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THE LOGIC OF QUANTUM MECHANICS

V. Felouzis

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Abstract. These notes, written for the Summer school in Operator Theory (Chios 2010) provide a brief and elementary introduction to the Logic of Quantum mechanics and its connections with the theory of operators in a Hilbert space.

1. Introduction. In order to examine a physical system $\mathfrak{S}$ we have to make *observations* that are measurements of some physical quantities related to the system (as the energy, the position or the momentum of some elements of the systems). These quantities which can be measured by an experiment are called the *observables of the system.* All the assertions about the system $\mathfrak{S}$ are the *propositions* of the system and the structure of the set of all propositions is called the *logic* of the system.

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Suppose that A is a physical quantity (observable) related with the system $\mathfrak{S}$. A typical elementary proposition concerning the system is of the form "*the observable A has a value in the set δ of real numbers*" for which we use the shorthand notation "$A \in \delta$". In all applications, the set δ will be a Borel subset of the real numbers $\mathbb{R}$. In Physics we must be able to tell something about the "truth" of a proposition of the form $A \in \delta$. But this depends of the "state" of the system in the moment that we make our measurements. Suppose now that we prepare an experiment in order to assign a truth value to the proposition $A \in \delta$. First, we must be able to repeat this experiment, so we have to specify the exact conditions of the experiment. These conditions consist the *state* of the system. But, even with the same conditions we do not assume that the results of the experiments are identical. As it is mentioned in [7, p. 6] there are two possible reasons to explain the fact that we cannot in general expect that in the repetitions of the same experiment we take the same values.

1) The conditions are insufficient to determine the exact value of the measurement of the observable. In this case we may assume that if we add complementary conditions in the preparation of the experiment this uncertainty can be removed and the results of the experiment will be uniquely determined. In classical mechanics is the only explanation for the uncertainty in the repeated trials of the same experiment.
2) The properties of the physical system are such that in every repeated trial of an experiment we shall take different values for the the observable, independently how well the experiment is prepared. This means that the uncertainty is an inner property of the system. This is the case of quantum mechanics.

In their seminal paper Birkhoff and non Neumann [3] proposed a mathematical foundation of quantum mechanics based in the concept of the "logic of a quantum mechanical system" which is the mathematical structure of the set all propositions related to the system. The set of all propositions of a system $\mathfrak{S}$ is called the *logic* of $\mathfrak{S}$ and it is denoted by $\mathscr{L}(\mathfrak{S})$. Roughly speaking, "the logical point of view of quantum mechanics" is to describe the structure of the sets $\mathscr{O}(\mathfrak{S})$ of all observables and $\mathscr{S}(\mathfrak{S})$ of all states of $\mathfrak{S}$, and even the dynamical laws of the system starting from the mathematical structure of the logic $\mathscr{L}(\mathfrak{S})$ of a quantum mechanical system $\mathfrak{S}$. For a very interesting discussion concerning the ideas related to the concept of Quantum Logic see [14].

In our presentation we follow mainly the approach on the subject of the Logic of Quantum Mechanics given in [6], [12] and [15].

2. Logics. The point of view of classical logics is to assume that *the set of all propositions has an algebraic structure* which is the structure of a *boolean algebra*. A boolean algebra B can be described using to basic operations, the implication and the negation. So, for every $a, b \in B$ corresponds a unique element $a \Rightarrow b$ of B, which can be interpreted as the proposition "a implies b" and for every $a \in B$ corresponds a unique element $\neg a$ which can interpreted as the "negation of the proposition a". The implication is a kind of order between the propositions (you can write $a \Rightarrow b$ as $a \leq b$). If $a \Rightarrow b$ then b is a proposition stronger of a, or a is a proposition more general of b. The axioms of an order are the following:

(O1) If $a \Rightarrow b$ and $b \Rightarrow c$ then $a \Rightarrow c$.

(O2) For every a we have $a \Rightarrow a$.

(O3) If $a \Rightarrow b$ and $b \Rightarrow a$ then $a = b$.

The basic axioms are the first and second one. The third axiom can be considered as a definition of the equality of propositions. A set P where a binary relation $\Rightarrow$ or $\leq$ is defined so that the axioms (O1), (O2) and (O3) are satisfied (if we replace $\Rightarrow$ by$\leq$) is called a *partially ordered set* or a *poset*. We shall always suppose that a poset has a least element denoted by 0 and a greatest element denoted by 1.

The *negation* is an operation $a \mapsto \neg a$ which is is connected with implication by the following axioms

(N1) If $a \Rightarrow b$ then $\neg b \Rightarrow \neg a$.

(N2) For every $a \in B$ we have $\neg(\neg a) = a$.

(N3) $\neg 1 = 0$ and $\neg 0 = 1$.

We wish also formate, given the propositions a and b the propositions "a and b" and "b or a", which are denoted by $a \wedge b$ and $a \vee b$ respectively. We define $a \wedge b$ as the *strongest* proposition c which has the property "$c \Rightarrow a$ and $c \Rightarrow b$". This means that $a \wedge b$ must have the following properties:

(Inf1) $(a \wedge b) \Rightarrow a$ and $(a \wedge b) \Rightarrow b$.

(Inf2) If for some proposition c happens that $c \Rightarrow a$ and $c \Rightarrow b$ then $c \Rightarrow (a \wedge b)$.

So, in the terminology of the theory of order $a \wedge b$ is the "*greatest lower bound* of a and b", or the " *infimum* of the set $\{a, b\}$".
We also define $a \wedge b$ as the *weakest* proposition c which has the property "$a \Rightarrow c$ and $b \Rightarrow c$". This means that $a \wedge b$ must have the following properties:

(Sup1) $a \Rightarrow (a \vee b)$ and $b \Rightarrow (a \vee b)$.

(Sup2) If for some c happens that $a \Rightarrow c$ and $b \Rightarrow c$ then $(a \vee b) \Rightarrow c$.

In the terminology of the theory of order, $a \vee b$ is the "*least upper bound* of a and

b", or the "*supremum* of the set $\{a,b\}$". In general, if Q is a (nonempty) subset of we denote by $\bigwedge Q$ to be the greatest element $a \in P$ (if a such element exists) with the property $a \Rightarrow x$ for every $x \in P$. Also, by $\bigvee Q$ we shall denote the least element $a \in P$ with the property $x \Rightarrow a$ for every $x \in P$ (if a such element exists). In the following, the order relation will be denoted by $\leq$ instead of $\Rightarrow$ and the negation with $a^{\perp}$ instead of $\neg a$.

Definition 2.1.

(1) *A poset* $(P, \leq, 0, 1)$ *with a negation (that is with an operation* $a \mapsto a^{\perp}$ *which satisfies the axioms* (N1), (N2) *and* (N3)*) is called an* orthoposet*.*

(2) *A poset (resp. orthoposet)* P *such that for every* $a, b \in P$ *there exist the* $a \vee b$ *and* $a \wedge b$ *is called a* lattice *(resp.* ortholattice*). If for every* $Q \subseteq P$*, the elements* $\bigwedge Q$ *and* $\bigvee Q$ *exist then* P *is called a* complete lattice*. Finally, if for every countable set* $Q \subseteq P$*, the elements* $\bigwedge Q$ *and* $\bigvee Q$ *exist then* P *is called a* σ-complete lattice*.*

(3) *A lattice such that for every three elements* a, b, c *we have the identity*

$$(D) \qquad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

is called a distributive lattice*.*

(4) *A distributive ortholattice is called a* boolean algebra*.*

The following proposition is a simple consequence of definitions.

Proposition 2.2. *Let* $(L, \leq)$ *be a lattice. Then*

(1) $a \vee a = a$ *and* $a \wedge a = a$.

(2) $a \vee b = b \vee a$ *and* $a \wedge b = b \wedge a$

(3) $a \leq b$ *if and only if* $a \vee b = b$.

(4) $a \leq b$ *if and only if* $a \wedge b = a$.

The basic assumption of classical mechanics is that *the set of propositions of any (classical) system has the structure of a Boolean algebra.*

The situation is very different from the point of view of quantum mechanics. First, is not clear that from the propositions a and b we can formate the proposition $a \wedge b$. This problem is discussed in many books of quantum mechanics; however, it can be overcome (see the discussion in [6, CHAPTER 5]). The second and more serious objection is that many experiments show that the distributive law does not hold. An example of a such situation is presented in [12, p. 22]:

We shall call a question *every experiment leading to an alternative of which the terms are "yes" and "no". (...) Let us suppose that we have a beam of photons. The experiment which consists in placing a polarizer in the beam defines a question. In fact it is possible to verify, by despatching photons one by one, that this experiment leads to a plain alternative: either the photon passes through, or it is absorbed. We shall define the proposition a_ϕ by specifying the orientation of the polarizer (the angle ϕ) and interpreting the passage of a photon as a "yes". Experience shows that, to obtain a photon prepared in such a way that "a_ϕ is true" it is sufficient to consider the photons which have traversed a first polarizer oriented at this angle ϕ. But experiment also shows that it is impossible to prepare photons capable of traversing with complete certainty a polarizer oriented at the angle ϕ as well as another oriented at an angle $\phi' \neq \phi \mod \pi$. In other words*

$$a_\phi \wedge a_{\phi'} = 0, \text{ if } \phi' \neq \phi \mod \pi$$

To summarize, the propositional lattice for the photon, $\mathcal{L} = \{a_\phi : \phi \in [0, \pi)\}$ evidently is not distributive.

