Comment on "Modified quantum-speed-limit bounds for open quantum dynamics in quantum channels"

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In a recent paper [Phys. Rev. A **95**, 052118 (2017)], the authors claim that our criticism, in Phys. Rev. A **94**, 052125 (2016), to some quantum speed limit bounds for open quantum dynamics that appeared recently in literature are invalid. According to the authors, the problem with our analysis would be generated by an artifact of the finite-precision numerical calculations. We analytically show here that it is not possible to have any inconsistency associated with the numerical precision of calculations. Therefore, our criticism of the quantum speed limit bounds continues to be valid.

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The essence of the quantum speed limit (QSL) theory consists in the estimation of the minimal time of evolution between an initial state and any state, achievable by the dynamics, that has a fixed fidelity value with respect to the chosen initial state. In principle, this estimation should be possible to calculate with only some information about the quantum dynamics of the system, that is, without knowing the whole evolution. Several expressions for this minimal time of evolution are available in the literature corresponding to generic unitary or nonunitary quantum dynamics. In Ref. [1], we show that the quantum speed limit (QSL) bounds $\tau_t^{\text{op,tr,HS}}$ presented in Ref. [2], τ_t^{quant} presented in Ref. [3], and τ_t^{av}, constructed by us and inspired by the previous ones but using quantum Fisher information concepts, do not cleave to the essence of the QSL theory.

In order to show our statement, we have tested the prediction for the minimal time of evolution given by the expressions τ_t^x ($x \equiv \text{op},\text{tr},\text{HS},\text{quant},\text{av}$) in a particular open quantum evolution for a qubit. We have chosen the following conditions: (i) the initial state, $\hat{\rho}_0$, is in the equator of Bloch's sphere and (ii) we fixed the value of the fidelity $F_B = \sqrt{2}/2$ between the initial state and any final state achievable by the dynamics. It happens that for the chosen dynamics there is only one state with a fidelity $F_B = \sqrt{2}/2$ with respect to an initial condition at the equator of the Bloch's sphere. This state is an asymptotic state of the dynamics, so the actual time of evolution to reach it is infinite. The asymptotic state is $\hat{\rho}_\infty = |z,-\rangle\langle z,-|$, which corresponds to the south pole of the Bloch's sphere. So the QSL bounds for the conditions specified are denoted by τ_∞^x.

It is easy to see analytically that all τ_∞^x diverge for the conditions specified. All these bounds have the structure

$$t \geqslant \tau_t^x = \frac{G_x(\hat{\rho}_0,\hat{\rho}_t)}{\mathcal{V}_t^x}, \tag{1}$$

where $x \equiv \text{op},\text{tr},\text{HS},\text{quant},\text{av}$, $G_{\text{op, tr, hs}}(\hat{\rho}_0,\hat{\rho}_t) = \sin^2\{\arccos[F_B(\hat{\rho}_0,\hat{\rho}_t)]\}$, $G_{\text{quant}}(\hat{\rho}_0,\hat{\rho}_t) = \sqrt{Q(\hat{\rho}_0,\hat{\rho}_t)/2}$, and $G_{av}(\hat{\rho}_0,\hat{\rho}_t) = \arccos[F_B(\hat{\rho}_0,\hat{\rho}_t)]$, with $F_B(\hat{\rho}_0,\hat{\rho}_t)$ being the Bures fidelity and $Q(\hat{\rho}_0,\hat{\rho}_t) = 2\|[\hat{\rho}_0,\hat{\rho}_t]\|_{\text{HS}}^2$ ($\|\ldots\|_{\text{HS}}$ is the Hilbert-Schmidt norm). The average velocities are defined as

$$\mathcal{V}_t^x = \frac{1}{t}\int_0^t f_x(t')dt', \tag{2}$$

where t is the actual time of evolution and $f_{\text{op,tr,HS}}(t) = \|\dot{\hat{\rho}}_t\|_{\text{op,tr,HS}}$, $f_{\text{quant}} = \|[\hat{\rho}_0,\dot{\hat{\rho}}_t]\|_{\text{HS}}$, and $f_{\text{av}}(t) \equiv \sqrt{\mathcal{F}_Q(t)/4}$. Here, we denoted $\|\ldots\|_{\text{op}}$ and $\|\ldots\|_{\text{tr}}$ as the operator norm and trace norm respectively and $\mathcal{F}_Q$ is the quantum Fisher information. Whenever the open quantum evolution has an asymptotic stationary state, $\hat{\rho}_\infty$, we have $\mathcal{V}_\infty^x \equiv \lim_{t\to+\infty}\mathcal{V}_t^x = 0$ (see at the end of the Comment). Since $G_x(\hat{\rho}_0,\ \hat{\rho}_\infty)$ is finite, then $\tau_\infty^x \equiv \lim_{t\to\infty}\tau_t^x = +\infty$. Therefore, the bounds τ_t^x are continuous, increasing function of the time t, which goes to infinity as the actual evolution time. This is an analytical demonstration that the behavior of the expressions τ_t^x as t approaches to infinite is dictated by the fact that the average velocities $\mathcal{V}_t^x$ go to zero while the quantities $G_x(\hat{\rho}_0,\hat{\rho}_t)$ remain finite all the way up to $t = +\infty$. There is no numerical precision involved in this result.

