A note on the star order in Hilbert spaces

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Abstract

We study the star order on the algebra $L(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$. We present a new interpretation of this order which allows to generalize to this setting many known results for matrices: functional calculus, semi lattice properties, shorted operators and orthogonal decompositions. We also show several properties for general Hilbert spaces regarding the star order and its relationship with the functional calculus and the polar decomposition, which were unknown even in the finite dimensional setting. We also study of the existence of strong limits for star-monotone sequences and nets.

Contents

1 Introduction

Given two $n \times n$ complex matrices A and B, Hestenes introduced in [9] the concept of $\star$-orthogonality, defined by the equations $A^*B = 0$ and $AB^* = 0$, where A^* (resp. B^*) denotes the transpose and component-wise conjugate of A (resp. B). In the same paper he defined and discussed the relation defined by $A \sim B$ if

$$A^*A = B^*A \quad \text{and} \quad AA^* = AB^* .$$

Later on, Drazin proved [5] that this relation on the set of square matrices, and even more generally in semigroups with involution, is in fact a partial ordering. This order, that we shall denote $\stackrel{\star}{\leq}$, is nowadays called star order (or $\star$-order). Although this order can be generalized to much more general setting, it has been studied specially in the space of complex matrices.

In this paper, we study the $\star$-order on the algebra $L(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$. Since many of the usual techniques used in finite dimensional spaces (as pseudoinverses or singular value decompositions) are not longer available for general Hilbert spaces, we introduce new techniques which allow us to show that almost all the known properties which hold for matrices can be generalized to operators acting on a Hilbert space $\mathcal{H}$, and to obtain simpler proofs. Indeed, several results of the articles [1], [2], [3], [6], [7], [8], and [10] concerning the $\star$-order are contained in this note if we consider finite dimensional spaces. On the other hand, we show several properties for general Hilbert spaces which were unknown even in the finite dimensional setting, particularly those results concerning the polar decomposition and those regarding the relationship between the $\star$-order and the functional calculus. We also study some questions that only have sense in the infinite dimensional setting, such as the existence of strong limits for $\star$-monotone sequences and nets.

The article is organized as follows: section 2 starts with the relationship between the $\star$-order and certain sets of projections. Roughly speaking, given two operators $A, B \in L(\mathcal{H})$, the relation $A \stackrel{\star}{\leq} B$ says that A is "a piece" of B. Concretely, it can be proved that $A \stackrel{\star}{\leq} B$ if and only if $A = P_{\overline{R(A)}} B = B P_{\overline{R(A^*)}}$, where $P_{\mathcal{S}}$ denotes the orthogonal projection onto the closed subspace $\mathcal{S}$. Since the relation $A \stackrel{\star}{\leq} B$ means that both equalities must hold, it is not true that for every projection P the inequality $PB \stackrel{\star}{\leq} B$ holds, even if $R(P) \subseteq \overline{R(B)}$. The main aim in the first part of section 2 is to give a characterization of those projections P such that there is an operator $A \in L(\mathcal{H})$ so that $A \stackrel{\star}{\leq} B$ and $P = P_{\overline{R(A)}}$ (and by symmetry also the set of those Q such that $Q = P_{\overline{R(A^*)}}$). We prove that the mentioned set consists of those projections P that satisfy

$$R(P) \subseteq \overline{R(B)} \quad \text{and} \quad P \cdot BB^* = BB^* \cdot P \ . \tag{1}$$

Moreover, we show that if P_1 and P_2 satisfy these properties, then $P_1 B \stackrel{\star}{\leq} P_2 B \iff P_1 \leq P_2$ (see Theorem 2.7). These facts can be viewed as a reformulation of the definition of the $\star$-order, and their proofs are quite simple. However, these criteria can be applied to obtain very short proofs of several results throughout the paper, and they are particularly useful to work in the context of general Hilbert spaces.

Then, we study what functions preserve the $\star$-order when they are applied to operators using some convenient functional calculi. These results are motivated by the work of Baksalary, Hauke, X. Liu and S.Liu [1], where these authors prove that the polynomial functions of the form $p(x) = x^{2^k}$ (with $k \in \mathbb{N}$) preserve the $\star$-order, provided that some technical requirement on the ranges of the matrices involved holds. Under a slightly less restrictive additional hypothesis, we prove that any function, that can be applied to the operators considered, preserves the $\star$-order. We conclude section 2 by studying the relationship between the $\star$-order and the polar decomposition. One of the main tools used in the finite dimensional setting is the singular value decomposition, but it is not available for general operators on a Hilbert space. We prove that the polar decomposition behaves very well with respect to the $\star$-order and it may be a natural substitute.

Section 3 is devoted to the lattice properties of $L(\mathcal{H})$ endowed with the relation $\stackrel{\star}{\leq}$. In the finite dimensional setting these properties have been studied by Hartwig and Drazin in

[8]. A similar approach can be pursue using the results proved in section 2 regarding the $\star$-order and the polar decomposition. Nevertheless, we prefer to follow a different way that leads to the results directly. Given $B \in L(\mathcal{H})$, we prove that there exists an order preserving (in both directions) bijection between the sets

1. $\mathcal{L}_B := \{A \in L(\mathcal{H}) : \ A \overset{\star}{\leq} B\}$ with the $\star$-order;
2. $\mathcal{P}_B := \{P = P^2 = P^* \in L(\mathcal{H}) \ : \ R(P) \subseteq \overline{R(B)} \ \text{ and } \ PBB^* = BB^*P\}$ with the usual order (or with the $\star$-order because on this set both orders coincide).

This immediately implies the existence of a minimum between two operators $A, B \in L(\mathcal{H})$, which is denoted by $A \wedge^\star B$. Several properties of this minimum are analyzed. Section 3 concludes with the study of some limits theorems. In the finite dimensional setting it is not difficult to see that any star monotone sequence is constant from some n_0 on. However, in the infinite dimensional setting, this is not longer true, and a natural question is whether or not a bounded star monotone sequence converges. In this last subsection of section 3 we answer positively this question and we also study the behavior of the minimum with respect to monotone sequences. We prove that it behaves well with respect to monotone decreasing sequences, but we exhibit a counterexamples which shows that the results proved for monotone decreasing sequences are not valid for monotone increasing ones. In some way, this is one of the obstruction to pursue the generalization to the infinite dimensional setting based on the finite dimensional case and (star) monotone increasing sequences consisting of operators with finite dimensional ranges (see also Remark 3.7).

In section 4, we study the so-called star shorted operator. This notion was introduced for matrices by Mitra in [10]. Recall that given a matrix A and two subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathbb{C}^n$, the star shorted operator, denoted by $\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})$, is defined as the star maximum of the set of matrices

$$\overset{\star}{\mathcal{M}}(A, \mathcal{S}, \mathcal{T}) = \big\{D \overset{\star}{\leq} A \ , \ R(D) \subseteq \mathcal{T} \ \text{ y } \ R(D^*) \subseteq \mathcal{S}\big\}.$$

We prove that this maximum also exists in the infinite dimensional setting, where the subspaces are asked to be closed, and we characterized this maximum as the minimum $A \wedge^\star (P_{\mathcal{T}} A P_{\mathcal{S}})$. We also prove that the modulus of $\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})$ can be characterized as a star shorted of $|A|$ with respect to some suitable subspaces.

