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Exact quantum dynamics of interacting spin systems subjected to controllable time dependent magnetic fields

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To my family and my Aurora,

always present

We are both actors and audience in the grate drama of life

N. Bohr

Nature is earlier than man, but man is earlier than natural science

C. F. F. Weizsäcker

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List of Publications

Papers discussed in the Thesis

- R. Grimaudo, A. Messina, and H. Nakazato, *Phys. Rev. A* **94** (2), 022108 (2016).
- R. Grimaudo, A. Messina, P.A. Ivanov, and N.V. Vitanov, *J. Phys. A: Math. and Theor.* **50** (17), 175301 (2017).
- R. Grimaudo, Y. Belousov, H. Nakazato, and A. Messina, *Ann. Phys. (NY)* **392**, 242 (2018).
- R. Grimaudo, A. S. M. de Castro, H. Nakazato, and A. Messina, *Ann. Phys. (Berlin)* **530** (12), 1800198 (2018).
- R. Grimaudo, A. S. M. de Castro, M. Kuś, and A. Messina, *Phys. Rev. A* **98** (3), 033835 (2018).
- R. Grimaudo, L. Lamata, E. Solano, and A. Messina, *Phys. Rev. A* **98** (4), 042330 (2018).
- R. Grimaudo, A. Isar, T. Mihaescu, I. Ghiu, and A Messina, *Res. Phys.* **13**, 102147 (2019).
- R. Grimaudo, A. S. M. de Castro, H. Nakazato, and A. Messina, *Phys. Rev. A* **99** (5), 052103 (2019).
- R. Grimaudo, N. V. Vitanov, and A. Messina, *Phys. Rev. B* **99** (17), 174416 (2019).
- R. Grimaudo, N. V. Vitanov, and A. Messina, *Phys. Rev. B* **99** (21), 214406 (2019).
- R. Grimaudo, V. I. Man'ko, M. A. Man'ko, and A. Messina, *Physica Scripta* **95** (2), 024004 (2019).

Further Papers

- L. A. Markovich, R. Grimaudo, A. Messina, and H. Nakazato, *Ann. Phys. (NY)* **385**, 522 (2017).
- T. Suzuki, H. Nakazato, R. Grimaudo, and A. Messina, *Sc. Rep.* **8** (1), 17433 (2018).
- A. Sergi, G. Hanna, R. Grimaudo, and A. Messina, *Symmetry* **10** (10), 518 (2018).
- A. Sergi, R. Grimaudo, G. Hanna, and A. Messina, *Physics* **1** (3), 402-411 (2019).

Preprint

- I. Ghiu, R. Grimaudo, T. Mihaescu, A. Isar, and A. Messina, *"Sudden death and revival of entanglement for two interacting spin-qubit systems", submitted to Proceedings of the London Royal Society A.*

Introduction

One of the fundamental aspects distinguishing human beings is certainly the Language. In fact, the development of a complex language may be interpreted as symptom of refined intellectual abilities. A well-structured language allows a better, deeper and unambiguous development of concepts and a consequent more effective communication. In this sense, the most prestigious and efficient language elaborated by man is certainly Mathematics which is also the one "spoken by Nature", as R. P. Feynman stated. The basic importance of the language for human societies resides in the fact that it is the basis of a clear communication. It therefore possesses a crucial role within Science, given that scientists must necessarily communicate unambiguously their results in order to collaborate and progress together in the knowledge of Nature.

Such an aspect, apparently trivial and obvious, assumes a crucial importance in Quantum Physics, the theory born in the early 1900s to describe the atomic and, more in general, the microscopic world. As N. Bohr and W. Heisenberg often underline in their works [1, 2], the greatest paradox of QM consists in the fact that it cannot do without Classical Physics (CP), although it is a more generalized theory. CP, in fact, well describes physical systems for which the involved actions are so great that the quantum nature of the action can be neglected; the laws of CP, indeed, can be derived through the limit $\hbar \to 0$. The assertion of the two fathers of QM means that, to interpret and to communicate the results obtained from experiments investigating the microscopic quantum world, we can only use the language and those specific concepts related to the CP. However this language and such concepts can only result inadequate, or at least incomplete, to describe the microscopic reality governed by different logic and laws. This fact is identified as the *vulnus*, that is, the source of those paradoxes existing in QM.