The previous analysis shows that all the τ_∞^x are inconsistent estimates of the minimum time of evolution for the conditions specified because the minimum time of evolution is not the actual time of evolution unless the path in the Hilbert space of density operators is a geodesic (this is not the case in our example). In Ref. [1], we confirm that the origin of the inconsistency is the fact that all the QSL bounds, τ_t^x, depend explicitly of the actual evolution time. In our example, the behavior of τ_t^x as a function of t is identical to the actual evolution time, that is, it grows indefinitely as the asymptotic fidelity $F_B = \sqrt{2}/2$ is reached (see Fig. 3 in our paper [1] and Fig. 1 in Ref. [4]).

The authors of Ref. [4] criticize our conclusion, arguing that the inconsistency arises "... as soon as the limit of resolution of a calculation program is achieved" (p. 052118-1). The problem with their conclusion is that they misunderstood the inconsistency we found in the estimation of the minimum time of evolution given by the QSL bounds τ_t^x. In Ref. [4], it is argued that while τ_t^x grows indefinitely as $t \to \infty$, a state $\hat{\rho}_{\tau^{\rm cri}}$ is achieved, at a finite time $\tau^{\rm cri}$,[1] which is indistinguishable from the asymptotic state $\hat{\rho}_\infty$, due to the finite precision of the computer machine used to calculate the evolved states.[2] Then the authors saw an inconsistency once it is attributed that finite estimates of the minimal time of evolution of the QSL theory should appear in relation to the time $\tau^{\rm cri}$ where the state $\hat{\rho}_{\tau^{\rm cri}}$ is almost indistinguishable from the stationary state $\hat{\rho}_\infty$. They said that "this exactly causes the inconsistent estimates that the final state is reached at a finite time but the QSL bound grows indefinitely" (p. 052118-3). We did not state or suggest in Ref. [1] that the minimal time of evolution could be associated with some $\tau^{\rm cri}$. To attribute some relation between the minimal time of evolution to a time associated with the precision of the machine used to calculate the evolved states is an absolute physical nonsense, so the "inconsistency" pointed out in Ref. [4] does not exist.

The confusion of the statements in Ref. [4] is based on the misunderstanding of the origin of the concept of QSL bound. This bound does not arise because a state $\hat{\rho}_{\tau^{\rm cri}}$, almost indistinguishable from $\hat{\rho}_\infty$, should be achieved in some finite time $\tau^{\rm cri}$. The QSL bound has a profound quantum mechanical origin and has to do with the fact that for the evolved states to depart from an initial state to some fixed amount of distinguishability takes at least some finite time. This is due to the fact that the instant speed of evolution, $\sqrt{\mathcal{F}_Q(t')/4}$, in the path of the evolved states in the Hilbert space of density operators is finite. The finite time that should be the minimal value in the evolution is known as the QSL bound between an initial state and any state with a fixed fidelity value.

We showed in Ref. [1] that the QSL bound that casts exactly this idea is the time $\tau^{\rm min}$ that was completely ignored in Ref. [4]. Indeed, in the example presented, this time is calculated as

$$\frac{\pi}{2} = \arccos\left[F_B(\hat{\rho}_0,\hat{\rho}_\infty)\right] = \int_0^{\tau_\infty^{\rm min}} \sqrt{\mathcal{F}_Q(t')/4}\, dt'. \tag{3}$$

In this way, $\tau_\infty^{\rm min}$ corresponds to the time the qubit takes to travel through the actual evolution path, with the instant velocity $\sqrt{\mathcal{F}_Q(t')/4}$, the same distance as the geodesic length $\frac{\pi}{2} = \arccos\left[F_B(\hat{\rho}_0,\hat{\rho}_\infty)\right]$. It is important to note that, generically, the actual path of evolution is not a geodesic path. The calculation in Eq. (3) means that the minimal time to reach $\hat{\rho}_\infty$ from $\hat{\rho}_0$ is the time the system takes to traverse, through the actual path of evolution with velocity $\sqrt{\mathcal{F}_Q(t')/4}$, the shortest distance between $\hat{\rho}_0$ and $\hat{\rho}_\infty$, given by the geodesic length.

