Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **40** (2014), 77–88

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

EMPIRICAL BAYES TEST FOR THE PARAMETER OF EXPONENTIAL-WEIBULL DISTRIBUTION UNDER NEGATIVE ASSOCIATED SAMPLES*

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Communicated by S. T. Rachev

Abstract. By using weighted kernel-type density estimator, the empirical Bayes test rules for parameter of Exponential-Weibull distribution are constructed and the asymptotically optimal property is obtained under negative associated samples. It is shown that the convergence rates of the proposed EB test rules can arbitrarily close to $O(n^{-\frac{1}{2}})$ under very mild conditions.

Introduction. Since H. Robbins [1, 2] introduced empirical Bayes (EB) approach, lately it has been studied extensively, the readers are referred to literature [3]–[7].

2010 *Mathematics Subject Classification:* 62C12, 62F12.

Key words: Negative associated samples; empirical Bayes test; asymptotic optimality; convergence rates.

*The research is supported by National Natural Science Foundation of China grants (No. 11061029), National Statistics Research Projects (No.2012LY178) and Guangdong Ocean University of Humanities and Social Sciences project (C13112).

However, few article investigations on EBT or EBE are studied under negative associated samples. Being different from previous researches, in this paper, we obtain a more ideal result about convergence rate. Firstly, we introduce the definition of negative associated (NA).

Definition 1.1 [11]. *Random variables* $X_1, X_2, \ldots, X_n$ $(n \geq 2)$ *are said to be NA, if for every pair of disjoint subsets* T_1 *and* T_2 *of* $\{1, 2, 3, \ldots, n\}$, $\mathrm{Cov}(f_1(X_i, i \in T_1), f_2(X_j, j \in T_1)) \leq 0$. *Here* f_1 *and* f_2 *are increasing or decreasing for every variable such that this covariance exists. Random variables sequence* $\{X_i, i \in n\}$ *is said to be NA, if for every natural number* $n \geq 2$, $X_1, X_2, \ldots, X_n$ $(n \geq 2)$ *are negative associated.*

Let X have a conditional density function for given θ

$$f(x|\theta) = e^{-x}\theta(1 - e^{-x})^{\theta-1}, \tag{1.1}$$

where θ is an unknown parameter, with sample space $\Omega = \{x \mid x > 0\}$ and parameter space $\Theta = \{\theta \mid \theta > 0\}$. In this paper, we discuss the following one-sided test problem

$$H_0 : \theta \leq \theta_0 \Leftrightarrow H_1 : \theta > \theta_0, \tag{1.2}$$

where θ_0 is a given positive constant.

For the hypothesis test problem (1.2), we have loss function

$$L_0(\theta, d_0) = a(\theta - \theta_0)^2 I_{(\theta>\theta_0)}, L_1(\theta, d_1) = a(\theta_0 - \theta)^2 I_{(\theta\leq\theta_0)}.$$

where $a > 0$,$d = \{d_0, d_1\}$ is the action space, d_0 and d_1 imply acceptance and rejection of H_0.

Assuming that the prior distribution $G(\theta)$ of θ is unknown, we obtain randomized decision function

$$\delta(x) = P(\text{accept } H_0|X = x). \tag{1.3}$$

Then, the risk function of $\delta(x)$ is shown by

$$\begin{aligned} R(\delta(x), G(\theta)) &= \int_\Theta \int_\Omega [L_0(\theta, d_0)f(x|\theta)\delta(x) + L_1(\theta, d_1)f(x|\theta)(1 - \delta(x))]dxdG(\theta) \\ &= a\int_\Omega \beta(x)\delta(x)dx + C_G, \end{aligned} \tag{1.4}$$

where

$$C_G = \int_\Theta L_1(\theta, d_1)dG(\theta), \beta(x) = \int_\Theta (\theta - \theta_0)^2 f(x|\theta)dG(\theta). \tag{1.5}$$

The marginal density function of X is given by

$$f_G(x) = \int_\Theta f(x|\theta)dG(\theta) = \int_\Theta e^{-x}\theta(1-e^{-x})^{\theta-1}dG(\theta).$$

By (1.5) and simple calculation, we have

$$\beta(x) = u_1(x)f_G^{(2)}(x) + u_2(x)f_G^{(1)}(x) + u_3(x)f_G(x), \tag{1.6}$$

where $f_G^{(1)}(x)$ and $f_G^{(2)}(x)$ are the first and the second order derivative of $f_G(x)$, and $u_1(x) = e^{2x}-2e^x+1, u_2(x) = (e^x-1)(3e^x-2\theta_0), u_3(x) = 2e^{2x}-(1+2\theta_0)e^x+\theta_0^2$.