Finally, section 5 is devoted to study the relationship between the $\star$-order and the notion of $\star$-orthogonality. We show that if an operator A admits a suitable $\star$-orthogonal decomposition then the minima and the shorted operators of A can be also decomposed in terms of the elements of that decomposition.

2 The star order

Notations

Given a Hilbert space $\mathcal{H}$, $L(\mathcal{H})$ denotes the algebra of bounded linear operators on $\mathcal{H}$, $L_{sa}(\mathcal{H})$ the real vector space of self-adjoint operators, and $L(\mathcal{H})^+$ the cone of positive operators. For an operator $A \in L(\mathcal{H})$, $R(A)$ denotes the range or image of A, $\ker A$ the nullspace of A, $\sigma(A)$ the spectrum of A, A^* the adjoint of A, $|A| = (A^*A)^{1/2}$ the modulus of A, and $\|A\|$ the usual norm of A.

The word *projection* is used exclusively for orthogonal projections. We denote by $\mathcal{P}(\mathcal{H}) = \{P \in L(\mathcal{H}) : P = P^* = P^2\}$, the set of all projections in $L(\mathcal{H})$. Throughout this paper, $\mathcal{S} \sqsubseteq \mathcal{H}$ means that $\mathcal{S}$ is a closed subspace of $\mathcal{H}$, and $P_\mathcal{S} \in \mathcal{P}(\mathcal{H})$ denotes the unique projection onto $\mathcal{S}$. For every $A \in L(\mathcal{H})$, we denote by

$$P_A = P_{\overline{R(A)}} \in \mathcal{P}(\mathcal{H}) \quad \text{and} \quad Q_A = P_{A^*} = P_{\overline{R(A^*)}} = I - P_{\ker A} \in \mathcal{P}(\mathcal{H}) \ .$$

Definition 2.1. Given $A, B \in L(\mathcal{H})$, we say that A is lower or equal than B with respect to the $\star$-order, which is denote by $A \overset{\star}{\leq} B$, if

1. $A^*A = A^*B = B^*A$;
2. $AA^* = BA^* = AB^*$. △

Remark 2.2. Given $A, B \in L(\mathcal{H})$, it is easy to see that $A \overset{\star}{\leq} B \iff A^* \overset{\star}{\leq} B^*$. Also

$$A \overset{\star}{\leq} B \implies AA^* \overset{\star}{\leq} BB^* \ , \quad A^*A \overset{\star}{\leq} B^*B \quad \text{and} \quad B - A \overset{\star}{\leq} B \ . \qquad \triangle$$

2.1 Star order and projections

In this section we describe the relationship between the $\star$-order and some subsets of the Grassmann manifold, viewed as the set of projections. We begin with the following two well known characterizations of the equalities that define the $\star$-order. Since these characterizations will be very important in the sequel, and for a sake of completeness, we include a short proof valid in our general setting.

Proposition 2.3. Let $A, B \in L(\mathcal{H})$. Then

$$BA^* = AA^* \iff A = BQ_A \iff A = BQ \text{ for some } Q \in \mathcal{P}(\mathcal{H}) \ . \tag{2}$$

Similarly, it holds that

$$B^*A = A^*A \iff A = P_A B \iff A = PB \text{ for some } P \in \mathcal{P}(\mathcal{H}) \ . \tag{3}$$

Proof. Indeed, given $x \in \overline{R(A^*)}$, there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ contained in $R(A^*)$ such that $u_n \xrightarrow[n\to\infty]{\|\cdot\|} x$. If we assume that $BA^* = AA^*$, then $Bu = Au$ for every $u \in R(A^*)$. Therefore

$$Bx = \lim_{n\to\infty} Bu_n = \lim_{n\to\infty} Au_n = Ax \ .$$

On the other hand, both $Ax = BP_{A^*}\,x = 0$ for every $x \in R(A^*)^\perp = \ker A$.

Suppose now that $A = BQ$ for some $Q \in \mathcal{P}(\mathcal{H})$. Then

$$BA^* = B(BQ)^* = BQB^* = (BQ)(QB^*) = AA^* \ .$$

The proof of (3) is almost the same. It can also be obtained from (2). □

Corollary 2.4. Let $A, B \in L(\mathcal{H})$ such that $A \overset{\star}{\leq} B$. Then $R(A) \subseteq R(B)$ and $R(A^*) \subseteq R(B^*)$.

Corollary 2.5. Let $A, B \in L(\mathcal{H})$ such that $A \overset{\star}{\leq} B$ and $B^2 = B$. Then $A^2 = A$.

Proof. Indeed, $A^2 = P_A\, B\, B\, Q_A = P_A\, B\, Q_A = P_A\, A = A$. □

The statement of Proposition 2.3 implies the following alternative description of the $\star$-order: Given $A, B \in L(\mathcal{H})$, then

$$A \overset{\star}{\leq} B \quad \iff \quad A \ = \ P_A\, B \ = \ B\, Q_A \ . \tag{4}$$

This description suggests that the relation $A \overset{\star}{\leq} B$ means that A is "a piece" of B. Nevertheless, Eq. (4) does not say exactly which pieces of B (of the type $A = PB$ for $P \in \mathcal{P}(\mathcal{H})$) are $\star$-smaller than B. In other words, the above results do not characterize which projections P and Q satisfy that $P = P_A$ or $Q = Q_A$ for some $A \overset{\star}{\leq} B$. This characterization is the main goal of this subsection and the first step in that direction is the next lemma which gives a one sided description (in terms of the action of projections) of the relation $A \overset{\star}{\leq} B$.

Lemma 2.6. Let $B \in L(\mathcal{H})$.

1. If $P \in \mathcal{P}(\mathcal{H})$ and $R(P) \subseteq \overline{R(B)}$, then $PB \overset{\star}{\leq} B \iff P\,BB^* = BB^*\,P$.
2. If $Q \in \mathcal{P}(\mathcal{H})$ and $R(Q) \subseteq \overline{R(B^*)}$, then $BQ \overset{\star}{\leq} B \iff Q\,B^*B = B^*B\,Q$.

Proof. Denote by $A = PB$. If $A \overset{\star}{\leq} B$, then $PBB^* = AB^* = BA^* = BB^*P$. Conversely, the identity $A = PB$ implies that $B^*A = A^*A$, by Proposition 2.3. On the other hand, since P commutes with BB^*, it holds that

$$BA^* = BB^*P = PBB^*P = AA^* \ .$$

The proof of the second statement follows mutatis mutandis. □

Theorem 2.7. *Let $A, B, C \in L(\mathcal{H})$. Then it holds that*

$$\begin{aligned} A \overset{\star}{\leq} B \quad &\iff A = P_A\, B \ , \quad P_A \leq P_B \ \ \textit{and} \ \ P_A \cdot BB^* = BB^* \cdot P_A \\ &\iff A = B\, Q_A \ , \quad Q_A \leq Q_B \ \ \textit{and} \ \ Q_A \cdot B^*B = B^*B \cdot Q_A \ . \end{aligned} \tag{5}$$

Moreover, if both $A \overset{\star}{\leq} B$ and $C \overset{\star}{\leq} B$, then

$$A \overset{\star}{\leq} C \ \Leftrightarrow \ P_A \leq P_C \ \Leftrightarrow \ Q_A \leq Q_C \ \Leftrightarrow \ R(A) \subseteq R(C) \ \Leftrightarrow \ \ker C \subseteq \ker A \ . \tag{6}$$

Proof. Both implications $\Rightarrow$ of (5) follow from Proposition 2.3, Corollary 2.4 and Lema 2.6. The reverse implications follow from Lema 2.6.

