathbb{R}^+$ and M-theory*, *JHEP* **03** (2014) 019 [arXiv:1212.1586] [INSPIRE].

JHEP01(2019)117

[59] D.S. Berman, H. Godazgar, M.J. Perry and P. West, *Duality Invariant Actions and Generalised Geometry*, *JHEP* **02** (2012) 108 [arXiv:1111.0459] [INSPIRE].

[60] M. Cederwall, J. Edlund and A. Karlsson, *Exceptional geometry and tensor fields*, *JHEP* **07** (2013) 028 [arXiv:1302.6736] [INSPIRE].

[61] J.A. Rosabal, *On the exceptional generalised Lie derivative for $d \geq 7$*, *JHEP* **09** (2015) 153 [arXiv:1410.8148] [INSPIRE].

[62] M. Cederwall and J.A. Rosabal, *E_8 geometry*, *JHEP* **07** (2015) 007 [arXiv:1504.04843] [INSPIRE].

[63] M. Cederwall and J. Palmkvist, *Extended geometries*, *JHEP* **02** (2018) 071 [arXiv:1711.07694] [INSPIRE].

[64] A. Coimbra, C. Strickland-Constable and D. Waldram, *$E_{d(d)} \times \mathbb{R}^+$ generalised geometry, connections and M-theory*, *JHEP* **02** (2014) 054 [arXiv:1112.3989] [INSPIRE].

[65] D.S. Berman, M. Cederwall, A. Kleinschmidt and D.C. Thompson, *The gauge structure of generalised diffeomorphisms*, *JHEP* **01** (2013) 064 [arXiv:1208.5884] [INSPIRE].

[66] A.G. Tumanov and P. West, *E11 and exceptional field theory*, *Int. J. Mod. Phys.* **A 31** (2016) 1650066 [arXiv:1507.08912] [INSPIRE].

[67] O. Hohm and H. Samtleben, *Exceptional Field Theory I: $E_{6(6)}$ covariant Form of M-theory and Type IIB*, *Phys. Rev.* **D 89** (2014) 066016 [arXiv:1312.0614] [INSPIRE].

[68] E. Musaev and H. Samtleben, *Fermions and supersymmetry in $E_{6(6)}$ exceptional field theory*, *JHEP* **03** (2015) 027 [arXiv:1412.7286] [INSPIRE].

[69] O. Hohm and H. Samtleben, *Exceptional field theory. III. $E_{8(8)}$*, *Phys. Rev.* **D 90** (2014) 066002 [arXiv:1406.3348] [INSPIRE].

[70] A. Baguet and H. Samtleben, *$E_{8(8)}$ Exceptional Field Theory: Geometry, Fermions and Supersymmetry*, *JHEP* **09** (2016) 168 [arXiv:1607.03119] [INSPIRE].

[71] C.D.A. Blair and E. Malek, *Geometry and fluxes of* SL(5) *exceptional field theory*, *JHEP* **03** (2015) 144 [arXiv:1412.0635] [INSPIRE].

[72] A. Abzalov, I. Bakhmatov and E.T. Musaev, *Exceptional field theory:* SO(5, 5), *JHEP* **06** (2015) 088 [arXiv:1504.01523] [INSPIRE].

[73] E.T. Musaev, *Exceptional field theory:* SL(5), *JHEP* **02** (2016) 012 [arXiv:1512.02163] [INSPIRE].

[74] D.S. Berman, C.D.A. Blair, E. Malek and F.J. Rudolph, *An action for F-theory:* $\mathrm{SL}(2)\mathbb{R}^+$ *exceptional field theory*, *Class. Quant. Grav.* **33** (2016) 195009 [arXiv:1512.06115] [INSPIRE].

[75] F. Ciceri, A. Guarino and G. Inverso, *The exceptional story of massive IIA supergravity*, *JHEP* **08** (2016) 154 [arXiv:1604.08602] [INSPIRE].

[76] D. Cassani, O. de Felice, M. Petrini, C. Strickland-Constable and D. Waldram, *Exceptional generalised geometry for massive IIA and consistent reductions*, *JHEP* **08** (2016) 074 [arXiv:1605.00563] [INSPIRE].

[77] A. Baguet, M. Magro and H. Samtleben, *Generalized IIB supergravity from exceptional field theory*, *JHEP* **03** (2017) 100 [arXiv:1612.07210] [INSPIRE].

JHEP01(2019)117

[78] G. Aldazabal, W. Baron, D. Marques and C. Núñez, *The effective action of Double Field Theory*, *JHEP* **11** (2011) 052 [*Erratum ibid.* **11** (2011) 109] [arXiv:1109.0290] [INSPIRE].

[79] D. Geissbuhler, *Double Field Theory and $N = 4$ Gauged Supergravity*, *JHEP* **11** (2011) 116 [arXiv:1109.4280] [INSPIRE].