Using (1.4), the Bayes test function is obtained as follows

$$\delta_G(x) = \begin{cases} 1, & \text{if } \beta(x) \le 0, \\ 0, & \text{if } \beta(x) > 0, \end{cases} \tag{1.7}$$

Further, we can get the minimum Bayes risk

$$R(G) = \inf_\delta R(\delta, G) = R(\delta_G, G) = a\int_\Omega \beta(x)\delta_G(x)dx + C_G. \tag{1.8}$$

When the prior distribution of $G(\theta)$ is known and $\delta(x) = \delta_G(x)$, $R(G)$ is achieved. However, when $G(\theta)$ is unknown, $\delta_G(x)$ cannot be made use of and we need to introduce EB method.

The rest of this paper is organized as follows. Section 2 presents an EB test. In Section 3, we obtain for the asymptotic optimality and the optimal rate of convergence of the EB test.

2. Construction of EB Test Function. Under the following condition, we need to construct the EB test function. Let $X_1, X_2, \ldots, X_n, X$ be a random variable sequence with common marginal density function $f_G(x)$, where $X_1, X_2, \ldots, X_n$ are historical samples, X is a present sample. Suppose that $X_1, X_2, \ldots, X_n$ are weakly stationary NA sequences, and the historical samples and the present sample are mutually independent. Assume $f_G(x) \in C_{s,\alpha},\ x \in R^1$, where

$$C_{s,\alpha} = \{g(x) \mid g(x) \text{ is a probability density function and has continuous } s\text{-th order derivative } g^{(s)}(x) \text{ with } |g^{(s)}(x)| \le \alpha,\ s \ge 2,\ \alpha > 0\}.$$

First we construct an estimator of $\beta(x)$.

Let $K_r(x)$ be a Borel measurable bounded function vanishing off $(0,1)$ such that

$$(A1)\qquad \frac{1}{t!}\int_0^1 y^t K_r(y)dy = \begin{cases} (-1)^t, & \text{when } t=r, \\ 0, & \text{when } t\neq r, t=0,1,2,\dots,s-1. \end{cases}$$

The ϕ-mixing coefficient statisfies

$$(A2)\qquad \sum_{i=1}^{\infty}\phi^{1/2}(i)<\infty.$$

The kernel estimation of $f_G^{(r)}(x)$ is defined by

$$f_n^{(r)}(x)=\frac{1}{nh_n^{(1+r)}}\sum_{j=1}^{n}K_r\left(\frac{x-X_j}{h_n}\right), \tag{2.1}$$

where $f_G^{(r)}(x)$ is the r order derivative of $f_G(x)$ and $f_G^{(0)}(x)=f_G(x)$.
Thus, the estimator of $\beta(x)$ is obtained by

$$\beta_n(x)=u_1(x)f_n^{(2)}(x)+u_2(x)f_n^{(1)}(x)+u_3(x)f_n(x). \tag{2.2}$$

Hence, the EB test function is defined by

$$\delta_n(x)=\begin{cases} 1, & \beta_n(x)\leq 0, \\ 0, & \beta_n(x)>0. \end{cases} \tag{2.3}$$

Let E stand for the mathematical expectation with respect to the joint distribution of $X_1,X_2,\dots,X_n$,
Hence, we get the overall Bayes risk of $\delta_n(x)$

$$R(\delta_n(x),G)=\int_a^b \beta(x)E[\delta_n(x)]dx+C_G. \tag{2.4}$$

If $\lim\limits_{n\to\infty}R(\delta_n,G)=R(\delta_G,G)$, then $\{\delta_n(x)\}$ is called the asymptotic optimality of the EB test function, and $R(\delta_n,G)-R(\delta_G,G)=O(n^{-q})$, where $q>0$; $O(n^{-q})$ is the asymptotic optimality convergence rates of the EB test function of $\{\delta_n(x)\}$. Before proving the theorems, we give a series of lemmas.

Let c,c_1,c_2,c_3,c_4 be different constants in different cases even in the same expression.

