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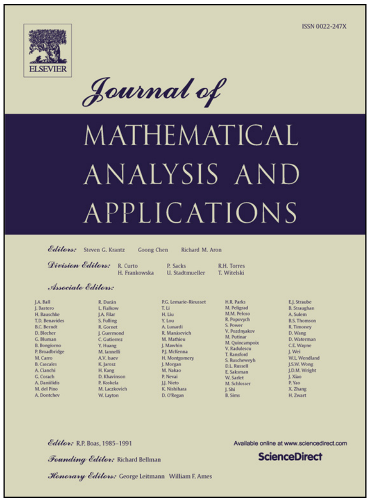

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DYNAMICS OF NON CONVOLUTION OPERATORS AND HOLOMORPHY TYPES

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Abstract. In this article we study the hypercyclic behavior of non convolution operators defined on spaces of analytic functions of different holomorphy types over Banach spaces. The operators in the family we analyze are a composition of differentiation and composition operators, and are extensions of operators in $H(\mathbb{C})$ studied by Aron and Markose in 2004. The dynamics of this class of operators, in the context of one and several complex variables, was further investigated by many authors. It turns out that the situation is somewhat different and that some purely infinite dimensional difficulties appear. For example, in contrast to the several complex variable case, it may happen that the symbol of the composition operator has no fixed points and still, the operator is not hypercyclic. We also prove a Runge type theorem for holomorphy types on Banach spaces.

Introduction

An operator $T : X \to X$ is said to be hypercyclic if there exists some vector $x \in X$, called hypercyclic vector of T, such that the orbit $Orb(x, T) := \{x, T(x), \ldots, T^n(x), \ldots\}$ is dense in X. The first examples of hypercyclic operators appeared in the works by Birkhoff [7] and MacLane [32]. Birkhoff's result states that the translation operator $T : H(\mathbb{C}) \to H(\mathbb{C})$ defined by $T(h)(z) = h(z + 1)$ is hypercyclic. Likewise, MacLane's result says that the differentiation operator on $H(\mathbb{C})$ is hypercyclic.

Several criteria to determine if an operator is hypercyclic have been studied. It is known that a large supply of eigenvectors implies hypercyclicity. In particular, if the eigenvectors associated to the eigenvalues of modulus less than 1 and the eigenvectors associated to the eigenvalues of modulus greater than 1 span dense subspaces, then the operator is hypercyclic. This result is due to Godefroy and Shapiro [25]. In the same article, they also prove that non-trivial convolution operators, i.e. operators that commute with translations which are not multiples of the identity, on the space of entire functions on $\mathbb{C}^n$ are hypercyclic. Birkhoff's translation operators and MacLane's differentiation operators are special examples of non-trivial convolution operators. This result has also been extended to some spaces of entire functions of infinitely many variables

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by several authors (see [3, 6, 11, 18–21, 29, 35, 38, 39]). Some of these extensions even hold for spaces of functions associated to very wide classes of holomorphy types. The Godefroy - Shapiro theorem has been improved by Bonilla and Grosse-Erdmann in [9]. They showed that non-trivial convolution operators are frequently hypercyclic (see also [21, 35]).

Recent work developed by Bayart and Matheron [5] provides other eigenvector criteria to determine whether a given continuous map $T : X \to X$ acting on a topological vector space X admits an ergodic probability measure, or a strongly mixing one. If the measure is strictly positive on any non void open sets of X, ergodic properties on T imply topological counterparts. In particular, if a continuous map $T : X \to X$ happens to be ergodic with respect to a Borel probability measure μ with full support, then almost every $x \in X$, relative to μ, has a dense orbit. Moreover, from Birkhoff's ergodic theorem, we can obtain frequent hypercyclicity.

In [4], Aron and Markose studied the following class of operators in the seek of examples of non convolution hypercyclic operators on $H(\mathbb{C}) : f \mapsto [z \to f(\lambda z + b)]$. This class of operators, or variants of it, has also been studied in [22, 28, 31, 36, 40]. In [36] we extended the work of [4] analyzing the hypercyclic behavior of some non-convolution operators on the space $H(\mathbb{C}^N)$. Given $\phi(z) = Az + b$ a diagonal affine linear map on $\mathbb{C}^N$ and $\alpha \in \mathbb{N}_0^N$, we proved that the operator $C_\phi \circ D^\alpha$ is hypercyclic if and only if either $\prod_i |A_{ii}|^{\alpha_i} \geq 1$ or ϕ has no fixed points. In this article we study the hypercyclic behavior of the analogous class of operators defined on the general context of holomorphy types on infinite dimensional Banach spaces. Some purely infinite dimensional difficulties appear, for example, the non existence of a fixed point of the affine maps does not longer guarantee the hypercyclicity of these operators.

A Runge approximation theorem valid for holomorphy types had to be developed, which we think is of independent interest. For the proof of the latter result, a classical theorem about the Bohr radius of an entire function was applied.

The structure of the article goes as follows. In the Preliminaries section we recall some concepts of linear dynamics. In Section 2 we recall the definition of holomorphy types and of the spaces of holomorphic functions associated to it. We introduce the family of operators that we will study and prove some properties there, including some Cauchy estimates valid for general holomorphy types, as well as a Runge type approximation theorem for holomorphic functions of a given type on Banach spaces. In the final section we prove our main results on the dynamics induced by the operators in the family that concerns us. Finally, we focus in the holomorphy type that determines the space $H_{bc}(E)$, of entire functions of compact type.

1. Preliminaries

In this section we state some results that ensure the existence of an invariant Borel probability measure with full support on complex separable Fréchet spaces, with respect to which an operator is strongly mixing. First we recall some basic definitions. In what follows X denotes a Fréchet space.

Definition 1.1. An operator T on X is called *mixing* if for every pair of non void open sets U and V, there exists $N \in \mathbb{N}$ such that $T^n U \cap V \neq \emptyset$ for every $n \geq N$.

A Borel probability measure on X is Gaussian if and only if it is the distribution of an almost surely convergent random series of the form $\xi = \sum_0^\infty g_n x_n$, where $(x_n) \subset X$ and (g_n) is a sequence of independent, standard complex Gaussian variables.

Definition 1.2. An operator $T \in \mathcal{L}(X)$ is *strongly mixing in the Gaussian sense* if there exists some Gaussian T-invariant probability measure μ on X with full support such that for any measurable sets A, $B \subset X$,

$$\lim_{n\to\infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B).$$

We will use the following result, which is a corollary of a theorem due to Bayart and Matheron (see [5]) and which states that the existence of sufficiently many eigenvectors associated to unimodular eigenvalues implies that the operator is strongly mixing in the Gaussian sense.

Theorem 1.3 (Bayart, Matheron)**.** *Let X be a complex separable Fréchet space, and let $T \in \mathcal{L}(X)$. Assume that for any set $D \subset \mathbb{T}$ such that $\mathbb{T} \setminus D$ is dense in $\mathbb{T}$, the linear span of $\bigcup_{\lambda \in \mathbb{T}-D} \ker(T - \lambda)$ is dense in X. Then T is strongly mixing in the Gaussian sense.*

The following result, which is [33, Theorem 1], says that operators satisfying the Frequent Hypercyclicity Criterion are strongly mixing with respect to an invariant Borel measure with full support.

Theorem 1.4 (Murillo-Arcila, Peris)**.** *Let X be a separable, Fréchet space and $T \in \mathcal{L}(X)$. Suppose that there exists a dense subspace $X_0 \subset X$ such that $\sum_{n\in\mathbb{N}} T^n x$ is unconditionally convergent for all $x \in X_0$. Suppose further that there exist a sequence of maps $S_k : X_0 \to X$, $k \in \mathbb{N}$ such that $T \circ S_1 = Id$, $T \circ S_k = S_{k-1}$ and $\sum_k S_k(x)$ is unconditionally convergent for all $x \in X_0$. Then there exist a Borel probability measure μ in X, T-invariant, such that the operator T is strongly mixing respect to μ.*

The hypotheses of Theorem 1.4 imply the corresponding ones of the Theorem 1.3. So both theorems allow us to conclude the existence of an invariant Gaussian probability measure for linear operators of full support which are strongly mixing. Finally, the next proposition states that the existence of such measures is preserved by linear conjugation.

Proposition 1.5. *Let X and Y be separable, Fréchet spaces and $T \in \mathcal{L}(X)$, $S \in \mathcal{L}(Y)$. Suppose that $SJ = JT$ for some linear mapping $J : X \to Y$ of dense range then, if T has an invariant Borel measure then so does S. Moreover, if T has an invariant Borel measure that is Gaussian, strongly mixing, ergodic or of full support, then so does S.*

2. Holomorphy types. Holomorphic functions of $\mathfrak{A}$-bounded type

From now on, E denotes a complex Banach space and $B_E(x,r)$ denotes the open ball of radius r and center $x \in E$. We denote by $\mathcal{P}^k(E)$ the Banach space of all continuous k-homogeneous polynomials from E to $\mathbb{C}$. The space $\mathcal{P}^0(E)$ is just $\mathbb{C}$.

We define, for each $P \in \mathcal{P}^k(E)$, $a \in E$ and $0 \leq j \leq k$ the polynomial $P_{a^j} \in \mathcal{P}^{k-j}(E)$ by

$$P_{a^j}(x) = \overset{\vee}{P}(a^j, x^{k-j}) = \overset{\vee}{P}(\underbrace{a, ..., a}_{j}, \underbrace{x, ..., x}_{k-j}),$$

where $\overset{\vee}{P}$ is the symmetric k-linear form associated to P. We write P_a instead of P_{a^1}.

Let us recall the definition of polynomial ideal [23, 24].