[80] M. Graña and D. Marques, *Gauged Double Field Theory*, *JHEP* **04** (2012) 020 [arXiv:1201.2924] [INSPIRE].

[81] G. Dibitetto, J.J. Fernandez-Melgarejo, D. Marques and D. Roest, *Duality orbits of non-geometric fluxes*, *Fortsch. Phys.* **60** (2012) 1123 [arXiv:1203.6562] [INSPIRE].

[82] D. Geissbuhler, D. Marques, C. Núñez and V. Penas, *Exploring Double Field Theory*, *JHEP* **06** (2013) 101 [arXiv:1304.1472] [INSPIRE].

[83] D.S. Berman and K. Lee, *Supersymmetry for Gauged Double Field Theory and Generalised Scherk-Schwarz Reductions*, *Nucl. Phys.* **B 881** (2014) 369 [arXiv:1305.2747] [INSPIRE].

[84] W. Cho, J.J. Fernández-Melgarejo, I. Jeon and J.-H. Park, *Supersymmetric gauged double field theory: systematic derivation by virtue of twist*, *JHEP* **08** (2015) 084 [arXiv:1505.01301] [INSPIRE].

[85] F. Hassler and D. Lüst, *Consistent Compactification of Double Field Theory on Non-geometric Flux Backgrounds*, *JHEP* **05** (2014) 085 [arXiv:1401.5068] [INSPIRE].

[86] R. Blumenhagen, F. Hassler and D. Lüst, *Double Field Theory on Group Manifolds*, *JHEP* **02** (2015) 001 [arXiv:1410.6374] [INSPIRE].

[87] F. Ciceri, G. Dibitetto, J.J. Fernandez-Melgarejo, A. Guarino and G. Inverso, *Double Field Theory at* SL(2) *angles*, *JHEP* **05** (2017) 028 [arXiv:1612.05230] [INSPIRE].

[88] A. Catal-Ozer, *Duality Twisted Reductions of Double Field Theory of Type II Strings*, *JHEP* **09** (2017) 044 [arXiv:1705.08181] [INSPIRE].

[89] D.S. Berman, E.T. Musaev, D.C. Thompson and D.C. Thompson, *Duality Invariant M-theory: Gauged supergravities and Scherk-Schwarz reductions*, *JHEP* **10** (2012) 174 [arXiv:1208.0020] [INSPIRE].

[90] G. Aldazabal, M. Graña, D. Marqués and J.A. Rosabal, *Extended geometry and gauged maximal supergravity*, *JHEP* **06** (2013) 046 [arXiv:1302.5419] [INSPIRE].

[91] E.T. Musaev, *Gauged supergravities in 5 and 6 dimensions from generalised Scherk-Schwarz reductions*, *JHEP* **05** (2013) 161 [arXiv:1301.0467] [INSPIRE].

[92] K. Lee, C. Strickland-Constable and D. Waldram, *Spheres, generalised parallelisability and consistent truncations*, *Fortsch. Phys.* **65** (2017) 1700048 [arXiv:1401.3360] [INSPIRE].

[93] W.H. Baron and G. Dall'Agata, *Uplifting non-compact gauged supergravities*, *JHEP* **02** (2015) 003 [arXiv:1410.8823] [INSPIRE].

[94] O. Hohm and H. Samtleben, *Consistent Kaluza-Klein Truncations via Exceptional Field Theory*, *JHEP* **01** (2015) 131 [arXiv:1410.8145] [INSPIRE].

[95] K. Lee, C. Strickland-Constable and D. Waldram, *New Gaugings and Non-Geometry*, *Fortsch. Phys.* **65** (2017) 1700049 [arXiv:1506.03457] [INSPIRE].

[96] E. Malek and H. Samtleben, *Dualising consistent IIA/ IIB truncations*, *JHEP* **12** (2015) 029 [arXiv:1510.03433] [INSPIRE].

JHEP01(2019)117

[97] E. Malek, *Half-Maximal Supersymmetry from Exceptional Field Theory*, *Fortsch. Phys.* **65** (2017) 1700061 [arXiv:1707.00714] [INSPIRE].

[98] G. Inverso, *Generalised Scherk-Schwarz reductions from gauged supergravity*, *JHEP* **12** (2017) 124 [arXiv:1708.02589] [INSPIRE].

[99] G. Aldazabal, M. Mayo and C. Nuñez, *Probing the String Winding Sector*, *JHEP* **03** (2017) 096 [arXiv:1611.04927] [INSPIRE].

[100] G. Aldazabal, D. Marques, C. Núñez and J.A. Rosabal, *On Type IIB moduli stabilization and $N = 4, 8$ supergravities*, *Nucl. Phys.* **B 849** (2011) 80 [arXiv:1101.5954] [INSPIRE].

[101] G. Dibitetto, A. Guarino and D. Roest, *How to halve maximal supergravity*, *JHEP* **06** (2011) 030 [arXiv:1104.3587] [INSPIRE].

JHEP01(2019)117