Lemma 2.1 [10]. *Let $\{X_i, 1 \le i \le n\}$ be negative associated random variables , with $EX_i = 0$ and $E|X_i|^2 < \infty$, $i = 1, 2, \cdots, n$. Then*

$$E|\sum_{i=1}^{n} X_i|^2 \le c \sum_{i=1}^{n} E|X_i|^2.$$

Lemma 2.2. *Let $f_n^{(r)}(x)$ be defined by (2.1). Assume that $(A1)$ holds. Then for every $x \in \Omega$:*

(I) *When $f_G^{(r)}(x)$ is a continuous function, $\lim_{n\to\infty} h_n = 0$ and $\lim_{n\to\infty} \frac{1}{nh_n^{2r+2}} = 0$, we get*

$$\lim_{n\to\infty} E|f_n^{(r)}(x) - f_G^{(r)}(x)|^2 = 0.$$

(II) *When $f_G(x) \in C_{s,a}$,taking $h_n = n^{-\frac{1}{2+2s}}$, for $0 < \lambda \le 1$, we get*

$$E|f_n^{(r)}(x) - f_G^{(r)}(x)|^{2\lambda} \le c \cdot n^{-\frac{\lambda(s-r)}{1+s}}.$$

P r o o f. Proof of (I). By the inequality for C_r, we have

$$E|f_n^{(r)}(x) - f_G^{(r)}(x)|^2 \le 2|Ef_n^{(r)}(x) - f_G^{(r)}(x)|^2 + 2Var(f_n^{(r)}(x)) := 2(I_1^2 + I_2), \tag{2.5}$$

where

$$\begin{aligned}
Ef_n^{(r)}(x) &= E\left[\frac{1}{nph_n^{1+r}} \sum_{i=1}^{n} K_r\left(\frac{x - X_i}{h_n}\right)\delta_i\right] \\
&= \frac{1}{h_n^{1+r}} EK_r\left(\frac{x - X_i}{h_n}\right) \\
&= h_n^{-(1+r)} E\left[K_r\left(\frac{x - X_1}{h_n}\right)\right] \\
&= h_n^{-(1+r)} \int_0^{\infty} K_r\left(\frac{x - y}{h_n}\right) f_G(y)dy \\
&= h_n^{-r} \int_0^1 K_r(u) f_G(x - h_n u)du.
\end{aligned}$$

We obtain the following Taylor expansion of $f_G(x - h_n u)$ in x

$$f_G(x-h_nu)-f_G(x) = \frac{f_G'(x)}{1!}(-h_nu)+\frac{f_G''(x)}{2!}(-h_nu)^2+\cdots+\frac{f_G^{(r)}(x-\xi h_nu)}{r!}(-h_nu)^r,$$

where $0 < \xi < 1$.

Since $f_G(x)$ is continuous in x and (A1), it is easy to see that

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} |Ef_n^{(r)}(x) - f_G^{(r)}(x)| = \lim_{n\to\infty} \left| \frac{1}{h_n^r} \int_0^1 K_r(u) f_G(x - h_n u) du - f_G^{(r)}(x) \right| \\ &\leq \frac{1}{r!} \int_0^1 u^r |K_r(u)| \lim_{n\to\infty} |f_G^{(r)}(x - \xi h_n u) - f_G^{(r)}(x)| du = 0. \end{aligned}$$

Hence, we have

$$\lim_{n\to\infty} I_1^2 = \lim_{n\to\infty} |Ef_n^{(r)}(x) - f_G^{(r)}(x)|^2 = 0. \tag{2.6}$$

Since $K_r(x)$ is a bounded variation function, there exists monotone bounded variation functions $K_{r1}(x)$ and $K_{r2}(x)$ such that $K_r(x) = K_{r1}(x) - K_{r2}(x)$.

It is easy to see that

$$\begin{aligned} f_n^{(r)}(x) - Ef_n^{(r)}(x) &= \frac{1}{nh_n^{(1+r)}} \sum_{j=1}^n \left[K_r\left(\frac{x - X_j}{h_n}\right) - EK_r\left(\frac{x - X_j}{h_n}\right) \right] \\ &= \frac{1}{nph_n^{(1+r)}} \left\{ \sum_{j=1}^n \left[K_{r1}\left(\frac{x - X_j}{h_n}\right) - EK_{r1}\left(\frac{x - X_j}{h_n}\right) \right] \right. \\ &\quad \left. - \sum_{j=1}^n \left[K_{r2}\left(\frac{x - X_j}{h_n}\right) - EK_{r2}\left(\frac{x - X_j}{h_n}\right) \right] \right\} \\ &= \frac{1}{nh_n^{(1+r)}} \left[\sum_{j=1}^n Y_j - \sum_{j=1}^n Z_j \right] =: \frac{1}{nh_n^{(1+r)}} [S_{n1} - S_{n2}], \end{aligned}$$

where

$$Y_j = K_{r1}\left(\frac{x - X_j}{h_n}\right) - EK_{r1}\left(\frac{x - X_j}{h_n}\right), \qquad S_{n1} = \sum_{j=1}^n Y_j,$$

$$Z_j = K_{r2}\left(\frac{x - X_j}{h_n}\right) - EK_{r2}\left(\frac{x - X_j}{h_n}\right), \qquad S_{n2} = \sum_{j=1}^n Z_j.$$

