ON THE DIRICHLET PROBLEM IN CYLINDRICAL DOMAINS FOR EVOLUTION OLEĬNIK–RADKEVIČ PDE'S: A TIKHONOV-TYPE THEOREM

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ABSTRACT. We consider the linear second order PDO's

$$\mathscr{L} = \mathscr{L}_0 - \partial_t := \sum_{i,j=1}^{N} \partial_{x_i}(a_{i,j}\partial_{x_j}) - \sum_{j=i}^{N} b_j \partial_{x_j} - \partial_t,$$

and assume that $\mathscr{L}_0$ has nonnegative characteristic form and satisfies the Oleĭnik–Radkevič rank hypoellipticity condition. These hypotheses allow the construction of Perron-Wiener solutions of the Dirichlet problems for $\mathscr{L}$ and $\mathscr{L}_0$ on bounded open subsets of $\mathbb{R}^{N+1}$ and of $\mathbb{R}^N$, respectively.

Our main result is the following Tikhonov-type theorem:
Let $\mathcal{O} := \Omega\times]0,T[$ be a bounded cylindrical domain of $\mathbb{R}^{N+1}$, $\Omega \subset \mathbb{R}^N$, $x_0 \in \partial\Omega$ and $0 < t_0 < T$. Then $z_0 = (x_0, t_0) \in \partial\mathcal{O}$ is $\mathscr{L}$-regular for $\mathcal{O}$ if and only if x_0 is $\mathscr{L}_0$-regular for Ω.

As an application, we derive a boundary regularity criterion for degenerate Ornstein–Uhlenbeck operators.

1. Introduction

We consider linear second order partial differential operators of the type

$$\mathscr{L}_0 := \sum_{i,j=1}^{N} \partial_{x_i}\left(a_{ij}\partial_{x_j}\right) + \sum_{j=1}^{N} b_j \partial_{x_j} \tag{1.1}$$

in an open set X of $\mathbb{R}^N$, $N \geq 2$, and their "evolution"counterpart in $X \times \mathbb{R}$

$$\mathscr{L} = \mathscr{L}_0 - \partial_t. \tag{1.2}$$

We assume $\mathscr{L}_0$ in (1.1) is of non totally degenerate Oleĭnik and Radkevič type, i.e., we assume

(H1) $a_{ij} = a_{ji}, b_i \in C^\infty(X, \mathbb{R})$ and

$$A(x) := (a_{ij}(x))_{i,j=1,\dots,N} \geq 0 \qquad \forall x \in X.$$

Moreover

$$\inf_X a_{11} =: \alpha > 0.$$

2010 *Mathematics Subject Classification.* 35H10; 35K65; 35J70; 35J25; 31D05; 35D99.
Key words and phrases. Dirichlet problem, Perron-Wiener solution, Boundary behavior of Perron-Wiener solutions, Hypoelliptic operators, Potential theory.

(H2) rank $\mathrm{Lie}\{X_1, \dots, X_N, X_0\}(x) = N \qquad \forall x \in X$, where,

$$X_i = \sum_{j=1}^{N} a_{ij}\partial_{x_j}, \ i = 1, \dots, N, \quad \text{and} \quad X_0 = \sum_{j=1}^{N} b_j \partial_{x_j}.$$

Hypotheses (H1) and (H2) imply that $\mathscr{L}_0$ is hypoelliptic in X (see [OR73]), that is:

$$\Omega \text{ open subset of } X,\ u \in \mathcal{D}'(\Omega), \mathscr{L}_0 u \in C^\infty(\Omega, \mathbb{R}) \implies u \in C^\infty(\Omega, \mathbb{R}).$$

The same assumptions (H1) and (H2) also imply that $\mathscr{L}_0 - \partial_t$ is hypoelliptic in $X \times \mathbb{R}$.

We will show in Section 2 that $\mathscr{L}_0$ and $\mathscr{L}_0 - \partial_t$ endow X and $X \times \mathbb{R}$, respectively, with a local structure of σ^*-harmonic space, in the sense of [3], Chapter 6. As a consequence, in particular, the Dirichlet problems

$$\begin{cases} \mathscr{L}_0 u = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = \varphi, \end{cases} \quad \text{and} \quad \begin{cases} (\mathscr{L}_0 - \partial_t) v = 0 \text{ in } \mathcal{O} := \Omega \times]0, T[, \\ v|_{\partial\mathcal{O}} = \psi, \end{cases}$$

have a generalized solution in the sense of Perron–Wiener, for every bounded open set $\Omega \subset\subset X$, for every $T > 0$, and for every $\varphi \in C(\partial\Omega, \mathbb{R})$ and $\psi \in C(\partial\mathcal{O}, \mathbb{R})$. We will denote such generalized solutions by, respectively,

$$H_\varphi^\Omega \quad \text{and} \quad K_\psi^{\mathcal{O}}.$$

As usual, we say that a point $x_0 \in \partial\Omega$ ($(x_0, t_0) \in \partial\mathcal{O}$) is $\mathscr{L}_0$-regular for Ω ($\mathscr{L}$-regular for $\mathcal{O}$) if

$$\lim_{x \longrightarrow x_0} H_\varphi^\Omega(x) = \varphi(x_0) \qquad \forall \varphi \in C(\partial\Omega, \mathbb{R})$$

$$\left(\lim_{(x,t) \longrightarrow (x_0, t_0)} K_\psi^{\mathcal{O}}(x, t) = \psi(x_0, t_0) \qquad \forall \psi \in C(\partial\mathcal{O}, \mathbb{R}) \right).$$

The aim of this paper is to prove the following theorem:

Theorem 1.1. *Let Ω be a bounded open set with $\overline{\Omega} \subseteq X$, and let $x_0 \in \partial\Omega$ and $t_0 \in]0, T[$. Then, x_0 is $\mathscr{L}_0$-regular for Ω if and only if (x_0, t_0) is $\mathscr{L}_0 - \partial_t$-regular for $\mathcal{O} := \Omega \times]0, T[$.*

When $\mathscr{L} = \Delta - \partial_t$ is the classical heat operator, our result re-establishes a theorem proved by Tikhonov in 1938 [Tik38]. Other proofs of the Tikhonov Theorem were given by Fulks in 1956 and in 1957 [Ful56, Ful57] and by Babuška and Výborný in 1962 [BV62]. Chan and Young extended the Tikhonov Theorem to parabolic operators with Hölder continuous coefficients in 1977 [CY77], and Arendt to parabolic operators with bounded measurable coefficients in 2000 [Are00]. The corresponding version for p-Laplacian-type evolution operators has been proved by Kilpeläinen and Lindqvist in 1996 [KL96] and by Banerjee and Garofalo in 2015 [BG15].

