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VIABILITY AND AN OLECH TYPE RESULT*

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Abstract. We study the existence of solutions of differential inclusions with upper semicontinuous right-hand side. The presented approach is based on the directionally continuous selections techniques developed by Bressan and on Srivatsa's Baire class 1 selectors for upper semicontinuous set-valued maps. We propose a concept for invariant ε-approximation of an upper semicontinuous set-valued map on the elements of a relatively open partitioning. We prove a result of Olech type where the assumptions on the set of lower semicontinuity is G_δ in contrast to the usual openness assumption. The proof is based on a generalization of invariant ε-approximations.

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1. Introduction. We study the existence of solutions of the differential inclusion

$$\dot{x} \in F(x),\ x(0) = x_0. \tag{1.1}$$

We assume that $F : R^n \rightrightarrows R^n$ is an upper semicontinuous bounded set-valued map with nonempty compact values.

A longstanding open problem is the existence of solution of differential inclusions with upper semicontinuous right-hand side with non-convex values. As is seen from the well known Filippov's examples, such a solution need not always exist.

The question whether solutions exist for continuous F with non-convex values was solved by Filippov in 1971 (cf. [13] and [14]). This was generalized to the case of lower semicontinuous right-hand side, by Bressan (1980) by means of the selection approach (cf. [4]), and by Lojasiewicz (1980) by means of Filippov's method (cf. [19]).

Bressan, Cellina and Colombo studied in 1989 the existence of solutions to differential inclusions with upper semicontinuous cyclically monotone right-hand sides (cf. [8]). These set-valued maps are exactly the upper semicontinous (u.s.c.) ones, the graph of which is contained in the subdifferential of a proper convex function. An extension of [8] is obtained by H. Benabdellah in [2] under the assumption that $F(x)$ is contained in the subdifferential of a locally Lipschitz and regular function. The last result along these lines, known to the authors, is obtained in 2005 by Bounkhel and Haddad (cf. [3]). A direct corollary of their result is that the differential inclusion $\dot{x} \in F(x)$ has a solution if the right-hand side is an u.s.c. map with compact values, the graph of which is contained in the graph of the proximal subdifferential of an uniformly regular l.s.c. function.

Another result (cf. [24]), obtained by Veliov in 1997, already proved to be very useful in studying invariance, stability and attainability properties of a given compact set with respect to a differential inclusion. It yields the existence of solutions of differential inclusions with right-hand side of the form

$$F(x) = \left\{\eta \in G(x) :\ D^{-}\psi(x;\eta) \le \phi(x)\right\},$$

where G is an u.s.c. map with compact convex values, ψ is a locally Lipschitz function, ϕ is an u.s.c. real-valued function, and D^{-} denotes the lower Dini derivative.

In 2005 we (together with Ts. Tsachev) studied the existence of ε-solutions of differential inclusions with upper semicontinuous right-hand side in [17], i.e. we investigated the existence of an absolutely continuous function $x : [0,T] \to R^n$ such that

$$\left| \begin{array}{l} \dot{x}(t) \in F(x(t)) + \varepsilon \bar{B} \quad \text{a.e. on } [0,T] \\ x(0) = x_0. \end{array} \right.$$

A new concept "colliding on a set" was defined. In the case when the admissible velocities do not "collide" on the set of discontinuities of the right-hand side, we expect that at least one trajectory emanates from every point. If the velocities do "collide" on the set of discontinuities of the right-hand side, the existence of solutions is not guaranteed. Let us state the definition for "not colliding on a set": Let D be an intersection of an open and a closed subset of R^n and let $F : D \Rightarrow R^n$ be an upper semicontinuous set-valued map with nonempty compact values which are uniformly bounded.

Definition 1.1. *Let us fix an arbitrary $\varepsilon > 0$. It is said that F does not ε-collide from D to the point $\hat{x}$ of D iff there exist a subset A of D and a multivalued map $G : A \Rightarrow R^n$ such that:*

(*i*) *the set A is an intersection of an open set and a closed set, and $\hat{x} \in A \cap \mathrm{cl}(\mathrm{int}\, A)$ (here $\mathrm{int}\, A$ denotes the interior of A relatively in D);*

(*ii*) *G is upper semicontinuous convex valued map with $G(x) \subset F(x) + \varepsilon B$ for each $x \in A$;*

(*iii*) *for each point $x \in \partial A \cap A$ (here ∂A denotes the boundary of A relatively in D) and for each $\zeta \in \hat{N}_A(x)$ there exists $v \in G(x)$ with*

$$\langle \zeta, v \rangle \leq 0 \tag{1.2}$$

(*here $\hat{N}_A(x)$ denotes the proximal normal cone to A at the point x*).

The meaning of Definition 1.1 is that the differential inclusion $\dot{x} \in G(x)$ with $x(0) = x_0$ has a solution on a small time interval whenever $x_0 \in A$ and D is open in R^n, i.e. the set A is weakly invariant (viable) with respect to the convex-valued ε-approximation G of F.

It is straightforward to verify that all lower semicontinuous set-valued maps as well as all upper semicontinuous convex-valued maps do not collide at any point.

Whenever the admissible velocities do "collide" on some set S, in order to have a solution of $\dot{x} \in F(x)$ which does not leave S, one has to assume that there exist tangent velocities to S. More precisely, for the right-hand sides under consideration, we assume the following **Basic assumption**: whenever the velocities "collide" on a set S there exist tangent velocities (belonging to the Clarke tangent cone to S) on a dense subset of S. Then we were able to prove existence of an ε-solution for every $\varepsilon > 0$.

The idea of the corresponding proof is to construct a "suitable" relatively open partitioning of D.

Definition 1.2. *Let X be a topological space and*

$$\mathcal{U} = \{U_\alpha : \ 1 \leq \alpha < \alpha_0\}$$

be a well ordered family of its subsets. It is said that $\mathcal{U}$ is a relatively open partitioning of X iff

(*i*) *U_α is contained in* $X \setminus \left(\bigcup_{\beta<\alpha} U_\beta \right)$ *and it is relatively open in it for every α;*

(*ii*) $X = \bigcup_{1\leq\alpha<\alpha_0} U_\alpha.$

Note that each element of a relatively open partitioning is an intersection of an open set and a closed set.

The origin of the techniques from [17] is the directionally continuous selections approach developed by Bressan and Srivatsa's Baire class 1 selectors for upper semicontinuous set-valued maps (cf. [21]). The present note is devoted to further development of these ideas. In Section 2 we propose a concept for invariant ε-approximation of an upper semicontinuous mapping on the elements of a relatively open partitioning. This definition is motivated by the main result of Veliov in [23]. From our point of view the properties collected in Proposition 2.1 are important for understanding of the approach proposed here and allow us to prove a better version of Theorem 5.1 in [17]. In Section 3 we generalize the structure developed in Section 2 in order to cope with the case when the local finiteness of the partitioning is not possible to achieve. Theorem 3.4 proved there reveals where the difficulties in the general problem appear. This theorem suggests a scheme which could be implemented in different particular cases (of course, if one can handle the "bad points" of the right-hand side) to ensure existence of solution of differential inclusions. Section 4 serves as an example of applying this general scheme. The obtained result is closely related to the results of Olech [1975] (cf.

[18]), Lojasiewicz [1985] (cf. [20]), T. Haddad, A. Jourani and L.Thibault (cf. [15]) and T. Haddad and L.Thibault (cf. [16]) (the last three papers are based on the Filippov's approach), but the assumptions on the set of lower semicontinuity is G_δ in contrast to the usual openness assumption. Let us point out that it is well known that every upper semicontinuous mapping with values in R^n is lower semicontinuous on a dense G_δ subset of its domain. This suggests that the idea of the proof of the main result from section 4 could be used in more general situations.

2. Viability – locally finite case. We start this section with the standard

Definition 2.1. *Let D be an intersection of an open subset and a closed subset of R^n and let $x : [0,T) \to R^n$ be an absolutely continuous function. It is said that the set D is invariant with respect to the curve $x(\cdot)$ if for each $t \in [0,T)$ with $x(t) \in D$ there exists $\delta > 0$ such that $x(\tau) \in D$ for each $\tau \in [t, t+\delta)$.*

We are going to construct "suitable" partitionings of the domain D of F, where "suitable" means that the elements of the partitioning should be weakly invariant with respect to $F + \varepsilon B$. In fact, usually they are going to be weakly invariant with respect to $F + \eta B$ for each $0 < \eta < \varepsilon$.

