Serdica

Mathematical Journal

Сердика

Математическо списание

For further information on
Serdica Mathematical Journal
which is the new series of
Serdica Bulgaricae Mathematicae Publicationes
visit the website of the journal http://www.math.bas.bg/~serdica
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

Serdica Math. J. **41** (2015), 415–430

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

CONVOLUTIONAL CALCULUS OF DIMOVSKI AND QR-REGULARIZATION OF THE BACKWARD HEAT PROBLEM*

Emilia Bazhlekova

Communicated by I. D. Iliev

Dedicated to Professor Ivan Dimovski on occasion of his 80th anniversary

Abstract. The final value problem for the heat equation is known to be ill-posed. To deal with this, in the method of quasi-reversibility (QR), the equation or the final value condition is perturbed to form an approximate well-posed problem, depending on a small parameter ε. In this work, four known quasi-reversibility techniques for the backward heat problem are considered and the corresponding regularizing problems are treated using the convolutional calculus approach developed by Dimovski (I.H. Dimovski, *Convolutional Calculus*, Kluwer, Dordrecht, 1990). For every regularizing problem, applying an appropriate bivariate convolutional calculus, a Duhamel-type representation of the solution is obtained. It is in the form of a convolution product of a special solution of the problem and the given final value function. A non-classical convolution with respect to the space variable is used. Based on the obtained representations, numerical experiments are performed for some test problems.

2010 *Mathematics Subject Classification:* 35C10, 35R30, 44A35, 44A40.

Key words: convolutional calculus, non-classical convolution, Duhamel principle, ill-posed problem, quasi-reversibility.

*The author is partially supported by Grant DFNI-I02/9/12.12.2014 from the Bulgarian National Science Fund and the Bilateral Research Project “Mathematical modelling by means of integral transform methods, partial differential equations, special and generalized functions” between BAS and SANU.

1. Introduction. Many practical situations lead to inverse problems in mathematical physics, in which one would like to determine causes for a desired or observed effect. One of the characterizing properties of many of the inverse problems is that they are usually ill-posed in the sense of Hadamard. This means that there is no solution that depend continuously on the data. The backward heat problem is known to be ill-posed [7]. In order to obtain stable approximate solutions, one has to regularize the problem. Many regularization procedures are available in the literature [6]. In the method of quasi-reversibility (QR) [7, 9, 1, 2], the equation or the final value condition is perturbed to form an approximate well-posed problem (regularizing problem), depending on a small parameter ε. An important step in a quasi-reversibility method is choosing of an optimal value of the regularizing parameter ε. Therefore, the solution of such problems requires both basic theory and experimentation and adaptation.

Let us note that many applications in engineering and medicine (e.g. in computer tomography, ultrasound, MRI, CAT, PET imaging in medicine) can be formulated as backward heat problems.

In [4], applying the convolutional calculus approach proposed by Dimovski, closed-form representations of the solutions of some problems arising in QR regularization of ill-posed problems are obtained. Such representations are convenient for fast numerical computation of the regularizing solutions at each point independently and for the establishing of the optimal value of the regularizing parameter.

Inspired by [4], in this paper the convolutional calculus approach is applied to find Duhamel-type representations of the solutions of four different regularizing problems for the backward heat problem. The obtained representations are used for calculating the numerical solution of some test problems. Several numerical experiments are performed in order to find the optimal value of the regularizing parameter ε. The four regularizing techniques are compared.

2. Forward and backward problems for the heat equation. Recall the forward problem (FP) for the one-dimensional heat equation:

$$\begin{aligned} & u_t = u_{xx}, \quad (x,t) \in (0,1) \times (0,T), \\ & u(0,t) = u(1,t) = 0, \quad 0 < t \le T, \\ & u(x,0) = f(x), \quad x \in [0,1]. \end{aligned} \tag{2.1}$$

The forward problem (2.1) is a well-posed problem in $L^2(0,1)$, which means that a unique solution $u(x,t)$ exists and it depends continuously on the initial data (the function $f(x)$). Denote the solution of (2.1) by $u(x,t;f)$. Its eigenfunction expansion is given by:

$$u(x,t;f) = \sum_{n=1}^{\infty} f_n \sin(n\pi x)\exp(-n^2\pi^2 t), \tag{2.2}$$

where f_n are the Fourier coefficients in the eigenfunction expansion of the initial function $f(x)$

$$f(x) = \sum_{n=1}^{\infty} f_n \sin(n\pi x) \tag{2.3}$$

and they can be computed by

$$f_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx, \quad n \in \mathbb{N}.$$

Consider now the backward problem (BP):

$$\begin{aligned} &u_t = u_{xx}, \quad (x,t) \in (0,1)\times(0,T),\\ &u(0,t) = u(1,t) = 0, \quad 0 < t \le T,\\ &u(x,T) = g(x), \quad x \in [0,1]. \end{aligned} \tag{2.4}$$

Given a final value $u(x,T) = g(x)$, we would like to determine a solution $u(x,t)$ (or approximation of it) for $t \in [0,T]$. In particular, we are looking for a function $f(x)$, such that the solution $u(x,t)$ of the forward problem (2.1) yields $u(x,T;f) = g(x)$.