To the best of our knowledge, the only Tikhonov-type theorem for second order "evolution"sub-Riemannian PDO's appearing in the literature is the result by Negrini [Neg83] in abstract β-harmonic spaces[1].

This paper is organised as follows. In Section 2, all the notions and results from Potential Theory that we need are briefly recalled. In particular, we recall the notion of σ^*-harmonic space and then we prove that $\mathscr{L}_0$ and $\mathscr{L}$ endow X and

[1]For a definition of β-harmonic spaces see [CC72].

$X \times \mathbb{R}$, respectively, with a local structure of σ^*-harmonic space. In this way, we derive the existence of a generalized solution in the sense of Perron–Wiener in both our settings. Section 3 is devoted to two key results for the proof of the main theorem (Theorem 1.1), which is the content of Section 4. Finally, combining our Tikhonov-type theorem with a corollary of the Wiener–Landis-type criterion for Kolmogorov-type operators proved in [KLT18], we establish a geometric boundary regularity criterion for degenerate Ornstein–Uhlenbeck operators.

2. $\mathscr{L}_0$-harmonic and $\mathscr{L}$-harmonic spaces

2.1. The σ^*-harmonic space. For the readers' convenience we recall the definition of σ^*-harmonic space supported on a an open set $E \subseteq \mathbb{R}^p$, $p \geq 2$, and refer to Chapter 6 of the monograph [BLU07] for details.

Let $\mathcal{H}$ be a sheaf of functions in E such that $\mathcal{H}(V)$ is a linear subspace of $C(V, \mathbb{R})$, for every open set $V \subseteq E$. The functions in $\mathcal{H}(V)$ are called *$\mathcal{H}$-harmonic in V*. The open set V is called *$\mathcal{H}$-regular* if

(i) $\overline{V} \subseteq E$ is compact;

(ii) for every $\varphi \in C(\partial V, \mathbb{R})$ there exists a unique function such that
$$h_\varphi^V(x) \to \varphi(\xi) \text{ as } x \to \xi, \text{ for every } \xi \in \partial V;$$

(iii) $h_\varphi^V \geq 0$ if $\varphi \geq 0$.

A lower semicontinuous function $u : W \longrightarrow]-\infty, \infty]$, $W \subseteq E$ open, is called *$\mathcal{H}$-superharmonic* if

(i) $u \geq h_\varphi^V$ in V for every $\mathcal{H}$-regular open set V with $\overline{V} \subseteq W$ and for every $\varphi \in C(\partial V, \mathbb{R})$ with $\varphi \leq u|_{\partial V}$;

(ii) $\{x \in W \mid u(x) < \infty\}$ is dense in W.

We denote by $\overline{\mathcal{H}}(W)$ the cone of the $\mathcal{H}$-superharmonic functions in W.

The couple $(E, \mathcal{H})$ is called a σ^*-harmonic space if the following axioms hold:

(A1) There exists a function $h \in \mathcal{H}(E)$ such that $\inf h > 0$.

(A2) If $(u_n)_{n\in\mathbb{N}}$ is a monotone increasing sequence of $\mathcal{H}$-harmonic functions in an open set $V \subseteq E$ such that
$$\{x \in V \mid \sup_{n\in\mathbb{N}} u_n(x) < \infty\}$$
is dense in Ω, then
$$u := \sup_V u_n \quad \text{is } \mathcal{H}\text{-harmonic in } V.$$

(A3) The family of the $\mathcal{H}$-regular open sets is a basis of the Euclidean topology on E.

(A4) For every $x, y \in E$, $x \neq y$, there exist two nonnegative $\mathcal{H}$-superharmonic and continuous functions u, v in E such that
$$u(x)v(y) \neq u(y)v(x).$$

(A5) For every $x_0 \in E$ there exists a nonnegative $\mathcal{H}$-superharmonic and continuous function S_{x_0} in E, such that $S_{x_0}(x_0) = 0$ and
$$\inf_{E \setminus V} S_{x_0} > 0$$
for every neighborhood V of x_0.

We now recall some crucial results in σ^*-harmonic space theory; first of all the definition of Perron–Wiener solution to the Dirichlet problem.

Let V be a bounded open set with $\overline{V} \subseteq E$, and let $\varphi : \partial V \longrightarrow \mathbb{R}$ be a bounded lower semicontinuous or upper semicontinuous function. Define

$$\overline{\mathcal{U}}_\varphi^V = \{u \in \overline{\mathcal{H}}(V) \mid \liminf_{x \longrightarrow \xi} u(x) \geq \varphi(\xi) \quad \forall \xi \in \partial V\}$$

and

$$H_\varphi^V =: \inf \overline{\mathcal{U}}_\varphi^V. \tag{2.1}$$

Then H_φ^V is $\mathcal{H}$-harmonic in Ω. It is called the generalized Perron–Wiener solution to the Dirichlet problem

$$\begin{cases} u \in \mathcal{H}(V), \\ u|_{\partial V} = \varphi. \end{cases}$$

We also have

$$H_\varphi^V =: \sup \underline{\mathcal{U}}_\varphi^V, \tag{2.2}$$

where,

$$\underline{\mathcal{U}}_\varphi^V = \{v \in \underline{\mathcal{H}}(V) \mid \limsup_{x \longrightarrow \xi} v(x) \leq \varphi(\xi) \quad \forall \xi \in \partial V\}.$$

Here $\underline{\mathcal{H}}(V) := -\overline{\mathcal{H}}(V)$ denotes the cone of the $\mathcal{H}$-*subharmonic* functions in V.