One of the most striking paradoxes is certainly the "double nature" of both particles and light. Both of them, depending on the experimental conditions, can either behave as material objects or give rise to wave phenomena. But the attribution of a "double nature" to a single element of reality may be interpreted, according to the canons of CP, as a symptom of the incompleteness of the theory. However, as claimed by Bohr and Heisenberg [1, 2], the descriptive non-univocity depends, actually, on the application of the 'classical' concepts of wave and particle (elaborated within the daily experience and which therefore belong to to the sphere of CP) to the microscopic reality that obeys laws and logic that are different and far from the classical ones. In order to solve this basic issue, Bohr introduced the so-called complementary principle. According to such a principle, mutually exclusive 'classical' descriptions must be considered complementary in order to have a complete and exhaustive view of the reality investigated by QM.

It is important to stress that these "complementary realities" are related to different experimental conditions and investigation methods. The existence of this interpretative dichotomy stems from the individuality of quantum phenomena, i.e. from the fact that in QM an event is not infinitely divisible as in CP. This is connected to the fact that in QM an objective separation between subject and object (as in CP) is impossible: the measuring instruments are active part of the phenomenon. This justifies the circumstance that different results can be obtained with the same equipment and under the same experimental conditions; such a circumstance gives rise to the intrinsic probabilistic character of QM.

At this point, it should be emphasized that interpretative problems and the emergence of paradoxical aspects occur only when the description of quantum events is developed in terms of common language, inevitably involving the use of classical 'ideas'. The same difficulties do not emerge, instead, within the mathematical formalism at the basis of QM which presents, rather, a high level of coherence and elegance. This is witnessed by the equivalence between the different approaches: the one of Hesienberg based on matrices, the one based on operators of Dirac and the one developed by Schrödinger based on the concept of wave function. This observation, as mentioned at the beginning, highlights how the existence of a clear, rigorous and synthetic language, as the mathematical one, is of fundamental importance in scientific investigations.

One, moreover, may expect that a language possessing transversality and versatility (that is, the possibility of being able to be used in different contexts), besides the features listed above, might be even more efficient. Within the quantum formalism, an exemplary language is the spin language. To date, in fact, the study of theoretical models involving interacting spin dynamical variables embraces a plethora of various physical scenarios [3, 4, 5, 6, 7, 8, 9, 10, 11]; such an aspect gives to the spin language a great importance from an applicative point of view. The spin formalism is used to describe, first, the intrinsic quantum degree of freedom (with no classical analogue) of elementary particles. It has the characteristics of an angular momentum, such as 1) the possibility of interacting with an electromagnetic field, as brought to light by the Stern and Gerlach's experiment in 1922 with Ag atoms; 2) to satisfy the superposition principle; this means that even composite structures such as nuclei, atoms and molecules show spin degrees of freedom resulting from the addition of the spins of the individual particles composing them; 3) to interact with other spins generating quantum correlations between particles or, more generally, between different elements. The in-depth analysis of the properties of interacting spin models, first of all the seminal Ising and Heisenberg models, allowed to explain and to forecast, from a theoretical point of view, important magnetic properties of matter and particularly remarkable phenomena such as quantum phase transitions.

The mathematical spin formalism, however, can potentially be used to describe, more in general, all those physical systems whose dynamics, under appropriate conditions, can be traced back to that of a finite N-level system. Although it may seem a purely theoretical expedient, the simplest case of a two-level system plays a fundamental role in Physics. Many physical scenarios, indeed, under conditions of experimental interest, reduce to an effective two-level system [9, 12, 13, 14, 15]. Moreover, in recent decades the interest towards quantum two-level systems has exponentially grown thanks to the idea of a possible quantum computer and thanks to the emergence of the Quantum Information and Quantum Computation fields. Any two-level system, effective or not, can be thought, indeed, as a potential qubit (quantum bit).

To date, among the physical systems referable to actual two-level systems that can be easily experimentally controlled, we find trapped atoms and ions and superconducting circuits. In the first case, the application of laser beams with appropriate frequencies allows both to generate a dynamic between a precise pair of energy levels of the atom/ion (generating an effective two-level system), and to generate interaction between several qubits in such a way that they can be used as a quantum information register [5, 6]. In the second case, instead, it is the assembly of different circuit elements that allows, under appropriate experimental conditions, the realization of the so-called superconducting qubits: think of the Cooper Pair Box or the Transmon Qubit, as charge qubits, or the SQUID, as example of flow qubit [16, 17]. In this way, circuit architectures at the base of the cQED, succeeding in reproducing the characteristic dynamics of quantum optical systems (such as, for example, charge qubits coupled to a coplanar waveguide resonator that acts as a resonant cavity), furnishes perfect candidates of building blocks for the construction of future quantum computers [18, 19]. In 2019 IBM introduced the IBM Q System 1, a prototype quantum computer built of 20 superconducting qubits. However, it is important to underline that a lot of efforts are still necessary to reach the realization of an efficient and well improved quantum computer. The most problematic and difficult aspect to be experimentally managed is the unavoidable interaction existing between quantum systems and the surrounding environment. The latter disturbs the dynamics of the system generating dephasing, that is, loss of quantum coherence [20, 21]. As we know, coherence is the fundamental feature required to exploit quantum phenomena for applicative and technological purposes.