Definition 2.2. *Let the set D be an intersection of an open subset and a closed subset of R^n and $F : \bar{D} \Longrightarrow R^n$ be a multi-valued mapping with nonempty values. Let $\varepsilon > 0$, $\mathcal{U} = \{U_\alpha : \ 1 \le \alpha < \alpha_0\}$ be a relatively open partitioning of D and*

$$\mathcal{G} := \{G_U : \ U \in \mathcal{U}\}$$

be a family of multi-valued maps. We say that the pair $(\mathcal{U}, \mathcal{G})$ is an invariant ε-approximation of F iff

(i) $G_{U_\alpha} : \bar{U}_\alpha \to R^n$ is upper semicontinuous multi-valued mapping with nonempty convex compact values for each $\alpha \in [1, \alpha_0)$;

(ii) for each $x \in \bar{U}_\alpha$ and for each $\alpha \in [1, \alpha_0)$ the intersection $G_{U_\alpha}(x) \cap T_{D_\alpha}(x)$ (where $D_\alpha = \bar{D} \setminus (\bigcup_{\beta<\alpha} U_\beta))$ is nonempty and

$$G_{U_\alpha}(x) \cap T_{D_\alpha}(x) \subset F(x) + \varepsilon \bar{B}.$$

(here $T_{D_\alpha}(x)$ denotes the the Bouligand tangent cone to A at the point x).

Note that if we denote by U_{α_0} the set $\bar{D} \setminus D$, the family $\mathcal{U} \cup \{U_{\alpha_0}\}$ is a relatively open partitioning of the closure $\bar{D}$ of the set D. Moreover, the condition (ii) guarantees that U_α is weakly invariant with respect to G_{U_α} for each $\alpha \in [1, \alpha_0)$.

Proposition 2.3. *Let D be an intersection of a closed set S and an open set U of R^n, $F : D \Longrightarrow R^n$ be uniformly bounded and $(\mathcal{U}, \mathcal{G})$ be an invariant ε-approximation of F. Then*

(*i*) *for each point $x_0 \in D$ there exists an ε-solution $\varphi(\cdot)$ of $\dot{x} \in F(x)$ starting from x_0 and defined on the interval $[0, T)$ ($T = +\infty$ or $\varphi(t)$ tends to a point in $R^n \setminus U$ whenever $t \to T$) such that each element of the partitioning $\mathcal{U}$ is invariant with respect to φ;*

(*ii*) *for each absolutely continuous function $\varphi : [0, T) \to D$ with respect to which the elements of $\mathcal{U}$ are invariant, the function*

$$t \to \alpha(t), \ \textit{where } \varphi(t) \in U_{\alpha(t)}$$

is monotone increasing;

(*iii*) *if $\mathcal{U}$ is locally finite, $\varphi_k : [0, T] \to D$, $k = 1, 2, \ldots$ is a sequence of ε-solutions of $\dot{x} \in F(x)$ with respect to which the elements of $\mathcal{U}$ are invariant and $\varphi : [0, T] \to D$ is its uniform limit, then φ is an ε-solution of $\dot{x} \in F(x)$.*

P r o o f. (i) For every point $x \in D$ there exists a unique index α_x such that $x \in U_{\alpha_x}$. According to Definition **??**, (*ii*) and applying Theorem 3.2.4 from [1] (cf., also, [23]) to $G_{U\alpha_x}$, we obtain the existence of $t_x > 0$ and an absolutely continuous function $\varphi_{x,\tau} : [\tau, \tau + t_x) \to U_{\alpha_x}$ which is well defined on $[\tau, \tau + t_x)$ (here τ is an arbitrary positive real) and is a solution of the following differential inclusion

$$\left| \begin{array}{l} \dot{x}(t) \in G_{U_{\alpha_x}}(x(t)), \ \text{ a.e. on } \ [\tau, \tau + t_x) \\ x(\tau) = x. \end{array} \right.$$

In fact, as $\varphi_{x,\tau}(t) \in U_{\alpha_x}$ for each $t \in [\tau, \tau + t_x)$ and $D_{\alpha_x} = D \setminus \left(\bigcup\limits_{\beta < \alpha_x} U_\beta \right)$, we have that

$$\dot{\varphi}_{x,\tau}(t) \in G_{U_{\alpha_x}}(\varphi_{x,\tau}(t)) \cap T_{D_{\alpha_x}}(\varphi_{x,\tau}(t)) \subset F(\varphi_{x,\tau}(t)) + \varepsilon \bar{B} \ \text{ a.e. on } \ [\tau, \tau + t_x).$$

In particular, for the point x_0 there exists an absolutely continuous function $\varphi_{x_0,0} : [0, t_{x_0}) \to U_{\alpha_{x_0}}$. We define inductively an absolutely continuous function $\varphi : [0, \infty) \to D$ as follows: We set $\varphi(t) := \varphi_{x_0,0}(t)$ for each $t \in [0, t_{x_0})$. Assume that φ is defined on some interval $[0, \hat{t})$. Then for each increasing sequence $\{t_n\}_{n=1}^{\infty} \to \hat{t}$ we have

$$\|\varphi(t_n) - \varphi(t_{n-1})\| \leq \int_{t_{n-1}}^{t_n} \|\dot{\varphi}(\tau)\| \, d\tau \leq c\,(t_n - t_{n-1}).$$

This means that the sequence $\{\varphi(t_n)\}_{n=1}^{\infty}$ is a Cauchy sequence. Hence, there exist $\varphi(\hat{t}) := \lim\limits_{t \to \hat{t}} \varphi(t)$. If $\varphi(\hat{t}) \in D$, then we define an absolutely continuous extension of the function φ by $\varphi_{\varphi(\hat{t}),\hat{t}}$ on the interval $\left[\hat{t}, \hat{t} + t_{\varphi(\hat{t})}\right)$. Let $[0, T)$ be the maximal interval on which $\varphi(\cdot)$ can be defined in this way. If $T \neq +\infty$, in a similar way we obtain that the point $\varphi(t)$ tends to the boundary of U as t tends to T. This completes the proof.

(ii) Let $\mathcal{U} = \{U_\alpha : \ 1 \leq \alpha < \alpha_0\}$ and let us define $W_\alpha := \bigcup\limits_{\beta < \alpha} U_\beta$ for each $\alpha \in [0, \alpha_0)$. It is straightforward to check that $\mathcal{W} = \{W_\alpha : \ 1 \leq \alpha < \alpha_0\}$ is an increasing family of open subsets of D. Clearly, the function $t \to \alpha(t)$ is constant on $[0, t_{x_0})$ for some $t_{x_0} > 0$ (because of the weak invariance) and the assertion holds true. Let us assume that the function $t \to \alpha(t)$ is monotone increasing on $[0, \hat{t})$. Let $0 < \hat{t} < T$. Then $\varphi(\hat{t}) \in W_{\alpha(\hat{t})+1}$. Since the set $W_{\alpha(\hat{t})+1}$ is open and $\varphi(\cdot)$ is continuous, there exists $t' \in [0, \hat{t})$ such that $\varphi(t) \in W_{\alpha(\hat{t})+1}$ for each $t \in [t', \hat{t}]$. Hence, $\alpha(t) \leq \alpha(\hat{t})$ for each $t \in [t', \hat{t}\,]$. The inductive assumption implies that $\alpha(t) \leq \alpha(t') \leq \alpha(\hat{t})$ for each $t \in [0, t')$. Thus $\alpha(t) \leq \alpha(\hat{t})$ for each $t \in [0, \hat{t}\,]$. Because of the definition of weak invariance and the definition of $\varphi(\cdot)$, there exists an interval $[\hat{t}, \hat{t} + \delta_{\hat{t}})$, $\delta_{\hat{t}} > 0$, such that $\alpha(t) = \alpha(\hat{t})$ for each $t \in [\hat{t}, \hat{t} + \delta_{\hat{t}})$. Hence, our function $\alpha(\cdot)$ remains monotone increasing on $[0, \hat{t} + \delta_{\hat{t}})$.

(iii) For almost all τ in $[0, T]$ (cf. Lemma 2 from [20]) the derivatives $\dot{\varphi}(\tau)$ and $\dot{\varphi}_k(\tau)$ exist and

$$\dot{\varphi}(\tau) \in \ \overline{\text{co}} \ \{\dot{\varphi}_k(\tau) : \ k \geq k_0\}$$

for any positive integer k_0.