A point $y \in \partial V$ is called $\mathcal{H}$-*regular* for V if

$$\lim_{x \longrightarrow y} H_\varphi^V(x) = \varphi(y) \qquad \forall \varphi \in C(\partial V, \mathbb{R}).$$

On the σ^*-harmonic space Bouligand Theorem holds. Indeed: *a point $y \in \partial V$ is $\mathcal{H}$-regular for V if and only if there exists a $\mathcal{H}$-barrier for V at y*, i.e., if there exists a function b $\mathcal{H}$-superharmonic in $V \cap W$, where W is a neighborhood of y, such that

(*i*) b is $\mathcal{H}$-superharmonic;
(*ii*) $b(x) > 0 \ \forall x \in V \cap W$ and $b(x) \longrightarrow 0$ as $x \longrightarrow y$.

For our purposes it is important to recall that if $y \in \partial V$ is $\mathcal{H}$-regular for V there exists a barrier function for V at y which is defined and $\mathcal{H}$-harmonic all over V.

Finally, we recall the *minimum principle* for $\mathcal{H}$-superharmonic functions.

Let V be a bounded open set with $\overline{V} \subseteq E$ and let $u \in \overline{\mathcal{H}}(V)$. If

$$\liminf_{x \longrightarrow y} u(x) \geq 0 \quad \forall y \in \partial V,$$

then $u \geq 0$ in V.

2.2. **The $\mathscr{L}_0$-harmonic space.** Let E be a bounded open subset of X such that $\overline{E} \subseteq X$. For every open set $V \subseteq E$ we let

$$\mathcal{H}(V) = \{u \in C^\infty(V, \mathbb{R}) \mid \mathscr{L}_0 u = 0 \text{ in } V\}.$$

Then, $V \longmapsto \mathcal{H}(V)$ is a a sheaf of functions such that $\mathcal{H}(V)$ is a linear subspace of $C(V, \mathbb{R})$.

If $u \in \mathcal{H}(V)$ we will say that u is $\mathcal{H}$-harmonic or $\mathscr{L}_0$-harmonic in V.

We have that

$$(E, \mathcal{H}) \text{ is a } \sigma^*\textit{-harmonic space.} \tag{2.3}$$

Before showing this statement we remark that a C^2-function u in a open set V is $\mathcal{H}$-superharmonic if and only if $\mathscr{L}_0 u \leq 0$ in V. This is a easy consequence of Picone's maximum principle (see e.g. [KP16], page 547). Now we are ready to prove (2.3).

(A1) is satisfied since the constant functions are $\mathscr{L}_0$-harmonic.

(A2) -(A4) are proved in [KP16]. We would like to stress that our operators $\mathscr{L}_0$ are contained in the class considered in [KP16] since the rank condition (H2) implies that both $\mathscr{L}_0$ and $\mathscr{L}_0 - \beta$, for every $\beta \geq 0$, are hypoelliptic.

The axiom (A5) follows from the following Lemma which seems to have an independent interest in its own right.

Lemma 2.1. *Let us consider a linear second order PDO of the kind*

$$\mathcal{L} := \sum_{i,j=1}^{N} a_{ij}\partial_{x_i x_j} + \sum_{j=1}^{N} b_j \partial_{x_j},$$

where $a_{ij} = a_{ji}, b_j$ are continuous functions in $\overline{Y}$, where Y is a bounded open subset of $\mathbb{R}^N$. Suppose

$$\inf_Y a_{11} := \alpha > 0 \quad \text{ and } \quad \sum_{j=1}^{N} a_{jj} > 0 \text{ in } Y^2.$$

Then, for every $x_0 \in Y$ there exists a function $h \in C^\infty(Y, \mathbb{R})$ such that

(*i*) *$h(x_0) = 0$ and $h(x) > 0$ for every $x \neq x_0$;*

(*ii*) *$\mathcal{L}h > 0$ in X.*

Proof. For the sake of simplicity we assume $x_0 = 0$. We define

$$h(x) = E(\lambda x_1) + (x_2^2 + \cdots + x_N^2), \quad x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N,$$

where $\lambda > 0$ will be fixed below. Moreover,

$$E(s) = \exp(\phi(s)) - \exp(\phi(0))$$

and

$$\phi(s) = \sqrt{1 + s^2}, \quad s \in \mathbb{R}.$$

We have:

$$\phi(0) = 1, \quad \phi(s) > 1 \quad \forall s \neq 0, \quad E(s) > 0 \quad \forall s \neq 0, \quad E(0) = 0,$$

$$\phi'(s) = \frac{s}{\sqrt{1+s^2}}, \quad \phi''(s) = \frac{1}{(1+s^2)^{\frac{3}{2}}}.$$

Hence

$$\phi'^2 + \phi'' = \frac{s^2}{1+s^2} + \frac{1}{(1+s^2)^{\frac{3}{2}}} \geq \frac{1}{2\sqrt{2}} \quad \forall s \in \mathbb{R}.$$

On the other hand

$$E' = \exp(\phi)\phi', \quad E'' = \exp(\phi)(\phi'^2 + \phi'').$$

Therefore, letting

$$\beta := \sup_X \sum_{j=1}^{N} |b_j| \qquad (< \infty) \quad \text{ and } \quad \lambda = \sup_{x \in \overline{X}} |x|,$$

[2] We don't require $(a_{ij})_{i,j=1,\ldots,N}$ to be nonnegative definite.