Assume that $A \overset{\star}{\leq} B$ and $C \overset{\star}{\leq} B$. Then we have that $A = P_A\, B$ and $C = P_C\, B$. If $A \overset{\star}{\leq} C$, then Corollary 2.4 shows that all the other conditions of Eq. (6) hold. Conversely, suppose that $P_A \leq P_C$. Then $P_A\, C = P_A\, P_C\, B = P_A\, B = A$, and

$$P_A\, CC^* = P_A\, P_C\, BB^* = P_A\, BB^* = BB^* P_A = BB^* P_C\, P_A = CC^* P_A \ .$$

Therefore, by Lema 2.6, we can conclude that $A = P_A\, C \overset{\star}{\leq} C$. The proof of the other case $Q_A \leq Q_C \implies A \overset{\star}{\leq} C$ is almost the same. □

The main advantage of Theorem 2.7 over Proposition 2.3 or Eq. (4) is that each equivalence involves only one projection. The price we have had to pay for this simplification does not seem to be very high, at least in the applications of this result that we shall consider, because the commutation relation is not too complicated to check in those applications. On the other side, the set of projections of a "commutant" subálgebra of $L(\mathcal{H})$ has excelent properties. The following results are direct consequences of Theorem 2.7:

Corollary 2.8. Let $V \in L(\mathcal{H})$ a partial isometry. Denote by $P = P_V$. Then

$$W \overset{\star}{\leq} V \iff W = QV \quad \text{for some} \quad Q \in \mathcal{P}(\mathcal{H}) \quad \text{such that} \quad Q \leq P \ .$$

Every such W is also a partial isometry and all them are $\star$-ordered by the inclusion of their final (resp. initial) spaces. □

Corollary 2.9. *Given $A \in L(\mathcal{H})$ and $B \in L(\mathcal{H})^+$, if $A \overset{\star}{\leq} B$, then also $A \in L(\mathcal{H})^+$.*

Proof. It is easy to see that the commutant of $BB^* = B^2$ coincides with $\{B\}'$. □

A more detailed analysis of the relation between the $\star$-order and the polar decomposition will be done in subsection 2.3. The next example shows that the above corollary is not longer true if we replace positive by self-adjoint or normal

Example 2.10. Consider the matrices $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Using that $P_A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $BB^* = 2I$, it is easy to se that and $A \overset{\star}{\leq} B$. Observe that $B = B^*$ but A is neither self-adjoint nor normal. △

2.2 Star order and functional calculi

In this subsection we study the problem of finding out those function that are monotone with respect to the $\star$-order, when they are applied using one of the following functional calculi: The Riesz functional calculus for holomorphic maps, and the continuous functional calculus for normal operators. The reader who is not familiarized with these topics is refereed to the excellent book by Conway [4]. The key remark to obtain these results is the following statement, which is a direct consequence of Eq. (4):

Proposition 2.11. *Let $A, B \in L(\mathcal{H})$ such that $P_A = P_{A^*} = Q_A$. Then, in terms of the orthogonal decomposition $\mathcal{H} = R(P_A) \oplus \ker P_A$, we have that*

$$A \overset{\star}{\leq} B \iff A = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_{11} & 0 \\ 0 & B_{22} \end{bmatrix} \ . \tag{7}$$

In such case $AB = BA$ and $\sigma(A) \subseteq \sigma(B) \cup \{0\}$.

Proof. Let $P = P_A = P_{A^*} = Q_A$. By Eq. (4), $A \overset{\star}{\leq} B \iff A = BP = PB$, which is a reformulation of Eq. (7). □

The announced results involving the $\star$-order and the functional calculi follows directly from of the above 2×2 decomposition. Note that the type of functions considered in each proposition only depends on the class of operator considered and the functional calculus defined on them, there is no other restriction. It should be mentioned that, besides their applications, these propositions may be interesting by themselves because they provide two different generalizations of Theorem 4.1 of [1], where a similar result is proved for functions of the form $f(t) = t^{2^k}$ for $k \in \mathbb{N}$.

Proposition 2.12. *Let $A, B \in L(\mathcal{H})$ such that $P_A = P_{A^*}$. Let f be a complex analytic function defined in a neighborhood of $\{0\} \cup \sigma(A) \cup \sigma(B)$ such that $f(0) = 0$. Then*

$$A \overset{\star}{\leq} B \quad \Longrightarrow \quad f(A) \overset{\star}{\leq} f(B) \ .$$

Moreover, if f is also injective, then $A \overset{\star}{\leq} B \iff f(A) \overset{\star}{\leq} f(B)$.

Proof. Using Proposition 2.11, and taking 2×2 matrices with respect to the decomposition $\mathcal{H} = R(P_A) \oplus \ker P_A$, we have that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \overset{\star}{\leq} \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix} = B \implies f(A) = \begin{bmatrix} f(A_1) & 0 \\ 0 & 0 \end{bmatrix} \overset{\star}{\leq} \begin{bmatrix} f(A_1) & 0 \\ 0 & f(B_1) \end{bmatrix} = f(B) \, ,$$

because $f(0) = 0$ and $f(A_1 \oplus B_1) = f(A_1) \oplus f(B_1)$. □

With almost the same proof, we have the following result regarding the continuous functional calculus for normal operators:

Proposition 2.13. Let A and B be normal operators. Then, for every continuous function $f : \{0\} \cup \sigma(A) \cup \sigma(B) \to \mathbb{C}$ satisfying that $f(0) = 0$, it holds that

$$A \overset{\star}{\leq} B \quad \Longrightarrow \quad f(A) \overset{\star}{\leq} f(B) \ .$$

Moreover, if f is also injective, then $A \overset{\star}{\leq} B \iff f(A) \overset{\star}{\leq} f(B)$. □

The condition $R(A) = R(A^*)$ may seem too restrictive, but as we shall see in the next example taking from [1], without this condition the above results are false even for $f(x) = x^2$:

Example 2.14. Let $A, B \in \mathcal{M}_2(\mathbb{C})$ be the matrices of Example 2.10, that is

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \overset{\star}{\leq} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = B \ .$$

However, since $A^2 = A$ and $B = 2I$, it is clear that $A^2 \overset{\star}{\nleq} B^2$. △

To conclude this subsection we give an useful application of these results.

Corollary 2.15. *Let $A, B \in L(\mathcal{H})$ such that $A \overset{\star}{\leq} B$. Then $|A| \overset{\star}{\leq} |B|$ and $|A^*| \overset{\star}{\leq} |B^*|$.*

Proof. By Remark 2.2 we know that $A^*A \overset{\star}{\leq} B^*B$ and $AA^* \overset{\star}{\leq} BB^*$. Then, using Proposition 2.13 with the function $f(t) = t^{1/2}$, we obtain the desired result. □

2.3 Star order and the polar decomposition

One of the main tools used to study the $\star$-order in finite dimensional spaces is the singular value decomposition, which in general is not longer available in the infinite dimensional setting. This subsection is devoted to the study of the relationship between the $\star$-order and the polar decomposition, which could be seen as a natural substitute of the singular value decomposition in arbitrary Hilbert spaces.