Let us fix $\bar{\tau}$ in $[0, T]$. Then there exists a neighbourhood V of $x(\bar{\tau})$ that intersects at most finitely many members of $\mathcal{U}$. Because V is open and $\|F(\cdot)\|$

is uniformly bounded, there exist a positive integer k_0 and a positive real δ such that $\varphi_k(t) \in V$ for each $t \in (\bar\tau - \delta, \bar\tau + \delta) \cap [0, T]$ and for each $k \geq k_0$.

Let $\alpha_1 < \alpha_2 < \cdots < \alpha_s$ be such that $V \cap U_\alpha = \emptyset$ whenever $\alpha \neq \alpha_s$, $s = 1, \dots, \bar s$. Then $\{\varphi_k(t) :\ t \in (\bar\tau - \delta, \bar\tau + \delta) \cap [0, T]\} \subset \bigcup_{s=1}^{\bar s} U_{\alpha_s}$ whenever $k \geq k_0$. We set

$$\Delta_s^k := \{t :\ \varphi_k(t) \in U_{\alpha_s}\},\ s = 1, 2, \dots, \bar s.$$

According to (ii) the sets Δ_s^k, $s = 1, 2, \dots, \bar s$, are intervals such that $t_{s_1} < t_{s_2}$ whenever t_{s_1} and t_{s_2} are arbitrary points from $\Delta_{s_1}^k$ and $\Delta_{s_2}^k$, respectively, with $s_1 < s_2$. Of course, some of the sets Δ_s^k can be empty.

Let us denote by t_s^k, $s = 1, 2, \dots, \bar s$, the points of transition of the trajectory $\varphi_k(\cdot)$ from the set $U_{\alpha_{s-1}}$ to the set U_{α_s}. We set $t_{s+1}^k := \min\{\bar\tau + \delta, T\}$. If the trajectory passes from U_{α_s} to U_{α_l} with $l \geq s + 2$, then we set $t_s^k = t_{s+1}^k = \cdots = t_{l-1}^k = t_l^k$. If the trajectory enters (leaves) V through U_{α_s} for $s > 1$ ($s \leq \bar s$), we set $t_1^k = t_2^k = \cdots = t_s^k$ $\left(t_{s+1}^k = t_{s+1}^k = \cdots = t_{s+1}^k\right)$.

Without loss of generality, we may think that each sequence $\left\{t_s^k\right\}_{k=k_0}^\infty$ is convergent and tends to the number t_s, $s = 1, 2, \dots, \bar s$. The inequalities $t_s^k \leq t_{s+1}^k$, imply that $t_s \leq t_{s+1}$, $s = 1, 2, \dots, \bar s$.

Let $\tau \in ((\bar\tau - \delta, \bar\tau + \delta) \cap [0, T]) \setminus \{t_s :\ s = 1, 2, \dots, \bar s\}$ and let $\dot\varphi(\tau)$ and $\dot\varphi_k(\tau)$, $k \geq k_0$, exist and $\dot\varphi_k(\tau) \in G_{U_{\alpha^k(\tau)}}(\varphi_k(\tau))$, where $\varphi_k(\tau) \in U_{\alpha^k(\tau)}$. Note that these conditions are satisfied for almost all $\tau \in (\bar\tau - \delta, \bar\tau + \delta) \cap [0, T]$. Then $\tau \in (t_{s-1}, t_s)$ for some s, and so $\tau \in (t_{s-1}^k, t_s^k)$ for all sufficiently large k. Thus the points $\varphi_k(\tau)$, $k \geq k_1 \geq k_0$, belong to one and the same element of the partitioning $\mathcal{U}$, say $U := U_{\alpha_{s-1}}$, for all sufficiently large $k \geq k_1 \geq k_0$. Using the upper semicontinuity of $G_U := G_{U_{\alpha_{s-1}}}$ at the point $\varphi(\tau) \in \bar U$, we obtain that for each $\eta > 0$ we have that $G_U(\varphi_k(\tau)) \subset G_U(\varphi(\tau)) + \eta \bar B$ for $k \geq k_\eta \geq k_1$. Thus

$$\dot\varphi(\tau) \in \overline{\text{co}}\ \{\dot\varphi_k(\tau) :\ k \geq k_\eta\} \subset \overline{\text{co}} \left(\bigcup_{k \geq k_\eta} G_{U_{\alpha^k(\tau)}}(\varphi_k(\tau)) \right) =$$

$$= \overline{\text{co}} \left(\bigcup_{k \geq k_\eta} G_U(\varphi_k(\tau)) \right) \subset \overline{\text{co}}\ \left(G_U(\varphi(\tau)) + \eta \bar B\right) = G_U(\varphi(\tau)) + \eta \bar B.$$

As $\eta > 0$ is arbitrary and $G_U(\varphi(\tau))$ is closed, we obtain that $\dot{\varphi}(\tau) \in G_U(\varphi(\tau))$.

According to Theorem 2.10 of [10] and $\bar{U} \subset D_{\alpha_{s-1}}$, we have that

$$\dot{\varphi}(\tau) \in T_{\bar{U}}\ (\varphi(\tau)) \subset T_{D_{\alpha_{s-1}}}(\varphi(\tau))$$

and so

$$\dot{\varphi}(\tau) \in\ G_U(\varphi(\tau)) \cap T_{D_{\alpha_{s-1}}}(\varphi(\tau)) \subset F(\varphi(\tau)) + \varepsilon\bar{B}.$$

This completes the proof. $\square$

Definition 2.4. *Let $\mathcal{U}^1$ and $\mathcal{U}^2$ be two relatively open partitionings of D. We say that $\mathcal{U}^2 \prec \mathcal{U}^1$ ($\mathcal{U}^2$ is a refinement of $\mathcal{U}^1$) if*

(*i*) *for every element $U^2 \in \mathcal{U}^2$ there exists an element $U^1 \in \mathcal{U}^1$ with $U^2 \subset U^1$;*

(*ii*) *if $\mathcal{U}^2 \ni U^2_{\beta_i} \subset U^1_{\alpha_i} \in \mathcal{U}^1$, $i = 1, 2$, with $\alpha_1 < \alpha_2$, then $\beta_1 < \beta_2$;*

Definition 2.5. *Let the set D be an intersection of an open subset and a closed subset of R^n and $F : \bar{D} \Rightarrow R^n$ be a multi-valued mapping with nonempty values. Let $(\mathcal{U}^1, \mathcal{G}^1)$ be an invariant ε_1-approximation of F and $(\mathcal{U}^2, \mathcal{G}^2)$ be an invariant ε_2-approximation of F, where $\mathcal{U}^1$ and $\mathcal{U}^2$ are relatively open partitionings of D. It is said that $(\mathcal{U}^2, \mathcal{G}^2)$ is a refinement of $(\mathcal{U}^1, \mathcal{G}^1)$ iff*

(*i*) $0 < \varepsilon_2 < \varepsilon_1$;

(*ii*) $\mathcal{U}^2 \prec \mathcal{U}^1$;

(*iii*) *if $\mathcal{U}^2 \ni U^2_\beta \subset U^1_\alpha \in \mathcal{U}^1$, then $G^2_\beta(x) \subseteq G^1_\alpha(x)$ for each $x \in \bar{U}^2_\beta$.*

Using these concepts and imposing additional assumptions, we can pass to the limit as $\varepsilon \to 0$ and obtain a solution of the considered differential inclusion. Using Proposition 2.3, we can drop the assumption on monotonicity of the approximate solutions in Theorem 5.1 of [17] and reformulate the respective result as follows:

Theorem 2.6. *Let the set D be an intersection of an open subset and a closed subset of R^n and $F : \bar{D} \Rightarrow R^n$ be a uniformly bounded multi-valued* *mapping with nonempty closed values. We assume that there exist invariant* $\frac{1}{k}$*-approximations* $(\mathcal{U}^k, \mathcal{G}^k)$ *of* F *such that* $\mathcal{U}^k$ *is a locally finite relatively open partitioning of* D*, and* $(\mathcal{U}^{k+1}, \mathcal{G}^{k+1})$ *is a refinement of* $(\mathcal{U}^k, \mathcal{G}^k)$*, for each* $k = 1, 2, \ldots$*.*

Then for every $x_0 \in D$ *there exist* $T > 0$ *and an absolutely continuous function* $x : [0, T] \to D$ *such that*

$$\left| \begin{array}{l} \dot{x}(t) \in F(x(t)) \quad a.e. \ on \ [0, T] \\ x(0) = x_0. \end{array} \right.$$

Proof. Let us fix an arbitrary point $x_0 \in D$ and an arbitrary positive integer k. According to Proposition 2.3, (i) and (ii), there exists some $\frac{1}{k}$-solution $x_k(\cdot)$ of F starting from x_0 and defined on some interval $[0, T_k)$ such that each U^k_α is weak invariant with respect to $\mathcal{G}^k$, and hence the function

$$t \to \alpha^k(t), \text{ where } x_k(t) \in U^k_{\alpha^k(t)}$$

is monotone increasing. Since $D = G \cap U$, where U is an open subset and G is a closed subset of R^n, and D is invariant with respect to x_k,

$$T_k \geq T := \frac{\operatorname{dist}(x_0, R^n \setminus U)}{c + 1} > 0,$$

where c is an upper bound for $\{\|F(x)\| : \ x \in D\}$.