we get

$$\begin{aligned}
\mathcal{L}h(x) &= \lambda^2 E''(\lambda x_1)a_{11}(x) + \lambda E'(\lambda x_1)b_1 + 2\sum_{j=2}^{N}(a_{jj}(x) + b_j(x)x_j) \\
&\geq \exp(\phi(\lambda x_1))\left(\frac{a_{11}(x)}{2\sqrt{2}}\lambda^2 - \lambda|b_1|\right) - 2\sum_{j=2}^{N}|b_j||x_j| \\
&\geq \lambda^2\left(\frac{\alpha}{2\sqrt{2}} - \frac{|b_1|}{\lambda}\right) - 2\beta\lambda \\
&\geq \lambda^2\left(\frac{\alpha}{2\sqrt{2}} - \frac{\beta}{\lambda}\right) - 2\beta\lambda.
\end{aligned}$$

If λ is big enough, this implies

$$\mathcal{L}h > 0 \text{ in } X.$$

Moreover

$$h(0) = E(0) = 0, \quad h(x) > 0 \quad \text{if } \ x > 0.$$

The proof is complete. □

2.3. **The $\mathscr{L}$-harmonic space.** Let $\widehat{E}$ be a bounded open subset of $X \times \mathbb{R}$ such that $\overline{\widehat{E}} \subseteq X \times \mathbb{R}$. For every open set $V \subseteq \widehat{E}$ we let

$$\mathcal{K}(V) = \{u \in C^\infty(V, \mathbb{R}) \mid \mathscr{L}u = 0 \text{ in } V\}.$$

Then, $V \longmapsto \mathcal{K}(V)$ is a a sheaf of functions making

$$(\widehat{E}, \mathcal{K}) \text{ a } \sigma^*\text{-}harmonic\ space.$$

This can be proved just by proceeding as in subsection 2.2. We call $\mathcal{K}$-harmonic or $\mathscr{L}$-harmonic in a open set V the solutions to $\mathscr{L}u = 0$ in V.

Here we prove some typical results of the present $\mathcal{K}$-harmonic space, that we will need in the proof of the main theorem of this paper. We first show a "parabolic"minimum principle for $\mathscr{L}$-subharmonic functions in cylindrical domains.

Proposition 2.2. *Let Ω be a bounded open subset of X such that $\overline{\Omega} \subseteq X$ and let $T > 0$. Consider the cylindrical domain $\mathcal{O} := \Omega \times]0, T[$ and define the "parabolic boundary"of $\mathcal{O}$ as follows*

$$\partial_p \mathcal{O} := (\Omega \times \{0\}) \times (\partial\Omega \times]0, T]).$$

Then, if $u \in \overline{\mathcal{K}}(\mathcal{O})$ is such that

$$\liminf_{z \longrightarrow \zeta} u(z) \geq 0 \quad \forall \zeta \in \partial_p \mathcal{O},$$

we have $u \geq 0$ in $\mathcal{O}$.

Proof. For every arbitrarily fixed $\widehat{T} \in]0, T[$ we let $\widehat{\mathcal{O}} = \Omega \times]0, \widehat{T}[$. We will prove that $u \geq 0$ in $\widehat{\mathcal{O}}$. Since $\widehat{T}$ is arbitrarily fixed in $]0, T[$, this will give the proof of our lemma. To this end, given any $\varepsilon > 0$, we define

$$u_\varepsilon(z) = u_\varepsilon(x, t) := u(x, t) + \frac{\varepsilon}{\widehat{T} - t}, \quad z \in \widehat{\mathcal{O}}.$$

Since u is $\mathcal{K}$-superharmonic in $\mathcal{O}$ and

$$\mathscr{L}\frac{\varepsilon}{\widehat{T}-t} = -\varepsilon\partial_t\frac{1}{\widehat{T}-t} = -\frac{\varepsilon}{(\widehat{T}-t)^2} < 0 \text{ in } \widehat{\mathcal{O}},$$

then u_ε is $\mathcal{K}$-superharmonic in $\mathcal{O}$. Moreover

$$\liminf_{z\longrightarrow\zeta} u_\varepsilon(z) \geq 0 \quad \forall\zeta\in\partial_p\widehat{\mathcal{O}},$$

and, for every $\xi\in\Omega$,

$$\liminf_{z\longrightarrow(\xi,\widehat{T})} u_\varepsilon(z) \geq u(\varepsilon,\widehat{T}) + \liminf_{t\nearrow\widehat{T}} \frac{\varepsilon}{\widehat{T}-t} = \infty.$$

By the minimum principle recalled in subsection 2.1, we have $u_\varepsilon \geq 0$ in $\widehat{\mathcal{O}}$. Letting ε go to zero we have $u_\varepsilon \geq 0$ in $\widehat{\mathcal{O}}$, thus completing the proof. □

Proposition 2.3. *Let $\Omega\subseteq X$ be open and let T_0 and $T\in\mathbb{R}$, such that $0<T_0<T$. Let $\mathcal{O} := \Omega\times]0,T[$ and $u:\mathcal{O}\longrightarrow\mathbb{R}$ be such that the restrictions $u|_{\Omega\times]0,T_0[}$ and $u|_{\Omega\times]T_0,T[}$ are $\mathcal{K}$-superharmonic. Then, if*

$$\liminf_{\substack{z\longrightarrow(\xi,T_0)\\(x,t)\in\mathcal{O}}} u(x,t) = \liminf_{\substack{z\longrightarrow(\xi,T_0)\\t<T_0\\(x,t)\in\mathcal{O}}} u(x,t) = u(\xi,T_0) \quad \forall\xi\in\Omega, \tag{2.4}$$

the function u is $\mathcal{K}$-superharmonic in $\Omega\times]0,T[$.

Proof. Since u is lower semicontinuous in $\Omega\times]0,T_0[$ and in $\Omega\times]T_0,T[$, the assumption (2.4) implies that u is lower semicontinuous in $\mathcal{O}=\Omega\times]0,T[$.