It is known that, in general, the process is irreversible, see e.g. [6, 7] and the BP is an ill-posed problem in the sense of Hadamard: solution does not exist for all $g(x) \in L^2(0,1)$ and even if $g(x)$ is smooth enough that a solution exists, there is no continuous dependence of $u(x,t)$ on $g(x)$.

Let us look at the eigenfunction expansion of the solution of BP. From (2.2) it follows:

$$g(x) = u(x,T;f) = \sum_{n=1}^{\infty} f_n \sin(n\pi x)\exp(-n^2\pi^2 T),$$

and thus

$$g_n = f_n \exp(-n^2\pi^2 T)$$

implying

$$f(x) = \sum_{n=1}^{\infty} g_n \sin(n\pi x) \exp(n^2\pi^2 T), \tag{2.5}$$

$$u(x,t) = \sum_{n=1}^{\infty} g_n \sin(n\pi x) \exp(n^2\pi^2 (T-t)). \tag{2.6}$$

Convergence of the series (2.5) and the series (2.6) for all $t \in [0,T]$ is achieved only if $|g_n| \exp(n^2\pi^2 T) < \infty$, which is a very restrictive condition on the function $g(x)$. Even if it is satisfied, small perturbations of $g(x)$ would lead to large deviations in the function $f(x)$, defined by (2.5). This illustrates the ill-posedness of the backward problem.

On the other hand, it is known [7] that for small ε, we can always find a function $f_\varepsilon(x)$ (there are an infinite number of such functions), such that

$$\|u(x,T;f_\varepsilon) - g(x)\|_{L^2(0,1)} \leq \varepsilon.$$

To solve the backward problem (2.4) means to find one $f_\varepsilon(x)$ yielding such approximation.

3. Quasi-reversibility methods. A key technique in the study of ill-posed problems is regularization of the problem. Instead of the original ill-posed problem, an approximate regularizing problem is considered, which is well-posed, and which depends on a small regularization parameter $\varepsilon > 0$.

Various regularization techniques for solving the backward heat problem are proposed so far. The quasi-reversibility method is introduced by Lattès and Lions [7]. In this section, four different regularizing techniques from this type are considered.

In the first two cases the backward problem is rewritten first as a forward problem (which is also ill-posed) for the function $v(x,t)$ defined by

$$v(x,t) = u(x, T-t)$$

and then a small perturbation is introduced in the equation. The first regularizing problem, which we consider, is proposed by Lattès and Lions [7], where the ill-posed problem is replaced by the following well-posed problem of twice higher order:

$$\frac{\partial v_\varepsilon}{\partial t} + \frac{\partial^2 v_\varepsilon}{\partial x^2} + \varepsilon \frac{\partial^4 v_\varepsilon}{\partial x^4} = 0, \quad (\varepsilon > 0)$$

$$v_\varepsilon(0,t) = v_\varepsilon(1,t) = (v_\varepsilon)_{xx}(0,t) = (v_\varepsilon)_{xx}(1,t) = 0,$$
$$v_\varepsilon(x,0) = g(x). \tag{3.1}$$

The second regularizing problem is proposed by Schowalter [9]. The considered perturbation in the equation is suggested by some fluid flow models.

$$\begin{aligned} &\frac{\partial v_\varepsilon}{\partial t} + \frac{\partial^2 v_\varepsilon}{\partial x^2} - \varepsilon\frac{\partial^3 v_\varepsilon}{\partial t \partial x^2} = 0, \quad (\varepsilon > 0)\\ &v_\varepsilon(0,t) = v_\varepsilon(1,t) = 0,\\ &v_\varepsilon(x,0) = g(x). \end{aligned} \tag{3.2}$$

Making again change of variables, the corresponding to (3.1) and (3.2) approximate solutions for the backward problem will be

$$u_\varepsilon(x,t) = v_\varepsilon(x, T-t).$$

The second two methods belong to the group of the so-called quasi-boundary-value methods, where a small perturbation is introduced in the final condition, leading to a nonlocal condition.