There exist different approaches to formally and theoretically treat the presence of an environment influencing the dynamics of a quantum system. Also in this case the spin language results of great importance. The most profitable approaches for the treatment of such a basic issue are essentially three. The first one, due to Gorini, Kossakowski, Lindblad and Sudarshan, is aimed at constructing the master equation, that is, the equation of motion governing the dynamics of the system which results after tracing out the degrees of freedom of the bath, whose presence and influence is reflected on the structure of the so called dissipator [22]. The second one is that based on the partial Wigner transform based on a Monte Carlo numerical approach and which therefore allows to treat very complex systems in presence of an environment [23]. This approach has been developed, indeed, within the bio-chemical-physical framework to study the noisy dynamics of macromolecules. The last approach, instead, due to Feshbach, is focused on the study of a quantum system of finite dimension coupled with a continuum of states that represents the environment [24]. Feshbach developed a method, based on projection operators, through which it is possible to derive an effective Hamiltonian dependent only on the dynamical variables of the system. In such a Hamiltonian the information of the presence of a noisy environment is present in the form of non-self-adjoint terms.

The characteristic of these non-self-adjoint Hamiltonians, so derived, is to have the possibility of possessing a spectrum of either real or complex conjugated eigenvalues. The matrices representing these Hamiltonians are called pseudo-Hermitian matrices [25]. Within this class we find the quasi-Hermitian matrices characterized by a completely real spectrum, which the famous PT-symmetric matrices (matrices which commute with the PT operator and therefore invariant by inversion of space and time variables) are part of [26]. The latter proved to be of particular usefulness in studying those open physical systems whose dynamics can be experimentally manipulated through the introduction of properly controlled energy-gain and loss mechanisms. The more peculiar physical feature possessed by these systems is the possibility of exhibiting a phase transition connected with the PT-symmetry breaking. Several examples of this kind of systems may be encountered in various fields [11, 27, 28, 29, 30]. The case of optical PT-symmetry systems is particularly interesting since it furnishes an example in which the formalism of quantum mechanics and in particular the spin language can be useful tools for investigating the dynamics of classical systems under appropriate experimental conditions. In fact, optical gain-loss systems consist mainly of coupled waveguides whose dynamics, under the scalar and paraxial approximation [10], obeys a Schrödinger equation where the time variable is replaced from the spatial one. In the specific case of two coupled waveguides, for example, the system consists of two coupled levels and therefore it can be described in terms of dynamical variables of a single spin-1/2.

In this thesis the study of the dynamics of interacting spin systems subjected to the action of classical external time-dependent fields is reported. The interest towards this kind of models, as explained above, lies in the fact that they can formally describe the dynamics of (or they may be implemented through) different physical scenarios of interest: mainly trapped ions/atoms and superconducting circuits. Moreover, the study of the dynamics of these systems in presence of time-dependent fields is physically relevant since it allows to understand how it is possible to experimentally operate in order to be able to control and to guide the time evolution of the same systems for specific applicative purposes.

The resolution of a dynamical problem related to a time-dependent Hamiltonian presents, in general, greater complications with respect to the one related to a time-independent Hamiltonian. Just think to the fact that the dynamical problem related to a two-level time-dependent Hamiltonian cannot be solved, if not formally, in its general form, i.e. when the time dependence of the Hamiltonian parameters is generic. In literature, in fact, there are no so many scenarios, identified by specific time dependences of the Hamiltonian parameters, whose related dynamical problem can be exactly solved. Among these scenarios, the most important examples, which must be counted for the seminal impact they had in Physics and elsewhere, are: 1) the Rabi model [31, 32] consisting of a spin-1/2 subjected to a static magnetic field and a rotating one orthogonal to the first. The main physical effect, namely the periodic oscillations of the system between the two levels (called Rabi oscillations), is at the basis of the well-known Nuclear Magnetic Resonance (NMR) technique. The latter had and has a fundamental role not only in Physics for the measurement of the magnetic moment of the nuclei of atoms, but also in medicine for tissue imaging in order to detect possible tumors. 2) The Landau-Majorana-Stuckelberg-Zener model [33] consisting in the application of a magnetic field linearly varying over time and an orthogonal constant one. Its effect is to allow a perfect population inversion in a two-level system (generalizable to the N-level case) through an adiabatic dynamic. 3) The STIRAP technique [34] that allows, by sending two laser pulses, an adiabatic transfer of population in a three-level system.