To prove that u is $\mathcal{K}$-harmonic in $\mathcal{O}$ we will show the following claim.
Claim. For every $z\in\mathcal{O}$ there exists a basis B_z of $\mathcal{K}$-regular neighborhoods of V such that

$$u(z)\geq K^V_\varphi(z) \qquad \forall\varphi\in C(\partial V,\mathbb{R}), u|_{\partial V}\geq\varphi.$$

Here K^V_φ denotes the unique $\mathcal{K}$-harmonic function in V, continuous up to ∂V and such that $K^V_\varphi|_{\partial V}=\varphi$.

From this Claim our assertion follows thanks to Corollary 6.4.9 in [BLU07].

If $z\in\Omega\times]0,T_0[$ or if $z\in\Omega\times]0,T[$, the Claim is satisfied since u is $\mathcal{K}$-superharmonic both in $\Omega\times]0,T_0[$ and in $\Omega\times]0,T[$. Then it remains to prove the Claim for every point $\zeta=(\xi,T_0),\xi\in\Omega$. Let $B_\rho=(V)$ be a basis of $\mathcal{K}$-regular neighborhoods of ζ such that $\overline{V}\subseteq\mathcal{O}$. Let $\varphi\in C(\partial V,\mathbb{R}),\varphi\leq u|_{\partial V}$. Then $u-K^V_\varphi$ is $\mathcal{K}$-superharmonic in $\Omega\times]0,T_0[$ and

$$\liminf_{z\longrightarrow z'} u(z)\geq u(z')-u(z')\geq 0 \qquad \forall z'\in\partial_p\Omega\times]0,T_0[.$$

Therefore, by Proposition 2.2,

$$u-K^V_\varphi\geq 0 \text{ in } V\cap\{t<T_0\}.$$

As a consequence, keeping in mind assumption (2.4),

$$u(\xi,T_0) = \liminf_{\substack{(x,t)\longrightarrow(\xi,\tau)\\t<T_0}} u(x,t) \geq \liminf_{\substack{(x,t)\longrightarrow(\xi,T_0)\\t<T_0}} K^V_\varphi(x,t) = K^V_\varphi(\xi,T_0),$$

that is,

$$u(\xi,T_0)\geq K^V_\varphi(\xi,T_0).$$

This completes the proof. □

3. Some preliminary results

The proof of our main theorem rests on the following two lemmata.

Lemma 3.1. *Let Ω be a bounded open set such that $\overline{\Omega} \subseteq X$, and let $\mathcal{O} := \Omega \times]0,T[$, $T \in \mathbb{R}, T > 0$. Let $\varphi : \partial\mathcal{O} \longrightarrow \mathbb{R}$ be upper semicontinuous and such that $t \longmapsto \varphi(x,t)$ is monotone decreasing, $\forall x \in \partial\Omega$ and*

$$\varphi(x,0) = M = \sup_{\partial\mathcal{O}} \varphi \qquad (M \in \mathbb{R}).$$

Then, the Perron solution $K_\varphi^{\mathcal{O}}$ is monotone decreasing w.r.t. the variable t: more precisely

$$t \longmapsto K_\varphi^{\mathcal{O}}(x,t) \text{ is monotone decreasing for every fixed } x \in \Omega.$$

Proof. For every fixed $\delta \in]0,T[$ let us define

$$h(x,t) = K_\varphi^{\mathcal{O}}(x,t) - K_\varphi^{\mathcal{O}}(x,t+\delta),\ x \in \Omega, 0 < t < T - \delta.$$

It is enough to prove that $h \geq 0$ in $\mathcal{O}_\delta := \Omega \times]0, T-\delta[$. To this end we show that, for every $u \in \overline{\mathcal{U}}_\varphi^{\mathcal{O}}$ and $v \in \underline{\mathcal{U}}_\varphi^{\mathcal{O}}$, the function

$$w(x,t) = u(x,t) - v(x,t+\delta)$$

is nonnegative in $\mathcal{O}_\delta$. Now, we have:

(a) w is $\mathcal{K}$-superharmonic in $\mathcal{O}_\delta$, since $u \in \overline{\mathcal{K}}(\mathcal{O})$ and $(x,t) \longmapsto v(x,t+\delta)$ is $\mathcal{K}$-subharmonic in $\mathcal{O}_\delta$ being $v \in \underline{\mathcal{K}}(\mathcal{O})$ and $\mathscr{L}$ translation invariant in the variable t.

(b) For every $\overline{x} \in \Omega$,

$$\begin{aligned}\liminf_{(x,t)\longrightarrow(\overline{x},0)} w(x,t) &\geq \liminf_{(x,t)\longrightarrow(\overline{x},0)} u(x,t) - \liminf_{(x,t)\longrightarrow(\overline{x},0)} v(x,t+\delta)\\ &\geq \varphi(\overline{x},0) - v(\overline{x},\delta)\\ &= M - v(\overline{x},\delta) \geq 0.\end{aligned}$$

We remark that $v \leq M$ in $\mathcal{O}$ since v is $\mathcal{K}$-subharmonic and

$$\limsup_{z\longrightarrow\zeta} v(z) \leq \varphi(\zeta) \leq M \quad \forall \zeta \in \partial\mathcal{O}.$$

Here we use the maximum principle for subharmonic functions.

(c) For every $\zeta = (\xi,\tau)$, $\xi \in \partial\Omega, 0 < \tau < T - \delta$,

$$\liminf_{(x,t)\longrightarrow(\xi,\tau)} w(x,t) \geq \varphi(\xi,\tau) - \varphi(\xi,\tau+\delta) \geq 0,$$

by hypotesis.

From (a), (b) and (c) and the minimum principle for superharmonic functions we get

$$w \geq 0 \text{ in } \mathcal{O}_\delta.$$

This completes the proof. □

With Lemma 3.1 at hand we can easily prove the following key result for our main theorem.