Without loss of generality we may think that the sequence $\{x_k(\cdot)\}_{k=1}^\infty$ is uniformly convergent on the interval $[0, T]$ to an absolutely continuous function denoted by $x(\cdot)$.

Let us fix an arbitrary positive integer m. Because $(\mathcal{U}^k, \mathcal{G}^k)$ is a refinement of $(\mathcal{U}^m, \mathcal{G}^m)$ whenever $k \geq m$, the elements of $\mathcal{U}^m$ are invariant with respect to the trajectories $x_k(\cdot)$ for $k \geq m$. According to Proposition 2.1 (iii), $x(\cdot)$ is a $\frac{1}{m}$-solution of $\dot{x} \in F(x)$, i.e.

$$\dot{x}(\tau) \in F(x(\tau)) + \frac{1}{m}\bar{B} \ \text{ for almost all } \ \tau \in [0, T].$$

Since $\dot{x}(\tau)$ does not depend on m, the set-valued map F is closed valued and the union of countably many sets of measure zero is a set of measure zero, we obtain that

$$\dot{x}(\tau) \in \ F(x(\tau)) \ \text{ for almost all } \tau.$$

This completes the proof. □

The above Theorem is a natural extension of the approach proposed by Bressan for studying the lower semi-continuous case as well as his "patchy vector fields approach". The difference is that the elements of the relatively open partitionings are weakly invariant in contrast to the strong invariance hypothesis imposed by Bressan. Another difference is that we do not impose any smoothness assumptions regarding the boundary of these elements and regarding the vector fields.

3. Viability – the general case. A natural question is the following one: Can we apply the above theorem whenever the Basic assumption is satisfied and thus the existence of an ε-solution of F is guaranteed for each $\varepsilon > 0$? At least, can we do it when F does not collide from D to any point?

A natural class to investigate when trying to answer this question is the class of monotone usco operators. The reason is that they do not collide. It is proved in [17] that for each positive integer k there exists an invariant $\frac{1}{k}$-approximation $(\mathcal{U}^k, \mathcal{G}^k)$ of F such that $\mathcal{U}^k$ is a finite family. Nevertheless, the existence of solution problem for $\dot{x} \in F(x)$ is open. Indeed, there exist monotone operators which can not be expressed as a sum of a cyclically monotone map and a continuous function (J. Borwein, private communication), thus the result of Bounkhel and Haddad [3] is not applicable.

The answer of the above question is no. Let us consider the following simple example:

$$F(x,y) := \begin{cases} \left(1 - \dfrac{1}{\ln\sqrt{x^2+y^2}}\right)\dfrac{(x,y)}{\sqrt{x^2+y^2}} - \\[2ex] - \dfrac{1}{\ln\sqrt{x^2+y^2}}\dfrac{(-y,x)}{\sqrt{x^2+y^2}} \quad \text{if } (x,y) \neq (0,0) \\[2ex] \big\{(p,q) :\ p^2+q^2 = 1\big\} \quad \text{if } (x,y) = (0,0) \end{cases}$$

The set-valued map $F : D \rightrightarrows R^2$, where $D := \big\{(p,q) :\ p^2+q^2 < 1\big\}$, is monotone and upper semicontinuous. Clearly, the differential inclusion $(\dot{x}, \dot{y}) \in F(x,y)$ has a solution starting from each point of D including the origin. It is remarkable that any solution starting from the origin revolves infinitely many times around it for arbitrary small time due to the fact that the integral $\displaystyle\int_0^{\varrho_o} \frac{\mathrm{d}\varrho}{\varrho \ln \varrho}$ is divergent. This observation shows that any attempt to find locally finite partitionings with weakly invariant (with respect to $F + \varepsilon \bar{B}$ for every $\varepsilon > 0$) elements for every monotone usco operator is doomed. This example demonstrates that the applicability of Theorem 2.6 is rather restricted. To obtain some extension of Theorem 2.6 one should give up either the refinement condition or the local finiteness of the relatively open partitionings. The first one means that the pieces of the partitionings possess a stronger invariance property. We opted for giving up the second one. To do this we need to develop a more complicated set structure than relatively open partitionings. It is based on countably many open partitionings each of them refining the previous one.

Definition 3.1. *Let the set D be an intersection of an open subset and a closed subset of R^n and $F : \bar{D} \Rightarrow R^n$ be a multi-valued mapping with nonempty values. Let us fix $\varepsilon > 0$ and let $\mathcal{U}$ be a σ-relatively open partitioning of D (that is $\mathcal{U} = \bigcup\limits_{m=1}^{\infty} \mathcal{U}^m$, where each $\mathcal{U}^m$ is a relatively open partitioning of D) such that $\mathcal{U}^{m+1} \prec \mathcal{U}^m$ for every positive integer m.*

We say that the pair $(\mathcal{U},\mathcal{G})$ is an invariant partial ε-approximation of F if there exist a disjunct subfamily **Green** $\mathcal{U}$ *of $\mathcal{U}$ and a family $\mathcal{G} := \{G_U : U \in$* **Green** $\mathcal{U}\}$ *such that*

- (i) *whenever $U \in$ (**Green** $\mathcal{U}) \cap \mathcal{U}^m$ then $U \in \mathcal{U}^k$ for every positive integer $k \geq m$.*
- (ii) *for every point $x \in \cup$ (**Green** $\mathcal{U}$) there is a neighborhood of x intersecting at most finitely many members of* **Green** $\mathcal{U}$*;*
- (iii) *the set $\cup$ (**Green** $\mathcal{U}$) is open in D;*
- (iv) *(**Green** $\mathcal{U}$,$\mathcal{G}$) is an invariant ε-approximation of F, i.e.*
  - $(iv.a)$ *$G_U : \bar{U} \to R^n$ is upper semicontinuous multi-valued mapping with nonempty convex compact values for each $U \in$* **Green** $\mathcal{U}$*;*
  - $(iv.b)$ *whenever $U_\alpha^m \in$ (**Green** $\mathcal{U}) \cap \mathcal{U}^m$ then for each $x \in \bar{U}_\alpha^m$ and for each $\alpha \in [1, \alpha_0^m)$ the intersection $G_{U_\alpha^m}(x) \cap T_{D_\alpha^m}(x)$ (where $D_\alpha^m = \bar{D} \setminus (\bigcup\limits_{\beta<\alpha} U_\beta^m)$) is nonempty and*

$$G_{U_\alpha^m}(x) \cap T_{D_\alpha^m}(x) \subset F(x) + \varepsilon \bar{B}.$$

Remark 3.2. There is a natural linear ordering $\prec$ in the family **Green** $\mathcal{U}$ inherited from the ordering in $\mathcal{U}_m$, $m = 1, 2, \ldots$. We have to point out that already (**Green** $\mathcal{U}, \prec$) is not well ordered.

Definition 3.3. *Let the set D be an intersection of an open subset and a closed subset of R^n and $F : \bar{D} \Rightarrow R^n$ be a multi-valued mapping with nonempty values. Let $\mathcal{U}^i$ be a σ- relatively open partitioning of D and $(\mathcal{U}^i, \mathcal{G}^i)$ be an invariant partial ε_i-approximation of F, $i = 1, 2$. It is said that $(\mathcal{U}^2, \mathcal{G}^2)$ is a refinement of $(\mathcal{U}^1, \mathcal{G}^1)$ iff*

- *(i)* $0 < \varepsilon_2 < \varepsilon_1$;
- *(ii)* *for every element $U^2 \in$ **Green** $\mathcal{U}^2$ there exists an element $U^1 \in$ **Green** $\mathcal{U}^1$ with $U^2 \subset U^1$;*
- *(iii)* *if* **Green** $\mathcal{U}^2 \ni U_i^2 \subset U_i^1 \in$ **Green** $\mathcal{U}^1$, $i = 1, 2$, *with* $U_2^1 \prec U_1^1$, *then* $U_2^2 \prec U_1^2$;
- *(iv)* *if* **Green** $\mathcal{U}^2 \ni U^2 \subset U^1 \in$ **Green** $\mathcal{U}^1$, *then* $G_{U^2}(x) \subseteq G_{U^1}(x)$ *for each* $x \in \bar{U}^2$.