NOTATION: Given $B \in L(\mathcal{H})$, we denote by $U_B \in L(\mathcal{H})$ the **unique** partial isometry such that $B = U_B\,|B|$ and $\ker U_B = R(|B|)^\perp = \ker|B| = \ker B$. We shall write that $B = U_B\,|B|$ is **the** polar decomposition of B. Observe that also $B = |B^*|U_B$.

Recall that, for every $B \in L(\mathcal{H})$, it holds that $R(|B|) = R(B^*)$ (without closures). Also $\overline{R(B^*)} = \ker B^\perp = \ker U_B{}^\perp = R(U_B^*)$. Therefore, in our notations,

$$P_{|B|} = Q_{|B|} = P_{B^*} = Q_B = Q_{U_B} \ . \tag{8}$$

Lemma 2.16. *Let $A, B \in L(\mathcal{H})$ such that $A \overset{\star}{\leq} B$. If $B = U|B|$ is* **a** *polar decomposition of B, then $A = U|A|$. In particular, $A = U_B|A|$.*

Proof. By Corollary 2.15, we have that $|A| \overset{\star}{\leq} |B|$. Then $|A| = |B|\,Q_{|A|}$ and, by Eq. (8), it holds that $U|A| = U|B|Q_{|A|} = BQ_A = A$. □

Theorem 2.17. *Let $A, B \in L(\mathcal{H})$. Then, the following statements are equivalent*

1. *$A \overset{\star}{\leq} B$;*
2. *$|A| \overset{\star}{\leq} |B|$ and $U_A \overset{\star}{\leq} U_B$.*

Proof. 1 ⇒ 2: By Corollary 2.15 we already know that $|A| \overset{\star}{\leq} |B|$. On the other hand, by Lemma 2.16, $U_A|A| = U_B|A|$. So U_A and U_B coincides on $R(|A|)$, and by continuity, on $\overline{R(|A|)} = R(Q_A) = R(Q_{U_A})$. Therefore $U_A = U_B\,P_{|A|} = U_B\,Q_A = U_B\,Q_{U_A}$. Also

$$|A| \overset{\star}{\leq} |B| \implies R(U_A^*) = R(Q_A) \subseteq R(Q_B) = R(U_B^*) \implies Q_{U_A} \leq Q_{U_B} \ .$$

So, by Corollary 2.8, we have that $U_A \overset{\star}{\leq} U_B$.

2 ⇒ 1: Denote by $Q = Q_A = Q_{|A|} = P_{|A|} = Q_{U_A}$. By Theorem 2.7, we have that

$$|A| \overset{\star}{\leq} |B| \implies |A| = |B|Q \ , \quad Q \leq Q_{|B|} \ \text{ and } \ QB^*B = Q|B|^2 = |B|^2Q = B^*BQ \ .$$

Then we get that $Q|B| = |B|Q = |A|$. Similarly, $U_A \overset{\star}{\leq} U_B \implies U_A = U_B\,Q$. Hence we have that $BQ = U_B|B|Q = U_BQ|A| = U_A|A| = A$, that $Q \leq Q_{|B|} = Q_B$ and $QB^*B = B^*BQ$. Therefore, by Theorem 2.7, $A \overset{\star}{\leq} B$. □

Remark 2.18. The same arguments used to prove 1 ⇒ 2 imply that, if $B = U|B|$ is a polar decomposition of B, then $U_B \overset{\star}{\leq} U$. Hence, in Theorem 2.17, the partial isometry U_B can be changed by any partial isometry U that can be used in the polar decomposition of B. △

3 Semi-lattice properties

This section is devoted to study the lower semi-lattice properties of $(L(\mathcal{H}), \overset{\star}{\leq})$. The key result to pursue these studies is Theorem 3.1.

3.1 Semi-lattice properties of $(L(\mathcal{H}), \overset{\star}{\leq})$

This study has already been done in the finite dimensional setting by Hartwig and Drazin in [8]. It is not difficult to see that, for example, the invertible operators are maximal elements with respect to the $\star$-order. So, it is enough to take two different invertible operators, say G_1 and G_2, to see that $L(\mathcal{H})$ can not be a lattice endow with $\overset{\star}{\leq}$. However, Hartwig and Drazin proved in [8] that the set of matrices endow with the $\star$-order is a lower semi-lattice, i.e., for every pair of matrices A and B, there exists the $\star$-maximum of the set

$$\{C \in L(\mathcal{H}) : \ C \overset{\star}{\leq} A \quad \text{and} \quad C \overset{\star}{\leq} B\} \ ,$$

They firstly prove the result for pairs of partial isometries and then reduce the general case to this particular case by using a smart trick based on the singular value decomposition. A similar approach can be done by replacing the singular value decomposition by the polar decomposition and using some of the results that have been proved in the previous section. However we prefer to pursue a direct approach based on the next useful reformulation of Theorem 2.7 that contains all we need to recover Hartwig-Drazin's result in our setting:

Theorem 3.1. Let $B \in L(\mathcal{H})$. Then, there is an order preserving (in both directions) bijection between the following ordered sets:

1. $\mathcal{L}_B := \{A \in L(\mathcal{H}) : \ A \overset{\star}{\leq} B\}$ with the $\star$-order;
2. $\mathcal{P}_B := \{P \in \mathcal{P}(\mathcal{H}) \ : \ R(P) \subseteq \overline{R(B)} \ \text{ and } \ PBB^* = BB^*P\}$ with the usual order (or with the $\star$-order because on this set both orders coincide).

This bijection is given by $\mathcal{L}_B \ni A \mapsto P_A$ and $\mathcal{P}_B \ni P \mapsto PB$. □

It is easy to see that $\mathcal{P}_B$ is a lattice, with respect to the usual order. Actually, $\mathcal{P}_B$ is the set of all projections of the von Neumann algebra $\mathcal{M}_B = P_B \ \{BB^*\}' \ P_B \subseteq L(R(P_B)\,)$, where $\{BB^*\}'$ is the commutant of BB^*.

Proposition 3.2. For every $B \in L(\mathcal{H})$ the set $\mathcal{L}_B$ is a lattice. Moreover, the ordered set $(L(\mathcal{H}), \overset{\star}{\leq})$ is a lower semi-lattice, i.e., for every pair $A, B \in L(\mathcal{H})$

$$\mathcal{L}(A, B) = \{C \in L(\mathcal{H}) : \ C \overset{\star}{\leq} A \quad \text{and} \quad C \overset{\star}{\leq} B\} \, . \tag{9}$$

has a $\star$-maximum, called the $\star$-minimum of A and B, and denoted by $A \wedge^\star B$.