Denote

$$Y_j' = K_{r1}\left(\frac{x - X_j}{h_n}\right) - EK_{r1}\left(\frac{x - X_j}{h_n}\right), \qquad S_{n1}' = \sum_{j=1}^n Y_j',$$

$$Z'_j = K_{r2}\left(\frac{x - X_j}{h_n}\right) - EK_{r2}\left(\frac{x - X_j}{h_n}\right), \qquad S'_{n2} = \sum_{j=1}^{n} Z'_j.$$

It is easy to see that $EY'_j = EZ'_j = 0$, $j = 1, 2, \cdots, n$.

Since $K_{r1}(\cdot)$ and $K_{r2}(\cdot)$ are monotone functions, respectively, Y'_j and Z'_j are monotone functions of X_j.

By Definition 1.1, we get that $\{Y'_j, 1 \le j \le n\}$ and $\{Z'_j, 1 \le j \le n\}$ are NA sequences.

Further, there exists some positive constant M, so that $K_{r1}(\cdot) \le M$ and $K_{r2}(\cdot) \le M$.

By Lemma 2.1, we have

$$\begin{aligned} E[f_n^{(r)}(x) - Ef_n^{(r)}(x)]^2 &= \frac{1}{n^2 h_n^{2(1+r)}} E[S_{n1} - S_{n2}]^2 \\ &\le \frac{c}{n^2 h_n^{2(1+r)}} (ES_{n1}^2 + ES_{n2}^2) \\ &\le \frac{c}{n^2 h_n^{2(1+r)}} \left\{\sum_{i=1}^{n} EY_i^2 + \sum_{i=1}^{n} EZ_i^2\right\} \\ &\le \frac{c}{n h_n^{2(1+r)}} \{EY_1^2 + EZ_1^2\} \\ &\le \frac{c}{n h_n^{2(1+r)}} \{EY_1'^2 + EZ_1'^2\} \\ &\le \frac{c}{n h_n^{2(1+r)}}. \end{aligned} \tag{2.7}$$

When $\dfrac{1}{n h_n^{2r+2}} \to 0$, we obtain

$$\lim_{n\to\infty} I_2 = 0. \tag{2.8}$$

Substituting (2.6) and (2.8) into (2.5), the proof of (I) is completed.

Proof of (II). Similarly to (2.5), we can show that

$$\begin{aligned} E|f_n^{(r)}(x) - f_G^{(r)}(x)|^{2\lambda} &\le 2[Ef_n^{(r)}(x) - f_G^{(r)}(x)]^{2\lambda} + 2[Var f_n^{(r)}(x)]^{\lambda} \\ &:= 2(J_1^{2\lambda} + J_2^{\lambda}). \end{aligned} \tag{2.9}$$

We obtain the following Taylor expansion of $f_G(x - h_n u)$ in x

$$f_G(x - h_n v) = f_G(x) + \frac{f'_G(x)}{1!}(-h_n v) + \frac{f''_G(x)}{2!}(-h_n v)^2 + \cdots + \frac{f_G^{(s)}(x - \xi h_n v)}{s!}(-h_n v)^s,$$

where $0 < \xi < 1$. Due to $(A1)$ and $f_G(x) \in C_{s,\alpha}$, we have

$$|Ef_n^{(r)}(x) - f_G^{(r)}(x)| \leq \int_0^1 |K_r(v)| h_n^{s-r} v^s \left| \frac{f_G^{(s)}(x - \xi h_n v)}{s!} \right| dv \leq c \cdot h_n^{s-r}.$$

When $h_n = n^{-\frac{1}{2+2s}}$, we have

$$J_1^{2\lambda} = |Ef_n^{(r)}(x) - f_G^{(r)}(x)|^{2\lambda} \leq c \cdot n^{-\frac{\lambda(s-r)}{s+1}}. \tag{2.10}$$