Definition 2.1. A Banach ideal of (scalar-valued) continuous k-homogeneous polynomial, $k \geq 0$, is a pair $(\mathfrak{A}_k, \|\cdot\|_{\mathfrak{A}_k})$ such that:

(i) For every Banach space E, $\mathfrak{A}_k(E) = \mathfrak{A}_k \cap \mathcal{P}^k(E)$ is a linear subspace of $\mathcal{P}^k(E)$ and $\|\cdot\|_{\mathfrak{A}_k(E)}$ is a norm on it. Moreover, $(\mathfrak{A}_k(E), \|\cdot\|_{\mathfrak{A}_k(E)})$ is a Banach space.

(ii) If $T \in \mathcal{L}(E_1, E)$ and $P \in \mathfrak{A}_k(E)$, then $P \circ T \in \mathfrak{A}_k(E_1)$ with

$$\|P \circ T\|_{\mathfrak{A}_k(E_1)} \leq \|P\|_{\mathfrak{A}_k(E)} \|T\|^k.$$

(iii) $z \mapsto z^k$ belongs to $\mathfrak{A}_k(\mathbb{C})$ and has norm 1.

We use the following version of the concept of holomorphy type, originally defined by Nachbin in [37] (see also [34]).

Definition 2.2. Consider the sequence $\mathfrak{A} = \{\mathfrak{A}_k\}_{k=0}^{\infty}$, where for each k, $\mathfrak{A}_k$ is a Banach ideal of k-homogeneous polynomials. We say that $\{\mathfrak{A}_k\}_k$ is a holomorphy type if for each $0 \leq l \leq k$ there exist a positive constant $c_{k,l}$ such that for every Banach space E, the following hold:

$$\text{if } P \in \mathfrak{A}_k(E),\ a \in E \text{ then } P_{a^l} \text{ belongs to } \mathfrak{A}_{k-l}(E) \text{ and } \|P_{a^l}\|_{\mathfrak{A}_{k-l}(E)} \leq c_{k,l} \|P\|_{\mathfrak{A}_k(E)} \|a\|^l. \tag{1}$$

Remark 2.3. Sometimes we require that the constants satisfy, for every k, l,

$$c_{k,l} \leq \frac{(k+l)^{k+l}}{(k+l)!} \frac{k!}{k^k} \frac{l!}{l^l}. \tag{2}$$

These constants are more restrictive than Nachbin's constants (the constants considered by Nachbin were of the form $c_{k,l} = \binom{k}{l} C^k$ for some fixed constant C), but on the other hand, the constants $c_{k,l}$ of every usual example of holomorphy type satisfy (2).

There is a natural way to associate to a holomorphy type $\mathfrak{A}$ spaces of holomorphic functions of bounded type on a Banach space E, namely the holomorphic functions that have a given $\mathfrak{A}$-radius of convergence at each point of E (see for example [11, 16, 18, 35]).

Definition 2.4. Let $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ be a holomorphy type, E be a Banach space, $x \in E$, and $0 < r \leq \infty$. We define the space of holomorphic functions of $\mathfrak{A}$-bounded type on $B(x,r)$ by

$$H_{b\mathfrak{A}}(B(x,r)) = \left\{ f \in H(B(x,r)) \ : d^k f(x) \in \mathfrak{A}_k(E) \text{ and } \limsup_{k\to\infty} \left\| \frac{d^k f(x)}{k!} \right\|_{\mathfrak{A}_k}^{1/k} \leq \frac{1}{r} \right\}.$$

The case $r = \infty$ correspond to the space of entire functions of bounded $\mathfrak{A}$-type, $H_{b\mathfrak{A}}(E)$.

We consider in $H_{b\mathfrak{A}}(B(x,r))$ the seminorms p_t^x, for $0 < t < r$, given by

$$p_t^x(f) = \sum_{k=0}^{\infty} \left\| \frac{d^k f(x)}{k!} \right\|_{\mathfrak{A}_k} t^k,$$

for all $f \in H_{b\mathfrak{A}}(B(x,r))$. It is easy to show that $\left(H_{b\mathfrak{A}}(B(x,r)), \{p_t^x\}_{0<t<r} \right)$ is a Fréchet space.

There are many usual spaces of entire functions of bounded type that may be constructed in this way.

Example 2.5. We mention the following spaces of entire functions: (i) $H_b(E)$ of all bounded type functions, (ii) of nuclear bounded type functions [27], (iii) of Hilbert-Schmidt type functions [17, 38], (iv) of compact bounded type functions [1, 3], (v) of weakly uniformly continuous functions on bounded sets [1], (vi) of extendible functions of bounded type [10] and (vii) of integral functions of bounded type [15].

Our objective is to define on $H_{b\mathfrak{A}}(E)$ analogues of the operators studied by Aron and Markose in [4] in the one variable case and to determine the dynamics they induce. It is clear that the space $H_{b\mathfrak{A}}(E)$ must be separable in order to support an hypercyclic operator. Since E' is a subspace of $H_{b\mathfrak{A}}(E)$, we need to restrict ourselves to the class of Banach spaces with separable dual space. On the other hand, if E' is separable and if we assume that the finite type polynomials are dense in each $\mathfrak{A}_k(E)$, it is a simple exercise to prove that $H_{b\mathfrak{A}}(E)$ is separable. Also, in order to be able to define partial derivatives we will assume that the space E has an unconditional basis. Recall (see for example [26]) that a basis $(e_s)_{s\in\mathbb{N}}$ is C-unconditional if there exist a positive constant C such that for every sequence of scalars $(a_n)_{n\in\mathbb{N}}$ and every sequence of scalars $(\varepsilon_n)_{n\in\mathbb{N}}$ of modulus at most 1, we have the inequality

$$\|\sum_{n=1}^{\infty} \varepsilon_n a_n e_n\| \leq C \|\sum_{n=1}^{\infty} a_n e_n\|.$$

Let E be a Banach space with basis $(e_s)_{s\in\mathbb{N}}$. Recall that the basis is shrinking if for every $e' \in E'$ the norm of the restriction of e' to the span of $(e_s)_{s\geq n}$, goes to 0 as $n \to \infty$.

Let E be a Banach space with unconditional basis $(e_s)_{s\in\mathbb{N}}$. The dual system of linear functionals associated to the basis $(e_s)_{s\in\mathbb{N}}$ is defined by the relation $e'_m(e_n) = \delta_{m,n}$. Let $\{P_s\}_{s\in\mathbb{N}}$ be the natural projections associated to the basis $(e_s)_{s\in\mathbb{N}}$. For every choice of scalars $(a_s)_{s\in\mathbb{N}}$ and for all integers $n < m$ we have $P_n^*(\sum_{s\leq m} a_s e'_s) = \sum_{s\leq n} a_s e'_s$. Since, $\|P_n^*\| = \|P_n\|$, we get that $(e'_s)_{s\in\mathbb{N}}$ is a basic sequence in E', whose basis constant is identical to that of $(e_s)_{s\in\mathbb{N}}$. Recall that

$(e'_s)_{s\in\mathbb{N}}$ form a basis of E' if and only if $(e_s)_{s\in\mathbb{N}}$ is shrinking [30, Proposition 1.b.1]. Note that if E has unconditional basis and E' is separable, then the basis is shrinking, because $\ell_1 \nsubseteq E$.

In order to apply the hypercyclicity criterion, we will need a convenient dense subset. In many cases, we will use the span of the monomials.

Definition 2.6. If $\beta = (\beta_i)_{i\in\mathbb{N}}$ is a finite multi-index, we define the monomial $z^\beta \in H_{b\mathfrak{A}}(E)$ as

$$z^\beta = \prod_i (e'_i)^{\beta_i}.$$

The next lemma tell us that under suitable assumptions on the holomorphy type, the monomials span a dense set in $H_{b\mathfrak{A}}(E)$. First recall the following definition from [11, 12].

Definition 2.7. We say that the holomorphy type $\mathfrak{A} = \{\mathfrak{A}_k\}_{k=0}^\infty$ is coherent if for each $k \geq 0$ there exist a positive constant d_k such that for every Banach space E, the following hold:

(3) if $P \in \mathfrak{A}_k(E)$, $\gamma \in E'$ then γP belongs to $\mathfrak{A}_{k+1}(E)$ and $\|\gamma P\|_{\mathfrak{A}_{k+1}(E)} \leq d_k \|P\|_{\mathfrak{A}_k(E)} \|\gamma\|_{E'}$.

Lemma 2.8. Let E be a Banach space with unconditional basis $(e_s)_{s\in\mathbb{N}}$ and let $\mathfrak{A}$ be a coherent holomorphy type such that finite type polynomials are dense in $\mathfrak{A}_k(E)$ for each k. Then, the linear span of the monomials is dense in $H_{b\mathfrak{A}}(E)$ if and only if the basis is shrinking.

Proof. Since $\mathfrak{A}$ is coherent, if $\varphi^1, \ldots, \varphi^n \in E'$, we get that

$$\begin{aligned}\|\varphi^1 \ldots \varphi^n\|_{\mathfrak{A}_n(E)} &\leq d_{n-1} \|\varphi^1\|_{\mathfrak{A}_1(E)} \|\varphi^2 \ldots \varphi^n\|_{\mathfrak{A}_{n-1}(E)} \\ &\leq \left(\prod_{j=1}^{n-1} d_j\right) \prod_{j=1}^n \|\varphi^j\|_{E'}.\end{aligned}$$

Since the basis of E is shrinking and $(e'_s)_{s\in\mathbb{N}}$ form a basis of E', we get that each $\varphi^j \in E'$ can be written as

$$\varphi^j = \sum_{s=1}^\infty a_s^{(j)} e'_s.$$

Now, given $\varepsilon > 0$ and $N > 0$, we fix $\varepsilon_i < \frac{\varepsilon}{nK^{i-1}\prod_{j=1}^{n-1} d_j \prod_{j\neq i} \|\varphi^j\|_{E'}}$ and $\xi^j := \sum_{s=1}^N a_s^{(j)} e'_s \in E'$ such that $\|\varphi^j - \xi^j\|_{E'} < \varepsilon_j$.