Proof. The fact that $\mathcal{L}_B$ is a lattice follows from Theorem 3.1. Consider the set

$$\mathcal{P}(A, B) = \Big\{ Q \in \mathcal{P}(\mathcal{H}) \cap \{A^*A, B^*B\}' : R(Q) \subseteq \overline{R(A)} \cap \overline{R(B)} \cap \ker(A^* - B^*) \Big\} \ .$$

It is clear that $\mathcal{P}(A, B) \subseteq \mathcal{P}_A \cap \mathcal{P}_B$ and hence it is also a lattice. Given $Q \in \mathcal{P}(A, B)$, the condition

$$R(Q) \subseteq \ker(A^* - B^*) \iff A^*Q = B^*Q \iff QA = QB \ ,$$

so that $QA = QB \in \mathcal{L}(A, B)$. On the other hand, any $C \in \mathcal{L}(A, B)$ satisfies the condition $C = P_C\, A = P_C\, B$, so that $P_C \in \mathcal{P}(A, B)$. Therefore

$$A \wedge^\star B = \max \mathcal{L}(A, B) = PA = PB \ , \quad \text{where} \quad P = P_{A\wedge^\star B} = \max \mathcal{P}(A, B) \ . \qquad \square$$

Next, we state some properties of $\star$-minima whose proofs are straightforward:

Proposition 3.3. Let $A, B, C \in L(\mathcal{H})$. Then

1. $A \wedge^\star B = B \wedge^\star A$ and $(A \wedge^\star B)^* = A^* \wedge^\star B^*$;
2. $(A \wedge^\star B) \wedge C = A \wedge^\star (B \wedge C)$;
3. $(A \wedge^\star B)(A \wedge^\star B)^* \overset{\star}{\leq} AA^* \wedge^\star BB^*$ and $|A \wedge^\star B| \overset{\star}{\leq} |A| \wedge^\star |B|$;
4. If A or B are positive then $A \wedge^\star B$ is also positive. $\square$

Observe that both inequalities of item 3 can be strict. Indeed, take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $A \wedge^\star B = 0$ but $0 \neq AA^* = |A| \overset{\star}{\leq} BB^* = B$.

3.2 Some limit theorems.

In finite dimensional spaces it is not difficult to see, by using simple arguments of dimension, that any sequence of operators which is non-decreasing (resp. non-increasing) with respect to the $\star$-order is constant from some n on. In the infinite dimensional setting the situation is different, and a natural question is whether or not a star-monotone sequence converge. The following theorems provide a positive answer to this question. Recall that a sequence $\{A_n\}_{n\in\mathbb{N}}$ in $L(\mathcal{H})$ converges strongly to $A \in L(\mathcal{H})$, which is denoted by $A_n \xrightarrow[n\to\infty]{\text{S.O.T.}} A$, if $\|A_n\, x - Ax\| \xrightarrow[n\to\infty]{} 0$ for every $x \in \mathcal{H}$. It is well known that every sequence (or net) in $L_{sa}(\mathcal{H})$ which is bounded and monotone must converge strongly.

Proposition 3.4. *Let $\{A_n\}_{n\in\mathbb{N}}$ be sequence in $L(\mathcal{H})$ which is $\star$-non-increasing. Then, there exists $A \in L(\mathcal{H})$ such that $A_n \xrightarrow[n\to\infty]{S.O.T.} A$, and $A \overset{\star}{\leq} A_n$ for every $n \in \mathbb{N}$. Moreover, if $B \in L(\mathcal{H})$ satisfies that $B \overset{\star}{\leq} A_n$ for every $n \in \mathbb{N}$, then $B \overset{\star}{\leq} A$.*

Proof. Let $P_n = P_{A_n}$. Note that, by Theorem 2.7, for every $n \in \mathbb{N}$ it holds that

$$P_n \geq P_{n+1} \quad \text{and} \quad P_n(A_1A_1^*) = (A_1A_1^*)P_n \ .$$

Let $P = \inf\{P_n : n \in \mathbb{N}\}$, so that $P_n \xrightarrow[n\to\infty]{\text{S.O.T.}} P$. Then $A_n = P_nA_1 \xrightarrow[n\to\infty]{\text{S.O.T.}} P\, A_1 =: A$. Clearly P commutes with $A_1A_1^*$. Hence $A \overset{\star}{\leq} A_1$. Moreover, since $A_n \overset{\star}{\leq} A_1$ and $P \leq P_n$, then Theorem 2.7 assures that $A \overset{\star}{\leq} A_n$. Similarly, if $B \overset{\star}{\leq} A_n$ for every $n \in \mathbb{N}$, then $P_B \leq P_n$ for every $n \in \mathbb{N}$, and therefore $P_B \leq P$. As before, this implies that $B \overset{\star}{\leq} A$ because both $A, B \overset{\star}{\leq} A_1$. $\square$

In a similar fashion, we get the next proposition:

Proposition 3.5. Let $\{A_n\}_{n\in\mathbb{N}}$ be sequence in $L(\mathcal{H})$ which is $\star$-non-decreasing and $\star$-bounded from above some $B \in L(\mathcal{H})$. Then, there exists $A \in L(\mathcal{H})$ such that $A \overset{\star}{\leq} B$, $A_n \xrightarrow[n\to\infty]{\text{S.O.T.}} A$, and $A_n \overset{\star}{\leq} A$ for every $n \in \mathbb{N}$. □

Remark 3.6. Note that in Proposition 3.4, as well as in Proposition 3.5, the sequences can be replaced by nets and the results remain true with almost the same proof. △

Remark 3.7. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence as in Proposition 3.5 and let $A \in L(\mathcal{H})$ be the limit of this sequence. Then, it is not difficult to prove that $|A_n| \xrightarrow[n\to\infty]{\text{S.O.T.}} |A|$ and $|A_n^*| \xrightarrow[n\to\infty]{\text{S.O.T.}} |A^*|$. Observe that, if all the operators A_n have finite dimensional range, so do the operators $|A_n|$ and $|A_n^*|$. Then, simple computations (involving mainly Proposition 2.11) show that $|A|$ (resp. $|A^*|$) have to be diagonalizable, i.e, there exists an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of $|A|$ (resp. $|A^*|$). This in particular implies that there are operators in $L(\mathcal{H})$ that can not be reached using star non-decreasing sequences of finite range operators, which is one of the obstructions to pursue the generalization to infinite dimensional Hilbert spaces based in the results for matrices and monotone sequences. △

The next result shows that the $\star$-minimum behaves well with respect to $\star$-monotone decreasing sequences.

Proposition 3.8. *Let $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ be two sequences in $L(\mathcal{H})$ which are $\star$-non-increasing and so that $A_n \xrightarrow[n\to\infty]{S.O.T.} A$ and $B_n \xrightarrow[n\to\infty]{S.O.T.} B$, for some operators $A, B \in L(\mathcal{H})$. Then, $A_n \wedge^\star B_n \xrightarrow[n\to\infty]{S.O.T.} A \wedge^\star B$.*

Proof. We shall use Proposition 3.4 several times. Since $A_{n+1}\wedge^\star B_{n+1} \overset{\star}{\leq} A_n\wedge^\star B_n$ for every $n \in \mathbb{N}$, then there exists $L \in L(\mathcal{H})$ such that $A_n \wedge^\star B_n \xrightarrow[n\to\infty]{\text{S.O.T.}} L$. Observe that $L \overset{\star}{\leq} A_n \wedge^\star B_n \overset{\star}{\leq} A_n$ for every $n \in \mathbb{N}$. Then $L \overset{\star}{\leq} A$. Analogously, $L \overset{\star}{\leq} B$. On the other hand, let $C \in L(\mathcal{H})$ such that $C \overset{\star}{\leq} A$ and $C \overset{\star}{\leq} B$. As $C \overset{\star}{\leq} A \wedge^\star B \leq A_n \overset{\star}{\leq} B_n$ for every $n \in \mathbb{N}$, we get that $C \overset{\star}{\leq} L$, which completes the proof. □

As the next example shows that a similar result for $\star$-non-decreasing sequences is not true.