The third regularizing problem is proposed by Clark and Oppenheimer [1]:

$$\begin{aligned} &\frac{\partial u_\varepsilon}{\partial t} = \frac{\partial^2 u_\varepsilon}{\partial x^2},\\ &u_\varepsilon(0,t) = u_\varepsilon(1,t) = 0,\\ &\varepsilon u_\varepsilon(x,0) + u_\varepsilon(x,T) = g(x), \quad \varepsilon > 0. \end{aligned} \tag{3.3}$$

As fourth regularizing problem we consider a modification of (3.3), proposed by Denche and Bessila [2]:

$$\begin{aligned} &\frac{\partial u_\varepsilon}{\partial t} = \frac{\partial^2 u_\varepsilon}{\partial x^2},\\ &u_\varepsilon(0,t) = u_\varepsilon(1,t) = 0,\\ &u_\varepsilon(x,T) - \varepsilon (u_\varepsilon)_t(x,0) = g(x), \quad \varepsilon > 0. \end{aligned} \tag{3.4}$$

All regularizing problems are defined in such a way that for $\varepsilon \to 0$ the original ill-posed problem is recovered. Hence, the degree of approximation should be better for small ε. On the other hand, for $\varepsilon \to 0$ the solution of the regularizing problem becomes very unstable. Therefore, an important issue in a regularization method is to find an optimal value for ε. For this, some numerical experimentation and adaptation is needed. Next we obtain closed-form solutions of the regularizing problems (3.1-3.4) which are convenient for this purpose.

4. Convolutional calculi for the regularizing problems. In this section, we develop bivariate convolutional calculus for the regularizing problems (3.1)–(3.4). The notion of convolution of a linear operator is basic in a convolutional calculus.

Definition 4.1. *Let X be a linear space, and let $L : X \to X$ be a linear operator. A bilinear, commutative and associative operation $* : X \times X \to X$, such that*

$$L(f*g) = (Lf)*g \quad for \quad any\ f, g \in X, \tag{4.1}$$

is said to be a convolution of the operator L.

Further, any linear operator $M : X \to X$, satisfying the relation

$$M(f*g) = (Mf)*g \quad for \quad any\ f, g \in X,$$

is said to be a multiplier of the convolution algebra $(X, *)$.

4.1. Space variable. With respect to the space variable, any of the four regularizing problems contains the square of differentiation, subjected to Dirichlet boundary conditions. Let $C_x = C([0,1])$ be the space of continuous functions on $[0,1]$. Let L be the right inverse operator of the operator $D = d^2/dx^2$, defined by $(Lf)(0) = (Lf)(1) = 0$. It has the explicit representation:

$$Lf(x) = \int_0^x (x-\xi)f(\xi)\,d\xi - x\int_0^1 (1-\xi)f(\xi)\,d\xi.$$

In the following theorem, which is proven in more general form in [3], a convolution of the operator L is given. Moreover, the operator L is represented as a convolution operator with respect to this convolution.

Theorem 4.2. *The operation*

$$(f \overset{x}{*} g)(x) = -\frac{1}{2}\int_0^1 \left[\int_x^\xi f(\xi + x - \eta)g(\eta)\,d\eta - \int_{-x}^\xi f_1(\xi - x - \eta)g_1(\eta)\,d\eta\right] d\xi,$$

where $f_1(x) = f(|x|)\mathrm{sgn}(x)$, $g_1(x) = g(|x|)\mathrm{sgn}(x)$, is a convolution of the operator L in C_x. Moreover, the representation holds

$$Lf = \{x\} \overset{x}{*} f. \tag{4.2}$$

Following [3], consider also the defining projector F, which is "responsible" for the boundary conditions. It is defined by $F := I - LD$. Note that identity $DL = I$ implies

$$FL = 0, \quad F^2 = F, \tag{4.3}$$

i.e. F is indeed a projection operator. From its definition we find the explicit representation of F:

$$Ff(x) = f(0)(1-x) + f(1)x. \tag{4.4}$$

The following useful properties are proven in [3].

$$D(f \overset{x}{*} g) = (Df) \overset{x}{*} g + D((Ff) \overset{x}{*} g), \quad F(f \overset{x}{*} g) = F((Ff) \overset{x}{*} g). \tag{4.5}$$

Denote by $\mathcal{M}_x$ the multiplicative set of all multipliers of $(C_x, \overset{x}{*})$. It is a commutative ring. Clearly, the operator L is a multiplier of the convolution algebra $(C_x, \overset{x}{*})$, $L \in \mathcal{M}_x$. Moreover, L is a non-divisor of 0. Indeed, $Lf = 0$ implies $DLf = 0$ i.e. $f = 0$. Hence, the multiplicative subset of $\mathcal{M}_x$ consisting of all non-zero non-divisors of 0 is nonempty. Let us denote it by $\mathcal{N}_x$. Then we consider the multiplier fractions

$$\frac{M}{N}, \; M \in \mathcal{M}_x, N \in \mathcal{N}_x$$

with the usual convention

$$\frac{M}{N} = \frac{M_1}{N_1} \text{ iff } MN_1 = M_1N.$$

We consider numbers, functions, multipliers and multiplier fractions as elements of a single algebraic system: the ring of multiplier fractions. For more detailed description of this procedure we refer to the recent work [10] and the references cited there.