Therefore, we need a more "flexible" structure for the set of propositions of a quantum system than the boolean algebra. Birkhoff and von Neumann proposed to replace the distributive law by the following *modular law*:

(ML) If $a \leq b$ and then for every c we have that $a \vee (b \wedge c) = b \wedge (a \vee c)$

which says that if a, b, c are three elements of a lattice and two of them are comparable then the distributive laws hold.

In fact, von Neumann [11] has also studied an important class of lattices which obey the modular law (the continuous geometries) and he proved a very deep representation theorem about them.

Von Neumann also, [10] had already presented a very elegant theory on the mathematical foundations of quantum mechanics based in the *-algebra $B(\mathcal{H})$ of all linear bounded operators $T : \mathcal{H} \to \mathcal{H}$. This presentation implies that the logic of the the quantum mechanics must be the complete lattice $(P(\mathcal{H}), \leq)$ of all projections $P : \mathcal{H} \to \mathcal{H}$ where $P \leq Q$ means $PQ = P$. As we shall see later, $(P(\mathcal{H}), \leq)$ is modular if and only if the dimension of $\mathcal{H}$ is finite. But $(P(\mathcal{H}), \leq)$ does satisfy a "weak modular law" the so-called "orthomodular law" (see Definition 2.3 below).

In these notes we shall assume that the set of all propositions of a system has the structure of an *orthomodular poset.*

Definition 2.3. *An orthoposet $(P, \leq, \perp)$ is called an* orthomodular poset *when*

(1) *If* $a \perp b$ *then* $a \vee b$ *exists.*

(2) $(P, \leq, \perp)$ *satisfies the following* orthomodular law

$$\text{(OM)} \qquad \textit{If } a \leq b^{\perp} \textit{ and } a \vee b = 1 \textit{ then } a = b^{\perp}.$$

An orthomodular lattice *is an orthomodular poset which is a lattice.*

Remark 2.4. In an orthomodular poset any finite orthogonal subset has a least upper bund and also $x \wedge y = (x^{\perp} \vee y^{\perp})^{\perp}$ exists whenever $y^{\perp} \leq x$. In the definition of an ortho- modular poset one may replace the implication (OM) by any one of the the following

$$\text{(OM)}_1 \qquad \text{If } a^{\perp} \leq b \text{ and } a \wedge b = 0 \text{ then } a = b^{\perp}.$$

$$\text{(OM)}_2 \qquad \text{If } a \leq b^{\perp} \text{ then } (a \vee b) \wedge b^{\perp} = a.$$

$$\text{(OM)3} \qquad \text{If } a \leq b \text{ then } a \vee (a^{\perp} \wedge b) = b.$$

If Q is a subset of an orthoposet we set

$$Q^{\perp} = \{q^{\perp} : q \in Q\}$$

Lemma 2.5 (The De-Morgan Laws). *Let* $(P, \leq, \perp)$ *be an orthoposet and* Q *a subset of* P. *Then*

(1) *If* $\bigvee Q$ *exists then* $(\bigvee Q)^{\perp} = \bigwedge Q^{\perp}$.

(2) *If* $\bigwedge Q$ *exists then* $(\bigwedge P)^{\perp} = \bigvee Q^{\perp}$.

Proof. Suppose that $\bigvee Q$ exists. Then,

(a) $\left(\bigvee Q\right)^{\perp}$ is a lower bound of $Q^{\perp}$. Indeed, if $q \in Q$ then $q \leq \bigvee Q$ and so $\left(\bigvee Q\right)^{\perp} \leq q^{\perp}$.

(b) $\left(\bigvee Q\right)^{\perp}$ is the greatest lower bound of $Q^{\perp}$. Indeed $p \in P$ is a lower bound of $Q^{\perp}$. then for every $q^{\perp} \in Q^{\perp}$ we have that $q^{\perp} \leq p$ so $p^{\perp} \leq \bigvee Q$ and so $\left(\bigvee Q\right)^{\perp} \leq p$.

By (a) and (b) we have that $(\bigvee Q)^{\perp} = \bigwedge Q^{\perp}$. The proof of the second assertion of the Lemma is similar. □

Axiom 1. *The logic $\mathscr{L}(\mathfrak{S})$ of a physical system $\mathfrak{S}$ is a σ-complete orthomodular lattice.*

Definition 2.6. *A σ-complete orthomodular lattice is called a* logic.

Now we shall introduce the important notion of orthogonality in a logic.

Definition 2.7. *If a, b are elements of a logic we shall say that the a is* orthogonal *to b, and we shall write $a \perp b$ if $a \leq b^{\perp}$.*

Note that if $a \leq b^{\perp}$ then $(b^{\perp})^{\perp} \leq a^{\perp}$ or equivalently $b \leq a^{\perp}$ which means that also $b \perp a$. So, if $a \perp b$ we shall say that a, b are *orthogonal.*

Proposition 2.8. *If $a \leq b$ there exists a unique element c of the logic such that $a \perp c$ and $a \vee c = b$.*

P r o o f. Indeed, if $c = b \wedge a^{\perp}$ then $c \leq a^{\perp}$ and so $c \perp a$. By the orthomodular law (OM) we have that $a \vee c = b$ and so a such c exists. If c' is an element of the logic such that $c' \perp a$ and $a \vee c' = b$ then – since $c' \leq a^{\perp}$ and $c' \leq b$– we have that $c' \leq a^{\perp} \wedge b = c$. But $a^{\perp} \leq c'$ and so $b \wedge a^{\perp} \leq b \wedge c' = c'$. Therefore $c' = c$ and so c is unique. $\square$

If $a \vee c = b$ with $a \perp c$ we shall write $b = a \oplus c$ and $c = b \ominus a$.

Proposition 2.9. *Let $(a_n)_{n=1}^{\infty}$ a sequence of elements of a logic L. If $b \perp a_n$ for every n then $b \perp \bigvee_n a_n$ and $b \perp \bigwedge_n a_n$.*

P r o o f. Since for every n we have $a_n \leq b^{\perp}$, then for every n, $b \leq a_n^{\perp}$ and so $b \leq \bigwedge_n a_n^{\perp}$ Therefore, $\bigvee_n a_n = \left(\bigwedge_n a_n^{\perp}\right)^{\perp} \leq b^{\perp}$. Also, since for every n we have $a_n \leq b^{\perp}$, we have that $\bigwedge_n a_n =\leq b^{\perp}$. $\square$

It is not difficult to show that

Proposition 2.10. *If a, b are elements of the logic L with $a \leq b$. Then the set $L(a,b) = \{x \in L : a \leq x \leq b\}$ is also a logic, with $0 = a$, $1 = b$ and $x^{\perp} = b \ominus x$.*

Definition 2.11. *Let L and M be two logics. A mapping $f : L \to M$ is called a* morphism *if for $x, x_i \in L$, $i = 1, 2, \ldots$ we have that*

(1) $f\left(\bigvee_{i=1}^{\infty} x_i\right) = \bigvee_{i=1}^{\infty} f(x_i).$

(2) $f(x^{\perp}) = f(x)^{\perp}$.

If a morphism f is 1-1 is called a monomorphism *and if it is 1-1 and onto is called an* isomorphism.

2.1. Algebras of sets. The standard example of a complete complemented and distributive lattice is the the set $\wp(X)$ of all subsets of a set X, with partial order the inclusion $\subseteq$ of sets. In this case if A, B are subsets of X. $A \vee B = A \cup B$ the union of the sets A and B and in general if $\mathcal{A}$ is any family of subsets of X,

$$\bigvee \mathcal{A} = \bigcup A = \{x \in X : \text{ there exists } A \in \mathcal{A} \text{ with } x \in A\}$$

Also, $A \wedge B = A \cap B$ the intersection of the sets A, and B and in general if $\mathcal{A}$ is any family of subsets of X,

$$\bigwedge \mathcal{A} = \bigcap A = \{x \in X : \text{ for every } A \in \mathcal{A} \text{ we have } x \in A\}$$

Finally, $A' = X \setminus A = \{x \in X : x \notin A\}$ is the usual complement of a set A. The proof of following proposition is left to reader.

Proposition 2.12. *Let X be set. Then $(\wp A, \subseteq)$ is a complete Boolean algebra where*

(1) $1 = X$, $0 = \emptyset$,

(2) $A' = X \setminus A$

(3) *If $\mathcal{A} \subseteq \wp(X)$ is any family of subspaces of X then the greatest lower bound $\bigwedge \mathcal{A}$ of the elements of $\mathcal{A}$ is the intersection $\bigcap A$ of the elements of $\mathcal{A}$. Also, $\bigvee \mathcal{A} = \bigcup A$.*

Definition 2.13. *An* algebra of subsets *of a set X is a family $\mathcal{A}$ of subsets of X such that*

(1) *If $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$*

(2) *If $A \in \mathcal{A}$ then $X \setminus A \in \mathcal{A}$.*

(3) $X \in \mathcal{A}$.

If an algebra of sets satisfies the following property

$(1)'$ *If $A_n \in \mathcal{A}$ for $n = 1, 2, \ldots$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.*

is called a σ-algebra of subsets *of* X.