Lemma 3.2. *Let Ω be a bounded open set such that $\overline{\Omega} \subseteq X$, and let $\mathcal{O} := \Omega \times]0,T[$, $T \in \mathbb{R}$, $T > 0$. Let $z_0 = (x_0,t_0) \in \partial\Omega \times]0,T[$ be a $\mathscr{L}$-regular boundary point.*

Then there exists a function $b \in \mathcal{K}(\mathcal{O})$ such that

(*i*) *b is an $\mathscr{L}$-barrier for $\mathcal{O}$ at z_0;*
(*ii*) *$t \longmapsto b(x,t)$ is monotone decreasing for every fixed $x \in \Omega$.*

Proof. Let Y be a bounded open set such that $\overline{\Omega} \subseteq Y \subseteq \overline{Y} \subseteq X$ and let $x_0 \in \Omega$. By Lemma 2.1 there exists a function $h \in C^\infty(Y, \mathbb{R})$ such that

(*a*) $h(x_0) = 0$ and $h(x) > 0 \quad \forall x \neq x_0$.
(*b*) $\mathscr{L}_0 h > 0$ in Ω.

For a fixed $\delta \in]0, T_0[$ let us define

$$\widehat{h} : \overline{\Omega} \times [0,T] \longrightarrow \mathbb{R}, \quad \widehat{h}(x,t) = \begin{cases} h(x) & \text{if } \ \delta < t \leq T, \\ M & \text{if } \ 0 \leq t \leq \delta, \end{cases}$$

where $M = \sup_{\overline{\Omega}} h$.

This function is $\mathscr{L}$-superharmonic in $\mathcal{O}_1 := \Omega \times]0, \delta[$ and in $\mathcal{O}_2 := \Omega \times]\delta, T[$ since

$$\mathscr{L}\widehat{h} = 0 \text{ in } \mathcal{O}_1 \quad \text{ and } \quad \mathscr{L}\widehat{h} = \mathscr{L}_0 h > 0 \text{ in } \mathcal{O}_2.$$

On the other hand,

$$\limsup_{\substack{(x,t) \longrightarrow (\xi,\delta) \\ t<\delta}} \widehat{h}(x,t) = M = \limsup_{(x,t) \longrightarrow (\xi,\delta)} \widehat{h}(x,t).$$

Then, by Proposition 2.3,

$$\widehat{h} \in \underline{\mathcal{K}}(\Omega \times]0,T[).$$

Moreover,

$$t \longmapsto \widehat{h}(x,t) \ \text{ is monotone decreasing,}$$

for every fixed $x \in \overline{\Omega}$.

Let us now put

$$b := K^{\mathcal{O}}_{\widehat{h}|\partial\mathcal{O}},$$

which is well defined and $\mathcal{K}$-harmonic in $\mathcal{O}$, since $\widehat{h}|_{\partial\mathcal{O}}$ is bounded and upper semi-continuous.

Moreover, by Lemma 3.1, $t \longmapsto b(x,t)$ is monotone decreasing for every fixed $x \in \Omega$.

It remains to show that b is an $\mathscr{L}$-barrier for $\mathcal{O}$ at z_0. To this end we first remark that

$$\widehat{h} \in \underline{\mathcal{U}}^{\mathcal{O}}_{\widehat{h}|\partial\mathcal{O}},$$

so that

$$\widehat{h} \leq b \text{ in } \mathcal{O}.$$

This implies $b > 0$ in $\mathcal{O}$ since $\widehat{h}$ is strictly positive.

On the other hand, since $\widehat{h}|_{\partial\mathcal{O}}$ is continuous in a neighborhood of z_0, and z_0 is $\mathscr{L}$-regular for $\mathcal{O}$,

$$\lim_{z \longrightarrow z_0} b(z) = \lim_{z \longrightarrow z_0} K^{\mathcal{O}}_{\widehat{h}|\partial\mathcal{O}}(z) = \widehat{h}(z_0) = \phi(x_0) = 0.$$

This completes the proof. □

4. Proof of Theorem 1.1

Let us keep the notation of Theorem 1.1 and split the proof in two steps.

(1) *If $x_0 \in \partial\Omega$ is $\mathscr{L}_0$-regular for Ω, then $z = (x_0, t_0)$ is $\mathscr{L}$-regular for $\mathcal{O}$.*

Indeed, the $\mathscr{L}_0$-regularity of x_0 implies the existence of a $\mathscr{L}_0$-harmonic barrier for Ω at x_0, i.e. a function $b_0 \in \mathcal{K}(\Omega)$ such that

$$b_0 > 0 \text{ in } \Omega \quad \text{ and } \quad b_0 \longrightarrow 0 \text{ as } x \longrightarrow x_0.$$

It follows that

$$\widehat{b}(x,t) = b_0(x), \quad (x,t) \in \mathcal{O},$$

is $\mathscr{L}$-harmonic in $\mathcal{O}$ ($\mathscr{L}\widehat{b} = \mathscr{L}_0 b_0 = 0$). Moreover,

$$\widehat{b} > 0 \text{ in } \mathcal{O} \quad \text{ and } \quad \widehat{b}(x,t) = b_0(x) \longrightarrow 0 \text{ as } (x,t) \longrightarrow (x_0, t_0).$$

Hence, $\widehat{b}$ is an $\mathscr{L}$-barrier function for $\mathcal{O}$ at z_0 and, as a consequence, z_0 is $\mathscr{L}$-regular for $\mathcal{O}$.

(2) *If $z = (x_0, t_0)$, $x_0 \in \Omega, 0 < t_0 < T$, is $\mathscr{L}$-regular for $\mathcal{O}$, then x_0 is $\mathscr{L}_0$-regular for Ω.*

Indeed, by Lemma 3.2, there exists a function $b \in \mathcal{K}(\mathcal{O})$ such that $b > 0$, $b(z) \longrightarrow 0$ as $z \longrightarrow z_0$ and

$$t \longmapsto b(x,t) \ \text{ is monotone decreasing } \quad \forall x \in \Omega.$$

It follows that, letting $b_0(x) = b(x, t_0)$,

$$\mathscr{L}_0 b_0 = \mathscr{L} b + \partial_t b = \partial_t b \leq 0 \text{ in } \Omega.$$

Hence, b_0 is $\mathscr{L}_0$-superharmonic in Ω. Moreover, $b_0 > 0$ in Ω and

$$b_0(x) = b(x, t_0) \longrightarrow 0 \text{ as } x \longrightarrow x_0.$$

Therefore, b_0 is an $\mathscr{L}$-barrier for Ω at x_0 , and x_0 is $\mathscr{L}_0$-regular.