In this ring the operator L can be identified by the function $\{x\}$, since $L = \{x\} \overset{x}{*}$, see (4.2). The algebraic inverse of L:

$$S := \frac{1}{L}$$

plays a basic role in the corresponding convolutional calculus. It can be considered as "algebraic differentiation operator". More precisely, for $f \in C^2([0,1])$

$$f'' = Sf - S\{(1-x)f(0)\} - f(1), \tag{4.6}$$

where $f(1)$ is to be considered as a numerical operator. Indeed, (4.4) implies

$$Lf''(x) = f(x) - Ff(x) = f(x) - f(0)(1-x) - f(1)x.$$

Since $\{x\} \equiv L$, multiplying by $S = 1/L$ we get (4.6).

4.2. Time variable. Let $C_t = C([0,T])$ be the space of continuous functions on $[0,T]$ and let $\Phi : C_t \to \mathbb{R}$ be a linear continuous functional, such that

$\Phi\{1\}=1$. Let l be the right inverse operator of d/dt, defined by $\Phi_\tau\{lf(\tau)\}=0$. Therefore it has explicit representation

$$lf(t)=\int_0^t f(\tau)\,d\tau-\Phi_\tau\left\{\int_0^\tau f(\sigma)\,d\sigma\right\}.$$

The following theorem is proven in [3].

Theorem 4.3. *The operation*

$$(f\overset{t}{*}g)(t)=\Phi_\tau\left\{\int_\tau^t f(t+\tau-\sigma)g(\sigma)\,d\sigma\right\}$$

is a convolution of the operator l in C_t. Moreover,

$$lf=\{1\}\overset{t}{*}f. \tag{4.7}$$

Hence, l is a multiplier of the convolution algebra $(C_t,\overset{t}{*})$. According to (4.7), it is a convolution operator which can be identified with the function $\{1\}$. We repeat the procedure from subsection 4.1 and define the ring of multiplier fractions, corresponding to convolution $\overset{t}{*}$. In this ring of multiplier fractions consider the algebraic inverse of l:

$$s:=\frac{1}{l}.$$

It is an "algebraic differential operator", satisfying:

$$f'=sf-\Phi\{f\},\quad f\in C_t^1, \tag{4.8}$$

where $\Phi\{f\}$ is not a constant function, but a numerical operator.

4.3. Bivariate operational calculus. Let $\Delta=[0,1]\times[0,T]$, and let $C(\Delta)$ be the space of continuous functions on Δ. Following [5] we construct a bivariate operational calculus, based on the calculi, developed in subsections 4.1 and 4.2. Define a bivariate convolution of $f(x,t)$ and $g(x,t)$ from $C(\Delta)$:

$$(f\overset{x,t}{*}g)(x,t)=\Phi_\tau\left\{\int_\tau^t f(x,t+\tau-\sigma)\overset{x}{*}g(x,\sigma)\,d\sigma\right\}.$$

Then the following separability property holds true: if

$$f(x,t)=f_1(x)f_2(t),g(x,t)=g_1(x)g_2(t),$$

then

$$\left(f\overset{x,t}{*}g\right)(x,t)=\left(f_1\overset{x}{*}g_1\right)(x)\left(f_2\overset{t}{*}g_2\right)(t).$$

The operators L and l are multipliers of $\overset{x,t}{*}$, such that

$$Llf = \{x\} \overset{x,t}{*} f. \tag{4.9}$$

We repeat the construction from subsection 4.1. Let $\mathcal{M}$ be the set of all multipliers of $(C, \overset{x,t}{*})$. It is a commutative ring. The multipliers L and l are non-zero non-divisors of 0. Let $\mathcal{N} \subset \mathcal{M}$ be the set of all non-zero non-divisors of 0 in $\mathcal{M}$. Let $\mathcal{N}^{-1}\mathcal{M}$ be the ring of multiplier fractions. Then

$$S = \frac{1}{L}\ , s = \frac{1}{l} \in \mathcal{N}^{-1}\mathcal{M}.$$

Consider numbers, functions, multipliers and multiplier fractions as elements of a single algebraic system: the ring of multiplier fractions.