Since the intersection of σ-algebras is a σ-algebra we may define

Definition 2.14. *The family* $\mathscr{B}(\mathbb{R})$ *of* Borel subsets *of* $\mathbb{R}$ *is the smallest family* $\mathscr{A}$ *of subsets of* $\mathbb{R}$ *which contains the open intervals.*

The family $\mathscr{B}(\mathbb{R})$ is very rich and contains all the interesting (from the physical point of view) subsets of $\mathbb{R}$, like all kinds of intervals (open, closed, semi-open, bounded or unbounded), also every open and closed set of reals, and much more.

3. Observables. Let X be an observable of physical system $\mathfrak{S}$. Then for every set δ of real numbers corresponds a proposition $X \in \delta$ which says that "*the value of observable belongs to the set* δ". For example if $\mathfrak{S}$ is a system consisting from a single particle, an electron and X is the total energy of $\mathfrak{S}$ then $X \in (90, +\infty) \cup (3, 0))$ is the proposition "*the electron has energy* > 90 *or* < 3 ".

Clearly, there exist sets δ of real numbers where $X \in \delta$ is impossible to be verified. For example if $\delta = \mathbb{R} \setminus \mathbb{Q}$ then $X(\delta)$ means that the observable takes only irrational numbers, and this proposition cannot be experimentally verified. Also, we cannot expect that $X = \pi$, *the total energy of the electron is equal to* π, can be verified by a finite number of observations, but we can accept this proposition since we may suppose that with verified this proposition making experiments of more and more accuracy. We shall suppose that $X(\delta)$ will be defined to a family of sets which contains the open intervals of $\mathbb{R}$ and it is closed under countable unions and complements. This family is the family $\mathscr{B}(\mathbb{R})$ of Borel subsets of $\mathbb{R}$. But the main reason that we use the Borel sets is that the main mathematical tool in quantum mechanics is the Lebesgue measure and the corresponding notion of Lebesgue integral. The Lebesgue measure $\lambda(\delta)$ of set cannot be defined for every subset of $\mathbb{R}$ but only for a family of subsets of $\mathbb{R}$ which contains the Borel sets. Also, for the same mathematical reasons, we shall not consider all the functions $f : \mathbb{R} \to \mathbb{R}$ but only the Borel functions.

So every A defines a function $A : \mathscr{B}(\mathbb{R}) \to \mathscr{L}(\mathfrak{S})$, where $A(\delta)$ is the proposition $A \in \delta$. We consider that every such correspondence determines completely an observable:

Definition 3.1. *Let* $\mathfrak{S}$ *be a physical system and* $\mathscr{L}(\mathfrak{S})$ *its logic. An* observable *of* $\mathfrak{S}$ *is function* $X : \mathscr{B}(\mathbb{R}) \to \mathscr{L}(\mathfrak{S})$ *such that*

(1) $X(\mathbb{R}) = 1$.

(2) *For every Borel subset* δ *of* $\mathbb{R}$ *we have that* $X(\mathbb{R} \setminus \delta) = X(\delta)^{\perp}$.

(3) *For every sequence* $(\delta)_{n=1}^{\infty}$ *of pairwise disjoint Borel subsets of* $\mathbb{R}$ *we have that* $X\left(\bigcup_{n=1}^{\infty}\delta_n\right) = \bigvee_{n=1}^{\infty} X(\delta_n)$.

Suppose that X is an observable which can take the values $\{0, 1, 2, 3\}$ and we wish define the observable X^2 which has as possible values the $\{0^2, 1^2, 2^2, 3^2\}$. In general, if $f : \mathbb{R} \to \mathbb{R}$ is a real valued function we want define an observable $Y = f \circ X$, such that if the value of X is in the set δ if and only if the value of Y belongs in the set $f(\delta) = \{f(t) : t \in \delta\}$. So, the value of Y will be in the Borel set δ if the value of X is in the set $f^{-1}(\delta) = \{t : f(t) \in \delta\}$. Since we demand that the value of X must be in a Borel set the function f must have the property that for every $\delta \in \mathscr{B}(\mathbb{R})$ the set $f^{-1}(\delta) \in \mathscr{B}(\mathbb{R})$.

Definition 3.2. *A function* $f : \mathbb{R} \to \mathbb{R}$ *is called a* Borel function *if for every Borel subset* δ *of* $\mathbb{R}$ *the set* $f^{-1}(\delta)$ *is also a Borel set.*

Definition 3.3. *For every observable* $X \in \mathscr{O}(\mathfrak{S})$ *and for every Borel function* $f : \mathbb{R} \to \mathbb{R}$ *we define an observable* $f \circ X$ *by*

$$f \circ X(\delta) = X\left(f^{-1}(\delta)\right)$$

Proposition 3.4. *For every observable* $X \in \mathscr{O}(\mathfrak{S})$ *and for every sequence* $(\delta_n)_{n=1}^{\infty}$ *of Borel sets we have that*

$$\text{If } \delta_1 \subseteq \delta_2 \text{ then } X(\delta_1) \leq X(\delta_2) \tag{1}$$

$$X\left(\bigcup_{n=1}^{\infty}\delta_n\right) = \bigvee_{n=1}^{\infty} X(\delta_n) \tag{2}$$

$$X\left(\bigcap_{n=1}^{\infty}\delta_n\right) = \bigwedge_{n=1}^{\infty} X(\delta_n) \tag{3}$$

Also, f *and* g *are Borel functions and* X *is an observable then*

$$(f \circ g) \circ X = f \circ (g \circ X)$$

Proof. Since $\delta_1 \subseteq \delta_2$ we have that $\delta_2 = \delta_1 \cup (\delta_2 \setminus \delta_1)$ and so $X(\delta_2) = X(\delta_1) \vee X(\delta_2 \setminus \delta_1)$. This proves (1).
Since for every n, $X(\mathbb{R} \setminus \delta_n) \perp X(\delta_n)$ and $X(\mathbb{R} \setminus \delta_n) \vee X(\delta_n) = 1$, by Lemma 2.5,

(2) implies (3) and so it is enough to prove (2). Let $\delta = \bigcup_n \delta_n$. By (1), for every n we have $X(\delta_n) \leq X(\delta)$ and so

$$\bigvee_n X(\delta_n) \leq X(\delta) \tag{4}$$

On the other hand, if we let $\varepsilon_1 = \delta_1$ and $\varepsilon_{n+1} = \delta_{n+1} \setminus \left(\bigcup_{i=1}^{n} \delta_i\right)$. The sets ε_n are pairwise disjoint, $\varepsilon_n \subseteq \delta_n$ and $\bigcup_n \varepsilon_n = \delta$. Therefore,

$$X(\delta) = \bigvee_n X(\epsilon_n) \leq \bigvee_n X(\delta_n) \tag{5}$$

By (4) and (5) we have (2). $\square$

Definition 3.5. *The* spectrum $\sigma(X)$ *of an observable is defined by*

$$\sigma(X) = \bigcap \{C : C \textit{ is closed and } X(C) = 1\}$$

An observable X is called bounded *if its spectrum is a compact set.*

Exercise 3.6. Prove that $X(\sigma(X)) = 1$ and so the spectrum of X is the smallest closed subset C of $\mathbb{R}$ which have the property $X(C) = 1$

Exercise 3.7. Prove that if $X(\{\lambda\})) \neq 0$ then $\lambda \in \sigma(X)$. A such λ is called a *strict value* of X. Therefore the strict values of an observable belong to the spectrum of the observable but the converse is not in general true.

4. States. A *probability measure* on $\mathbb{R}$ or simply a *probability* on $\mathbb{R}$ is a function $\mathrm{P} : \mathscr{B}(\mathbb{R}) \to [0, 1]$ such that for every Borel subset δ of real numbers corresponds a real number $P(\delta)$ such that

(1) $\mathrm{P}(\emptyset) = 0$.
(2) For every Borel set δ, $\mathrm{P}(\mathbb{R} \setminus \delta) = 1 - \mathrm{P}(\delta)$.
(3) For every sequence of pairwise disjoint Borel sets $(\delta_n)_{n=1}^{\infty}$ we have that $\mathrm{P}\left(\bigcup_{n=1}^{\infty} \delta_n\right) = \sum_{n=1}^{\infty} \mathrm{P}(\delta_n)$.

By $\mathcal{M}(\mathbb{R})$ we shall denote the set of all probability measures of $\mathbb{R}$.

For every function $f : \mathbb{R} \to \mathbb{R}$ and for every subset δ of $\mathbb{R}$ we have that $f^{-1}(\mathbb{R} \setminus \delta) = \mathbb{R} \setminus f^{-1}(\delta)$. Also, if $(\delta_n)_{n=1}^{\infty}$ is sequence of pairwise disjoint sets the sets $f^{-1}(\delta_n)$ are pairwise disjoint and $f^{-1}\left(\bigcup_{n=1}^{\infty} \delta_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(\delta_n)$.

These remarks imply that if $\mathrm{P} \in \mathcal{M}(\mathbb{R})$ and f is Borel function then the mapping $f \circ \mathrm{P} : \mathscr{B}(\mathbb{R}) \to [0,1]$ defined by

$$f \circ \mathrm{P}(\delta) = \mathrm{P}\left(f^{-1}(\delta)\right), \quad \delta \in \mathscr{B}(\mathbb{R}$$

is a probability.