Note that $\prod_{j=1}^n \xi^j$ is a linear combination of monomials because,

$$\prod_{j=1}^n \xi^j = \prod_{j=1}^n \sum_{s=1}^N a_s^{(j)} e'_s = \sum_{s_1,\ldots,s_n=0}^N a_{s_1}^{(1)} \ldots a_{s_n}^{(n)} \prod_{j=1}^n e'_{s_j}.$$

Also, since $(e'_s)_{s\in\mathbb{N}}$ is a basis, there exist a constant K such that

$$\|\xi^j\|_{E'} = \left\|\sum_{s=1}^N a_s^{(j)} e'_s\right\|_{E'} \leq K \|\varphi^j\|_{E'}.$$

Thus, we have

$$\begin{aligned}\|\varphi^1\dots\varphi^n-\xi^1\dots\xi^n\|_{\mathfrak{A}_n(E)} &\le \sum_{i=1}^n \left\|\left(\prod_{j=1}^{i-1}\xi^j\right)\left(\prod_{j=i}^{n}\varphi^j\right)-\left(\prod_{j=1}^{i}\xi^j\right)\left(\prod_{j=i+1}^{n}\varphi^j\right)\right\|_{\mathfrak{A}_n(E)}\\
&= \sum_{i=1}^n \left\|\left(\prod_{j=1}^{i-1}\xi^j\right)(\varphi^i-\xi^i)\left(\prod_{j=i+1}^{n}\varphi^j\right)\right\|_{\mathfrak{A}_n(E)}\\
&\le \sum_{i=1}^n\left(\prod_{j=1}^{n-1}d_j\right)\left(\prod_{j=1}^{i-1}\|\xi^j\|_{E'}\right)\|\varphi^i-\xi^i\|_{E'}\left(\prod_{j=i+1}^{n}\|\varphi^j\|_{E'}\right)\\
&\le \sum_{i=1}^n\left(\prod_{j=1}^{n-1}d_j\right)K^{i-1}\left(\prod_{j\neq i}\|\varphi^j\|_{E'}\right)\|\varphi^i-\xi^i\|_{E'}\end{aligned}$$

So, we get that $\|\varphi^1\dots\varphi^n-\xi^1\dots\xi^n\|_{\mathfrak{A}_n(E)}<\varepsilon$.

Reciprocally, suppose that the linear span of the monomials of degree k is dense in $\mathfrak{A}_k(E)$, for all $k\in\mathbb{N}$. Since the norm of $\mathfrak{A}_1(E)$ coincides with the norm in E', the linear span of the monomials of degree 1 is dense in E'. But, this means that $(e_s')_{s\in\mathbb{N}}$ is a basis of E'. □

In the spaces *ii*), *iii*) and *iv*) of Example 2.5 finite type polynomials are dense for arbitrary Banach spaces. For the other spaces of holomorphic functions appearing in Example 2.5 finite type polynomial are dense for specific Banach spaces (see [14, 35]).

Let us also remark that holomorphy types in which finite type polynomials are dense are called α-β-holomorphy types in [16] and π_1 holomorphy types in [6, 18–21]

Now we define the family of operators we will study and prove that they are bounded on $H_{b\mathfrak{A}}(E)$. Let E be a Banach space with a C-unconditional shrinking basis, $(e_s)_{s\in\mathbb{N}}$. Let $\mathfrak{A}$ be a holomorphy type such that the finite type polynomials are dense in each $\mathfrak{A}_k(E)$. Fix a finite multi-index $\alpha=(\alpha_i)_{i\in\mathbb{N}}$, $|\alpha|=m$, which counts how many times the operator T partially differentiates in each variable, where the partial derivative in the $s-$th variable is

$$D^{e_s}f(z)=\lim_{h\to 0}\frac{f(z+he_s)-f(z)}{h}.$$

Also, for fix vectors, $\lambda=(\lambda_j)_{j\in\mathbb{N}}\in\ell_\infty$ and $b=\sum_{j\in\mathbb{N}}b_je_j\in E$, let λb denote the vector $\lambda b=\sum_{j\in\mathbb{N}}\lambda_jb_je_j\in E$. The operator $T:H_{b\mathfrak{A}}(E)\to H_{b\mathfrak{A}}(E)$ is defined by

$$Tf(z)=D^\alpha f(\lambda z+b). \tag{4}$$

Proposition 2.9. *Let $\mathfrak{A}$ be a holomorphy type with constants as in* (2)*, and T defined as in* (4)*, then T is a continuous linear operator on $H_{b\mathfrak{A}}(E)$. Moreover, for each $f\in H_{b\mathfrak{A}}(E)$, $x\in E$, and $r,\varepsilon>0$,*

$$p_r^x(Tf)\le\frac{C(\alpha)}{\varepsilon^{|\alpha|}}p^{\lambda\cdot x+b}_{rC\|\lambda\|_\infty+\varepsilon}(f), \tag{5}$$

where $C(\alpha)$ is a positive constant depending only on α, which can be taken equal to $e^{|\alpha|+1}(\prod_{\alpha_i\neq 0}\alpha_i)^{1/2}$.

Observe that we can think T as a composition of three operators. Indeed, let $\Lambda : x \mapsto \lambda\cdot x$ be the coordinate-wise multiplication operator on E, which satisfies $\|\Lambda\| \leq C\|\lambda\|_\infty$. Then Λ induces a composition operator $M_\lambda : H_{b\mathfrak{A}}(E) \to H_{b\mathfrak{A}}(E)$, defined by $M_\lambda(f) = f\circ\Lambda$. Then,

$$Tf = M_\lambda \circ \tau_b \circ D^\alpha(f),$$

where $\tau_b : H_{b\mathfrak{A}}(E) \to H_{b\mathfrak{A}}(E)$ is the translation operator defined by $\tau_b(f)(z) = f(z+b)$.

To prove the above proposition we will show that the three operators are continuous on $H_{b\mathfrak{A}}(E)$. For the partial differentiation operator D^α we will need two lemmas. The first one, which should be well known, shows that partial differentiation coincides with taking differentials and the second is a generalization of the Cauchy inequalities to holomorphy types.

Lemma 2.10. Let E be a Banach space with basis $(e_n)_n$ and let $f \in H(E)$ be a holomorphic function on E. Then

(i) $D^{e_s}f(z) = df(z)(e_s)$,
(ii) $d[d^kf(\cdot)^\vee(e_{s_1},\dots,e_{s_k})](z)(e_l) = [d^{k+1}f(z)]^\vee(e_l, e_{s_1},\dots,e_{s_k})$.
(iii) $D^\alpha f(z) = [d^{|\alpha|}f(z)]^\vee(e_1^{\alpha_1},\dots,e_l^{\alpha_l})$, if $\alpha_j = 0$ for every $j > l$.

Proof. To prove (i), we write $f(w) = \sum_{k\geq 0}\frac{d^kf(z)}{k!}(w-z)$. Thus we have that

$$\begin{aligned}\frac{f(z+he_s)-f(z)}{h} &= \frac{1}{h}\sum_{k\geq1}\frac{d^kf(z)}{k!}(he_s) = \sum_{k\geq1}h^{k-1}\frac{d^kf(z)}{k!}(e_s)\\ &= df(z)(e_s) + h\left[\sum_{k\geq2}h^{k-2}\frac{d^kf(z)}{k!}(e_s)\right]\underset{h\to0}{\longrightarrow} df(z)(e_s).\end{aligned}$$

The last assertion is true because since $f\in H(E)$, it satisfies that $\limsup\left\|\frac{d^kf(z)}{k!}\right\|^{1/k} < \infty$, and thus $\sum_{k\geq2}h^{k-2}\frac{d^kf(z)}{k!}(e_s)$ is a power series with positive radius of convergence.

For (ii), observe that if $z\in E$,

$$\begin{aligned}\sum_{k=0}^\infty\frac{d^kf(a)}{k!}(x) = f(x+a) &= \sum_{j=0}^\infty\frac{d^jf(z)}{j!}(x+a-z)\\ &= \sum_{j=0}^\infty\sum_{k=0}^j\binom{j}{k}\left[\frac{d^jf(z)}{j!}\right]^\vee((a-z)^{j-k},x^k)\\ &= \sum_{k=0}^\infty\sum_{j=k}^\infty\binom{j}{k}\left[\frac{d^jf(z)}{j!}\right]^\vee((a-z)^{j-k},x^k).\end{aligned}$$

Thus we get that

$$d^kf(a)(x) = k!\sum_{j=1}^\infty\binom{j}{k}\left[\frac{d^jf(z)}{j!}\right]^\vee((a-z)^{j-k},x^k),$$

from which we conclude that if $g(z) = d^k f(z)^\vee(e_{s_1}, \ldots, e_{s_k})$, then

$$\begin{aligned}\frac{g(z+he_l)-g(z)}{h} &= \frac{1}{h}\left[k!\sum_{j=k}^{\infty}\binom{j}{k}\left[\frac{d^j f(z)}{j!}\right]^\vee((he_l)^{j-k}, e_{s_1}, \ldots, e_{s_k}) - d^k f(z)^\vee(e_{s_1}, \ldots, e_{s_k})\right] \\ &= \frac{1}{h}\left[k!\sum_{j=k+1}^{\infty}\binom{j}{k}h^{j-k}\left[\frac{d^k f(z)}{k!}\right]^\vee(e_l^{j-k}, e_{s_1}, \ldots, e_{s_k})\right] \\ &\underset{h\to 0}{\longrightarrow} \quad [d^{k+1}f(z)]^\vee(e_l, e_{s_1}, \ldots, e_{s_k}),\end{aligned}$$

because f has positive radius of convergence at z.