Example 3.9. Let $\{\mathcal{S}_n\}_{n\in\mathbb{N}}$ and $\{\mathcal{T}_n\}_{n\in\mathbb{N}}$ be two increasing sequences of closed subspaces of $\mathcal{H}$ such that $P_{\mathcal{S}_n} \xrightarrow[n\to\infty]{\text{S.O.T.}} I$ and $P_{\mathcal{T}_n} \xrightarrow[n\to\infty]{\text{S.O.T.}} I$ but $\mathcal{S}_n \cap \mathcal{T}_n = \{0\}$ for each $n \in \mathbb{N}$. Then both sequences $\{P_{\mathcal{S}_n}\}_{n\in\mathbb{N}}$ and $\{P_{\mathcal{T}_n}\}_{n\in\mathbb{N}}$ are $\star$-increasing. However, $P_{\mathcal{S}_n} \wedge^\star P_{\mathcal{T}_n} = 0$ for every $n \in \mathbb{N}$, and $I \wedge^\star I = I \neq 0$. △

4 The star shorted

In finite dimensional spaces the study of shorted operators related with the $\star$-order was carried out by Mitra [10]. The key tool used by Mitra was the singular value decomposition. In the infinite dimensional setting, this approach is only available for compact operators. So, in order to generalize Mitra's results to any operator on an arbitrary Hilbert space we need to develop a new approach. The main goal of this section is to characterize the star-shorted as a $\star$-minimum of two operators.

Theorem 4.1. *Let $A \in L(\mathcal{H})$ and let $\mathcal{S}, \mathcal{T} \sqsubseteq \mathcal{H}$. Then, the set of operators*

$$\overset{\star}{\mathcal{M}}(A, \mathcal{S}, \mathcal{T}) = \left\{ D \in L(\mathcal{H}) : \ D \overset{\star}{\leq} A \ , \ R(D) \subseteq \mathcal{T} \quad y \quad R(D^*) \subseteq \mathcal{S} \right\}$$

has a $\star$-maximum given by $S = A \ \wedge^{\star} \ \big(P_{\mathcal{T}} \, A \, P_{\mathcal{S}}\big)$.

The next technical lemma is part of the proof of Theorem 4.1, but we write it separately because we think that it could be interesting by itself.

Lemma 4.2. *Let $A, B \in L(\mathcal{H})$ such that $A \overset{\star}{\leq} B$ and $\mathcal{S}, \mathcal{T} \sqsubseteq \mathcal{H}$ two subspaces such that $R(A) \subseteq \mathcal{T}$ and $R(A^*) \subseteq \mathcal{S}$. Then $A \overset{\star}{\leq} P_{\mathcal{T}} B P_{\mathcal{S}}$.*

Proof. Note that $P_A (P_{\mathcal{T}} B P_{\mathcal{S}}) = P_A B P_{\mathcal{S}} = A P_{\mathcal{S}} = A$. Similarly, it holds that $(P_{\mathcal{T}} B P_{\mathcal{S}}) Q_A = A$. Then, using Proposition 2.3, we conclude that $A \overset{\star}{\leq} P_{\mathcal{T}} B P_{\mathcal{S}}$. □

Proof of Theorem 4.1: Let $S = A \ \wedge^{\star} \ \big(P_{\mathcal{T}} \, A \, P_{\mathcal{S}}\big)$. By Lemma 4.2, if $D \in \overset{\star}{\mathcal{M}}(A, \mathcal{S}, \mathcal{T})$ then $D \overset{\star}{\leq} P_{\mathcal{T}} \, A \, P_{\mathcal{S}}$ and $D \overset{\star}{\leq} A$. Thus, $D \overset{\star}{\leq} S$. Now, it is enough to prove that $S \in \overset{\star}{\mathcal{M}}(A, \mathcal{S}, \mathcal{T})$. Since $S \overset{\star}{\leq} P_{\mathcal{T}} \, A \, P_{\mathcal{S}}$, then

$$R(S) \subseteq R(P_{\mathcal{T}} \, A \, P_{\mathcal{S}}) \subseteq \mathcal{T} \quad \text{y} \ R(S^*) \subseteq R(P_{\mathcal{S}} \, A^* \, P_{\mathcal{T}}) \subseteq \mathcal{S} \ .$$

On the other hand, $S \overset{\star}{\leq} A$. So, $S \in \overset{\star}{\mathcal{M}}(A, \mathcal{S}, \mathcal{T})$. □

Definition 4.3. Given $A \in L(\mathcal{H})$ and $\mathcal{S}, \mathcal{T} \sqsubseteq \mathcal{H}$, the maximum of the set $\overset{\star}{\mathcal{M}}(A, \mathcal{S}, \mathcal{T})$, whose existence is guaranteed by the above theorem, is called $\star$-shorted operator of A with respect to the subspaces $\mathcal{S}$ and $\mathcal{T}$, and it is denoted by $\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})$. If $\mathcal{S} = \mathcal{T}$ we abbreviate $\overset{\star}{\mathcal{M}}(A, \mathcal{S}, \mathcal{T}) = \overset{\star}{\mathcal{M}}(A, \mathcal{S})$ and $\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T}) = \overset{\star}{\Sigma}(A, \mathcal{S})$. △

The next Proposition summarizes some properties of the star shorted operator which are direct consequences of the definition and the properties of the $\star$-order.

Proposition 4.4. *Let $A, B \in L(\mathcal{H})$ and $\mathcal{S}, \mathcal{T}, \mathcal{U}, \mathcal{V} \sqsubseteq \mathcal{H}$. Then*

1. $\overset{\star}{\Sigma}\left(\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T}), \mathcal{S}, \mathcal{T}\right) = \overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})$.
2. $\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T}) = \overset{\star}{\Sigma}\left(A, \mathcal{S} \cap \overline{R(A^*)}, \mathcal{T} \cap \overline{R(A)}\right)$.
3. $\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})^* = \overset{\star}{\Sigma}(A^*, \mathcal{T}, \mathcal{S})$.
4. *If A is selfadjoint, then $\overset{\star}{\Sigma}(A, \mathcal{S})$ is selfadjoint.*
5. *If A is positive then $\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})$ is positive, even if $\mathcal{S} \neq \mathcal{T}$.*
6. *If $A \overset{\star}{\leq} B$, $\mathcal{S} \subseteq \mathcal{U}$ and $\mathcal{T} \subseteq \mathcal{V}$, then $\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T}) \overset{\star}{\leq} \overset{\star}{\Sigma}(A, \mathcal{U}, \mathcal{V})$.* □

The characterization of the $\star$-shorted operator as a maximum allow to prove the following result in a fairly standard way:

Proposition 4.5. *Let $A, B \in L(\mathcal{H})$ and $\mathcal{S}, \mathcal{T}, \mathcal{U}, \mathcal{V} \sqsubseteq \mathcal{H}$. Then*

$$\overset{\star}{\Sigma}\left(\overset{\star}{\Sigma}(A,\mathcal{S},\mathcal{T}),\mathcal{U},\mathcal{V}\right) = \overset{\star}{\Sigma}(A,\mathcal{S}\cap\mathcal{U},\mathcal{T}\cap\mathcal{V}) \ .$$