From these examples we see how the investigation of microscopic systems describable by time-dependent Hamiltonian models is of fundamental importance. It can lead, indeed, to the identification of exactly solvable models under suitable temporal scenarios and the identification of peculiar physical phenomena that may be of significant interest under an applicative point of view. In this respect, it is worth to underline that in recent years we have witnessed the appearance in literature of several methods aimed at identifying precisely solvable dynamical problems related to two-level time-dependent Hamiltonians [35, 36, 37, 38]. The specific interest towards the simplest system of a single qubit, besides for the physical reasons previously exposed, stems also from the fact that, as shown by the Group Theory, the solution of the dynamical problem of one spin-J (N-level system) subjected to a time-dependent magnetic field is traceable back to the solution of the analogous problem related to a spin-1/2 [39, 40].

This latter approach, however, is not valid and generally not applicable to time-dependent Hamiltonian models involving interacting qudits systems. In this case, therefore, another type of approach, that allows to exploit the knowledge of exactly solvable scenarios of single spin-1/2, is necessary. Such an issue is the research guide line of the Ph.D. work whose results are reported in this thesis.

The first step of our approach towards the exact resolution of the dynamics of interacting spin systems consists in the analysis of the symmetries of the models under scrutiny. This allows the identification of constants of motion and therefore of dynamically invariant Hilbert subspaces. In this way, it is possible to decompose the general dynamical problem into sub-problems defined in smaller Hilbert spaces and therefore to trace back the resolution of the original problem to the resolution of a set of independent and relatively simpler dynamical problems. In case of two-dimensional subspaces, we can describe the dynamics of the system in terms of an effective single spin-1/2 Hamiltonian. Thus, we can use the exactly solvable time scenarios existing in literature to derive, within that subspace, an exact solution of the dynamical problem of the interacting spin system. It is important to underline that the physical meaning of such a procedure emerges only when the time-dependence of the effective Hamiltonian parameters and the solutions are interpreted in terms of the 'true' physical variables of the interacting spin system. In the case, instead, of subspaces of greater dimension it is always possible to refer to the solutions valid for single spin-1/2 scenarios if the Hamiltonian related to the subspace under scrutiny presents a SU(2) symmetry. If this circumstance does not occur, it is always possible to describe the dynamics of the system within the invariant subspaces in terms of a fictitious system of interacting spins or in terms of a single spin described by a non-linear effective Hamiltonian. Obviously, such s case may be difficult to be treated analytically; however, it is interesting to note that such a dynamical decomposition, in view of a more efficient resolution of the problem, results crucial also for a numerical approach.

The thesis is organized as follows. In Chapter 1 it has been brought to light the mathematical difficulties encountered in solving the dynamical problem related to two-level time-dependent Hamiltonians. Such a problem, indeed, cannot be analytically solved when the time-dependence of the Hamiltonian parameters is generic. However, the Group Theory gives us the formal structure (dependent on two complex time-dependent functions) of the time evolution operator, solution of the time-dependent Schrödinger equation. On the basis of such a structure, in Ref. [38] an approach aiming at identifying specific relations between the Hamiltonian parameters in order to make solvable the dynamical problem, has been proposed. In the first chapter, besides getting new exactly solvable scenarios, we emphasized the underlying physical meaning of such relations, such as, for example, for the generalized (time-dependent) resonant Rabi condition.

In Chapter 2 we apply the technique based on the dynamical decomposition, briefly discussed above, to the case of two interacting qubits subjected to local time-dependent fields. We show that when the latter linearly vary over time, LMSZ transitions can occur, although an indispensable orthogonal field is absent. The possibility of such an effect stems from the presence of the coupling existing between the two spins which, in the two independent fictitious two-level problems, acts exactly as the necessary orthogonal field generating the well-known avoided crossing. The applicative interest of such a physical effect relies on the fact that, as we show, it is possible to generate entangled states of the two spin-1/2's by appropriately setting the slope of the ramp field. Moreover, thanks to the dynamical decomposition, we can analytically study the time evolution of the quantum correlations getting established between the qubits, such as the entanglement and, for some specific initial conditions, the quantum discord.