(iii) follows from (i) and (ii). □

Using the previous lemma it is immediate to translate a result in [34, Lemma 4.5] to conclude that D^α is a bounded operator on $H_{b\mathfrak{A}}(E)$.

Lemma 2.11 (Cauchy estimates for holomorphy types)**.** *Let $\mathfrak{A}$ be a holomorphy type with constants as in* (2)*, E be a Banach space with basis $(e_n)_n$. Then for $f \in H_{b\mathfrak{A}}(E)$, $x \in E$ and $r, \varepsilon > 0$, we have*

$$p_r^x(D^\alpha f) \le \frac{C(\alpha)}{\varepsilon^{|\alpha|}} p_{r+\varepsilon}^x(f),$$

where $C(\alpha)$ is a positive constant depending only on α, we can take $C(\alpha)$ to be equal to $e^{|\alpha|+1}(\prod_{\alpha_i \neq 0}\alpha_i)^{1/2}$.

Proof of Proposition 2.9. Let us first show estimates for M_λ in terms of the seminorms p_r^x. If $f = \sum_k P_k$ then

$$\|M_\lambda(P_k)\|_{\mathfrak{A}_k(E)} = \|P_k \circ \Lambda\|_{\mathfrak{A}_k(E)} \le \|P_k\|_{\mathfrak{A}_k(E)}(C\|\lambda\|_\infty)^k,$$

and

$$\frac{d^j(M_\lambda f)(x)}{j!} = \sum_{k\ge j}\binom{k}{j}(M_\lambda P_k)_{x^{k-j}}.$$

Thus, using that $\mathfrak{A}_j$ is an ideal, we have

$$\begin{aligned}p_r^x(M_\lambda f) = \sum_{j\ge 0} r^j \left\|\frac{d^j(M_\lambda f)(x)}{j!}\right\|_{\mathfrak{A}_j(E)} &= \sum_{j\ge 0} r^j \left\|\sum_{k\ge j}\binom{k}{j}(M_\lambda P_k)_{x^{k-j}}\right\|_{\mathfrak{A}_j(E)} \\ &= \sum_{j\ge 0} r^j \left\|M_\lambda\left(\sum_{k\ge j}\binom{k}{j}(P_k)_{\Lambda(x)^{k-j}}\right)\right\|_{\mathfrak{A}_j(E)} \\ &\le \sum_{j\ge 0}(rC\|\lambda\|_\infty)^j \left\|\sum_{k\ge j}\binom{k}{j}(P_k)_{(\Lambda(x))^{k-j}}\right\|_{\mathfrak{A}_j(E)} = p_{rC\|\lambda\|_\infty}^{\Lambda(x)}(f).\end{aligned}$$

For the translation operator we have,

$$\begin{aligned} p_r^x(\tau_b f) &= \sum_{j\geq 0} \left\| \frac{d^j(\tau_b f)(x)}{j!} \right\|_{\mathfrak{A}_j(E)} r^j \\ &= \sum_{j\geq 0} r^j \left\| \frac{d^j f(x+b)}{j!} \right\|_{\mathfrak{A}_j(E)} = p_r^{x+b}(f). \end{aligned}$$

To finish the proof just apply the Cauchy inequalities (Lemma 2.11) together with the estimates for M_λ and τ_b. □

2.1. **A Runge type approximation theorem.** We will also need in the next section a version of Runge's Theorem for the space $H_{b\mathfrak{A}}(E)$. Recall that the classical Runge's approximation theorem states that if K is a compact subset of $\mathbb{C}$ and $\mathbb{C} \setminus K$ is connected then, any function that is holomorphic in a neighbourhood of K can be uniformly approximated on K by polynomial functions.

In [29], A. Hallack proved that the translations are hypercyclic in the space $H_{bc}(E)$ of entire functions of compact bounded type. For that, the following version of Runge's approximation theorem is proved: Let B_1 and B_2 be two disjoint closed balls in a complex Banach space E. If f is a holomorphic and bounded function in a uniform neighborhood of $B_1 \cup B_2$, then f can be uniformly approximated by polynomials on $B_1 \cup B_2$.

We wish to have a result in which if f is a holomorphic function of a given type $\mathfrak{A}$ on two disjoint balls, then f may be approximated by polynomials, but in the topology of $H_{b\mathfrak{A}}$. In order to obtain a result in this direction, we need to work with multiplicative holomorphy types, which is a slightly more restrictive concept than that of coherent sequence.

Definition 2.12. Let $\{\mathfrak{A}_k\}_k$ be a sequence of polynomial ideals. We say that $\{\mathfrak{A}_k\}_k$ is multiplicative at E if there exist constants $c_{k,l} > 0$ such that for each $P \in \mathfrak{A}_k(E)$ and $Q \in \mathfrak{A}_l(E)$, we have that $PQ \in \mathfrak{A}_{k+l}(E)$ and

$$\|PQ\|_{\mathfrak{A}_{k+l}(E)} \leq c_{k,l} \|P\|_{\mathfrak{A}_k(E)} \|Q\|_{\mathfrak{A}_l(E)}.$$

All the examples mentioned in Example 2.5 are multiplicative with constants as in (2), see [13, 34].

Remark 2.13. Suppose that $\mathfrak{A}$ is multiplicative with constants $c_{k,l}$ as in (2), and E is a Banach space. Then, each seminorm p_s^x is "almost" multiplicative in the following sense. Given $\varepsilon > 0$ there exists a constant $c = c(\varepsilon, s) > 1$ such that

$$p_s^x(fg) \leq c p_s^x(g) p_{s+\varepsilon}^x(f). \tag{6}$$

This was proved in [34, Proposition 3.3].

Finally we recall Bohr's inequality [8] for analytic functions defined on $\mathbb{C}$.

Theorem 2.14. [Bohr's inequality] For every analytic function in $H(\mathbb{C})$, $f(z) = \sum_{n\geq 0} a_n z^n$ and any $0 \leq r \leq 1/3$, the following inequality holds

$$\sum_{n\geq 0} |a_n| r^n \leq \sup_{|z|\leq 1} \left| \sum_{n\geq 0} a_n z^n \right|.$$

Now we can state and prove Runge's approximation theorem for $H_{b\mathfrak{A}}(E)$.

Theorem 2.15. Let $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ be a holomorphy type, E be a Banach space, $x \in E$, and r, s and δ be positive real numbers. Suppose that finite type polynomials are dense in each $\mathfrak{A}_k$ and that $\mathfrak{A}$ is multiplicative with constants as in (2). If f is a holomorphic function of $\mathfrak{A}$-bounded type on the disjoint balls $B(0, r + \delta)$ and $B(a, s + \delta)$, then there are polynomials in $H_{b\mathfrak{A}}(E)$ which approximate f in $H_{b\mathfrak{A}}(B(0, r/3))$ and $H_{b\mathfrak{A}}(B(a, s/3))$.

Proof. Let $\varepsilon > 0$. First, since f is holomorphic of $\mathfrak{A}$-bounded type in $B(0, r)$ and $B(a, s)$, there exist polynomials P_1 and P_2 in $H_{b\mathfrak{A}}(E)$ such that $p^0_{r/3}(P_1 - f) < \varepsilon/3$ and that $p^a_{s/3}(P_2 - f) < \varepsilon/3$.

Let $M = c(p^0_{r/3+1}(P_1) + p^0_{r/3+1}(P_2) + p^a_{s/3+1}(P_1) + p^a_{s/3+1}(P_2))$, where c is a positive constant such that $p^0_{r/3}(gh) \leq cp^0_{r/3+1}(g)p^0_{r/3}(h)$ and $p^a_{s/3}(gh) \leq cp^a_{s/3+1}(g)p^a_{s/3}(h)$ for any polynomials g, h in $H_{b\mathfrak{A}}(E)$, (see Remark 2.13).

By the Hahn-Banach separation theorem, there exists $\varphi \in E'$ such that $K_1 = \overline{\varphi(B(0, r))}$ and $K_2 = \overline{\varphi(B(a, s))}$ are disjoint convex compact sets in $\mathbb{C}$. In fact, since K_1 and K_2 are closed, convex, balanced sets in $\mathbb{C}$, we get that $K_1 = \overline{D}(0, r\|\varphi\|)$ and $K_2 = \overline{D}(\varphi(a), s\|\varphi\|)$. Now we can apply Runge's Theorem to the compact set $K = K_1 \cup K_2 \subset \mathbb{C}$ and find a polynomial $q \in \mathbb{C}[z]$ such that $|q(z) - 1| < \frac{\varepsilon}{3M}$ for every $z \in K_1$ and $|q(z)| < \frac{\varepsilon}{3M}$ for every $z \in K_2$.