By (2.7), when $h_n = n^{-\frac{1}{2+2s}}$, we have

$$J_2^{\lambda} \leq c_1[(h_n^{2r+2})^{-1}]^{\lambda} \leq c \cdot n^{-\frac{\lambda(s-r)}{1+s}}. \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.9), the proof of (II) is completed. □

Lemma 2.3 [6]**.** *Let $R(\delta_G, G)$ and $R(\delta_n, G)$ be defined by* (8) *and* (13)*. Then*

$$0 \leq R(\delta_n, G) - R(\delta_G, G) \leq c \int_a^b |\beta(x)| P(|\beta_n(x) - \beta(x)| \geq |\beta(x)|) dx.$$

3. Asymptotic optimality and convergence rates of empirical Bayes test.

Theorem 3.1. *Let $f_n^{(r)}(x)$ be defined by* (2.4)*. Assume that* $(A1)$*–*$(A2)$ *and the following regularity conditions hold*:
(i) $h_n > 0$, $\lim\limits_{n\to\infty} h_n = 0$, $\lim\limits_{n\to\infty} nh_n^5 = \infty$,
(ii) $\int_\Theta |\theta - 1| dG(\theta) < +\infty$, $\int_\Theta |(\theta - 1)(\theta - 2)| dG(\theta) < +\infty$;
(iii) *If $f_G^{(2)}(x)$ is continuous function, we have* $\lim\limits_{n\to\infty} R(\delta_n, G) = R(\delta_G, G)$.

Proof. By Lemma 2.2, we have

$$0 \leq R(\delta_n, G) - R(\delta_G, G) \leq a \int_\Omega |\beta(x)| p(|\beta_n(x) - \beta(x)| \geq |\beta(x)|) dx.$$

Writing $Q_n(x) = |\beta(x)| p(|\beta_n(x) - \beta(x)| \geq |\beta(x)|)$, we obtain $Q_n(x) \leq |\beta(x)|$.

It follows that

$$f_G^{(1)}(x) = -f_G(x) + \int_\Theta \frac{\theta - 1}{e^x - 1} f(x|\theta) dG(\theta),$$

$$f_G^{(2)}(x) = f_G(x) - 3\int_\Theta \frac{\theta - 1}{e^x - 1} f(x|\theta) dG(\theta) + \int_\Theta \frac{(\theta - 1)(\theta - 2)}{(e^x - 1)^2} f(x|\theta) dG(\theta),$$

$$\beta(x) = (\theta_0 - 1)^2 f_G(x) + (3 - 2\theta_0) \int_\Theta (\theta - 1) f(x|\theta) dG(\theta) + \int_\Theta (\theta - 1)(\theta - 2) f(x|\theta) dG(\theta).$$

Again by (1.6) and the Fubini theorem, we can get

$$\begin{aligned}
\int_\Omega |\beta(x)| dx &= \int_\Omega \int_\Theta |(\theta_0 - 1)^2 + (3 - 2\theta_0)(\theta - 1) + (\theta - 1)(\theta - 2)| f(x|\theta) dG(\theta) dx \\
&\le \int_\Omega \int_\Theta |(\theta_0 - 1)^2| f(x|\theta) dG(\theta) dx \\
&\quad + \int_\Omega \int_\Theta |(3 - 2\theta_0)(\theta - 1)| f(x|\theta) dG(\theta) dx \\
&\quad + \int_\Omega \int_\Theta |(\theta - 1)(\theta - 2)| f(x|\theta) dG(\theta) dx \\
&\le (\theta_0 - 1)^2 + |(3 - 2\theta_0)| \int_\Theta |\theta - 1| dG(\theta) \\
&\quad + \int_\Theta |(\theta - 1)(\theta - 2)| dG(\theta) < +\infty.
\end{aligned}$$

Applying the domain convergence theorem, then

$$0 \le \lim_{n\to\infty} R(\delta_n, G) - R(\delta_G, G) \le \int_\Omega [\lim_{n\to\infty} Q_n(x)] dx, \tag{3.1}$$