From (4.6) and (4.8) we obtain the following properties necessary for problem algebraization:

$$\left\{\frac{\partial^2 u}{\partial x^2}\right\} = Su - S\{(1-x)u(0,t)\} - [u(1,t)]_x, \tag{4.10}$$

$$\left\{\frac{\partial u}{\partial t}\right\} = su - [\Phi_\tau\{u(x,\tau)\}]_t\,, \tag{4.11}$$

where $[.]_x$ and $[.]_t$ denote numerical operators with respect to x and t, i.e.

$$[f(t)]_x u(x,t) = f(t) \overset{t}{*} u(x,t), \quad [g(x)]_t u(x,t) = g(x) \overset{x}{*} u(x,t).$$

The following identities, implied by (4.2), (4.7) and (4.9) are also useful:

$$L \equiv [x]_t, \quad L^2 = L[x]_t \equiv \left[\frac{x^3 - x}{6}\right]_t, \quad Ll \equiv \{x\}. \tag{4.12}$$

5. Duhamel-type representations. Based on the developed convolutional calculi, in this section we find Duhamel-type representations of the solutions of the forward problem and the regularizing problems. In order to exhibit the full scope of the convolutional method, we consider the extended inhomogeneous variants of the equations. Note that nonhomogeneous boundary conditions can also be considered. Usually they can be incorporated in an appropriate "forcing" function. Therefore, without loss of generality, we take homogeneous boundary conditions and arbitrary forcing function.

First, consider the forward problem

$$u_t = u_{xx} + h(x,t), \qquad (x,t) \in (0,1) \times (0,T),$$

$$u(0,t) = u(1,t) = 0, \quad 0 < t \le T, \tag{5.1}$$
$$u(x,0) = f(x), \quad x \in [0,1].$$

In case of nonhomogeneous boundary conditions we set

$$\overline{u}(x,t) = u - Fu = u(x,t) - u(0,t)(1-x) - u(1,t)x.$$

Then $\overline{u}$ satisfies the same problem with homogeneous boundary conditions and new functions $\overline{f}$ and $\overline{h}$. So, without loss of generality, we consider problem (5.1). Applying identities (4.10) and (4.11) with $\Phi\{f\} = f(0)$ we rewrite the problem in algebraic form in the ring of multiplier fractions as follows:

$$su - [f(x)]_t = Su + h.$$

The algebraic solution is

$$u = \frac{[f(x)]_t}{s-S} + \frac{h}{s-S} = S\frac{L}{s-S}[f(x)]_t + Ss\frac{Ll}{s-S}h.$$

Now, using identities (4.12) and properties (4.10) and (4.11), it follows that the solution u has the following representation:

$$u = \frac{\partial^2}{\partial x^2}\left(U \overset{x}{*} f\right) + \frac{\partial^3}{\partial x^2 \partial t}\left(V \overset{x,t}{*} h\right), \tag{5.2}$$

where $U(x,t)$ is a solution of (5.1) with $h \equiv 0$ and $f(x) = x$, and $V(x,t)$ is a solution of (5.1) with $f \equiv 0$ and $h(x,t) = x$. Further, depending on the smoothness of the given functions $f(x)$ and $h(x,t)$ we can differentiate the convolution products in (5.2) applying (4.5).

Next we find similar Duhamel-type representations of the solutions of the regularizing problems. Consider the first regularizing problem

$$\begin{aligned} &\frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial x^2} + \varepsilon\frac{\partial^4 v}{\partial x^4} = h(x,t), \quad (\varepsilon > 0) \\ &v(0,t) = v(1,t) = v_{xx}(0,t) = v_{xx}(1,t) = 0, \\ &v(x,0) = g(x). \end{aligned} \tag{5.3}$$

If the problem is with nonhomogeneous boundary conditions, then set

$$\overline{v} = v - Fv - LFDv,$$

where operators D, L and F are defined in subsection 4.1. From the properties (4.3) of the defining projector F it follows that $F\overline{v} = 0$ and $FD\overline{v} = 0$, which means that $\overline{v}$ satisfies homogeneous boundary conditions and equation with a new forcing function $\overline{h}(x,t)$.