Extended Euler curves. Let the set D be an intersection of an open subset and a closed subset of R^n, $x_0 \in D$ and $F : \bar{D} \Rightarrow R^n$ be a uniformly bounded multi-valued mapping with nonempty values. Let $\mathcal{U}^k$ be a σ-relatively open partitioning of D and $(\mathcal{U}^k, \mathcal{G}^k)$ be an invariant partial ε_k -approximation of F for every $k = 1, 2, \ldots, m$, such that $(\mathcal{U}^{k+1}, \mathcal{G}^{k+1})$ is a refinement of $(\mathcal{U}^k, \mathcal{G}^k)$ for each $k = 1, 2, \ldots, m-1$. Let x_0 belong to the set D. We call an absolutely continuous function $x_m : [0, T] \to D$ (starting from x_0) an ε_m-extended Euler curve subordinated to $(\mathcal{U}^k, \mathcal{G}^k)$, $k = 1, 2, \ldots, m$, if there exist countably many knots $x_m(t_i)$, $i \in I$ (I well ordered) with $t_i < t_j$ whenever $i < j$, such that $t_{i+1} - t_i < \varepsilon_m$ and whenever $x_m(t_i) \in U$ for some $U \in$ **Green** $\mathcal{U}^k \setminus \left(\bigcup\limits_{l>k} \textbf{Green } \mathcal{U}^l \right)$ then $x_m(t) \in U$ for each $t \in [t_i, t_{i+1})$ and $\dot{x}_m(t) \in G_U(x_m(t))$ for almost every $t \in [t_i, t_{i+1})$.

The below written argument convince us that for arbitrary $x_0 \in D$ and arbitrary invariant partial ε_k-approximations $(\mathcal{U}^k, \mathcal{G}^k)$ of F, $k = 1, 2, \ldots, m$, such that $(\mathcal{U}^{k+1}, \mathcal{G}^{k+1})$ is a refinement of $(\mathcal{U}^k, \mathcal{G}^k)$ for each $k = 1, 2, \ldots, m-1$, there exists an ε_m-extended Euler curve subordinated to $(\mathcal{U}^k, \mathcal{G}^k)$, $k = 1, 2, \ldots, m$, starting from x_0 and defined on some interval $[0, T]$ with $T > 0$.

We set $(\mathcal{U}^0, \mathcal{G}^0)$ to be $\mathcal{U}^0 = \{D\}$, **Green** $\mathcal{U}^0 = \{D\}$ and $\mathcal{G}^0 = \{G\}$, where $G(x) = (c+1)B$ for each point $x \in D$ and c is an upper bound of $\|F(x)\|$, $x \in D$.

For every point $x \in D$ there exists unique maximal index $k_x \in \{0, 1, \ldots, m\}$ such that $x \in \mathbf{Green}\ \mathcal{U}^{k_x}$. Then there exists unique set $U_x \in \mathbf{Green}\ \mathcal{U}^{k_x}$ such that $x \in U_x$. According to Definition 3.1 and the result of [23], there exist $t_x > 0$ and an absolutely continuous function $\varphi_{x,\tau} : [\tau, \tau + t_x) \to U_x$ which is well defined on $[\tau, \tau + t_x)$ (here τ is an arbitrary positive real) and is a solution of the following differential inclusion

$$\left| \begin{array}{l} \dot{x}(t) \in G^{k_x}_{U_x}(x(t)) \ \text{ a.e. on } \ [\tau, \tau + t_x), \\ x(\tau) = x, \\ x(t) \in U_x \text{ for each } t \in [\tau, \tau + t_x). \end{array} \right.$$

We define inductively an absolutely continuous function $x_m : [0, T) \to D$ as follows: We set $x_m(t) := \varphi_{x_0,0}(t)$ for each $t \in [0, \min\{\varepsilon_m, t_{x_0}\})$. Let us have defined x_m on some interval $[0, \hat{t})$. Then for each increasing sequence $\{t_n\}_{n=1}^{\infty} \to \hat{t}$ we have

$$\|x_m(t_n) - x_m(t_{n-1})\| \le \int_{t_{n-1}}^{t_n} \|\dot{x}_m(\tau)\|\, d\tau \le c\,(t_n - t_{n-1}).$$

This means that the sequence $\{x_m(t_n)\}_{n=1}^{\infty}$ is a Cauchy sequence. Hence, there exist $x_m(\hat{t}) := \lim\limits_{t \to \hat{t}} x_m(t)$. Then we define an absolutely continuous extension of the function x_m by $\varphi_{x_m(\hat{t}),\hat{t}}$ on the interval $\Big[\hat{t}, \hat{t} + \min\{\varepsilon_m, t_{x_m(\hat{t})}\}\Big)$. Let $[0, T_m)$ be the maximal interval on which x_m can be defined in this way. Since $D = G \cap U$, where U is an open subset and G is a closed subset of R^n, and D is invariant with respect to x_m,

$$T_m \ge T := \frac{\operatorname{dist}(x_0, R^n \setminus U)}{c + 1} > 0. \tag{3.1}$$

Note that each ε_m-extended Euler curve subordinated to $(\mathcal{U}^k, \mathcal{G}^k)$, $k = 1, 2, \ldots, m$, is an ε_k-extended Euler curve subordinated to $(\mathcal{U}^l, \mathcal{G}^l)$, $l = 1, 2, \ldots, k$, for each $k \le m$.

Theorem 3.4. *Let the set D be an intersection of an open subset and a closed subset of R^n, $x_0 \in D$ and $F : \bar{D} \Rightarrow R^n$ be a uniformly bounded multi-valued mapping with nonempty closed values. Let $\varepsilon_k \longrightarrow 0$ be a sequence of positive reals tending to zero, $\mathcal{U}^k$ be a σ-relatively open partitioning of D and $(\mathcal{U}^k, \mathcal{G}^k)$ be an invariant partial ε_k-approximation of F for every $k = 1, 2, \ldots$ such that $(\mathcal{U}^{k+1}, \mathcal{G}^{k+1})$ is a refinement of $(\mathcal{U}^k, \mathcal{G}^k)$ for each $k = 1, 2, \ldots$. Then there exists* *an absolutely continuous function* $x : [0,T] \to D$ *satisfying* $x(0) = x_0$ *and such that* $\dot{x}(t) \in F(x(t))$ *almost everywhere in the set*

$$[0,T] \bigcap \left(\bigcap_{m=1}^{\infty} \left\{ \tau : \ x(\tau) \in \left(\bigcup \textbf{Green } \mathcal{U}^m \right) \right\} \right).$$

Proof. For each positive integer k we denote by $x_k(\cdot)$ an ε_k -extended Euler curve subordinated to $(\mathcal{U}^s, \mathcal{G}^s)$, $s = 1, 2, \dots, k$, starting from x_0. Without loss of generality we may think that the sequence $\{x_k(\cdot)\}_{k=1}^{\infty}$ is uniformly convergent on the interval $[0,T]$ to an absolutely continuous function denoted by $x(\cdot)$.

Clearly, it is sufficient to prove that $\dot{x}(\tau) \in F(x(\tau)) + \varepsilon_m \bar{B}$ for almost every τ from the set $\bigcap \left\{ t : \ x(t) \in \bigcup \textbf{Green } \mathcal{U}^m \right\}$ for each positive integer m. Indeed, let us fix an arbitrary positive integer m and let

$$\bar{\tau} \in [0,T] \bigcap \left\{ t : \ x(t) \in \bigcup \textbf{Green } \mathcal{U}^m \right\}.$$

Then there exists a neighbourhood W of $x(\bar{\tau})$ that intersects at most finitely many members of $\mathcal{U}^m$ and $W \subset \bigcup \textbf{Green } \mathcal{U}^m$ (cf. (i) and (ii) of Definition 3.1). Then we can find an open neighbourhood V of $x(\bar{\tau})$ and a positive real r such that $\bar{V} + rB \subset W$. Because the set V is open and the map F is bounded, there exist a positive integer $k_0 \geq m$ and a positive real δ such that $x_k(t) \in V$ for each $t \in (\bar{\tau} - \delta, \bar{\tau} + \delta) \cap [0,T]$ and for each $k \geq k_0$. Moreover, we may think that k_0 is so large that $c\,\varepsilon_{k_0} < r$.