Consider $h = q \circ \varphi \in H_{b\mathfrak{A}}(E)$. Suppose that $q(z) = \sum_{j=0}^m a_j z^j$, then applying Bohr's inequality in $\mathbb{C}$, we get that

$$p^0_{r/3}(h - 1) = |a_0 - 1| + \sum_{j=1}^m \left(\frac{r}{3}\right)^j |a_j| \|\varphi\|^j \leq \sup_{z\in\overline{D}(0,r\|\varphi\|)} |q(z) - 1| \leq \frac{\varepsilon}{3M}.$$

Suppose also that $q(z) = \sum_{j=0}^m b_j(z - \varphi(a))^j$. Applying Bohr's inequality again we get that

$$p^a_{s/3}(h) = \sum_{j=0}^m \left(\frac{s}{3}\right)^j |b_j| \|\varphi\|^j \leq \sup_{z\in\overline{D}(\varphi(a),s\|\varphi\|)} |q(z)| \leq \frac{\varepsilon}{3M}.$$

Finally, define $P = P_1 h + P_2(1 - h) \in H_{b\mathfrak{A}}(E)$, thus

$$\begin{aligned} p^0_{r/3}(P - f) &\leq p^0_{r/3}(P_1(h - 1)) + p^0_{r/3}(P_1 - f) + p^0_{r/3}(P_2(1 - h)) \\ &< cp^0_{r/3+1}(P_1)p^0_{r/3}(h - 1) + \frac{\varepsilon}{3} + cp^0_{r/3+1}(P_2)p^0_{r/3}(1 - h) \\ &\leq \varepsilon, \end{aligned}$$

and

$$\begin{aligned} p^a_{s/3}(P-f) &\le p^a_{s/3}(P_1h) + p^a_{s/3}(P_2-f) + p^a_{s/3}(P_2h)) \\ &< cp^a_{s/3+1}(P_1)p^a_{s/3}(h) + \frac{\varepsilon}{3} + cp^a_{s/3+1}(P_2)p^a_{s/3}(h) \\ &\le \varepsilon. \end{aligned}$$

□

3. Dynamics of non convolution operators in $H_{b\mathfrak{A}}(E)$

In this section we are concerned with the hypercyclic behavior of the family of operators $T = M_\lambda \circ \tau_b \circ D^\alpha$ defined in the previous section. If $\lambda_j = 0$ for some j, then we have that $d(T^n f)(\cdot)(e_j) = 0$, for every $n \in \mathbb{N}$ and every $f \in H_{b\mathfrak{A}}(E)$. Since, the application $g \mapsto dg(\cdot)(e_j)$ is continuous, we conclude that the orbit of f under T cannot be dense. We will thus always suppose that $\lambda_j \neq 0$ for all $j \in \mathbb{N}$.

The next result describes the hypercyclicity of the operator $Tf = M_\lambda \circ \tau_b \circ D^\alpha(f)$ in terms of the parameters involved. Let us denote $\lambda^\alpha = \prod_i \lambda_i^{\alpha_i}$. When no coordinate of the map $\phi(z) = (\lambda_j z_j + b_j)_j$ is a translation, we denote $\zeta := (b_1/(1-\lambda_1), b_2/(1-\lambda_2), b_3/(1-\lambda_3), \dots)$ the sequence in $\mathbb{C}^{\mathbb{N}}$ formed by the fixed points of every coordinate of the map ϕ. It is worth to notice that if $b_i = 0$ and $\lambda_i = 1$, then the fixed point of the i-coordinate of ϕ is 0, thus we will assume that in this case, $\zeta_i = 0$. Our main theorem reads as follows.

Theorem 3.1. Let E be a Banach space with a 1-unconditional shrinking basis, $(e_s)_{s\in\mathbb{N}}$. Let $\mathfrak{A}$ be a multiplicative holomorphy type with constants as in (2), such that finite type polynomials are dense in each $\mathfrak{A}_k(E)$ for every k. Let T be the operator on $H_{b\mathfrak{A}}(E)$, defined by $Tf(z) = M_\lambda \circ \tau_b \circ D^\alpha f(z)$, with $\alpha \neq 0$ and $\lambda_i \neq 0$ for all $i \in \mathbb{N}$.

a) If $|\lambda^\alpha| \ge 1$, then T is strongly mixing in the Gaussian sense.

b) If $\|\lambda\|_\infty = 1$ and $b_i \neq 0$ and $\lambda_i = 1$ for some $i \in \mathbb{N}$, then T is mixing.

c) If $\|\lambda\|_\infty = 1$, no coordinate of ϕ is a translation and $\zeta \notin E''$, then T is mixing.

d) If $|\lambda^\alpha| < 1$ and $\zeta \in E''$, then T is not hypercyclic.

Remark 3.2. Note that the assumptions on E are, in some sense, necessary. Indeed, the unconditionality of the basis is needed to guarantee the continuity of M_λ. On the other hand, if E is a Banach space with a non shrinking unconditional basis then E contains a isomorphic copy of ℓ_1 and thus, $H_{b\mathfrak{A}}(E)$ cannot be separable.

Note also that, from case *d*), we see that, in contrast to the finite dimensional case [36], it is possible that the affine map $M_\lambda \circ \tau_b$ does not have a fixed point in E and that the operator fails to be hypercyclic.

Since the basis of E, $(e_s)_{s\in\mathbb{N}}$, is shrinking the bidual of E can be identified (see [30, Proposition 1.b.2]) with the sequences of complex numbers $(z_1, z_2, z_3, \dots)$ such that

$$\sup_n \left\| \sum_{i=1}^{n} z_i e_i \right\| < \infty.$$

This correspondence is given by

$$z'' \leftrightarrow (z''(e_1'), z''(e_2'), z''(e_3'), \dots),$$

and the norm of z'' is equivalent to $\sup_n \| \sum_{i=1}^{n} z''(e_i')e_i\|$.

We will divide the proof of the theorem in several cases. Lemmas 3.3 and 3.4 prove case a), cases b), c) and d) are proven in Lemma's 3.5, 3.7 and 3.6, respectively. Let $A := \{n \in \mathbb{N} : \lambda_n = 1\}$ and $B := \{n \in \mathbb{N} : \lambda_n \neq 1\}$. If $w \in \mathbb{C}^{\mathbb{N}}$, we write $w_A = (w_i)_{i\in A}$ and $w_B = (w_i)_{i\in B}$. We have that $\mathbb{N} = A \dot{\cup} B$. We can also decompose $E = E(A) + E(B)$ (where, for $C \subset \mathbb{N}$, we denote by $E(C)$ to the closure of the subspace spanned by the e_j's with $j \in C$). We will show that the conditions of the hypercyclicity criteria are satisfied with dense subspaces of the form $span\{e^{\gamma} z^{\beta} : \gamma \in E', \beta \in \mathbb{N}^{(\mathbb{N})}, \gamma_B = 0, \beta_A = 0\}$. Since the basis $(e_s)_{s\in\mathbb{N}}$ is shrinking, we can think of the elements of E' as sequences in $(e_s')_{s\in\mathbb{N}}$, the dual system of the basis of E. The vectors γ appearing in the dense subspaces will only have finite non-zero coordinates.

Lemma 3.3. Let E be a Banach space with unconditional shrinking basis, $(e_s)_{s\in\mathbb{N}}$. Let $\mathfrak{A}$ be a coherent holomorphy type such that the finite type polynomials are dense in each $\mathfrak{A}_k(E)$. Suppose that $|\lambda^{\alpha}| \geq 1$ and $\alpha_B = 0$. Then T is strongly mixing in the Gaussian sense.

Proof. We show that T satisfies the conditions of Theorem 1.3. Take a function of the form $e^{\gamma} z^{\beta}$, with $\gamma_B = 0$ and $\beta_A = 0$. Define $c_\beta \in E$ as, $c_\beta(n) = 0$ if $\beta_n = 0$, and $c_\beta(n) = \frac{b_n}{\lambda_n - 1}$ if $\beta_n \neq 0$ and define τ_β as the translation operator by c_β (note that c_β has finite non zero coordinates). Then $\tau_\beta^{-1} \circ T \circ \tau_\beta(e^{\gamma} z^{\beta}) = \gamma^{\alpha} e^{\langle \gamma_A, b_A \rangle} \lambda^{\beta} e^{\gamma} z^{\beta}$. Therefore,

$$T(\tau_\beta e^{\gamma} z^{\beta}) = \gamma^{\alpha} e^{\langle \gamma_A, b_A \rangle} \lambda^{\beta}\, \tau_\beta e^{\gamma} z^{\beta},$$

that is, the functions $\tau_\beta e^{\gamma} z^{\beta}$ are eigenvectors of T.

Let $D \subset \mathbb{S}^1$ a dense subset. It is enough to prove that

$$span\{\tau_\beta e^{\gamma} z^{\beta} : \text{ with } \gamma_B = 0, \beta_A = 0, \gamma^{\alpha} e^{\langle \gamma_A, b_A \rangle} \lambda^{\beta} \in D\}$$

is dense in $H_{b\mathfrak{A}}(E)$. Define $f_\beta(\gamma) = \gamma^{\alpha} e^{\langle \gamma_A, b_A \rangle} \lambda^{\beta}$. For each β finite with $\beta_A = 0$, the function f_β is holomorphic on $E(A)'$ and not constant. By [35, Lemma 2.4], we get that $\{e^{\gamma} : f_\beta(\gamma) \in D\}$ spans a dense subspace in $H_{b\mathfrak{A}}(E(A))$, for each β finite with $\beta_A = 0$. Also, note that for $k \in \mathbb{N}_0$

$$span\{\tau_\beta(z^{\beta}) : |\beta| \leq k\} = span\{z^{\beta} : |\beta| \leq k\}.$$

This is clear for $k = 0$, because both sets are $\mathbb{C}$, and if $|\beta| = k$

$$(z - c_\beta)^{\beta} = \prod_i \sum_{j=0}^{\beta_i} {z_i}^{\beta_i - j} {c_\beta}_i^{\beta_i} \binom{\beta_i}{j} = z^{\beta} + g(z),$$

where g has monomials of degree $< k$. Also, note that $\tau_{\beta_B}(z^{\beta_A} z^{\beta_B}) = z^{\beta_A}\tau_{\beta_B}(z^{\beta_B})$, and so $span\{\tau_{\beta_B}(z^{\beta_A} z^{\beta_B})\}$ is dense in $H_{b\mathfrak{A}}(E)$, because by Lemma 2.8, the monomials span a dense subspace of $H_{b\mathfrak{A}}(E)$. Gathering the previous observations we get that the eigenvectors of T with eigenvalues in D span a dense subspace in $H_{b\mathfrak{A}}(E)$. Thus, we have seen that the conditions of Theorem 1.3 are satisfied, and so the operator T is strongly mixing in the Gaussian sense. □

To finish the case when $|\lambda^\alpha| \geq 1$, it remains to prove when T differentiates in some coordinate with $\lambda_n \neq 1$.