To rewrite problem (5.3) in algebraic form in the ring of multiplier fractions we use identities (4.10) and 4.11) with $\Phi\{f\} = f(0)$, which imply:

$$\begin{aligned} \frac{\partial v}{\partial t} &= sv - [v(x,0)]_t = sv - [g(x)]_t, \\ \frac{\partial^2 v}{\partial x^2} &= Sv - S\{(1-x)v(0,t)\} - [v(1,t)]_x = Sv, \\ \frac{\partial^4 v}{\partial x^4} &= S\left(\frac{\partial^2 v}{\partial x^2}\right) - S\left\{(1-x)\frac{\partial^2 v}{\partial x^2}(0,t)\right\} - \left[\frac{\partial^2 v}{\partial x^2}(1,t)\right]_x = S^2 v. \end{aligned} \tag{5.4}$$

Inserting these identities in the equation and using the initial/boundary conditions we get the algebraic problem

$$sv - [g(x)]_t + Sv + \varepsilon S^2 v = h.$$

The algebraic solution is then

$$v = v_1 + v_2, \text{ where } v_1 = \frac{[g(x)]_t}{s + S + \varepsilon S^2}, \; v_2 = \frac{h}{s + S + \varepsilon S^2}. \tag{5.5}$$

Rewrite v_1 and v_2 as follows

$$v_1 = S^2 \frac{L^2}{s + S + \varepsilon S^2}[g(x)]_t, \quad v_2 = Ss\frac{Ll}{s + S + \varepsilon S^2}h.$$

These representations together with the identities (4.12), (5.4), and (5.5) give

$$v_1 = \frac{\partial^4}{\partial x^4}\left(\Omega \overset{x}{*} g\right), \quad v_2 = \frac{\partial^3}{\partial x^2 \partial t}\left(V \overset{x,t}{*} h\right), \tag{5.6}$$

where $\Omega(x,t)$ is a solution of (5.3) with $h \equiv 0$ and $g(x) = \dfrac{x^3 - x}{6}$, and $V(x,t)$ is a solution of (5.3) with $g \equiv 0$ and $h(x,t) = x$.

Depending on the smoothness of the functions $g(x)$ and $h(x,t)$ we can further differentiate the convolution products applying (4.5). In particular, if $g \in C_x^2$ and $g(0) = g(1) = 0$ then applying (4.5), we get

$$v_1 = \frac{\partial^2}{\partial x^2}\left(\Omega \overset{x}{*} g''\right). \tag{5.7}$$

It remains to simplify the obtained representation, noting that it contains differentiation of the convolution (which is integration operation). Denote

$$f \overset{x}{\tilde{*}} g := \frac{d^2}{dx^2}(f \overset{x}{*} g). \tag{5.8}$$

Then, after some calculations, we obtain that the new operation $\overset{x}{\widetilde{*}}$ has the following form

(5.9)
$$(f \overset{x}{\widetilde{*}} g)(x) = -\frac{1}{2}\frac{d}{dx}\left[\int_x^1 f(1+x-\eta)g(\eta)\,d\eta + \int_{-x}^1 f_1(1-x-\eta)g_1(\eta)\,d\eta\right],$$

where $f_1(x) = f(|x|)\mathrm{sgn}(x)$, $g_1(x) = g(|x|)\mathrm{sgn}(x)$.

Denote also

$$(u \overset{x,t}{\widetilde{*}} v)(x,t) := \int_0^t u(x,t-\tau) \overset{x}{\widetilde{*}} v(x,\tau)\,d\tau.$$

Then (5.6), (5.7), and (5.8) imply

$$v = v_1 + v_2, \quad \text{where} \quad v_1 = \Omega \overset{x}{\widetilde{*}} g'', \quad v_2 = \frac{\partial}{\partial t}\left(V \overset{x,t}{\widetilde{*}} h\right). \tag{5.10}$$

The second regularizing problem (3.2) is solved analogously. Note that till now in this section, for the forward problem and the first two regularizing problems we take $\Phi\{f\} = f(0)$.

Consider now the third regularizing problem (3.3). Now we take functional

$$\Phi\{f\} = \frac{\varepsilon f(0) + f(T)}{\varepsilon + 1},$$

which corresponds to the nonlocal initial condition and satisfies the normalization assumption $\Phi\{1\} = 1$. After algebraization of the nonhomogeneous problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + h(x,t), \\ &u(0,t) = u(1,t) = 0, \\ &\varepsilon u(x,0) + u(x,T) = g(x), \quad \varepsilon > 0, \end{aligned} \tag{5.11}$$

by the use of (4.10) and (4.11) we obtain

$$su - \frac{[g(x)]_t}{\varepsilon + 1} = Su + h.$$

The algebraic solution is

$$u = \frac{[g(x)]_t}{(\varepsilon+1)(s-S)} + \frac{h}{s-S} = S^2 \frac{L^2}{(\varepsilon+1)(s-S)}[g(x)]_t + Ss\frac{Ll}{s-S}h.$$

Now, using identities (4.12) and properties (4.10) and (4.11), it follows that the solution u has the following representation:

$$u = \frac{\partial^4}{\partial x^4}\left(\Omega \overset{x}{*} g\right) + \frac{\partial^3}{\partial x^2 \partial t}\left(V \overset{x,t}{*} h\right),$$

where $\Omega(x,t)$ is a solution of (5.11) with $h(x,t) \equiv 0$ and $g(x) = (x^3 - x)/6$, and $V(x,t)$ is a solution of (5.11) with $g(x) \equiv 0$ and $h(x,t) = x$. It only remains to differentiate the convolution products and represent them in terms of the new operations $\overset{x}{\widetilde{*}}$ and $\overset{x,t}{\widetilde{*}}$.