Suppose that X is an observable and δ a Borel set. The truth or the falsity of the proposition $X(\delta)$ depends of the "state" of the system. Also, basic principle of quantum mechanics (and of classical statistic mechanics too) is the probabilistic nature of quantum mechanical systems. That means that in general we cannot say that a proposition a is true or false but to calculate the probability $p(a)$ of truth of the proposition. So if our system "is in a state s" then for every observable X and every Borel set δ we may find a number $0 \leq s(X,\delta) \leq 1$ which is "*the probability that the value of X is in δ.*"

Consider now the function $\mathrm{P}_X : \mathscr{B}(\mathbb{R}) \to [0,1]$, where $\mathrm{P}_X(\delta) = s(X,\delta)$. It is clear (since it is sure that the observable will take some value) that $\mathrm{P}_X(\mathbb{R}) = 1$ and $\mathrm{P}_X(\emptyset) = 0$. Also if δ is Borel set and $(\delta)_{n=1}^{\infty}$ a sequence of pairwise disjoint Borel subsets of $\mathbb{R}$ with $\bigcup_{n=1}^{\infty} \delta_n = \delta$ we must have that $s(X,\delta) = \sum_{n=1}^{\infty} s(X,\delta_n)$.

So, *a state s determines for every observable X a probability measure $\mathrm{P}_X \in \mathcal{M}(\mathbb{R})$ by $\mathrm{P}_X(\delta) = s(X,\delta_n)$.*

Definition 4.1. *Let $\mathfrak{S}$ be a physical system and $\mathscr{O}(\mathfrak{S})$ the set of all its observables. A* state *of $\mathfrak{S}$ is function $s : \mathscr{O}(\mathfrak{S}) \to \mathcal{M}(\mathbb{R})$ which assigns to every observable $X \in \mathscr{O}(\mathfrak{S})$ a probability measure $\mathrm{P}_X^s \in \mathcal{M}(\mathbb{R})$ such that*

$$\mathrm{P}_{f\circ X}^s = f \circ \mathrm{P}_X^s \tag{6}$$

for every Borel function $f : \mathbb{R} \to \mathbb{R}$.

There is a very simple manner to describe the states of a system as *measures defined on the logic* of the system.

Definition 4.2. *A* measure on a logic *L is a function $\mu : L \to [0,1]$ such that*

(1) $\mu(0) = 0$ *and* $\mu(1) = 1$.

(2) *If* $(a_n)_{n=1}^{\infty}$ *is a sequence of elements of* L *such that* $a_n \perp a_m$ *if* $n \neq m$ *then* $\mu\left(\bigvee_{n=1}^{\infty} a_n\right) = \sum_{n=1}^{\infty} \mu(a_n)$.

Every measure μ on the logic $\mathscr{L}(\mathfrak{S})$ defines a state s_μ. Indeed, if $X \in \mathscr{O}(\mathfrak{S})$ is an observable of the system we define the function $\mathrm{P}_X^\mu : \mathscr{B}(\mathbb{R}) \to [0,1]$ by

$$\mathrm{P}_X^\mu(\delta) = \mu\left(X(\delta)\right) \tag{7}$$

Suppose that $(\delta_n)_{n=1}^{\infty}$ is a sequence of pairwise disjoint Borel sets and $\delta = \bigcup_{n=1}^{\infty} \delta_n$.

Then $X(\delta) = \bigvee_{n=1}^{\infty} X(\delta_n)$ and so

$$\mathrm{P}_X^\mu(\delta) = \mu\left(\bigvee_{n=1}^{\infty} X(\delta_n)\right) = \sum_{n=1}^{\infty} \mu\left(X(\delta_n)\right) = \sum_{n=1}^{\infty} \mathrm{P}_X^\mu(\delta_n)$$

Also $1 = \mathrm{P}_X^\mu(\mathbb{R}) = \mu\left(X(\delta \cup (\mathbb{R} \setminus \delta)\right) = \mu\left(X(\delta) \vee X(\mathbb{R} \setminus \delta)\right) = \mathrm{P}_X^\mu(\delta) + \mathrm{P}_X^\mu(\mathbb{R} \setminus \delta)$. This shows that P_X^μ is a probability measure. Finally, for every Borel function f and every Borel set δ we have that

$$\mathrm{P}_{f \circ X}^\mu(\delta) = \mu\left(f \circ X(\delta)\right) = \mu\left(X(f^{-1}\delta)\right) = \mathrm{P}_X^\mu(f^{-1}(\delta) = f \circ \mathrm{P}_X^\mu(\delta)$$

Therefore the mapping $X \mapsto \mathrm{P}_X^\mu$ defines a state.

In order to prove that every state s defines a measure on the logic of the system we shall use the notion of a *question* due to Mackey [9].

Definition 4.3. *A* question *is an observable* Q *such that* $Q(\{0,1\}) = 1$.

If $Q(\{1\}) = a_Q$ then the proposition a_Q defines completely the question Q. Indeed, for every Borel set δ

$$Q(\delta) = \begin{cases} a_Q & \text{if } 1 \in \delta \\ a_Q^\perp & \text{if } 1 \notin \delta \end{cases} \tag{8}$$

Conversely every proposition a defines a question Q_a by

$$Q_a(\delta) = \begin{cases} a & \text{if } 1 \in \delta \\ a^\perp & \text{if } 1 \notin \delta \end{cases} \tag{9}$$

and the mapping $a \mapsto Q_a$ is bijection between the propositions of the system with the questions.

Let $X \mapsto \mathrm{P}_X$ be a state. We define for every $a \in \mathscr{L}(\mathfrak{S})$

$$\mu(a) = \mathrm{P}_{Q_a}(\{1\})$$

It is not hard to show that μ is a measure on the logic of the system (for details see [15, p. 50–51]

5. Boolean algebras–classical logics. In classical statistical mechanics the states s are probability measures on a topological space $\mathcal{X}$, which is called the *phase space* and the observables are Borel functions $f : \mathcal{X} \to \mathbb{R}$. If X is an observable and δ is a Borel set of real numbers then $s\left(X^{-1}(\delta)\right)$ is the probability that, when the system is in the state s, the value of X lies inside the set δ. In this section we shall see that this exactly the case if logic $\mathcal{L}$ of a system $\mathfrak{S}$ is distributive, that is a Boolean algebra. For this reason we shall call a system $\mathfrak{S}$ with a a Boolean algebra as logic a *classical system* and a system $\mathfrak{S}$ which its logic is not Boolean a *quantum system*. So, we can say that the property that distinguishes classical and quantum systems is the distributivity law.

The distributive lattices are in fact sublattices of the lattice $\wp(\mathcal{X})$ of all subsets of a set X, which means that for every distributive lattice $\mathcal{L}$ there exists a set $\mathcal{X}$ and a function $f : \mathcal{L} \to \wp(\mathcal{X})$ such that for any $a, b \in \mathcal{L}$ we have that $a \leq b$ if and only if $f(a) \subseteq f(b)$. This implies that $f(a \vee b) = f(a) \cup f(b)$ and $f(a \wedge b) = f(a) \cap f(b)$ which means that $\mathcal{L}$ is (lattice) isomorphic to a sublattice of $(\wp(\mathcal{X}), \subseteq)$. For Boolean algebras we have a stronger result, the *Stone's Representation Theorem.* In order, to describe this important theorem we need some notions. A topological space $\mathcal{X}$ is called a *totally disconnected space* if every open subset of $\mathcal{X}$ is the union of a family of subsets of $\mathcal{X}$ which are in the same time open and closed. A compact Hausdorff totally disconnected space is called a *Stone space.* For every topological space the family $\mathrm{OC}(\mathcal{X})$ of all open and closed subsets of $\mathcal{X}$ is Boolean algebra.

Definition 5.1.

(1) *A* filter *of a lattice L is a subset F of L such that*

(*a*) $0 \notin F$.

(*b*) *If $a \in F$ and $a \leq b$ then $b \in F$.*

(c) *If $a, b \in F$ then $a \wedge b \in F$.*

A filter which is not properly contained in any other filter is called a maximal filter*.*

(2) *The set $\mathscr{M}(\mathbb{B})$ of all maximal filters of a Boolean algebra $\mathbb{B}$ is denoted by $\mathscr{M}(\mathbb{B})$ and it is called the* Stone space *of $\mathbb{B}$.*

(3) *In the Stone space $\mathcal{X} = \mathscr{M}(\mathbb{B})$ we define a topology which has as basis the sets of the form*

$$\mathcal{X}_a = \{F \in \mathscr{M}(\mathbb{B}) : a \in F\}, \quad a \in \mathbb{B}$$

This means that the open sets of this topology are the unions of sets of the form $\mathcal{X}_a$. This topology is called the Stone Topology *of the boolean algebra $\mathbb{B}$.*

Remark 5.2. Note that a filter of a distributive lattice is maximal if and only if $a \vee b \in F$ then $a \in F$ or $b \in F$. This is a characteristic property of distributivity.