Lemma 3.4. Let E be a Banach space with unconditional shrinking basis, $(e_s)_{s\in\mathbb{N}}$. Let $\mathfrak{A}$ be a coherent holomorphy type such that the finite type polynomials are dense in each $\mathfrak{A}_k(E)$. Suppose that $|\lambda^\alpha| \geq 1$ and $\alpha_B \neq 0$. Then T is strongly mixing in the Gaussian sense.

Proof. We will show that T satisfies the conditions of Theorem 1.4. Let $D := \{n \in \mathbb{N} : \lambda_n \neq 1, \alpha_n \neq 0\}$, note that D is a finite set. Then T is topologically conjugate to

$$T_0 f(z) = D^\alpha f(\lambda z + \tilde{b})$$

through a translation, where $\tilde{b}_n = b_n$ for all $n \notin D$ and $\tilde{b}_n = 0$ for all $n \in D$. Indeed, defining $c \in E$ as $c_n = 0$ for $n \notin D$ and $c_n = \frac{-b_n}{\lambda_n - 1}$ for $n \in D$, we get that $T_0 \circ \tau_c = \tau_c \circ T$. We may thus, by Proposition 1.5, assume that $b_n = 0$ for every n such that $\lambda_n \neq 1$ and $\alpha_n \neq 0$. So we can split $\mathbb{N}$ into three disjoint sets,

$$A := \{n \in \mathbb{N} : \lambda_n = 1\},$$
$$C := \{n \in \mathbb{N} : \lambda_n \neq 1, \alpha_n = 0\},$$
$$D := \{n \in \mathbb{N} : \lambda_n \neq 1, \alpha_n \neq 0\}.$$

Note that $|\lambda^\alpha| = |\lambda_D^{\alpha_D}| \geq 1$. Define the subspace

$$X_0 = span\{e^\gamma z^\beta : \gamma \in E',\ \gamma_D = \gamma_C = 0,\ \beta_A = 0\}.$$

We can see that X_0 is dense in $H_{b\mathfrak{A}}(E)$ proceeding as in the previous lemma. We have that

$$T(e_{\gamma_A} z^{\beta_D} z^{\beta_C}) = {\gamma_A}^{\alpha_A} e^{\langle \gamma_A, b_A\rangle} e_{\gamma_A} \frac{\beta_D!}{(\beta_D - \alpha_D)!} z^{\beta_D - \alpha_D} \lambda_D^{\beta_D - \alpha_D} (\lambda z + b)^{\beta_C}.$$

Denote $L(z) = \lambda z + b$, then we can write $(\lambda z + b)^{\beta_C} = C_L(z^{\beta_C})$ with C_L the composition operator associated to L. We also have

$$T^n(e_{\gamma_A} z^{\beta_D} z^{\beta_C}) = {\gamma_A}^{n\alpha_A} e^{n\langle \gamma_A, b_A\rangle} e_{\gamma_A} \frac{\beta_D!}{(\beta_D - n\alpha_D)!} z^{\beta_D - n\alpha_D} \lambda_D^{n\beta_D - \frac{n(n-1)}{2}\alpha_D} {C_L}^n(z^{\beta_C}).$$

Then $\sum_n T^n(e_{\gamma_A} z^{\beta_D} z^{\beta_C})$ is unconditionally convergent because it is a finite sum.

Define a sequence of operators S_n on X_0 by

$$S_n(e_{\gamma_A} z^{\beta_D} z^{\beta_C}) = \frac{\beta_D!}{{\gamma_A}^{n\alpha_A} e^{n\langle \gamma_A, b_A\rangle} (\beta_D + n\alpha_D)! \lambda_D^{n\beta_D + \frac{n(n+1)}{2}\alpha_D}} e_{\gamma_A} z^{\beta_D + n\alpha_D} (C_{L^{-1}})^n (z^{\beta_C}),$$

where $L^{-1}(z) = \frac{z-b}{\lambda}$. The operators S_n are defined so that they satisfy $T \circ S_1 = Id$ and $T \circ S_n = S_{n-1}$ on X_0.

Observe that, if $\|z\|_E \leq R$

$$|C_{L^{-1}}{}^n(z^{\beta_C})| \leq \left(\frac{1}{|\lambda_C{}^{\beta_C}|}\right)^n \left(\|z\|_E + \|b_C\|_E \frac{|\lambda_C|^n + 1}{|\lambda_C - 1|}\right)^{\beta_C} \leq M^{n|\beta|} \frac{R^{|\beta|}}{|\lambda_C^{\beta_C}|^n},$$

where M is a positive constant depending only on λ_C and b_C. Thus, since $|\lambda_D{}^{\alpha_D}|^{n(n+1)/2} \geq 1$, we have

$$|S_n(e_{\gamma_A} z^{\beta_D} z^{\beta_C})| \leq \frac{K^n}{(\beta_D + n\alpha_D)!} \frac{1}{|\lambda_C^{\beta_C}|^n}.$$

Since $S_n(e_{\gamma_A} z^{\beta_D} z^{\beta_C})$ depends only on a finite number of variables, this implies unconditionally convergence in $H_{b\mathfrak{A}}(E)$. In fact, suppose that $Q \in \mathfrak{A}_k(E)$ depends only on N variables and consider the following diagram

$$\begin{tikzcd}
E \arrow[r, "Q"] \arrow[d, "\pi"'] & \mathbb{C} \\
\mathbb{C}^N \arrow[ur, "\widetilde{Q}"'] &
\end{tikzcd}$$

Since $\mathfrak{A}$ is a Banach ideal, we get that

$$\|Q\|_{\mathfrak{A}_k(E)} \leq \|\widetilde{Q}\|_{\mathfrak{A}_k(\mathbb{C}^N)} \|\pi\|^k \leq N^k \|\pi\|^k \|\widetilde{Q}\|_\infty = N^k \|\pi\|^k \|Q\|_\infty.$$

Then $\sum_n S_n(e_{\gamma_A} z^{\beta_D} z^{\beta_C})$ is unconditionally convergent in $H_{b\mathfrak{A}}(E)$. Thus, we have proved that T satisfies the conditions of Theorem 1.4. $\square$

The two previous lemmas prove that T is strongly mixing in the Gaussian sense, in the case $|\lambda^\alpha| \geq 1$. In order to study the hypercyclicity of the operator when $|\lambda^\alpha| < 1$, we need to apply the version of Runge's Theorem for the space $H_{b\mathfrak{A}}(E)$ proved in the previous section.

Denote by $\phi(z) = \lambda z + b$ for $z \in E$ and $\phi_i(z) = \lambda_i z + b_i$ for $z \in \mathbb{C}$. In the next Lemma we will prove case $b)$ of Theorem 3.1, which is the case that one coordinate of the map ϕ is a translation.

Lemma 3.5. Let E be a Banach space with a 1-unconditional shrinking basis, $(e_s)_{s\in\mathbb{N}}$. Let $\mathfrak{A}$ be a multiplicative holomorphy type with constants as in (2), such that the finite type polynomials are dense in each $\mathfrak{A}_k(E)$. Let $T : H_{b\mathfrak{A}}(E) \to H_{b\mathfrak{A}}(E)$ be defined by $Tf = M_\lambda \circ \tau_b \circ D^\alpha(f)$, and suppose that $\|\lambda\|_\infty = 1$ and that there exists some coordinate with $\lambda_k = 1$ and $b_k \neq 0$. Then T is a mixing operator.

Proof. If $|\lambda^\alpha| = 1$ this is implied by Lemma's 3.3 and 3.4. Thus, we may suppose that $|\lambda^\alpha| < 1$.

We want to show that T is a mixing operator, i.e, for every pair of open sets U and V in $H_{b\mathfrak{A}}(E)$, there exists a positive integer n_0 for which $T^n U \cap V \neq \emptyset$, for all $n \geq n_0$. Without loss of generality we can suppose that

$$U = \{h \in H_{b\mathfrak{A}}(E) \text{ such that } p_r^0(h - f) < \delta\} \quad \text{and} \quad V = \{h \in H_{b\mathfrak{A}}(E) \text{ such that } p_r^0(h - g) < \delta\},$$

for f, $g \in H_{b\mathfrak{A}}(E)$ and r, δ positive numbers. Since, E has a shrinking basis, by Lemma 2.8, we can assume that f is a finite linear combination of monomials. Define an inverse for T over the span of the monomials by integrating each monomial and denote it by S.

Applying n times inequality (5) with $\varepsilon_j = 2^{-j}$ at the j-step, $j = 1, \dots, n$, we get that for all $x \in E$

$$p_r^x(T^n f) \leq C(n, \alpha) p_{r+1}^{\phi^n(x)}(f).$$

Thus,

$$p_r^0(T^n q - f) = p_r^0(T^n(q - S^n f)) \leq C(n, \alpha) p_{r+1}^{\phi^n(0)}(q - S^n f).$$

The fact that $\lambda_k = 1$ and $b_k \neq 0$, implies that $(\phi^n(0))_k = nb_k$. Since E has a 1-unconditional basis we get that,

$$\|\phi^n(0)\|_E \geq n|b_k|.$$

Then if n is such that $|\phi^n(0)_k| > 6r + 5$ we get that, $B(0, 3r+1) \cap B(\phi^n(0), 3(r+1)+1) = \emptyset$. Thus, by Theorem 2.15, there exists a polynomial $q \in H_{b\mathfrak{A}}(E)$, such that

$$p_r^0(q - g) < \delta \text{ and } p_{r+1}^{\phi^n(0)}(q - S^n f) < \frac{\delta}{C(n, \alpha)}.$$

Then, we get that for all $n \in \mathbb{N}$ such that $n|b_k| > 6r + 5$, there exists a polynomial $q \in H_{b\mathfrak{A}}(E)$, such that

$$p_r^0(q - g) < \delta \text{ and } p_r^0(T^n q - f) < \delta.$$

So, we have proved that there is a positive integer n_0 for which $T^n U \cap V \neq \emptyset$, for all $n \geq n_0$. □

Now will we take care of the cases *c*) and *d*) of our main theorem, in which no coordinate of the function ϕ is a translation. Note that the fact that $\phi_i(z) = \lambda_i z + b_i$ is not a translation implies that ϕ_i has a fixed point at $b_i/(1 - \lambda_i)$ (or at 0 if $\lambda_i = 1$ and $b_i = 0$).