Proof. Let $\mathcal{M}_1 = \overset{\star}{\mathcal{M}}(\overset{\star}{\Sigma}(A,\mathcal{S},\mathcal{T}),\mathcal{U},\mathcal{V})$ and $\mathcal{M}_2 = \overset{\star}{\mathcal{M}}(A,\mathcal{S}\cap\mathcal{U},\mathcal{T}\cap\mathcal{V})$. It is enough to show that $\mathcal{M}_1 = \mathcal{M}_2$. If $D \in \mathcal{M}_1$, then, $D \overset{\star}{\leq} \overset{\star}{\Sigma}(A,\mathcal{S},\mathcal{T})$, so $R(D) \subseteq \mathcal{S}$ and $R(D^*) \subseteq \mathcal{T}$. Thus $R(D) \subseteq \mathcal{S}\cap\mathcal{T}$ and $R(D^*) \subseteq \mathcal{T}\cap\mathcal{V}$. But, $D \overset{\star}{\leq} \overset{\star}{\Sigma}(A,\mathcal{S},\mathcal{T}) \overset{\star}{\leq} A$, therefore $D \in \mathcal{M}_2$.

On the other hand, if $D \in \mathcal{M}_2$, then, $D \overset{\star}{\leq} A$, $R(D) \subseteq \mathcal{S}$, and $R(D^*) \subseteq \mathcal{T}$ therefore $D \overset{\star}{\leq} \overset{\star}{\Sigma}(A,\mathcal{S},\mathcal{T})$. Moreover, $R(D) \subseteq \mathcal{U}$ and $R(D^*) \subseteq \mathcal{V}$, therefore $D \in \mathcal{M}_1$. ☐

Corollary 4.6. Let $A \in L(\mathcal{H})^+$ and $\mathcal{S}, \mathcal{T} \sqsubseteq \mathcal{H}$. Then $\overset{\star}{\Sigma}(A,\mathcal{S},\mathcal{T}) = \overset{\star}{\Sigma}(A,\mathcal{S}\cap\mathcal{T})$.

Proof. By Proposition 4.5, as $B := \overset{\star}{\Sigma}(A,\mathcal{S},\mathcal{T}) \geq 0$ we get that

$$B = \overset{\star}{\Sigma}(B,\mathcal{S},\mathcal{T}) = \overset{\star}{\Sigma}(B^*,\mathcal{S},\mathcal{T}) = \overset{\star}{\Sigma}\left(\overset{\star}{\Sigma}(A,\mathcal{T},\mathcal{S}),\mathcal{S},\mathcal{T}\right) = \overset{\star}{\Sigma}(A,\mathcal{S}\cap\mathcal{T},\mathcal{S}\cap\mathcal{T}) \ ,$$

which completes the proof. ☐

Proposition 4.7. *Let $A \in L(\mathcal{H})$ and $\mathcal{S},\mathcal{T} \sqsubseteq \mathcal{H}$ such that $\mathcal{S} \subseteq R(Q_A)$ and $\mathcal{T} \subseteq R(P_A)$. Then, if $A = U|A| = |A^*|U$ is* **the** *polar decomposition of A, it holds that*

$$|\overset{\star}{\Sigma}(A,\mathcal{S},\mathcal{T})| = \overset{\star}{\Sigma}(|A|,\mathcal{S},U^*(\mathcal{T})) \ \textit{and}\ |\overset{\star}{\Sigma}(A,\mathcal{S},\mathcal{T})^*| = \overset{\star}{\Sigma}(|A^*|,U(\mathcal{S}),\mathcal{T}) . \tag{10}$$

Proof. By Corollary 4.6, $\overset{\star}{\Sigma}(|A|,\mathcal{S},U^*(\mathcal{T})) = \overset{\star}{\Sigma}(A,\mathcal{S}\cap U^*(\mathcal{T}))$. Also, Lemma 2.16 assures that $U^*\overset{\star}{\Sigma}(A,\mathcal{S},\mathcal{T}) = |\overset{\star}{\Sigma}(A,\mathcal{S},\mathcal{T})|$. Then, to prove the first identity of (10) it is enough to prove that the map $\Gamma : \overset{\star}{\mathcal{M}}(A,\mathcal{S},\mathcal{T}) \to \overset{\star}{\mathcal{M}}(|A|,\mathcal{S}\cap U^*(\mathcal{T}))$, given by $\Gamma(B) = U^*B = |B|$, is an $\star$-order-preserving bijection. Let $B \in \overset{\star}{\mathcal{M}}(A,\mathcal{S},\mathcal{T})$. Then $U^*B = |B| \overset{\star}{\leq} |A|$,

$$R(U^*B) = U^*(R(B)) \subseteq U^*(\mathcal{T}) \ , \qquad \text{and} \qquad R(U^*B) = R(|B|) \subseteq \mathcal{S} \ .$$

Hence, $\Gamma(B) \in \overset{\star}{\mathcal{M}}(|A|,\mathcal{S}\cap U^*(\mathcal{T}))$. By Corollary 2.15, it is order preserving, and it is injective because $U\Gamma(B) = B$ by Lemma 2.16. On the other hand, to prove that Γ is onto, let $C \in \overset{\star}{\mathcal{M}}(|A|,\mathcal{S}\cap U^*(\mathcal{T}))$ and define $B = UC$. Observe that, since $\mathcal{T} \subseteq R(P_A)$, then

$$R(B) = U(R(C)) \subseteq UU^*(\mathcal{T}) = P_A(\mathcal{T}) = \mathcal{T} \ .$$

Also $R(B^*) \subseteq R(C) \subseteq \mathcal{S}$. Using that $R(C) \subseteq R(|A|)$ and $R(U^*U) = \overline{R(|A|)}$, then $U^*B = C$. Also $BB^* = UCCU^* = UC|A|U^* = BA^*$ and

$$B^*B = CC = C|A| = CU^*U|A| = B^*A \ ,$$

which shows that $B \overset{\star}{\leq} A$, and therefore $B \in \overset{\star}{\mathcal{M}}(A,\mathcal{S},\mathcal{T})$. Since $\Gamma(B) = U^*B = C$, then Γ is onto. The other equality of (10) follows in a similar way. ☐

Remark 4.8. Note that, by item 2 of Proposition 4.4, the conditions $\mathcal{S} \subseteq \overline{R(A^*)}$ and $\mathcal{T} \subseteq \overline{R(A)}$ asked in Proposition 4.7 are not too severe. △

5 Star orthogonal decompositions

As we have already mentioned, the definition of the $\star$-order has been motivated by the notion of $\star$-orthogonality introduced by Hestenes in [9]. So, it is natural to expect that this relation plays an important role in the different aspects of the $\star$-order. In this section we shall study how the $\star$-orthogonality is related with the $\star$-minima and the $\star$-shorted operators. We show that, if an operator admits a "suitable decomposition", the $\star$-minima as well as the $\star$-shorted can be computed in terms of that decomposition. In the next definition we say what we understand by a suitable decomposition:

Definition 5.1. Given $A \in L(\mathcal{H})$, we shall say that the family of operators $\{A_i\}_{i\in I}$, each $A_i \in L(\mathcal{H})$, is a star orthogonal decomposition ($\star$-OD) of A if

1. $A_i^*A_k = 0 = A_iA_k^*$ for every $i \neq k$;
2. The spectra $\sigma\Big(A_i\, A_i^*\Big|_{R(P_{A_i})}\Big)$ are pairwise disjoint.
3. $A = \sum_{i\in I} A_i$, where in the case of decompositions with infinitely many operators, the serires converges in the strong operator topology. △

Example 5.2. A typical example, which actually motivates Definition 5.1, is the singular value decomposition of compact operators. Given a compact operator $A \in L(\mathcal{H})$, its singular value decomposition can be written as:

$$A = \sum_{i=1}^{\infty} s_i\, U_i \ , \tag{11}$$

where the numbers s_i are the singular values of A, and the operators U_i are partial isometries of finite rank. In this case, $\{s_i\, U_i\}_{i\in\mathbb{N}}$ is a $\star$-OD of A. Decompositions like (11) are sometimes called Penrose's decompositions (see [8]). △

Lemma 5.3. *Let $A \in L(\mathcal{H})$ and let $\{A_i\}_{i\in I}$ a $\star$-OD of A. Then*

1. $AA^* = \sum_{i\in I} A_iA_i^*$ *and* $A^*A = \sum_{i\in I} A_i^*A_i$.
2. *The projections P_{A_i} are pairwise orthogonal, and $P_A = \sum_{i\in I} P_{A_i}$.*
3. *For every $i \in I$, it holds that $P_{A_i} \in \mathcal{P}_A$, so that $A_i = P_{A_i} A \overset{\star}{\leq} A$.*

Proof. Straightforward. □

Proposition 5.4. *Let $A, B \in L(\mathcal{H})$ such that $A \overset{\star}{\leq} B$, and let $\{B_i\}_{i\in I}$ a $\star$-OD of B. Then the sequence $\{A \wedge^\star B_i\}_{i\in I}$ is a $\star$-OD of A.*

Proof. By Theorem 2.7, P_A commutes with BB^*. So, by Lema 5.3 and the second condition in Definition 5.1, P_A commutes with each $B_iB_i^*$ and each P_{B_i}. Denote by $P_i = P_AP_{B_i} \in \mathcal{P}_{B_i}$, and $A_i = P_iB_i \overset{\star}{\leq} B_i$ (by Theorem 2.7). In particular,

$$\sigma\Big(A_i\, A_i^*\Big|_{R(P_{A_i})}\Big) \subseteq \sigma\Big(B_i\, B_i^*\Big|_{R(P_{B_i})}\Big) \implies \sigma\Big(A_i\, A_i^*\Big|_{R(P_{A_i})}\Big) \quad \text{are disjoint} \ .$$

On the other hand, straightforward computations also show that $A_i^* A_k = 0 = A_i A_k^*$ for every $i \neq k$. Finally, as $P_A \leq P_B = \sum_{i\in I} P_{B_i}$, then $P_A = \sum_{i\in I} P_i$ and $A = P_A\, B = \sum_{i\in I} A_i$.

Therefore, $\{A_i\}_{i\in I}$ a $\star$-OD of A. Thus, it is enough to prove that each $A_i = A \wedge^\star B_i$. Fix $i \in I$, and observe that $P_{A\wedge^\star B_i} \leq P_A$ and $P_{A\wedge^\star B_i} \leq P_{B_i}$. So $P_{A\wedge^\star B_i} \leq P_i$. Since $A \wedge^\star B_i \overset{\star}{\leq} B_i$ and also $A_i \overset{\star}{\leq} B_i$, Theorem 2.7 assures that $A \wedge^\star B_i \overset{\star}{\leq} A_i$. The other inequality follows by Lemma 5.3 and the fact that $A_i \overset{\star}{\leq} B_i$. □

Corollary 5.5. *Let $A, B \in L(\mathcal{H})$, and let $\{B_i\}_{i\in I}$ an $\star$-OD of B. Then the sequence $\{A \wedge^\star B_i\}_{i\in I}$ is a $\star$-OD of $A \wedge^\star B$.*

Proof. It is a direct consequence of Proposition 5.4 and the fact that

$$(A \wedge^\star B) \wedge^\star B_i = A \wedge^\star (B \wedge^\star B_i) = A \wedge^\star B_i \ . \qquad \square$$

Lemma 5.6. *Let $A \in L(\mathcal{H})$ and $\{A_i\}_{i\in I}$ a $\star$-OD of A. For every $i \in I$, let $\{A_{ij}\}_{j\in J_i}$ be a $\star$-OD of A_i . Then $\{A_{ij} : \ i \in I\ ,\ j \in J_i\}$ is an $\star$-OD of A.*

Proof. Straightforward. □

As a consequence of this lemma and Corollary 5.5 we obtain the following result:

Proposition 5.7. Let $A, B \in L(\mathcal{H})$. Suppose that $\{A_i\}_{i\in I}$ is a $\star$-OD of A and $\{B_j\}_{j\in J}$ is a $\star$-OD of B. Then $\{A_i \wedge^\star B_j\}_{(i,j)\in I\times J}$ is an $\star$-OD of $A \wedge^\star B$. In particular,

$$A \wedge^\star B = \sum_{(i,j)\in I\times J} A_i \wedge^\star B_j \ . \qquad \square$$

Finally, we state the relationship between $\star$-OD and shorted operators:

Proposition 5.8. *Let $A \in L(\mathcal{H})$, $\mathcal{S}, \mathcal{T} \sqsubseteq \mathcal{H}$ and suppose that $\{A_i\}_{i\in I}$ is a $\star$-OD of A. Then, the sequence $\{\overset{\star}{\Sigma}(A_i\,, \mathcal{S}, \mathcal{T})\}_{i\in I}$ is a $\star$-OD of $\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})$.*

Proof. Let $B_i = \overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T}) \wedge^\star A_i$, for every $i \in I$. Then, by Proposition 5.4, $\{B_i\}_{i\in I}$ is a $\star$-OD of $\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})$. Therefore, it is enough to prove that $B_i = \overset{\star}{\Sigma}(A_i\,, \mathcal{S}, \mathcal{T})$ for each $i \in I$. So, fix $i \in I$. On one hand, Lemma 5.3 assures that $A_i \overset{\star}{\leq} A$, and hence $\overset{\star}{\Sigma}(A_i\,, \mathcal{S}, \mathcal{T}) \overset{\star}{\leq} \overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})$ for every $i \in I$. On the other hand, by definition, $\overset{\star}{\Sigma}(A_i\,, \mathcal{S}, \mathcal{T}) \leq A_i$. Therefore each $\overset{\star}{\Sigma}(A_i\,, \mathcal{S}, \mathcal{T}) \overset{\star}{\leq} B_i$. Conversely, by its definition $B_i \overset{\star}{\leq} A_i$. Moreover, as $B_i \overset{\star}{\leq} \overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})$, $R(B_i) \subseteq \mathcal{T}$ and $R(B_i^*) \subseteq \mathcal{S}$. So, $B_i \overset{\star}{\leq} \overset{\star}{\Sigma}(A_i\,, \mathcal{S}, \mathcal{T})$, which concludes the proof. □

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