We can now state (without a proof[1]) the Stone's Representation Theorem:

Theorem 5.3 (Stone's Representation Theorem)**.** *Let $\mathbb{B}$ be a Boolean algebra. Then the Stone space $\mathcal{X} = \mathscr{M}(\mathbb{B})$ of $\mathbb{B}$ is a compact Hausdorff totally disconnected topological space and the map $a \mapsto \mathcal{X}_a$ is a lattice isomorphism of $\mathbb{B}$ with the Boolean algebra* $\mathrm{OC}(\mathcal{X})$ *of all open and closed subsets of $\mathcal{X}$.*

If the logic $\mathcal{L}$ of a system $\mathfrak{S}$ is a Boolean algebra then it is (by the definition of the logic) a Boolean σ-algebra and then if $\mathcal{X}$ is the Stone space of $\mathcal{L}$ we can find a σ-algebra $\mathcal{A}$ of Borel subsets of $\mathcal{X}$ and a σ-homeomorphism $h : \mathcal{A} \to \mathcal{L}$, from $\mathcal{A}$ onto $\mathcal{L}$ (this is the *Theorem of Loomis*, see [15] for the proof). In fact, if $OC(\mathcal{X})$ is the boolean algebra of open and closed subsets of $\mathcal{X}$ then $\mathcal{L}$ is isomorphic to $OC(\mathcal{X})$ and we take as $\mathcal{A}$ to be the smallest σ-algebra of subsets of $\mathcal{X}$ which contains $OC(\mathcal{X})$. Using Loomis theorem we can show that the states in the classical case are the usual probability measures.

6. Simultaneous observability and the center of a logic. Let L be a logic. A *sublattice* of L is subset L' of L such that if $a, b \in L'$ then $a \vee b$ and $a \wedge b$ belong to L'. Since the intersection of sublattices is again a or every subset S of L there exists the smallest sublattice $\mathrm{Lat}(S)$ of L that contains S.

[1] for a proof see the classical book of G. Birkhoff [2]

Definition 6.1. *Let $(L, \leq, \perp)$ be a logic. Two propositions $a, b \in L$ are said to be* simultaneously verifiable *if there exist $a_1, a_2, c \in L$, pairwise orthogonal such that $a = a_1 \vee c$ and $b = b_1 \vee c$. If a, b are simultaneously verifiable we shall write $a \leftrightarrow b$ and if they are not simultaneously verifiable shall write $a \not\leftrightarrow b$*

Proposition 6.2. *Let $(L, \leq, \perp)$ be a logic and $a, b \in L$. The following are equivalent:*

(1) $a \leftrightarrow b$.

(2) *The sublattice of L generated by $a, b, a^\perp, b^\perp$ is distributive.*

(3) $a = (a \wedge b) \vee (a \wedge b^\perp)$.

(4) $a \wedge (a^\perp \vee b) = a \wedge b$.

(5) $b^\perp = (a \wedge (a^\perp \vee b^\perp)) \vee (a^\perp \wedge b^\perp)$.

(6) $a \wedge (a^\perp \vee b^\perp) \perp b \wedge (a^\perp \vee b^\perp)$.

(7) $(a \wedge b) \vee (a^\perp \wedge b) \vee (a \wedge b^\perp) \vee (a^\perp \wedge b^\perp) = 1$.

Definition 6.3. *Let L be logic and $M \subseteq L$.*

(1) *The* commutant *of M is the set*

$$C(M) = \{a \in L : \text{ for every } x \in L, a \leftrightarrow x\}$$

(2) *The* center *of a L is the set*

$$C(L) = \{a \in L : \text{ for every } x \in L, a \leftrightarrow x\}$$

(3) *Clearly, $\{0, 1\} \subseteq C(L)$ and if $C(L) = \{0, 1\}$ we shall say that L has a* trivial center.

Definition 6.4. *Let $(L_1, \leq_1, \perp_1)$, $(L_2, \leq_2, \perp_2)$ be two logics. The direct sum $L_1 \oplus L_2$ of L_1, L_2 is the logic $(L_1 \times L_2, \leq, \perp)$ where if $x_1, y_1, x \in L_1$ and $x_2, y_2, y \in L_2$,*

(1) $(x_1, x_2) \leq (y_1, y_2)$ *if* $x_1 \leq_1 y_1$ *and* $x_2 \leq_2 y_2$.

(2) $(x, y)^\perp = (x^\perp, y^\perp)$.

Proposition 6.5. *Let $(L_1, \leq_1, \perp_1)$, $(L_2, \leq_2, \perp_2)$ be two logics and $L_1 \oplus L_2$ their direct sum. If $(x_1, x_2), (y_1, y_2) \in L_1 \oplus L_2$ then $(x_1, x_2) \leftrightarrow (y_1, y_2)$ if and only if $x_1 \leftrightarrow y_1$ and $x_2 \leftrightarrow y_2$. Therefore*

$$C(L_1 \oplus L_2) = \{(x, y) : x \in C(L_1) \text{ and } y \in C(L_2)\}$$

In particular, the center if a direct sum of logics is never trivial.

Proof. This follows easily from the facts that

$$(x_1, x_2) \vee (y_1, y_2) = (x_1 \vee y_2, y_1 \vee y_2), \quad (x_1, x_2) \wedge (y_1, y_2) = (x_1 \wedge y_2, y_1 \wedge y_2). \quad \square$$

Definition 6.6. *We say that a logic L is* reducible *if there exist logics L_1 and L_2 such that L is isomorphic to $L_1 \oplus L_2$. If L is not reducible we shall say that L is* irreducible.

Let L be a logic. If $a \in L$ we denote by $[0, a]$ the logic where $[0, a] = \{x \in L : 0 \leq x \leq a\}$, the order relation is the usual one and the complement of a $x \in [0, a]$ as a member of $[0, a]$ is defined to be $x^\perp \wedge a$ (see also Proposition 2.10).

Proposition 6.7. *A logic L is irreducible if and only if its center is trivial.*

Proof. Suppose that L has not trivial center and choose $c \in C(L)$ such that $c \neq 0$ and $c \neq 1$. Let $L_1 = [0, c]$ and $L_1 = [0, c^\perp]$. Since for every $x \in L$ we have that $x \leftrightarrow x$ we shall have that

$$x = (x \wedge c) \vee (x \wedge c^\perp)$$

and therefore the mapping $x \mapsto (x \wedge c, x \wedge c^\perp)$ is an isomorphism from L onto $L_1 \oplus L_2$. The converse follows immediately from Proposition 6.5. $\square$

In the classical case the center of the logic is equal to whole logic. In the pure quantum case the center is trivial. In physics there exists a large number of intermediate cases in which the center is strictly smaller than the whole lattice but contains nontrivial propositions. In that case we say that the system possesses *superselection rules*.

7. Concrete logics.

7.1. The Hilbert space formulation of quantum mechanics. In the Hilbert space formulation of quantum mechanics the observables are simply self-adjoint linear operators on a separable Hilbert space $\mathcal{H}$. The spectrum $\sigma(T)$

of a linear operator ? is the set of all $\lambda \in \mathbb{R}$ such that the operator $T - \lambda I$ is not invertible. If the spectrum of an operator (observable) is finite then the observable can have only finite many values and if the spectrum is discrete then the observable has a discrete set of possible values. Note that, since the operators are self-adjoint, their spectrum is a subset of the real line. In general the observables T cannot be defined for every $x \in \mathcal{H}$ but their domain $D(T)$ is a linear subspace of $\mathcal{H}$ which is dense in $\mathcal{H}$, in the sense that every $x \in \mathcal{H}$ is a limit of a sequence of vectors belonging to the domain $D(T)$ of the observable.

Also, the observables are not in general bounded. An observable T is said to be *bounded* as a linear operator if there is a constant C such that $\|Tx\| \leq C$ for every $x \in D(T)$ with $\|x\| \leq 1$. Note that T is bounded if and only if it is uniformly continuous as a function and therefore we can extend T in $\overline{D(T)}$ and therefore we can always assume that the domain of T is a closed subspace of $\mathcal{H}$. Since we also assume that the domain of observables is a dense subset of $\mathcal{H}$ in the case of bounded observables their domain is the whole space $\mathcal{H}$. If a linear operator $T : \mathcal{H} \to \mathcal{H}$ is bounded then there exists a unique linear and bounded operator $T^* : \mathcal{H} \to \mathcal{H}$ which has the property

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

for every $x, y \in H$, which called the *adjoint operator* of T. A bounded linear operator is called *self-adjoint* if $T = T^*$. But if T is not bounded it is more difficult to define what means that T is self-adjoint. We shall give the elements of the theory of Hilbert spaces and the linear operators in the next paragraphs.

Example 7.1. In the simplest case of a particle moving on a straight line the model is the one dimensional Hilbert space $L^2(\mathbb{R}, \mu)$ (of the equivalent classes) of all square integrable functions $f : \mathbb{R} \to \mathbb{C}$ and μ is the usual Lebesgue measure. A Lebesgue measurable function $f : \mathbb{R} \to \mathbb{C}$ is said to be *square integrable* if the (Lebesgue integral) $\displaystyle\int_{-\infty}^{+\infty} |f(x)|^2 dx < +\infty$. We consider that two measurable functions $f, g : \mathbb{R} \to \mathbb{C}$ as *equal* if $f = g$ almost everywhere in the sense that the set $\{x \in \mathbb{R} : f(x) \neq g(x)\}$ of all points where the two functions have different values has Lebesgue measure equal to zero.