Recall that we denote by $\zeta = (b_1/(1-\lambda_1), b_2/(1-\lambda_2), b_3/(1-\lambda_3), \dots) \in \mathbb{C}^{\mathbb{N}}$ the sequence of the fixed points of each ϕ_i. We are going to consider the cases in which $\zeta \in E''$ and $\zeta \notin E''$. We start with the case $\zeta \in E''$.

Lemma 3.6. Let E be a Banach space with a shrinking basis, $(e_s)_{s\in\mathbb{N}}$. Let $X \subset H_b(E)$ be a Fréchet space of holomorphic functions of bounded type. Suppose that $T : X \to X$, $Tf = M_\lambda \circ \tau_b \circ D^\alpha(f)$, with $|\lambda^\alpha| < 1$ and $\zeta \in E''$ is well defined. Then T is not hypercyclic.

Proof. Let us denote the Aron-Berner extension [2], defined on $H_b(E)$ by $AB : H_b(E) \to H_b(E'')$. Also denote by q_r the seminorms on $H_b(E)$,

$$q_r(f) = \sum_{k\geq 0} r^k \|P_k\|_{\mathcal{P}^k(E)},$$

for $f = \sum_k P_k$. Recall that AB is a continuous map and

$$q_r\left(AB\left(\sum_k P_k\right)\right) = \sum_{k\geq 0} r^k \|AB(P_k)\|_{\mathcal{P}^k(E'')} = \sum_{k\geq 0} r^k \|P_k\|_{\mathcal{P}^k(E)} = q_r\left(\sum_k P_k\right).$$

Finally, denote the evaluation at ζ by $\delta_\zeta : H_b(E'') \to \mathbb{C}$, $\delta_\zeta(g) = g(\zeta)$. If $g = \sum_k P_k \in H_b(E'')$, we get that

$$|g(\zeta)| = |\sum_{k\geq 0} P_k(\zeta)| \leq \sum_{k\geq 0} \|\zeta\|_{E''}^k \|P_k\|_{\mathcal{P}^k(E'')} = q_{\|\zeta\|}(g).$$

Under this assumptions, we can prove that no orbit of T can be dense. First recall that every orbit of T has the following form:

$$T^n f(z) = \lambda^{\frac{n(n-1)}{2}\alpha} D^{n\alpha} f\left(\phi^n(z)\right) = \lambda^{\frac{n(n-1)}{2}\alpha} D^{n\alpha} f\left(\lambda^n z + b\frac{1-\lambda^n}{1-\lambda}\right).$$

Thus, since ϕ^n is an affine map and since the Aron-Berner extension of a composition of a function with an affine map is the composition of the Aron-Berner extensions, we get that

$$\begin{aligned}\delta_\zeta AB(D^{n\alpha} f\circ\phi^n) &= AB(D^{n\alpha} f)(AB(\phi^n)(\zeta)) = AB(D^{n\alpha} f)\Big(\delta_\zeta(z\mapsto AB(\lambda^n z + b\frac{1-\lambda^n}{1-\lambda})\Big)\\ &= AB(D^{n\alpha} f)\Big(\lambda^n\zeta + b\frac{1-\lambda^n}{1-\lambda}\Big) = AB(D^{n\alpha} f)(\zeta).\end{aligned}$$

Now, we are able to show that T is not hypercyclic,

$$\begin{aligned}|\delta_\zeta AB(T^n f)| &= |\lambda^\alpha|^{\frac{n(n-1)}{2}} |\delta_\zeta AB(D^{n\alpha} f\circ\phi^n)| = |\lambda^\alpha|^{\frac{n(n-1)}{2}} |AB(D^{n\alpha} f)(\zeta)|\\ &\leq |\lambda^\alpha|^{\frac{n(n-1)}{2}} q_{\|\zeta\|}(D^{n\alpha} f)\\ &\leq |\lambda^\alpha|^{\frac{n(n-1)}{2}} e^{n|\alpha|+1} n^{|\alpha|/2}\left(\prod_{\alpha_i\neq 0}\alpha_i\right)^{1/2} q_{\|\zeta\|+1}(f) \underset{n\to\infty}{\to} 0,\end{aligned}$$

where we used the Cauchy inequalities for the current holomorphy type in the last inequality.

Since, $\delta_\zeta \circ AB$ is a surjective continuous map, then no orbit of T can be dense in $H_b(E)$. Thus, T is not hypercyclic. □

The last case it remains to be shown is when $\zeta \notin E''$. We will restrict ourselves to the case $\|\lambda\|_\infty = 1$. Note that if $\|\lambda\|_\infty < 1$, then $\zeta \in E$, thus T is not hypercyclic. If $\|\lambda\|_\infty > 1$, then the inequality (5) for the operator in $H_{b\mathfrak{A}}(E)$ is not useful for us, because we are not able to prove that ϕ is runaway. If we restrict to $\|\lambda\|_\infty = 1$, we can prove that the operator is mixing in $H_{b\mathfrak{A}}(E)$. Furthermore, if $\mathfrak{A}$ is the sequence of ideals of approximable polynomials, that is, when $H_{b\mathfrak{A}}(E) = H_{bc}(E)$, the space of entire functions of compact bounded type, we can dispense the condition on $\|\lambda\|_\infty$. This will be proved at the end of this section.

Lemma 3.7. Let E be a Banach space with a 1-unconditional shrinking basis, $(e_s)_{s\in\mathbb{N}}$. Let $\mathfrak{A}$ be a multiplicative holomorphy type with constants as in (2), such that the finite type polynomials are dense in each $\mathfrak{A}_k(E)$. Let $T : H_{b\mathfrak{A}}(E) \to H_{b\mathfrak{A}}(E)$ be defined by $Tf = M_\lambda \circ \tau_b \circ D^\alpha(f)$, and suppose that $\|\lambda\|_\infty = 1$ and that $\zeta \notin E''$. Then T is a mixing operator.

Proof. By Lemma's 3.3 and 3.4, it remains to prove the case $|\lambda^\alpha| < 1$.

Just like in the proof of Lemma 3.5, we fix a pair of open sets U and V in $H_{b\mathfrak{A}}(E)$. We will show the existence of a positive integer k_0 for which $T^k U \cap V \neq \emptyset$, for all $k \geq k_0$. Without loss of generality we can suppose that

$$U = \{h \in H_{b\mathfrak{A}}(E) \text{ such that } p_r^0(h - f) < \delta\} \quad \text{and} \quad V = \{h \in H_{b\mathfrak{A}}(E) \text{ such that } p_r^0(h - g) < \delta\},$$

for f, $g \in H_{b\mathfrak{A}}(E)$ and r, δ positive numbers. Since, E has a shrinking basis, by Lemma 2.8, we can assume that f is a finite linear combination of monomials. Define an inverse for T over the span of the monomials by integrating each monomial and denote it by S.

Applying (5) several times, with $\varepsilon = 2^{-j}$ at each step we get that for all $x \in E$

$$p_r^x(T^k f) \leq C(k, \alpha) p_{r+1}^{\phi^k(x)}(f).$$

Thus,

$$p_r^0(T^k q - f) = p_r^0(T^k(q - S^k f)) \leq C(k, \alpha) p_{r+1}^{\phi^k(0)}(q - S^k f).$$

It is enough to show that the sequence $\phi^k(0)$ is not bounded, because in that case, there exists some $k_0 \in \mathbb{N}$ such that the balls $B(0, 3r+1)$ and $B(\phi^k(0), 3(r+1)+1)$ are disjoint for all $k \geq k_0$. By an application of Theorem 2.15, it follows that $T^k U \cap V \neq \emptyset$ for all $k \geq k_0$.

A simple calculation shows that

$$\phi^k(0) = \sum_{j \in \mathbb{N}} b_j \frac{\lambda_j^k - 1}{\lambda_j - 1} e_j,$$

if $\lambda_j \neq 1$, and $\phi^k(0)_j = 0$ if $\lambda_j = 1$ (recall that we are assuming here that there are no translations, so if $\lambda_j = 1$ then $b_j = 0$).

Note that we can decompose $\mathbb{N} = N_1 \cup N_2$, in two disjoint subsets with

$$N_1 = \{n \in \mathbb{N} : |\lambda_n| = 1\} \text{ and } N_2 = \{n \in \mathbb{N} : |\lambda_n| < 1\}.$$

Define then for $i = 1, 2$ the vector ζ^i with $\zeta_n^i = \zeta_n$ for $n \in N_i$ and $\zeta_n^i = 0$ for $n \notin N_i$. Note that $\zeta = \zeta^1 + \zeta^2$.

We will divide the proof in two cases. First we will prove that the sequence $\phi^k(0)$ is not bounded if $\zeta^1 \notin E''$, and then we will do so if $\zeta^2 \notin E''$.