The fourth regularizing problem (3.4) is treated in a similar way, taking

$$\Phi\{f\} = f(T) - \varepsilon f'(0).$$

Particular solutions U, V and Ω can be found using eigenfunction expansion.

The main results of this section, concerning the forward problem and the four regularizing problems, are summarized in the following two theorems.

Theorem 5.1. *The solution of the forward problem* (2.1) *with initial function $f(x) \in C^1([0,1])$ has the representation*

$$u(x,t) = -\frac{1}{2}\frac{\partial}{\partial x}\left[\int_x^1 U(1+x-\eta,t) f(\eta)\, d\eta \right.$$
$$\left. + \int_{-x}^1 U(|1-x-\eta|,t) f(|\eta|) \operatorname{sgn}((1-x-\eta)\eta)\, d\eta\right],$$

where $U(x,t)$ is a particular solution of (2.1) *with $f(x) = x$ and is given by the series*

$$U(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin(n\pi x)\exp(-n^2\pi^2 t).$$

Theorem 5.2. *If $g(x) \in C^2([0,1])$ and $g(0) = g(1) = 0$ then the regularizing solutions, corresponding to regularizing problems* (3.1)–(3.4) *have the representation:*

$$u_\varepsilon(x,t) = -\frac{1}{2}\frac{\partial}{\partial x}\left[\int_x^1 \Omega_\varepsilon(1+x-\eta,t) g''(\eta)\, d\eta \right.$$
$$\left. + \int_{-x}^1 \Omega_\varepsilon(|1-x-\eta|,t) g''(|\eta|) \operatorname{sgn}((1-x-\eta)\eta)\, d\eta\right],$$

with

$$\Omega_\varepsilon(x,t) = \frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin(n\pi x)G_{n,\varepsilon}(t),$$

where in the four different cases of regularization (3.1)–(3.4) *we have respectively*

$$G^{I}_{n,\varepsilon}(t) = \exp\left(n^2\pi^2(1-\varepsilon n^2\pi^2)(T-t)\right), \quad G^{II}_{n,\varepsilon}(t) = \exp\left(\frac{n^2\pi^2(T-t)}{1+\varepsilon n^2\pi^2}\right),$$

$$G^{III}_{n,\varepsilon}(t) = \frac{\exp(-n^2\pi^2 t)}{\varepsilon + \exp(-n^2\pi^2 T)}, \qquad G^{IV}_{n,\varepsilon}(t) = \frac{\exp(-n^2\pi^2 t)}{\varepsilon n^2\pi^2 + \exp(-n^2\pi^2 T)}.$$

6. Numerical experiments. To get an impression of any of the four regularizing techniques, we work in the following way:

– Fix a final time T and choose a regularizing problem from (3.1-3.4);

– Given $g(x)$, apply Theorem 5.2. to solve the regularizing problem and to find approximation of the initial function $f_\varepsilon(x) = u_\varepsilon(x,0)$ for several values of $\varepsilon \to 0$;

– Applying Theorem 5.1, solve the forward problem with initial function $f_\varepsilon(x)$, and find

$$g_\varepsilon(x) = u(x,T;f_\varepsilon)$$

– For any ε study the behavior of f_ε and the error $\|g-g_\varepsilon\|$ and find the optimal value of ε.

We performed numerical computations for $T = 0.1$ and $g(x) = x(1-x)$ and compared the four methods. We found out that quasi-boundary-value methods (3.3) and (3.4) give better approximations of the desired final function $g(x)$ than methods (3.1) and (3.2), which is consistent with the error estimates given in [1, 2].

The presented plots on Figures 1 and 2 are obtained by the quasi-boundary-value method (3.3). On Fig. 1 initial functions $f_\varepsilon(x)$ and final functions $g_\varepsilon(x)$ are plotted for different values of ε and $T = 0.1$. The instability of the functions f_ε for $\varepsilon \to 0$ is clearly seen. Note that for the first two regularizing techniques this instability was more severe. On Fig. 2 approximate initial functions $f_\varepsilon(x)$ are plotted for different values of T and corresponding optimal values of ε. As expected, when T grows it becomes more difficult to inverse the problem. More precisely, to get a good approximation of $g(x)$ we have to take smaller value of ε, which in turn leads to more unstable behavior of f_ε.