5. An application to degenerate Ornstein–Uhlenbeck operators

In $\mathbb{R}^N$ let us consider the partial differential operator

$$L_0 = \operatorname{div}(A\nabla) + \langle Bx, \nabla \rangle, \tag{5.1}$$

where $A = (a_{ij})_{i,j=1,\dots,N}$ and $B = (b_{ij})_{i,j=1,\dots,N}$ are $N \times N$ real constant matrices, $x = (x_1, \dots, x_N)$ is the point of $\mathbb{R}^N$, div, ∇ and $\langle\ ,\ \rangle$ denote the divergence, the Euclidean gradient and the inner product in $\mathbb{R}^N$, respectively.

We suppose that the matrix A is symmetric, positive semidefinite and that it assumes the following block form

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix},$$

A_0 being a $p_0 \times p_0$ strictly positive definite matrix with $1 \leq p_0 \leq N$. Moreover, we assume the matrix B to be of the following type

$$B = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ B_1 & 0 & \dots & 0 & 0 \\ 0 & B_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & B_r & 0 \end{bmatrix}, \tag{5.2}$$

where B_j is a $p_{j-1} \times p_j$ block with rank p_j $(j = 1, 2, ..., r)$, $p_0 \geq p_1 \geq ... \geq p_r \geq 1$ and $p_0 + p_1 + ... + p_r = N$.

Finally, letting

$$E(s) := \exp(-sB), \quad s \in \mathbb{R},$$

we assume that the following condition is satisfied

$$C(t) = \int_0^t E(s)AE^T(s)\, ds \text{ is strictly positive definite for every } t > 0.$$

As it is quite well known this condition implies the hypoellipticity of L, see [LP94]. In that paper it is proved that the evolution counterpart of L_0, i.e. the operator

$$L = L_0 - \partial_t \text{ in } \mathbb{R}^{N+1},$$

is left translation invariant and homogeneous of degree two on the homogeneous group

$$\mathbb{K} = (\mathbb{R}^{N+1}, \circ, \delta_\lambda)$$

with composition law $\circ$ defined as follows

$$(x, t) \circ (x', t') = (x' + E(t')x, t + t')$$

and dilation $\delta_\lambda, \lambda > 0$, of this kind

$$\begin{aligned} \delta_\lambda : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}^{N+1}, \quad \delta_\lambda(x,t) &= \delta_\lambda(x^{(p_0)}, x^{(p_1)}, \dots, x^{(p_r),t}) \\ &:= (\lambda x^{(p_0)}, \lambda^3 x^{(p_1)}, \dots, \lambda^{2r+1} x^{(p_n)}, \lambda^2 t), \end{aligned}$$

where $x^{(p_i)} \in \mathbb{R}^{p_i},\ i = 0, \dots, r$.

The natural number $q := Q + 2$, with

$$Q := p_0 + 3p_1 + \dots + (2r+1)p_r, \tag{5.3}$$

is the homogenous dimension of $\mathbb{K}$. In what follows we will write

$$\delta_\lambda(z) = \delta_\lambda(x, t) = (D_\lambda(x), \lambda^2 t),$$

where,

$$D_\lambda(x) = (\lambda x^{(p_0)}, \lambda^3 x^{(p_1)}, \dots, \lambda^{2r+1} x^{(p_n)}, \lambda^2 t).$$

Obviously, $(D_\lambda)_{\lambda>0}$ is a group of dilations in $\mathbb{R}^N$. The natural number Q in (5.3) is the homogeneous dimension of $\mathbb{R}^N$ w.r.t. the group $(D_\lambda)_{\lambda>0}$.

The operator L has a fundamental solution Γ given by

$$\Gamma(z_0, z) := \gamma(z^{-1} \circ z_0), \quad z,\ z_0 \in \mathbb{R}^{N+1},$$

where $\circ$ is the composition law in $\mathbb{K}$, z^{-1} denotes the opposite of z in $\mathbb{K}$ and, for a suitable $C_Q > 0$,

$$\gamma(x,t) = \begin{cases} 0 & \text{if } \ t \le 0, \\ \\ \frac{C_Q}{t^Q} \exp\left(-\frac{1}{4} \left| D_{\frac{1}{\sqrt{t}}}(x) \right|_C^2 \right) & \text{if } \ t > 0, \end{cases}$$

where,

$$|y|_C^2 = \langle C^{-1}(1)y, y \rangle,$$

see again [LP94].

It is quite easy to recognise that our Tikhonov-type theorem applies to the operators L_0 and L. Hence, if Ω is a bounded open subset of $\mathbb{R}^N$, $x_0 \in \partial\Omega$ and $t_0 \in]-T, T[, T > 0$, we have:

$$x_0 \ \textit{is } L_0\textit{-regular for } \Omega$$

$$\text{if and only if}$$

$$z_0 = (x_0, 0) \ \textit{is } L\textit{-regular for } \mathcal{O}_T := \Omega \times] - T, T[.$$