Since the nonempty elements of the family $V \bigcap \mathcal{U}^m$ are finitely many, this family is a relatively open partitioning of $V \cap D$. Let us denote its nonempty elements by V_s, $s = 1, 2, \dots, s(m)$. Then $\{x_k(t) : \ t \in (\bar{\tau} - \delta, \bar{\tau} + \delta) \cap [0,T]\} \subset \bigcup_{s=1}^{s(m)} V_s$ whenever $k \geq k_0$.

According to the definition of extended Euler curves, we denote by $\bar{x}_k := x_k(\bar{t}_k)$ the last knot of $x_k(\cdot)$ which is before $x_k(\bar{\tau} - \delta)$ for each $k \geq k_0$. Then the following inequality holds true

$$\|x_k(\bar{\tau} - \delta) - \bar{x}_k\| \leq c(\bar{\tau} - \delta - \bar{t}_k) \leq c\,\varepsilon_k \leq c\,\varepsilon_{k_0} < r.$$

Therefore, $\bar{x}_k \in V + rB \subset W \subset \bigcup \textbf{Green } \mathcal{U}^m$. For each $t \in (\bar{\tau} - \delta, \bar{\tau} + \delta) \cap [0,T]$ there exists uniquely defined $s_t^k \in \{1, 2, \dots, s(m)\}$ with $x_k(t) \in V_{s_t^k}$. Thus from $\bar{x}_k \in \bigcup$ **Green** $\mathcal{U}^m$ and from $x_k(\cdot)$ being an ε_m-extended curve (according to the remark at the end of the definition of extended Euler curves) we obtain that for each $t \in (\bar{\tau}-\delta, \bar{\tau}+\delta) \cap [0,T]$ there exists uniquely defined $s_t^k \in \{1, 2, \dots, s(m)\}$ with $x_k(t) \in V_{s_t^k}$ and such that $x_k(\cdot)$ is a solution of the following differential inclusion

$$\left| \begin{array}{l} \dot{x}_k(\tau) \in G^m_{V_{s_t^k}}(x_k(\tau)) \ \text{ a.e. on } \ [t, t+\delta_t^k) \text{ for some } \delta_t^k > 0, \\ x_k(\tau) \in V_{s_t^k} \text{ for each } \tau \in [t, t+\delta_t^k). \end{array} \right.$$

The above written relations imply that

$$\dot{x}_k(\tau) \in G^m_{V_{s_t^k}}(x_k(\tau)) \cap T_{D \setminus \left(\cup_{s<s_t^k} V_s\right)}(x_k(\tau)) \subset F(x_k(\tau)) + \varepsilon_m \bar{B} \ \text{ a.e. on } \ [t, t+\delta_t^k).$$

Hence $\{x_k(\cdot)\}_{k \geq k_0}$ is a sequence of ε_m-solutions of $\dot{x} \in F(x)$ with respect to which the elements of $V \cap \mathcal{U}^m$ are weakly invariant. As $V \cap \mathcal{U}^m$ is locally finite, then Proposition 2.1 (iii) yields that $x(\cdot)$ is an ε_m-solution of $\dot{x} \in F(x)$ on $(\bar{\tau} - \delta, \bar{\tau} + \delta) \cap [0,T]$, thus having the same conclusion almost everywhere in the relatively open (hence measurable) subset $[0,T] \bigcap \left\{ t : \ x(t) \in \bigcup \textbf{Green } \mathcal{U}^m \right\}$ of $[0,T]$. This completes the proof. $\square$

4. An Olech type result. The following assertion is closely related to the results of Olech [1975] (cf. [18]), Lojasiewicz [1985] (cf. [20]), T. Haddad, A. Jourani and L. Thibault (cf. [15]) and T. Haddad and L. Thibault (cf. [16]). In our assumptions the set of lower semicontinuity of one of the summands is G_δ in contrast to the usual openness assumption. Our theorem is not a generalization of some of the above mentioned results because we consider for simplicity the autonomous case.

Theorem 4.1. *Let F and G be bounded mappings defined on an intersection D of an open subset and a closed subset of R^n. We assume that there exist a sequence of positive reals ε_k, $k = 1, 2, \dots$, tending to zero as k tends to infinity and relatively open (in D) sets D_k such that F is ε_k-lower semicontinuous on D_k (we assume that $D_1 = D$). Let F be upper semicontinuous and convex valued on the set $D \setminus \bigcap_{k=1}^{\infty} D_k$ and let G be upper semicontinuous and convex valued on D. We assume that the following tangential assumption holds true:*

for each positive integer k, for each $x \in D_k$ and for each $y \in F(x)$ there exists $y' \in F(x) \cap \bar{B}(y, \varepsilon_k) \cap (T_D(x) - G(x))$.

Then the differential inclusion $\dot{x} \in F(x)+G(x)$ has a local solution starting at each point of the set D and remaining in D.

Remark 4.2. The tangential assumption from the above formulated theorem reduces to the standard assumptions in the purely lower semicontinuous case as well as in the purely upper semicontinuous convex valued case.

Proof. We follow the construction of Bressan (cf. Lemma 6.2 in [11], p. 67). First, we define the "ice-cream cone"

$$K_\delta := \left\{(x,t) \in R^{n+1} : \ \|x\| \le (c+1)t, \ t \in [0,\delta)\right\},$$

where c is an upper bound of $\{\|y\| : \ y \in F(x) + G(x), \ x \in D\}$.

Without loss of generality we may assume that $D_{k+1} \subset D_k$ and $\varepsilon_{k+1} \le \frac{1}{2}\,\varepsilon_k$ for every $k = 1, 2, \dots$.

Let x_0 be an arbitrary point in D. We choose T to be an element of the interval $\left(0, \dfrac{\operatorname{dist}(x_0, R^n \setminus U)}{c+1}\right)$ (where $D = K \cap U$, K closed and U open).

Let us fix an arbitrary positive integer k. Then the set D_k is open in D and F is ε_k -l.s.c. at each point of D_k. The sets $\widetilde{D} := D \times [0,T]$, $\widetilde{D}_k := D_k \times [0,T]$ are intersections of a closed subset and an open subset of R^{n+1}. We set $z = (x,t) \in R^{n+1}$, $\widetilde{F}(z) := F(x) \times \{1\}$, $\widetilde{G}(z) := G(x) \times \{1\}$ and consider the differential inclusion $\dot{z} \in \widetilde{F}(z) + \widetilde{G}(z)$, $z \in \widetilde{D}$.

Let us consider the compact set

$$M_s^k := \left\{x \in D_k : \ \operatorname{dist}\ \left(x, \bar{D} \setminus D_k\right) \ge \frac{1}{s},\ \|x\| \le s\right\} \times [0,T]$$

for each positive integer s. The family $\{M_s^k : \ s = 1, 2, \dots\}$ of compact sets is increasing and its union is $\widetilde{D}_k$. For each $z \in M_s^k$ we consider a neighborhood $W(z)$ of the form $M_s^k \cap (\tilde{z} + K_\delta)$, where $\tilde{z} \in D_k \times [\tilde{\tau}, T]$ with $\tilde{\tau} < 0$, and z is in the relative interior of $W(z)$ with respect to M_s^k. Moreover, we choose $W(z)$ to be so small that $F(x') + \varepsilon_k \bar{B} \supset F(x)$ for each point $z' = (x', t') \in \overline{\widetilde{D}_k \cap (\tilde{z} + K_\delta)}$, where $z = (x,t)$ and $\bar{B}$ is the unit ball of R^n. It is possible because F is ε_k-l.s.c. at each point of D_k. Clearly the closed ball $\bar{B}(y, \varepsilon_k)$ centered at an arbitrary element y of $F(x) \times \{1\}$ with radius ε_k has nonempty intersection with $\widetilde{F}(z')$ for every point $z' \in \overline{\widetilde{D}_k \cap (\tilde{z} + K_\delta)}$.