The linear space $L^2(\mathbb{R}, \mu)$ is a Hilbert space with respect the inner product $\langle f, g\rangle = \displaystyle\int_{-\infty}^{+\infty} f(x)\overline{g(x)}dx$ and with norm $\|f\| = \displaystyle\int_{-\infty}^{+\infty} |f(x)|^2 dx < +\infty$. If ϕ is a measurable function $\phi : \mathbb{R} \to \mathbb{C}$ then ϕ can be considered as a linear operator $T_\phi : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ where $T_\phi(f) = \phi f$ for every $f \in L^2(\mathbb{R})$. If ϕ is not a bounded function then T_ϕ cannot be defined on all the space $L^2(\mathbb{R})$ and it is not in general bounded. An operator of the form T_ϕ is called a *multiplication operator*.

The *observable O of the position* of the particle is the multiplication operator T_ϕ defined by the identity function $\phi : \mathbb{R} \to \mathbb{C}$, that is $\phi(x) = x$ for every $x \in \mathbb{R}$. So $X(f)(x) = xf(x)$ for every $x \in \mathbb{R}$.

The domain of this operator is the set

$$D(X) = \{f \in L^2(\mathbb{R}) : xf(x) \in L^2(\mathbb{R})\}$$

Since $D(X)$ contains all the square integrable functions with compact support and therefore is a dense subset of all $L^2(\mathbb{R})$. Also $D(X) \neq L^2(\mathbb{R})$ since the function

$$f(x) = \begin{cases} 1 & \text{if } x \in [-1,1] \\ \dfrac{1}{x} & \text{if } |x| > 1 \end{cases}$$

is a square integrable function but the function $xf(x)$ is not.

The position is not a bounded operator. To see this we simply consider the characteristic functions f_n of the intervals $[n, n+1]$, $n \in \mathbb{N}$ that is

$$f_n(x) = \begin{cases} 1 & \text{if } x \in [n, n+1] \\ 0 & \text{if } x \notin [n, n+1] \end{cases}$$

and $n = 0, 1, 2, \ldots$. Clearly $f_n \in L^2(\mathbb{R})$, $\|f_n\| = 1$ but

$$\|X(f_n)\|^2 = \int_n^{n+1} x^2 dx = \frac{(n+1)^3 - n^3}{3} > n^2 + n$$

Note also that X is a self-adjoint operator, the spectrum of X is the whole real line but X has no eigenvalues. Indeed, suppose that there exists a $\lambda \neq 0$ which is an eigenvalue of X. Then for some $f \in D(X)$ with $f \neq 0$ we must have that $X(f) = \lambda f$ which means that $xf(x) = \lambda f(x)$ almost everywhere. Since $f \neq 0$ the set $\{x : f(x) \neq 0\}$ has measure $\neq 0$ and therefore $\lambda = x$ for at least two different values of x! A contradiction.

The theory of unbounded operators is much more complicated than that of bounded ones. In the following paragraphs we shall give some basic notions and facts of this theory, in order to examine the so called Standard Quantum Logic, which is the logic of all projections of a separable Hilbert space.

7.2. The standard quantum logic. *In this paragraph and in the following paragraphs by the term Hilbert space we shall always mean a **separable** Hilbert space.*

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$. The set $B(\mathcal{H})$ of the bounded linear operators $T:\mathcal{H}\to\mathcal{H}$, is a $*$-algebra with respect the operation $(S,T)\mapsto S+T$, $(S,T)\mapsto ST$, where ST is the usual composition of operators and involution $T\mapsto T^*$ where T^* is the unique linear bounded operator with the property, for every $x,y\in\mathcal{H}$

$$\langle Tx,y\rangle=\langle x,T^*y\rangle$$

A *projection* is an element $P\in B(\mathcal{H})$ such that $P=P^*=P^2$.

The set $S(\mathcal{H})$ of all closed subspaces of $\mathcal{H}$ is also a complete ortholattice lattice with respect the partial order of the usual set inclusion $\subseteq$, where and if $M,N\in S(\mathcal{H})$ then

$$M\wedge N=M\cap N,\quad M\vee N=\bigcup\{V\in S(\mathcal{H}):M\cup N\subseteq V\}$$

and complement

$$M^\perp=\{x\in\mathcal{H}:x\perp M\}$$

where $x\perp M$ means $\langle x,y\rangle=0$ for every $y\in M$. The fact that $(S(\mathcal{H}),\subseteq)$ is a complete lattice follows easily from the fact that the intersection of closed subspaces is a closed subspace and therefore if $\mathcal{F}$ is a family of closed subspaces then their greatest lower bound and least upper bound are

$$\bigwedge\mathcal{F}=\bigcap\{S:S\in\mathcal{F}\},\bigvee\mathcal{F}=\bigcap\left\{S:S\supseteq\bigcup\mathcal{F}\right\}$$

The fact that $M\mapsto M^\perp$ is an orthocomplement ia a consequence of the following basic geometric property of Hilbert spaces.

Lemma 7.2. *Let $\mathcal{H}$ be a Hilbert space and M a subspace of $\mathcal{H}$. Then for every $x\in\mathcal{H}$ there exists a unique $x_M\in M$ such that*

$$\|x-x_M\|=\operatorname{dist}(x,M)=\inf\{\|x-y\|:y\in M\}\ \textit{and}\ x-x_M\perp M$$

Proof. (Halmos) Let $\delta=\operatorname{dist}(x,M)=\inf\{\|x-y\|:y\in M\}$. We choose a sequence $(y_n)_n$ of elements of M such that $\|x-y_n\|\to\delta$. Since $\mathcal{H}$ is an inner product space holds the parallelogram law:

$$\|a+b\|^2+\|a-b\|^2=2(\|a\|^2+\|b\|^2)$$

Let $n, m \in \mathbb{N}$. Since $\dfrac{y_n + y_m}{2} \in M$ we have that $\left\| \dfrac{y_n + y_m}{2} - x \right\| \leq \delta$ and by the parallelogram low we obtain

$$\begin{aligned} \|y_n - y_m\|^2 &= 2\left(\|y_n - x\|^2 + \|y_m - x\|^2 - 2\left\| \frac{y_n + y_m}{2} - x \right\|^2 \right) \\ &\leq 2\left(\|y_n - x\|^2 + \|y_m - x\|^2 - 2\delta^2 \right) \to 2(\delta^2 + \delta^2 - 2\delta^2) \\ &= 0 \end{aligned}$$

So $(y_n)_n$ is a Cauchy sequence and since $\mathcal{H}$ is a complete it converges to an element $x_M \in M$. The continuity of the norm implies that $\|x - x_M\| = \delta$.
In order to prove that $z = x - x_M \perp M$ we consider $y \in M$ with $y \neq 0$. Let $\lambda \in \mathbb{R}$. Then $\|z\| = \delta$ and since $\lambda\langle z, y\rangle y \in M$ we have $\|z + \lambda\langle z, y\rangle y\| \geq \delta$ and so $\|z + \lambda\langle z, y\rangle y\|^2 - \|z\|^2 \geq \delta^2 - \delta^2 = 0$ which implies that

$$|\langle z, y\rangle|^2 \left(\lambda^2 \|y\|^2 + 2\lambda \right) \geq 0$$

and if we choose the value of λ to be $\lambda = -\dfrac{1}{\|y\|^2}$ we shall have that $-|\langle z, y\rangle|^2 \geq 0$ and therefore $\langle z, y\rangle = 0$.

Finally, suppose that there exist $x_1, x_2 \in M$ such that $x - x_1 \perp M$ and $x - x_2 \perp M$. Then $\langle x_2 - x, -x_1\rangle = 0$ and $\langle x - x_1, x_2\rangle = 0$. Adding, we obtain that $\langle x_1 - x_2, x_1 - x_2\rangle = 0$ and so $x_1 = x_2$. Therefore, x_M is unique. □

Lemma 7.3. $M^{\perp\perp} = M$

Proof. Note that

(1) $M \cap M^{\perp} = \{0\}$, since if $x \in M \cap M^{\perp}$ then $\langle x, x\rangle = 0$ and so $x = 0$.
(2) $M \subseteq M^{\perp\perp}$

Suppose that there exists $x \in M^{\perp\perp} \setminus M$. Clearly, $x \neq 0$. Let $x_M \in M$ such that $x - x_M \in M^{\perp}$. Since $x_M \in M$ we have that $x_M \in M^{\perp\perp}$ and therefore $x - x_M \in M^{\perp\perp} \cap M^{\perp}$. So, by (1), $x = x_M = 0$, a contradiction. □

By the previous lemma we have that

Proposition 7.4. $(P(\mathcal{H}), \leq)$ *is complete ortholattice.*

Remark 7.5. The prove that $M^{\perp\perp} = M$ we used in Lemma 7.2 the fact that a Hilbert space is a complete inner product space

If $\mathcal{H}$ is an inner product space which is not complete then it is not in general true that for every closed subspace M of $\mathcal{H}$we shall have $M^{\perp\perp} = M$. Let us denote by $S(\mathcal{H})$ the complete lattice of all closed subspaces of $\mathcal{H}$ and let

$$C(\mathcal{H}) = \left\{M \in S(\mathcal{H}) : M^{\perp\perp} = M\right\}$$

Then $(C(\mathcal{H}), \subseteq)$ is an ortholattice. Amemiya and Araki [1] proved that $(C(\mathcal{H}), \subseteq)$ is orthomodular if and only if $\mathcal{H}$ is a Hilbert space, and so if and only if $C(\mathcal{H}) = S(\mathcal{H})$.