If Theorem 3.1 holds, we only need to prove $\lim_{n\to\infty} Q_n(x) = 0 \quad a.s.x.$ By Markov's and Jensen's inequalities, then

$$\begin{aligned}
Q_n(x)) &\le E|\beta_n(x) - \beta(x)| \le |u_1(x)| E|f_n^{(2)}(x) - f_G^{(2)}(x)| \\
&\quad + |u_2(x)| E|f_n^{(1)}(x) - f_G^{(1)}(x)| + |u_3(x)| E|\widehat{f}_G(x) - f_G(x)| \\
&\le |u_1(x)| [E|f_n^{(2)}(x) - f_G^{(2)}(x)|^2]^{1/2} + |u_2(x)| [E|f_n^{(1)}(x) - f_G^{(1)}(x)|^2]^{1/2} \\
&\quad + |u_3(x)| [E|f_n(x) - f_G(x)|^2]^{1/2}.
\end{aligned}$$

Again by Lemma 2.1(I), for fixed $x \in \Omega$, when $r = 0, 1, 2$

$$
\begin{aligned}
0 &\le \lim_{n\to\infty} Q_n(x) \\
&\le |u_1(x)|[\lim_{n\to\infty} E|f_n^{(2)}(x) - f_G^{(2)}(x)|^2]^{1/2} \\
&\quad +|u_2(x)|[\lim_{n\to\infty} E|f_n^{(1)}(x) - f_G^{(1)}(x)|^2]^{1/2} \\
&\quad +|u_3(x)|[\lim_{n\to\infty} E|f_n(x) - f_G(x)|^2]^{1/2} = 0.
\end{aligned} \tag{3.2}
$$

Substituting (3.2) into (3.1), the proof of Theorem 3.1 is completed. □

Theorem 3.2. *Let $\widehat{f}_G^{(r)}(x)$ be defined by* (2.4). *Assume that* (*A*1)–(*A*2) *and the following regularity conditions hold:* $f_G(x) \in C_{s,\alpha}$ *for* $0 < \lambda \le 1$, *and*

$$(B1) \qquad \int_\Omega e^{(m\lambda x)}|\beta(x)|^{1-\lambda}dx < +\infty, \qquad m = 0, 1, 2, \quad \textit{when } h_n = n^{-\frac{1}{s+1}}.$$

We have

$$R(\delta_n, G) - R(\delta_G, G) = O(n^{-\frac{\lambda(s-3)}{2(s+2)}}), \quad \textit{where} \ \ s \ge 3.$$

Proof. By Lemma 2.2 and Markov's inequalities,

$$
\begin{aligned}
0 &\le R(\delta_n, G) - R(\delta_G, G) \le \int_\Omega |\beta(x)|^{1-\lambda} E|\beta_n(x) - \beta_G(x)|^\lambda dx \\
&\le c_1 \int_\Omega |\beta(x)|^{1-\lambda}|u_1(x)|E|f_n^{(2)}(x) - f_G^{(2)}(x)|dx \\
&\quad +c_2 \int_\Omega |\beta(x)|^{1-\lambda}|u_2(x)|E|f_n^{(1)}(x) - f_G^{(1)}(x)|dx \\
&\quad +c_3 \int_\Omega |\beta(x)|^{1-\lambda}|u_3(x)|E|f_n(x) - f_G(x)|dx = A_n + B_n + C_n.
\end{aligned} \tag{3.3}
$$

By Lemma 2.2 (II) and condition (B1), we get

$$A_n \le c_1 n^{-\frac{\lambda(s-3)}{2s+4}} \int_\Omega |\beta(x)|^{1-\lambda}|u_1(x)|^\lambda dx \le c_4 n^{-\frac{\lambda(s-3)}{2s+4}}, \tag{3.4}$$

$$B_n \le c_2 n^{-\frac{\lambda(s-1)}{2s+4}} \int_\Omega |\beta(x)|^{1-\lambda}|u_2(x)|^\lambda(x)dx \le c_5 n^{-\frac{\lambda(s-1)}{2s+4}}, \tag{3.5}$$

$$C_n \leq c_3 n^{-\frac{\lambda(s+1)}{2s+4}} \int_\Omega |\beta(x)|^{1-\lambda} |u_3(x)|^\lambda(x)dx \leq c_6 n^{-\frac{\lambda(s+1)}{2s+4}}. \tag{3.6}$$

Substituting (3.4)–(3.6) into (3.3), we get $R(\delta_n, G) - R(\delta_G, G) = O(n^{-\frac{\lambda(s-3)}{2(s+2)}})$, The proof of Theorem 3.2 is completed. $\square$

Remark. When $\lambda \to 1$, and $s \to \infty$, $O\left(n^{-\frac{\lambda(s-3)}{2(s+2)}}\right)$ is arbitrarily close to $O(n^{-\frac{1}{2}})$.

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Received September 21, 2013