The compactness of M_s^k implies that there exist finitely many z_i, $i = 1, 2, \ldots, m(k, s)$, with $M_s^k = \bigcup_{i=1}^{m(k,s)} W(z_i)$. Let $W(z_i) = M_s^k \cap (\tilde{z}_i + K_{\delta_i})$, where $\tilde{z}_i = (\tilde{x}_i, \tilde{t}_i)$. We consider the reals $\{t_i := \tilde{t}_i + \delta_i : \ i = 1, \ldots, m(k, s)\}$. Without loss of generality we may assume that this finite sequence is non decreasing. We set

$$V_1^{k,s} := \{z = (x, t) \in \widetilde{D}_k : \ t < t_1\} \setminus \left(\bigcup_{i=1}^{m(k,s)} (\tilde{z}_i + K_{\delta_i}) \right).$$

The set $V_1^{k,s}$ is open in $\widetilde{D}_k$ because the finitely many sets $\tilde{z}_i + K_{\delta_i}$, $i = 1, \ldots, m(k, s)$, are closed in $\{z = (x, t) : \ t < t_1\}$. Next, we define

$$(4.1) \quad V_{i+1}^{k,s} := \{z = (x, t) \in \widetilde{D}_k : \ t < t_1\} \bigcap \left((\tilde{z}_i + K_{\delta_i}) \setminus \left(\bigcup_{j=i+1}^{m(k,s)} (\tilde{z}_j + K_{\delta_j}) \right) \right)$$

for $i = 1, \ldots, m(s, k)$. Each set $V_i^{k,s}$ is relatively open in $\widetilde{D} \setminus \left(\bigcup_{j<i} V_j^{k,s} \right)$. We proceed in the same way in the next strips $\{z = (x, t) : \ t \in [t_p, t_{p+1})\}$, $p = 1, \ldots, m(k, s) - 1$, by setting the first element in the strip to be

$$\{z = (x, t) \in \widetilde{D}_k : \ t \in [t_p, t_{p+1})\} \setminus \left(\bigcup_{i=1}^{m(k,s)} (\tilde{z}_i + K_{\delta_i}) \right)$$

and the next elements are constructed recursively as in (4.1) but replacing $t < t_1$ by $t \in [t_p, t_{p+1})$. We put the last element of this relatively open partitioning $\mathcal{V}^{k,s}$ to be the set $\widetilde{D} \setminus \widetilde{D}_k$.

Our goal is to construct for each positive integer k an invariant partial $3\varepsilon_k$-approximation $(\mathcal{U}^k, \mathcal{G}^k)$ of $\widetilde{F} + \widetilde{G}$ such that $\cup$ $(\mathbf{Green}\, \mathcal{U}^k) = \widetilde{D}_k$.

We set $\mathcal{U}^{1,1}$ to be equal to $\mathcal{V}^{1,1}$ and $(\mathbf{Green}\, \mathcal{U}^1) \cap \mathcal{U}^{1,1}$ to be the family of nonempty elements of $\mathcal{V}^{1,1}$ which are contained in some "ice-cream cone". Let us have constructed $\mathcal{U}^{1,s}$ and $(\mathbf{Green}\, \mathcal{U}^1) \cap \mathcal{U}^{1,s}$ for some positive integer s. We fix an arbitrary element $U \in \mathcal{U}^{1,s}$. If U belongs to $(\mathbf{Green}\, \mathcal{U}^1) \cap \mathcal{U}^{1,s}$, we leave it as it is. If U does not belong to $(\mathbf{Green}\, \mathcal{U}^1) \cap \mathcal{U}^{1,s}$, then $\mathcal{V}^{1,s+1} \cap U$ is a relatively open partitioning of U. Proceeding in this way with all elements of $\mathcal{U}^{1,s}$ and using the lexicographic order (cf. [22]) we obtain the relatively open partitioning $\mathcal{U}^{1,s+1}$

of $\widetilde{D}$. We set (**Green** $\mathcal{U}^1$) $\cap\, \mathcal{U}^{1,s+1}$ to be the family of nonempty elements of $\mathcal{U}^{1,s+1}$ which are contained in some "ice-cream cone". In this way we obtain the σ-relatively open partitioning $\mathcal{U}^1 = \bigcup_{s=1}^{\infty} \mathcal{U}^{1,s}$ and its subfamily **Green** $\mathcal{U}^1$. Let V be an arbitrary element of **Green** $\mathcal{U}^1$. Then it is contained in some "ice-cream cone", and therefore there is some element $y_V \in R^n$ such that the closed ball $\bar{B}(y_V, \varepsilon_1)$ centered at y_V with radius ε_1 has nonempty intersection with $F(x)$ for every $z = (x,t)$ in the closure of V. We define the value of G_V^1 at each point $z = (x,t)$ of $\bar{V}$ to be $(\bar{B}(y_V, 2\varepsilon_1) + G(x)) \times \{1\}$. Clearly, the map $G_V^1 : \bar{V} \to R^{n+1}$ is upper semicontinuous with compact convex values. Also, $y_V \in F(x) + \varepsilon_1 \bar{B}$ and $F(x) \subset T_D(x) - G(x) + \varepsilon_1 \bar{B}$ imply that $G_V^1(z) \cap T_{\widetilde{D}}(z) \neq \emptyset$ for each $z \in \bar{V}$. Moreover, note that each ice-cream cone is strongly invariant with respect to the trajectories of G_V^1 because of the choice of c and the fact that without loss of generality we may think that $3\varepsilon_1 < 1$. Thus the set V is weakly invariant with respect to G_V^1. We have for every $z = (x,t) \in \bar{V}$ that

$$G_V^1(z) = (\bar{B}(y_V, 2\varepsilon_1) + G(x)) \times \{1\} = (y_V + 2\varepsilon_1 \bar{B} + G(x)) \times \{1\} \subset$$

$$\subset (F(x) + \varepsilon_1 \bar{B} + 2\varepsilon_1 \bar{B} + G(x)) \times \{1\} = \widetilde{F}(z) + \widetilde{G}(z) + 3\varepsilon_1 \bar{B}',$$

where $\bar{B}'$ is the closed unit ball of R^{n+1}.

We put $\mathcal{G}^1 := \{G_V^1 \;:\; V \in \textbf{Green}\, \mathcal{U}^1\}$. We have just checked that the so obtained couple $(\mathcal{U}^1, \mathcal{G}^1)$ is an invariant partial $3\varepsilon_1$-approximation of $\widetilde{F} + \widetilde{G}$.

Let us assume that we have constructed the σ - relatively open partitioning $\mathcal{U}^k := \bigcup_{s=1}^{\infty} \mathcal{U}^{k,s}$, the family **Green** $\mathcal{U}^k$ and $\mathcal{G}^k := \{G_U^k \;:\; U \in \textbf{Green}\, \mathcal{U}^k\}$ such that $(\mathcal{U}^k, \mathcal{G}^k)$ is an invariant partial $3\varepsilon_k$-approximation of $\widetilde{F} + \widetilde{G}$ for some positive integer k. We set $\mathcal{U}^{k+1,1}$ to be equal to $\mathcal{U}^{k,1} \cap \mathcal{V}^{k+1,1}$ and (**Green** $\mathcal{U}^{k+1}$) $\cap\, \mathcal{U}^{k+1,1}$ to be the family of nonempty elements of **Green** $\mathcal{U}^k \cap \mathcal{U}^{k,1}$ which are contained in some "ice-cream cone" coming from $\mathcal{V}^{k+1,1}$. Let us have constructed $\mathcal{U}^{k+1,s}$ and (**Green** $\mathcal{U}^{k+1}$) $\cap\, \mathcal{U}^{k+1,s}$ for some positive integer s. We fix an arbitrary element $U \in \mathcal{U}^{k+1,s}$. If U belongs to (**Green** $\mathcal{U}^{k+1}$) $\cap\, \mathcal{U}^{k+1,s}$, we leave it as it is. If U does not belong to (**Green** $\mathcal{U}^{k+1}$) $\cap\, \mathcal{U}^{k+1,s}$, then $\mathcal{U}^{k,s+1} \cap \mathcal{V}^{k+1,s+1} \cap U$ is a relatively open partitioning of U. Proceeding in this way with all elements of $\mathcal{U}^{k+1,s}$ and using the lexicographic order (cf. [22]) we obtain the relatively open partitioning $\mathcal{U}^{k+1,s+1}$ of $\widetilde{D}$. We set (**Green** $\mathcal{U}^{k+1}$) $\cap\, \mathcal{U}^{k+1,s+1}$ to be the family of nonempty elements of $\mathcal{U}^{k+1,s+1}$ which are contained in some "ice-cream cone" coming from $\mathcal{V}^{k+1,s+1}$. Since each element V of **Green** $\mathcal{U}^{k+1}$ is contained in some "ice-cream cone" $z_V + K_{\delta_V}$ with $z_V = (x_V, t_V)$, and in some element U of (**Green** $\mathcal{U}^k$) $\cap$ $\mathcal{U}^{k,s+1}$, we can find an element y_V that belongs to $F(x_V) \cap \bar{B}(y_U, \varepsilon_k)$ such that $\bar{B}(y_V, \varepsilon_{k+1})$ has nonempty intersection with $F(x')$ for every point $z' = (x', t')) \in U$. We define the value of G_V^{k+1} at each point of $\bar{V}$ to be $(\bar{B}(y_V, 2\varepsilon_{k+1}) + G(x)) \times \{1\}$. Clearly, as in the case $k = 0$, the map $G_V^{k+1} : \bar{V} \to R^{n+1}$ is upper semicontinuous with compact convex values and $G_V^{k+1}(z) \cap T_{\widetilde{D}}(z) \neq \emptyset$ for each $z \in \bar{V}$, so the set V is weakly invariant with respect to G_V^{k+1}. Also, for every $z = (x, t) \in \bar{V}$ we have $G_V^{k+1}(z) \subset \widetilde{F}(z) + \widetilde{G}(z) + 3\varepsilon_{k+1}\bar{B}'$. Therefore, $(\mathcal{U}^{k+1}, \mathcal{G}^{k+1})$ is an invariant $3\varepsilon_{k+1}$ approximation of $\widetilde{F} + \widetilde{G}$. Moreover, for each point $z' \in \bar{V}$ we have