If $x \in \mathcal{H}$ then for every $M \in S(\mathcal{H})$ there exists a unique decomposition of x

$$x = x_M + x_M^{\perp}$$

with $x_M \in M$ and $x_M^{\perp} \in M^{\perp}$. It is easy to see that the mapping

$$P_M : \mathcal{H} \to \mathcal{H}, \;\; P_M(x) = x_M$$

is a projection such that for every $M \in S(\mathcal{H})$ and every $Q \in P(\mathcal{H})$ we have

$$P_M(\mathcal{H}) = M, \;\; P_{Q(\mathcal{H})} = Q \tag{10}$$

and

$$P_M \leq P_N \text{ if and only if } M \subseteq N \tag{11}$$

By (10) and (11) we see that the mapping $M \mapsto P_M$ is a bijection from $S(\mathcal{H})$ onto $P(\mathcal{H})$ which respects the orders and therefore $S(\mathcal{H})$n and $P(\mathcal{H})$ as isomorphic as otholattices and since $(S(\mathcal{H}), \subseteq)$ is a complete otholattice the same holds for $(P(\mathcal{H}), \leq)$ respect the partial order

$$P \leq Q \text{ if and only if } P(\mathcal{H}) \subseteq Q(\mathcal{H})$$

The lattice operations in $(P(\mathcal{H}), \leq)$ have interesting connections with the algebraic operations of $B(\mathcal{H})$. We present a list of them

Proposition 7.6.

(i) $P \leq Q$ *if and only if* $PQ = P$.

(ii) $P^{\perp} = 1 - P$.

(iii) $P \perp Q$ *if and only if* $PQ = 0$

(iv) $PQ = QP$ *implies that* $P \wedge Q = PQ$ *and* $P \vee Q = P + Q - PQ$.

(v) $P \leftrightarrow Q$ *if and only if* $PQ = PQ$.

(vi) $P \perp Q$ *implies that* $P \vee Q = P + Q$.

Theorem 7.7. *$(P(\mathcal{H}), \leq)$ is a logic.*

Proof. By Proposition 7.4, $(P(\mathcal{H}), \leq)$ is a complete ortholattice. It remains to show that in $(P(\mathcal{H}), \leq)$ holds the orthomodular law:

$$\textit{If } P \leq Q \textit{ then } Q = P \vee (Q \wedge P^{\perp}).$$

Since, $P \leq Q$ we have that $Q(1-P) = (1-P)Q = Q - P$ and so by (iv) $Q \wedge P^{\perp} = Q(1-P) = Q - P$. By (vi) we shall have that

$$P \vee (Q \wedge P^{\perp}) = P + (Q - P) = P \qquad \square$$

Definition 7.8. *Let $\mathcal{H}$ We shall refer to logic of the form $(P(\mathcal{H}), \leq)$ as the* standard logic on *$\mathcal{H}$.*

Theorem 7.9. *$(P(\mathcal{H}), \leq)$ is a modular logic if and only if $\mathcal{H}$ is a finite dimensional vector space.*

Proof. If $\mathcal{H}$ is finite dimensional then every subspace of $\mathcal{H}$ is closed and we know that the lattice of all subspaces of a linear space is modular.

We shall show that if $\mathcal{H}$ is an infinite dimensional Hilbert space we can find closed subspaces M, N of $\mathcal{H}$ such that $M \cap N = \{0\}$ and $M + N$ is not closed. If $L = \overline{M+N} = M \vee N$ then $M \vee (L \wedge N) = M \cap N = \{0\}$ but $L \wedge (M \vee N) = L$. So, the modular law does not hold.

In order to construct the subspaces M and N we consider two sequences $(e_n)_{n=1}^{\infty}$, $(f_n)_{n=1}^{\infty}$ of elements of $\mathcal{H}$ such that $\langle e_i, f_j\rangle = 0$, $\langle e_i, e_j\rangle = \delta_{ij}$, $\langle f_i, f_j\rangle = \delta_{ij}$, where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$, and two sequences of real numbers $(\lambda_n)_{n=1}^{\infty}$, $(\mu_n)_{n=1}^{\infty}$ such that $\sum\limits_{n=1}^{\infty} \lambda_n^2 < +\infty$ and $\mu_n \to 1$, $\mu_n \neq 0$ for every n (take $\lambda_n = \sin\dfrac{1}{n}, \mu_n = \cos\dfrac{1}{n}$). Then M is smallest closed subspace of $\mathcal{H}$ containing the vectors $(f_n)_{n=1}^{\infty}$ and N is the smallest closed subspace of $\mathcal{H}$ containing the vectors $g_n = \lambda_n e_n + \mu_n f_n$, $n = 1, 2, \ldots$. $\square$

7.3. Linear operators and observables of the standard logic. Let $\mathcal{H}$ be a Hilbert space. A *linear operator* is a mapping T from a linear subspace D of $\mathcal{H}$, called the *domain* of T into $\mathcal{H}$. If S, T are linear operators then we define their sum $S+T$ to be the operator denoted by $S+T$ with $D(S+T) = D(S) \cap D(T)$ and $(S+T)x = Sx + Tx$ for every $x \in D(S) \cap D(T)$. Also we define the operator ST to be the operator with domain to be the set

$$D(ST) = \{x \in D(T) : Tx \in D(S)\} \text{ and } (ST)x = S(Tx)$$

Finally, for every linear operator T the adjoint operator T^* of T, is the operator where

(1) The domain $D(T^*)$ of T is the set of all $x \in H$ such that there exists $y \in \mathcal{H}$ such that $\langle Tw, x\rangle = \langle w, y\rangle$ for every $w \in D(T)$.

(2) We set $T^*x = y$, where $x \in D(T^*)$ and y is defined in (1) .

Therefore, we have

$$\langle Tx, y\rangle = \langle x, T^*y\rangle, \text{ if } x \in D(T), y \in D(T^*)$$

Definition 7.10. *A linear operator T is said to be*

(1) densely defined *if its domain $D(T)$ is a dense subset of $\mathcal{H}$*

(2) closed *if for every sequence $(x_n)_n$ of elements of $D(T)$ such that $\lim_n x_n$ and $\lim_n Tx_n$ exist then $x \in D(T)$ and $\lim_n Tx_n = T\left(\lim_n x_n\right)$.*

(3) closable *if for every sequence $(x_n)_n$ of elements of $D(T)$ such that $\lim_n x_n = 0$ and $\lim_n Tx_n$ exist then $\lim_n Tx_n = 0$.*

If T is closable and non-closed then we can extend T to a closed operator $\bar{T}$, where $x \in D(\bar{T})$ if there exists a sequence $(x_n)_n$ of elements of $D(T)$ such that $\lim_n x_n = x$ and $y = \lim_n Tx_n$ exists. In that case[2] we set $\bar{T}x = y$. The operator $\bar{T}$ is closed, it is the minimal closed extension of T and it is called, the *closure* of T. We have the following result which connects the closable operators and the second adjoint operator [4, p. 70]:

Proposition 7.11. *If T is a densely defined operator then T^* is densely defined if and only if T is closable. In that case T^{**} exists and $T^{**} = \bar{T}$. So, if T is a densely defined closed operator we have that $T = T^{**}$.*

Definition 7.12. *An operator T is called*

(1) symmetric *if for every $x, y \in D(T)$,*

$$\langle Tx, y\rangle = \langle x, Ty\rangle$$

(2) self-adjoint *if $T = T^*$.*

[2]The operator $\bar{T}$ is well defined. Indeed, suppose that $\lim_n x_n = \lim_n x'_n = x = x$, $y = \lim_n Tx_n$ and $y' = \lim_n Tx'_n$. Then $y - y' = \lim_n T(x_n - x'n)$ and since T is closable then $y - y' = 0$ or $y = y'$

Clearly, a linear operator T is symmetric if and only if $T \subseteq T^*$. Also, a self-adjoint linear operator is symmetric but the converse is not always true.

In order to examine how the observables of a standard logic $P(\mathcal{H})$ are connected with the self-adjoint linear operators on $\mathcal{H}$ we need the notion of the spectral measure.

Let X be a set and $\mathcal{A}$ a σ-algebra of subsets of X. We shall refer to the pair $(X, \mathcal{A})$ as a measurable space.

Definition 7.13. *Let $\mathcal{H}$ be a Hilbert space, $P(\mathcal{H})$ the lattice of all projections of $\mathcal{H}$ and $(X, \mathcal{A})$ a measurable space. A mapping $E : \mathcal{A} \to P(\mathcal{H})$ satisfying the following properties*

(1) (*Countable Additivity*) *If $(\delta_n)_{n=1}^{\infty}$ is a sequence of pairwise disjoint elements of $\mathcal{A}$ then*

$$E\left(\bigcup_{n=1}^{\infty} \delta_n\right) = \sum_{n=1}^{\infty} E(\delta_n)$$

(2) (*Completeness*)

$$E(X) = 1$$

is called a spectral measure. *We shall refer to $(X, \mathcal{A}, \mathcal{H}, E)$ as a* spectral measure space.

Definition 7.14. *Let $(T_n)_{n=1}^{\infty}$ be a sequence of bounded linear operators. A bounded linear operator T is called the*

(1) uniform limit *of $(T_n)_{n=1}^{\infty}$ and we shall write $T = \lim_{n\to\infty} T_n$ if*

$$\lim_{n\to\infty} T_n = T$$

(2) strong limit *of $(T_n)_{n=1}^{\infty}$ and we shall write $T = s - \lim_{n\to\infty} T_n$ if*

$$\lim_{n\to\infty} T_n x = Tx$$

for every $x \in \mathcal{H}$.