Suppose first that $\zeta^1 \notin E''$. Denote by $|||z||| = \sup_k \left\| \sum_{i=1}^k z_i e_i \right\|$, which is an equivalent norm in E. Suppose that there exists some positive constant C such that $\left|\left|\left|\phi^k(0)\right|\right|\right| \leq C$ for all $k \in \mathbb{N}$. Then we get that, for every N,

$$\frac{1}{N} \sum_{j=1}^N \left|\left|\left|\phi^j(0)\right|\right|\right| \leq C.$$

We will show that this leads to a contradiction. Since $\zeta^1 \notin E''$, let $A \in \mathbb{N}$ be a finite subset on N_1 such that $\lambda_n \neq 1$ if $n \in A$ and such that

$$\left\| \sum_{l \in A} \frac{b_l}{\lambda_l - 1} e_l \right\| \geq 2C.$$

Since E has a 1-unconditional basis, we get that

$$\begin{aligned}\frac{1}{N}\sum_{j=1}^{N}|||\phi^j(0)||| \geq \frac{1}{N}\sum_{j=1}^{N}\left\|\sum_{l\in\mathbb{N}}(\lambda_l^j-1)\frac{b_l}{\lambda_l-1}e_l\right\| &\geq \frac{1}{N}\sum_{j=1}^{N}\left\|\sum_{l\in A}(\lambda_l^j-1)\frac{b_l}{\lambda_l-1}e_l\right\| \\ &\geq \left\|\frac{1}{N}\sum_{j=1}^{N}\sum_{l\in A}(\lambda_l^j-1)\frac{b_l}{\lambda_l-1}e_l\right\| \\ &= \left\|\sum_{l\in A}\frac{b_l}{\lambda_l-1}e_l\left[\frac{1}{N}\sum_{j=1}^{N}(\lambda_l^j-1)\right]\right\|\end{aligned}$$

Since $|\lambda_l|=1$ and $\lambda_l\neq 1$ for all $l\in N_1$, we can write $\lambda_l=e^{i\rho_l}$. Thus, if $l\in A$, we get that, for every N,

$$\begin{aligned}\frac{1}{N}\left|\sum_{j=1}^{N}(\lambda_l^j-1)\right| = \left|\left(\frac{1}{N}\sum_{j=1}^{N}\lambda_l^j\right)-1\right| &= \left|\frac{1}{N}\frac{e^{i(N+1)\rho_l}-e^{2i\rho_l}}{e^{i\rho_l}-1}-1\right| \\ &\geq 1-\frac{1}{N}\left|\frac{e^{i(N+1)\rho_l}-e^{2i\rho_l}}{e^{i\rho_l}-1}\right|\end{aligned}$$

Now, given $\eta>0$, we can fix $K\in\mathbb{N}$ such that

$$\frac{1}{K}\left|\frac{e^{i(K+1)\rho_l}-e^{2i\rho_l}}{e^{i\rho_l}-1}\right| \leq \frac{2}{K\min_{l\in A}|e^{i\rho_l}-1|}\leq\eta.$$

Finally, we get that for $l\in A$

$$\frac{1}{K}\left|\sum_{j=1}^{K}(\lambda_l^j-1)\right|\geq 1-\eta,$$

which means that

$$\begin{aligned}\frac{1}{K}\sum_{j=1}^{K}|||\phi^j(0)||| &\geq \left\|\sum_{l\in A}\frac{b_l}{\lambda_l-1}e_l\left[\frac{1}{K}\sum_{j=1}^{K}(\lambda_l^j-1)\right]\right\| \\ &\geq \left\|\sum_{l\in A}(1-\eta)\frac{b_l}{\lambda_l-1}e_l\right\| \\ &> (1-\eta)2C.\end{aligned}$$

It follows that the sequence $\phi^k(0)$ is not bounded.

Now we assume that $\zeta_2\notin E''$. If $j\in N_2$, we have that $|\lambda_j|<1$, which implies that

$$\lim_{k\to\infty}\phi^k(0)_j=\lim_{k\to\infty}b_j\frac{\lambda_j^k-1}{\lambda_j-1}=\frac{b_j}{1-\lambda_j}=\zeta_j^2.$$

Suppose that $\phi^k(0)$ is bounded. It follows that $\phi^k(0)$ has a w^*-accumulation point $z \in E''$ and that

$$\lim_{k\to\infty} \phi^k(0)_j = \zeta_j^2 = z_j,$$

for all $j \in N_2$. It follows that $\zeta^2 \in E''$, which is a contradiction. This proves that the sequence $\phi^k(0)$ is not bounded, hence the operator T is mixing as we wanted to prove. $\square$

3.1. **Holomorphic functions of compact bounded type.** In this section we deal with the case in which $\mathfrak{A} = \mathcal{A}$, is the sequence of ideals of approximable polynomials. Then $H_{b\mathcal{A}}(E)$ is the space $H_{bc}(E)$ of entire functions on E of compact type that are bounded on bounded subsets of E. The space $H_{bc}(E)$ is endowed with the topology of uniform convergence on bounded sets of E. Hence, we consider the following family of seminorms that generates the topology of this space. Given a bounded set $A \subset E$ and $f \in H_{bc}(E)$, we define

$$p_A(f) = \sup_{z\in A} |f(z)|.$$

Our objective is to prove the following strengthen version of Theorem 3.1, where in the statements (b) and (c) of our main theorem we may drop the condition $\|\lambda\|_\infty = 1$, thus, completely characterizing the hypercyclicity of T. We will just point out the changes needed to prove this case.

As we mentioned previously, $\mathcal{A}$ is a multiplicative holomorphy type in which the finite type polynomials are dense in each $\mathcal{A}_k(E)$.

Theorem 3.8. Let E be a Banach space with a 1-unconditional shrinking basis, $(e_s)_{s\in\mathbb{N}}$. Let T be the operator on $H_{bc}(E)$, defined by $Tf(z) = M_\lambda \circ \tau_b \circ D^\alpha f(z)$, with $\alpha \neq 0$ and $\lambda_i \neq 0$ for all $i \in \mathbb{N}$. Then,

a) If $|\lambda^\alpha| \geq 1$ then T is strongly mixing in the Gaussian sense.

b) If for some $i \in \mathbb{N}$ we have that $b_i \neq 0$ and $\lambda_i = 1$, then T is mixing.

c) If $\zeta := (b_1/(1-\lambda_1), b_2/(1-\lambda_2), b_3/(1-\lambda_3), \dots) \notin E''$, then T is mixing.

d) In any other case, T is not hypercyclic.

The key point to prove this new statements is that under this assumptions the affine symbol ϕ will result to be runaway. Then, applying Theorem 2.15 we will be able to prove that the operator is mixing. During this section E will denote a Banach space with separable dual and suppose that $(e_s)_{s\in\mathbb{N}}$ is a 1-unconditional shrinking basis. In order to prove that the operator T is mixing on $H_{bc}(E)$ we need to give bounds for $p_A(D^\alpha f)$ in terms of $p_A(f)$, eventually by enlarging if necessary the set A. For this we will assume that the space E is of the form $\mathbb{C}^N \times F$, and that α only have nonzero coordinates in corresponding to the coordinates of $\mathbb{C}^N$.

Remark 3.9. Let $A = A_1 \times A'$ be a bounded subset of $E = \mathbb{C}^N \times F$ and suppose that $\alpha_i = 0$ for every $i > N$. If $f \in H_{bc}(E)$ and $z = (z_1, \dots, z_N, z') \in E$, then

$$D^{\alpha}f(z_1,\dots,z_N,z') = \frac{\alpha!}{(2\pi i)^N}\int_{|w_1-z_1|=r_1}\cdots\int_{|w_N-z_N|=r_N}\frac{f(w_1,\dots,w_N,z')}{\prod_{i=1}^N(w_i-z_i)^{\alpha_i+1}}\,dw_1\dots dw_N.$$

Therefore, we can estimate the seminorm of $D^{\alpha}f$ over $A = B(z_1,r_1)\times\cdots\times B(z_N,r_N)\times A'$, where $B(z_j,r_j)$ denotes the closed disk of center $z_j\in\mathbb{C}$ and radius r_j. Fix positive real numbers $\varepsilon_1,\dots,\varepsilon_N$, then

$$p_A(D^{\alpha}f) \le \frac{\alpha!}{(2\pi)^N}\frac{p_{(A_1+\varepsilon,A')}(f)}{\varepsilon_1^{\alpha_1+1}\dots\varepsilon_N^{\alpha_N+1}}. \tag{7}$$

The case b) follows the lines of the case b) of [36, Theorem 3.4]. Actually the same proof remains valid adapting the bounded sets to this case. To prove the case c) we proceed in a similar way to the proof of it counterpart on Theorem 3.1. We can decompose the hole space E in two subspaces corresponding to the different sizes of the modulus of λ_i. Decompose $\mathbb{N}=N_1\cup N_2$, into two disjoint subsets with

$$N_1=\{n\in\mathbb{N}:|\lambda_n|\le 1\}\ \text{ and }\ N_2=\{n\in\mathbb{N}:|\lambda_n|>1\}.$$

We have that $E=E(N_1)+E(N_2)$. Define for i, $i=1,2$ the vector ζ^i with $\zeta^i_n=\zeta_n$ for $n\in N_i$ and $\zeta^i_n=0$ for $n\notin N_i$. Note that $\zeta=\zeta^1+\zeta^2$. If $\zeta^1\notin E''$, then following the lines of the proof of part c) in Theorem 3.1, we can conclude that ϕ is runaway, so that the operator $C_\phi\circ D^\alpha$ is mixing. Otherwise, if $\zeta^2\notin E''$ and since $|\lambda_i|>1$ for every $i\in N_2$, we can consider $\phi_2^{-1}:E(N_2)\to E(N_2)$. It is easy to see that ζ^2 is the fixed point of ϕ_2 and that $\phi_2(z)=\frac{1}{\lambda}(z-b)$. Since, $|\lambda_i|>1$ for every $i\in N_2$, we may again follow the proof of part c) of Theorem 3.1 to conclude that ϕ_2 is runaway. Now, since the topology on $H_{bc}(E)$ is the topology of uniform convergence on bounded sets, we get that ϕ is runaway and thus $C_\phi\circ D^\alpha$ is mixing by Theorem 2.15.

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