On the other hand, in [KLT18, Corollary 1.3] it is proved that

$$z_0 \ \textit{is } L\textit{-regular for } \mathcal{O}_T$$

if, for a $\mu \in]0, 1[$, the following condition holds:

$$\sum_{k=1}^{\infty} \frac{|\mathcal{O}^c_{T,k}(z_0)|}{\mu^{\alpha(k)\frac{Q+2}{Q}}} = \infty, \tag{5.4}$$

where $\alpha(k) = k \log k$, $|\cdot|$ denotes the Lebesque measure in $\mathbb{R}^{N+1}$ and

$$\mathcal{O}^c_{T,k}(z_0) = \left\{ z \neq \mathcal{O}_T \ : \ \left(\frac{1}{\mu} \right)^{\alpha(k)} \le \Gamma(z_0, z) \le \left(\frac{1}{\mu} \right)^{\alpha(k+1)} \right\}.$$

We express now this condition in a more explicit form. To this end we let

$$A^c_k(x_0) = \left\{ (x,t) \in \mathbb{R}^{N+1} \mid x \notin \Omega, \gamma(z^{-1} \circ (x, 0)) \ge \left(\frac{1}{\mu} \right)^{\alpha(k)} \right\}. \tag{5.5}$$

Then,

$$\begin{aligned} \mathcal{O}^c_{T,k}((x_0, 0)) &= (A_k(x_0) \smallsetminus A_{k+1}(x_0)) \cup \left\{ \gamma = \left(\frac{1}{\mu} \right)^{\alpha(k+1)} \right\} \\ &\supseteq A_k(x_0) \smallsetminus A_{k+1}(z_0). \end{aligned}$$

Hence, denoting for the sake of brevity,

$$d_k = |A_k(z_0)| \quad \text{ and } \quad \nu = \mu^{\frac{(Q+2)}{Q}},$$

condition (5.4) is satisfied if

$$\sum_{k=1}^{\infty} \frac{d_k - d_{k+1}}{\nu^{\alpha(k)}} = \infty. \tag{5.6}$$

On the other hand, for every $p \in \mathbb{N}$,

$$\sum_{k=1}^{\infty} \frac{d_k - d_{k+1}}{\nu^{\alpha(k)}}$$

$$= \frac{d_1}{\nu^{\alpha(1)}} + d_2\left(\frac{1}{\nu^{\alpha(2)}} - \frac{2}{\nu^{\alpha(1)}}\right) + \cdots + d_p\left(\frac{1}{\nu^{\alpha(p)}} - \frac{2}{\nu^{\alpha(p-1)}}\right) - \frac{d_{p+1}}{\nu^{\alpha(p)}}$$

$$\leq (1 - \nu^{\log 2}) \sum_{k=1}^{p} \frac{d_k}{\nu^{\alpha(k)}} - \frac{d_{p+1}}{\nu^{\alpha(p)}}.$$

Then, since $\dfrac{d_{p+1}}{\nu^{\alpha(p)}} \longrightarrow 0$ as $p \to \infty$ (as we will see later) condition (5.6) is satisfied if

$$\sum_{k=1}^{\infty} \frac{d_k}{\mu^{\alpha(k)}} = \infty. \tag{5.7}$$

Keeping in mind the very definition of Γ, we have that $A_k(x_0)$ is equal to the following set

$$\left\{(x,t) \in \mathbb{R}^{N+1} \mid x \in \Omega^c, t < 0, \left|D_{\frac{1}{\sqrt{|t|}}}(x_0 - E(|t|x))\right|_C^2 < 2Q \log \frac{(C_Q \mu^{\alpha(k)})^{\frac{2}{Q}}}{t}\right\},$$

whereby, with the change of variables $y := x_0 - E(|t|)x$, $\tau = -t$, we get

$$d_k = \left|\left\{(y,\tau) \mid \tau > 0,\ y \in x_0 - E(\tau)(\Omega^c), \left|D_{\frac{1}{\sqrt{|\tau|}}}\right|_C^2 < 2Q \log \frac{R_k}{\tau}\right\}\right|. \tag{5.8}$$

Here $R_k = (C_Q \mu^{\alpha(k)})^{\frac{2}{Q}}$ and $\Omega^c := \mathbb{R}^{N+1} \setminus \Omega$.

Therefore,

$$\begin{aligned} d_k &\leq \left|\left\{(y,\tau) \mid \tau > 0, \left|D_{\frac{1}{\sqrt{|\tau|}}}\right|_C^2 < 2Q \log \frac{R_k}{\tau}\right\}\right| \\ &\quad \text{(using the change of variables } y = D_{\sqrt{R_k}}(\xi), \tau = R_k s) \\ &= R_k^{\frac{Q+2}{Q}} \left|\left\{(\xi,s) \mid s > 0, \left|D_{\sqrt{\frac{1}{s}}}(\xi)\right| \leq 2Q \log \frac{1}{s}\right\}\right|. \end{aligned}$$

Hence, for a suitable dimensional constant $C_Q^* > 0$,

$$d_k \leq C_Q^* \mu^{\alpha(k)\frac{Q+2}{Q}} = C_Q^* \nu^{\alpha(k)}.$$

Then,

$$0 \leq \frac{d_{p+1}}{\nu^{\alpha(p)}} \leq C_Q^* \mu^{\alpha(p+1)-\alpha(p)} \longrightarrow 0 \text{ as } p \longrightarrow \infty,$$

since $0 < \mu < 1$ and $\alpha(p+1) - \alpha(p) = p \log \frac{p+1}{p} + \log(p+1) \longrightarrow \infty$.

We have completed the proof of the following criterion:
Let L be the Ornstein–Uhlenbeck-type operator in (5.1) *and let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set. Then, a point $x_0 \in \partial\Omega$ is L-regular for Ω if*

$$\sum_{k=1}^{\infty} \frac{d_k(\Omega, x_0)}{\mu^{\alpha(k)\frac{Q+2}{2}}} = \infty, \tag{5.9}$$

where $d_k(\Omega, x_0) := d_k$ is defined in (5.8).

We note that condition (5.9) holds if Ω satisfies the exterior cone-type condition introduced in [Kog19]. Geometric boundary regularity criteria for wide classes of hypoelliptic evolution operators are also established in [Man97], [LU10], [LTU17] and [Kog17].

Acknowledgment

The author has been partially supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

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