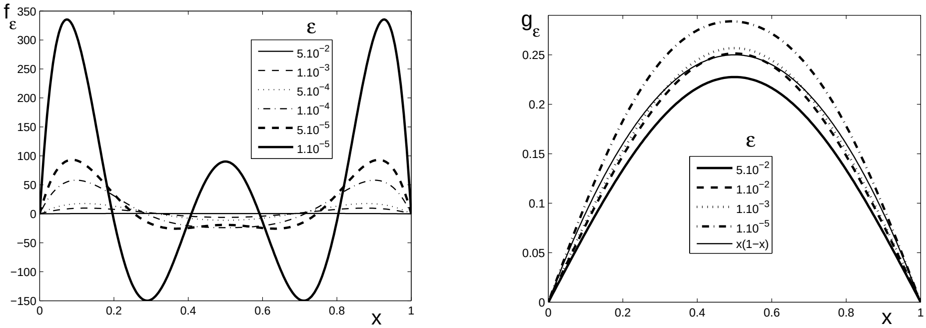

Fig. 1. Results for regularizing problem (3.3) with $T = 0.1$, $g(x) = x(1-x)$. Initial function $f_\varepsilon(x)$ (left) and final function $g_\varepsilon(x)$ (right) are plotted for different values of ε

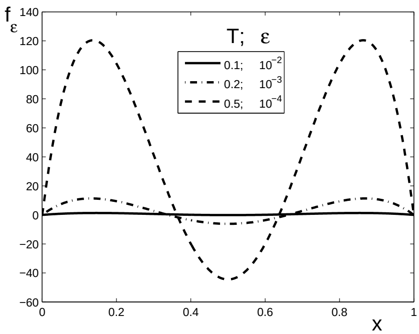

Fig. 2. Results for regularizing problem (3.3) with $g(x) = x(1-x)$. Initial function $f_\varepsilon(x)$ is plotted for different values of T and corresponding optimal values of ε

7. Conclusion and generalizations. The obtained Duhamel-type representations of the solutions are compact and have the same form for different regularizing problems. They are very convenient for fast and efficient numerical computation of the solutions of the regularizing problems in each point independently. This allows us to make easily many numerical experiments with different ε, T and $g(x)$ in order to get insight in the behavior of f_ε and g_ε, to compare different regularizing methods and to find the optimal parameter ε for each of them.

Feasible generalizations include adding noise to the final data $g(x)$, considering time-fractional backward heat equation and problems in two spatial dimensions [8, 11].

REFERENCES

[1] G. W. Clark, S. F. Oppenheimer. Quasireversibility methods for non-well-posed problems. *Electron. J. Differ. Equ.*, **8** (1994), No 08, 9 p. (electronic only).

[2] M. Denche, K. Bessila. A modified quasi-boundary value method for ill-posed problems. *J. Math. Anal. Appl.* **301**, *2* (2005), 419–426

[3] I. H. Dimovski. Convolutional Calculus. 2nd ed. Mathematics and Its Applications: East European Series, vol. **43**. Dordrecht, Kluwer Academic Publishers, 1990.

[4] I. H. Dimovski, S. L. Kalla, I. Ali. Operational calculus approach to PDE arising in QR-regularisation of ill-posed problems. *Math. Comput. Modelling* **35**, *7–8* (2002), 835–848.

[5] I. H. Dimovski, Y. T. Tsankov. Operational calculi for multidimensional nonlocal evolution boundary value problems. *AIP Conf. Proc.* **1410** (2011), 167–180.

[6] H. W. Engl, M. Hanke, A. Neubauer. Regularization of Inverse Problems. Mathematics and its Applications, vol. **375**. Dordrecht, Kluwer Academic Publishers, 1996.

[7] R. Lattès, J. L. Lions. The Method of Quasi-reversibility. Applications to partial differential equations. New York, Elsevier, 1969

[8] M. Li, X. Xiong. On a fractional backward heat conduction problem: Application to deblurring. *Comput. Math. Appl.* **64**, *8* (2012), 2594–2602.

[9] R. E. Showalter. The final value problem for evolution equations. *J. Math. Anal. Appl.* **47** (1974), 563–572.

[10] Yu. Tsankov. Operational Calculi for Boundary Value Problems. PhD thesis, Bulgarian Academy of Sciences, Sofia, 2014 (in Bulgarian).

[11] M. Yang, J. Liu. Solving a final value fractional diffusion problem by boundary condition regularization. *Appl. Numer. Math.* **66** (2013), 45–58.

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria
e-mail: e.bazhlekova@math.bas.bg

Received August 27, 2014