(3) weak limit *of $(T_n)_{n=1}^{\infty}$ and we shall write $T = w - \lim_{n\to\infty} T_n$ if*

$$\lim_{n\to\infty} \langle T_n x, y\rangle = \langle Tx, y\rangle$$

for every $x, y \in \mathcal{H}$.

Proposition 7.15. *Let $(X, \mathcal{A}, \mathcal{H}, E)$ be a spectral measure space and $\delta_1, \delta_2, \ldots$ a sequence of elements of $\mathcal{A}$. Then*

(1) $E(\delta_1 \cap \delta_2) = E(\delta_1)E(\delta_2)$.

(2) *If* $\delta_1 \subseteq \delta_2 \ldots$ *then* $E\left(\bigcup_{n=1}^{\infty} \delta_n\right) = s - \lim_{n\to\infty} E(\delta_n)$.

(3) *If* $\delta_1 \supseteq \delta_2 \ldots$ *then* $E\left(\bigcap_{n=1}^{\infty} \delta_n\right) = s - \lim_{n\to\infty} E(\delta_n)$

Every spectral measure E generates a family of finite complex measures on the σ-algebra $\mathcal{A}$:

Definition 7.16. *Let $(X, \mathcal{A}, \mathcal{H}, E)$ be a spectral measure space. Then for every $x, y \in \mathcal{H}$ we define a measure $\mu_{x,y} : \mathcal{A} \to \mathbb{C}$ by*

$$\mu_{x,y}(\delta) = \langle E(\delta)x, y\rangle$$

In particular if $x = y$ we set

$$\mu_x(\delta) = \langle E(\delta)x, x\rangle = \langle E(\delta)x, E(\delta)x\rangle = \|E(\delta)x\|^2$$

For a detailed proof of the following theorem we we refer to [4].

Theorem 7.17. *Let $(X, \mathcal{A}, \mathcal{H}, E)$ be a spectral measure space. Then for every $\phi \in S(X, E)$ corresponds a closed dense linear operator T_ϕ on $\mathcal{H}$ denoted by $\int_X \phi dE$. The mapping $\phi \mapsto T_\phi$ has the following properties:*

(1)

$$D(T_\phi) = \left\{ x \in \mathcal{H} : \int |\phi|^2 d\mu_x < \infty \right\}$$

(2) *For every $x \in D(T_\phi)$ and $y \in \mathcal{H}$*

$$\langle T_\phi x, y\rangle = \int \phi \, d\mu_{x,y}$$

(3) *If $\phi, \psi \in S(X, E)$ and $\alpha, \beta \in \mathbb{C}$ then*

(a) $D(\alpha T_\phi + \beta T_\psi) = D(T_\phi) \cap D(T_\psi)$, $D(T_\phi T_\psi) = D(T_\phi) \cap D(T_\psi)$.

(*b*) $\alpha T_\phi + \beta T_\psi \subset T_{\alpha\phi+\beta\psi}$, $T_\phi T_\psi \subset T_{\phi\psi}$.

(*c*) $D(T_\phi^*) = D(T_\phi$ *and* $T_\phi^* = T_{\bar{\phi}}$

(*d*) $T_{\alpha\phi+\beta\psi} = \overline{\alpha T_\phi + \beta T_\psi}$.

(*e*) $T_{\phi\psi} = \overline{T_\phi T_\psi} = \overline{T_\psi T_\phi}$.

If we consider the logic of a quantum system to be the lattice of projections of a Hilbert space $\mathcal{H}$ then the observables E are the spectral measure spaces of the form $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathcal{H}, E)$. By Theorem 7.17 to every observable E corresponds a self-adjoint operator $T_E = \displaystyle\int_{\mathbb{R}} t dE$ and for every Borel function f we have that $T_{f(E)} = \displaystyle\int_{\mathbb{R}} f(t) dE$. A very important theorem of spectral theory, the so called spectral theorem for self-adjoint operators, shows that the converse statement also holds: For every self-adjoint operator T there exists a spectral measure E on $\mathbb{R}$ such that $T = \displaystyle\int_{\mathbb{R}} t dE$. So we have the following (see also [15])

Theorem 7.18. *If the $\mathcal{L}$ is the logic of all projections af a Hilbert space $\mathcal{H}$ then there exists a bijection $E \mapsto T_E$ from the set all observables of $\mathcal{L}$ onto the set of all self-adjoint operators on $\mathcal{H}$. The bounded observables correspond to the bounded self-adjoint of aperators on $\mathcal{H}$. Two bounded observables are simultaneously observable if and only if the corresponding bounded operators commute.*

7.4. States of the standard logic. Let $\mathcal{H}$ be a Hilbert space, $\mathcal{L}$ the lattice of all projections on $\mathcal{H}$ and $x \in \mathcal{H}$ be a vector with $\|x\| = 1$. If $P \in \mathcal{L}$

$$s_x(P) = \langle Px, x\rangle = \|Px\|^2 \tag{12}$$

The last equation holds because, since P is a projection then $P = P^2 = P^*$ and so we shall have that $\langle Px, x\rangle = \langle P^2x, x\rangle = \langle Px, P^*x\rangle = \langle Px, Px\rangle = \|Px\|^2$. Therefore if $(P_i)_{i=1}^\infty$ is a family of pairwise orthogonal elements of $\mathcal{L}$ then we shall have

$$s_x\left(\bigvee_{i=1}^\infty P_i\right) = \left\|\bigvee_{i=1}^\infty P_i x\right\|^2 = \sum_{i=1}^\infty \|P_i x\|^2 = \sum_{i=1}^\infty s_x(P_i).$$

This shows that the mapping $s_x : \mathcal{L} \to [0, +1]$ is a state of $\mathcal{L}$. To give a general description of the states in the case where the logic $\mathcal{L}$ is the lattice of all projection on $\mathcal{H}$ we need the notion of a trace class operator.

Let $T : \mathcal{H} \to \mathcal{H}$ is a bound linear operator then there exist two possibilities. Either for every orthonormal basis $(e_i)_{i=1}^{\infty}$ of $\mathcal{H}$ we shall have $\sum_{i=1}^{\infty} |\langle Te_i, e_i\rangle| = +\infty$ or for every orthonormal basis $(e_i)_{i=1}^{\infty}$ of $\mathcal{H}$ we shall have $\sum_{i=1}^{\infty} |\langle Te_i, e_i\rangle| < +\infty$. In the second case the series $\sum_{i=1}^{\infty} \langle Te_i, e_i\rangle$ is absolutely convergent and its value of is independent of the choice of the basis of $\mathcal{H}$.

Definition 7.19. *A bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is called a* trace class operator *if for every for every orthonormal basis $(e_i)_{i=1}^{\infty}$ of $\mathcal{H}$ we have that $\sum_{i=1}^{\infty} |\langle Te_i, e_i\rangle| < +\infty$. The number* $\mathrm{tr}(T) = \sum_{i=1}^{\infty} \langle Te_i, e_i\rangle$*, which is independent of the choice of the basis $(e_i)_{i=1}^{\infty}$ of $\mathcal{H}$, is called the* trace of T. *The set of all trace class operators is denoted by $S_1(\mathcal{H})$.*

The set $\mathcal{S}_1$ of all trace class operators is an ideal of $B(\mathcal{H})$, which means that

(i) it is a (non closed) subspace of $B(\mathcal{H})$ and

(ii) for every $A \in \mathcal{S}_1$ $T \in B(\mathcal{H})$ we have that $AT \in \mathcal{S}_1$ and $TA \in \mathcal{S}_1$.

Moreover $\mathrm{tr}(AT) = \mathrm{tr}(TA)$.

Definition 7.20. *Let $\mathcal{H}$ be a Hilbert space. A bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is called a* density operator *or a* von Neumann operator *is*

(1) *It is positive, i.e. for every $x \in \mathcal{H}$ we have that $\langle Tx, x\rangle \geq 0$.*
(2) *T is of trace class.*
(3) $\mathrm{tr}(T) = 1$.

Every von Neumann operator T is of the form

$$T = \sum \lambda_i P_i$$

where P_n are pairwise orthogonal projections of dimension 1 and λ_i ate positive numbers with sum equal to 1. Therefore, if T is a von Neumann operator then the function $s_T : \mathcal{L} \to [0, +1]$ given by $s_T(P) = \mathrm{tr}(PT)$ is a state.

The following important theorem of Gleason shows that if $\dim \mathcal{H} \geq 3$ then every state is of the form s_T where T is a von Neumann operator.

Theorem 7.21 (Gleason)**.** *If $\mathcal{H}$ is a Hilbert space with* $\dim\mathcal{H} \geq 3$ *and $\mathcal{L}$ is the logic of all projections on $\mathcal{H}$ then for every state s of $\mathcal{L}$ there exists a von Neumann operator T such that for every projection $P \in \mathcal{L}$ we have that* $s(P) = \mathrm{tr}(PT)$*.*

The proof of Gleason's theorem is complicated. We refer to [15] for a proof.

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V. Felouzis
Department of Mathematics
University of the Aegean
832 00 Karlovasi – Samos, Greece
e-mail: `felouzis@aegean.gr`

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