As we analytically demonstrated earlier, all the QSL bounds τ_t^x, in our example, diverge when $t \to \infty$. So, they cannot give a finite estimate of the minimal time of evolution, within the conditions specified. For this reason, these bounds do not cleave to the essence of the QSL theory. In Ref. [4], there is an attempt to fix the behavior of τ_t^x as $t \to \infty$ in order to introduce some plateau that could correspond to the finite estimation of the minimal time of evolution. Here, we emphasize that these modifications lack any physical sense. This is because the authors define the modified QSL bounds as

$$t \geqslant \tilde{\tau}_t^x = \begin{cases} \frac{G_x(\hat{\rho}_0,\hat{\rho}_t)}{\mathcal{V}_t^x} & \text{if } t < \tau^{\rm cri} \\ \frac{G_x(\hat{\rho}_0,\hat{\rho}_{\tau^{\rm cri}})}{\tilde{\mathcal{V}}^x_{\tau^{\rm cri}}} & \text{if } t \geqslant \tau^{\rm cri} \end{cases}, \tag{4}$$

where the new average velocities are defined as

$$\tilde{\mathcal{V}}^x_{\tau^{\rm cri}} = \frac{1}{\tau^{\rm cri}} \int_0^{\tau^{\rm cri}} f_x(t')dt' \,. \tag{5}$$

Here, $\tau^{\rm cri}$ is an arbitrary, externally fixed time that the authors associate to the finite numerical precision of calculation, for example, of the fidelity of the initial and the evolved state. This implies that the QSL bound $\tilde{\tau}_t^x$ could have an arbitrary value depending on the machine precision, which is clearly absurd.

Appendix. In order to see that $\mathcal{V}_\infty^x = 0$, it is enough to show that $\int_0^\infty f_x(t')dt'$ is finite. For the cases $x = \text{op,tr,HS}$, due to the fact that $\|\dot{\hat{\rho}}_t\|_{\rm op} \leqslant \|\dot{\hat{\rho}}_t\|_{\rm HS} \leqslant \|\dot{\hat{\rho}}_t\|_{\rm tr}$, with $\dot{\hat{\rho}}_t$ being the tangent vector in the evolution path on the space of density operators [5], we have

$$\int_0^\infty \|\dot{\hat{\rho}}_t\|_{\rm op} dt \leqslant \int_0^\infty \|\dot{\hat{\rho}}_t\|_{\rm HS} dt \leqslant \int_0^\infty \|\dot{\hat{\rho}}_t\|_{\rm tr} dt. \tag{6}$$

However, the integral $\int_0^\infty \|\dot{\hat{\rho}}_t\|_{\rm tr} dt'$ is the length of the path connecting $\hat{\rho}_0$ and $\hat{\rho}_\infty$, measured by the trace norm, which is clearly finite. When $x = \text{quant}$, we have $\|[\hat{\rho}_0,\dot{\hat{\rho}}_t]\|_{\rm HS} \leqslant 2\|\hat{\rho}_0\dot{\hat{\rho}}_t\| \leqslant \|\dot{\hat{\rho}}_t\|_{\rm HS}$, where we use the triangle and Cauchy-Schwarz inequalities. Therefore, in this case we have $\int_0^\infty \|[\hat{\rho}_0,\dot{\hat{\rho}}_t]\|_{\rm HS} dt \leqslant \int_0^\infty \|\dot{\hat{\rho}}_t\|_{\rm HS} \leqslant \int_0^\infty \|\dot{\hat{\rho}}_t\|_{\rm tr} dt$. For the case $x = \text{av}$, the integral $\int_0^\infty \sqrt{\mathcal{F}_Q(t')/4} dt'$ is the finite length of the actual path of evolution between the initial state and the asymptotic stationary one measured by Bures length [6].

[1] See p. 3, second column in Ref. [4].

[2] For this conclusion, the authors invoked a numerical calculation of the decoherence function or the trace distance, between the evolved state $\hat{\rho}_t$ and the asymptotic state $\hat{\rho}_\infty$, which saturate at very small value, different from zero, from a time $\tau^{\rm cri}$ onward.

[1] N. Mirkin, F. Toscano, and D. A. Wisniacki, Phys. Rev. A **94**, 052125 (2016).

[2] S. Deffner and E. Lutz, Phys. Rev. Lett. **111**, 010402 (2013).

[3] J. Jing, L.-A. Wu, and A. del Campo, Sci. Rep. **6**, 38149 (2016), arXiv:1510.01106.

[4] X. Liu, W. Wu, and C. Wang, Phys. Rev. A **95**, 052118 (2017).

[5] S. Braunstein, C. Caves, and G. Milburn, Ann. Phys. **247**, 135 (1996).

[6] M. M. Taddei, B. M. Escher, L. Davidovich, and R. L. deMatos Filho, Phys. Rev. Lett. **110**, 050402 (2013).