$$G_V^{k+1}(z') := (\bar{B}(y_V, 2\varepsilon_{k+1}) + G(x')) \times \{1\} \subset (\bar{B}(y_U, \varepsilon_k + 2\varepsilon_{k+1}) + G(x')) \times \{1\} \subset$$

$$\subset (\bar{B}(y_U, 2\varepsilon_k) + G(x')) \times \{1\} = G_U^k(z').$$

Therefore the so constructed $(\mathcal{U}^{k+1},\mathcal{G}^{k+1})$ is a refinement of $(\mathcal{U}^k,\mathcal{G}^k)$.

We show that for each positive integer k there exists an ε_k -extended Euler curve $z_k(\cdot)$ subordinated to $(\mathcal{U}^s, \mathcal{G}^s)$, $s = 1, 2, \ldots, k$, starting from $z_0 = (x_0, 0)$, and such that

$$\dot{z}_k(t) \in \widetilde{F}(\bar{B}(z_k(t), \varepsilon_k)) + \widetilde{G}(\bar{B}(z_k(t), \varepsilon_k)) + 3\varepsilon_k \bar{B}' \tag{4.2}$$

almost everywhere on $[0, T]$. Indeed, we have to specify the decision rule for the inductive construction of an extended Euler curve starting from a knot $z_k(\tau)$, $\tau \in [0, T)$ which does not belong to $\cup$**Green** $\mathcal{U}^k$. Let $z_k(\tau)$ belong to $\cup$**Green** $\mathcal{U}^l \setminus \cup_{s>l}$ ($\cup$**Green** $\mathcal{U}^s$) for some nonnegative integer $l < k$. Let $z_k(\tau) \in U \in$ **Green** $\mathcal{U}^l$. Then $G_U^l(z)$ has the appearance $(\bar{B}(y_U, 2\varepsilon_l) + G(x)) \times \{1\}$, where $\bar{B}(y_U, \varepsilon_l) \cap F(x) \neq \emptyset$ for each $z = (x, t) \in U$. Let us denote by $\bar{y}_x$ an arbitrary point from $\bar{B}(y_U, \varepsilon_l) \cap F(x)$. According to the tangential assumption, we have that there exists

$$y_x \in F(x) \cap \bar{B}(\bar{y}_x, \varepsilon_l) \cap (T_D(x) - G(x)).$$

Since $y_x \in \bar{B}(y_U, 2\varepsilon_l)$ we obtain that

$$\begin{aligned} &\emptyset \neq \Big(\Big(\big(\bar{B}(y_U, 2\varepsilon_l) \cap F(x)\big) + G(x)\Big) \cap T_D(x)\Big) \times \{1\} \subset \\ &\subset G_U^l(z) \cap \Big(\widetilde{F}(z) + \widetilde{G}(z)\Big) \cap T_{\widetilde{D}}(z) \end{aligned} \tag{4.3}$$

Since $z_k(\tau)$ does not belong to $\cup$**Green** $\mathcal{U}^k$, $\widetilde{F}(z_k(\tau))$ is a convex subset of R^{n+1}. The upper semicontinuity of F implies the existence of an open neighbourhood V of $z_k(\tau)$ such that $V \subset \bar{B}(z_k(\tau), \varepsilon_k)$ and $\widetilde{F}(z) \subset \widetilde{F}(z_k(\tau)) + \varepsilon_k \bar{B}$ for each $z \in V$. So, $\overline{\text{co}}\ \widetilde{F}(z) \subset \widetilde{F}(z_k(\tau)) + \varepsilon_k \bar{B}$, and hence $\overline{\text{co}}\ \left(\widetilde{F}(z) + \widetilde{G}(z)\right) \subset \widetilde{F}(z_k(\tau)) + \widetilde{G}(z) + \varepsilon_k \bar{B}$ for each $z \in V$.

We consider the differential inclusion

$$\begin{aligned} &\dot{z}(t) \in G_U^l(z(t)) \cap \overline{\text{co}}\ \left(\widetilde{F}(z(t)) + \widetilde{G}(z(t))\right) \\ &z(\tau) = z_k(\tau). \end{aligned}$$

There exists so small $t_{z_k(\tau)} > 0$ that the solution of the above written differential inclusion exists on $[\tau, \tau + t_{z_k(\tau)})$ and remains in $U \cap V$ (because of (4.3)). Thus our decision rule for extending $z_k(\cdot)$ on $[\tau, \tau + t_{z_k(\tau)})$ is to take an arbitrary solution of the above written differential inclusion.

Without loss of generality we may think that the sequence $\{z_k(\cdot)\}_{k=1}^{\infty}$ is uniformly convergent on the interval $[0, T]$ to an absolutely continuous function denoted by $z(\cdot)$.

Let us fix an arbitrary positive integer m and let

$$\bar{\tau} \in [0, T] \setminus \left\{ t : \ z(t) \in \bigcup \textbf{ Green } \mathcal{U}^m \right\}.$$

Then, for almost all such $\bar{\tau}$ we have

$$\dot{z}(\bar{\tau}) \in \ \overline{\text{co}}\ \{\dot{z}_k(\tau) : \ k \geq k_0\}.$$

As $\widetilde{F}$ and $\widetilde{G}$ are upper semicontinuous at $z(\bar{\tau})$ by assumption, there is a neighborhood W of $z(\bar{\tau})$ such that $\widetilde{F}(z) \subset \widetilde{F}(z(\bar{\tau})) + \varepsilon_m \bar{B}'$ and $\widetilde{G}(z) \subset \widetilde{G}(z(\bar{\tau})) + \varepsilon_m \bar{B}'$ for every $z \in W$. Let k_0 be so big that $z_k(\bar{\tau}) + \varepsilon_k \bar{B}' \subset W$ whenever $k \geq k_0$. Thus, using the convexity of $\widetilde{F}(z(\bar{\tau})) + \widetilde{G}(z(\bar{\tau}))$, $k_0 \geq m$ and (4.2), we have

$$\dot{z}(\bar{\tau}) \in \ \overline{\text{co}}\ \{\dot{z}_k(\bar{\tau}) : \ k \geq k_0\} \subset$$

$$\subset \overline{\text{co}} \left(\bigcup_{k \geq k_0} \left(\widetilde{F}(\bar{B}(z_k(\bar{\tau}), \varepsilon_k)) + \widetilde{G}(\bar{B}(z_k(\bar{\tau}), \varepsilon_k)) + 3\varepsilon_k \bar{B}' \right) \right) \subset$$

$$\subset \overline{\text{co}} \left(\bigcup_{k \geq k_0} \left(\widetilde{F}(z(\bar{\tau})) + \varepsilon_m \bar{B}' + \widetilde{G}(z(\bar{\tau})) + \varepsilon_m \bar{B}' + 3\varepsilon_k \bar{B}' \right) \right) \subset$$

$$\subset \widetilde{F}(z(\bar{\tau})) + \widetilde{G}(z(\bar{\tau})) + 5\varepsilon_m \bar{B}'.$$

This, combined with the conclusion of Theorem 3.4, completes the proof. □

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