In Chapter 3 we consider the same model for two qutrits. Also in this case, we demonstrate the occurrence of coupling-based LMSZ transitions, from which it is possible to generate entangled state for the two-qutrit system. In the same chapter we study also two interacting qudits subjected to a homogeneous field and a system of N spin-qubits subjected to local time-dependent fields and coupled through N-order interaction terms (interactions involving at the same time all the spins in the system). Such a type of interactions can be efficiently implemented through the trapped ion [41] and quantum superconducting circuit [8, 42] technologies. We show how it is possible, thanks to these exotic interactions, to 'propagate' to all the spins in the chain the dynamics of a single (ancilla) qubit addressed through the local application of an external time-dependent field.

In Chapter 4, instead, we prove that the same approach can be successfully exploited for physical systems living in infinite-dimensional Hilbert spaces. In our case the Hamiltonian model consists in a quantum harmonic oscillator bi-linearly coupled to a quantum Glauber amplifier (formally described as a quantum inverted harmonic oscillator) where the Hamiltonian parameters are considered time-dependent. Also in this case the identification of SU(2)-symmetry subspaces plays a pivotal role in solving analytically the dynamical problem. The dynamics of the system is moreover compared to that of the analogous physical system composed by two standard quantum harmonic oscillators. Exact solutions of the dynamical problem of the latter system can be found too, thanks to the same symmetry-based dynamical decomposition method.

In Chapter 5 we take into account time-dependent two-dimensional non-Hermitian SU(1,1)-symmetry matrices. Their physical relevance stems from the fact that they are pseudo-Hermitian matrices and quasi-Hermitian in precise regions of the parameter space. A subclass of them corresponds moreover to the well-known PT-symmetric matrices. Our contribution consists in the generalization of the approach reported in Ref. [38] to such matrices getting, in this way, a method to identify exactly solvable time scenarios. We show that a possible application of such results can be found in coupled waveguide systems.

Finally, in the conclusive section we report some final consideration about the work of the thesis and possible new directions for future further works.

Chapter 1

Schrödinger Equation and Two-Level Dynamical Systems

1.1 Schrödinger equation

In quantum mechanics a physical system, according to the Dirac formalism, can be mathematically represented by a ray-vector in a Hilbert space indicated by a ket-state: $|\Psi\rangle$. The dynamics of the physical system is described by the time evolution of its initial state. That is, if the system starts from the state $|\Psi(t_0)\rangle$, its state at time t is represented by

$$|\Psi(t,t_0)\rangle = U(t,t_0)|\Psi(t_0)\rangle. \tag{1.1}$$

Here $U(t,t_0)$ represents the time evolution operator responsible of the dynamical evolution of the system. Such an operator possesses three main properties. 1) It must be a unitary operator in view of the probability conservation, that is,

$$|\langle\Psi(t_0)|\Psi(t_0)\rangle|^2 = |\langle\Psi(t,t_0)|\Psi(t,t_0)\rangle|^2 = 1, \tag{1.2}$$

which is possible only if

$$U^\dagger(t,t_0) = U^{-1}(t,t_0), \tag{1.3}$$

2) It satisfies the so called semigroup property:

$$U(t_2,t_0) = U(t_2,t_1)U(t_1,t_0), \tag{1.4}$$

that is, the time evolution of the system from the initial instant t_0 to the time instant t_2 is equivalent to propagate the state firstly from t_0 to $t_1 < t_2$ and then from t_1 to t_2. 3) Finally, the time evolution operator must satisfy of course

$$U(t_0,t_0) = \mathbb{1}. \tag{1.5}$$

By considering the infinitesimal time evolution operator $U(t+dt,t)$, it is possible to see [43] that all the previous requirements are satisfied by the operator:

$$U(t+dt,t) = 1 - i\frac{H}{\hbar}dt. \tag{1.6}$$

For such a construction one borrows from classical physics the idea that the Hamiltonian function of a physical system generates its time evolution.

By exploiting the composition rule, it is easy to get [43]

$$U(t+dt)-U(t)=-i\frac{H}{\hbar}dtU(t) \tag{1.7}$$

that is

$$i\hbar\dot{U}(t)=HU(t), \tag{1.8}$$

where, for convenience, we put $t_0=0$. This is the well known Schrödinger equation for the time evolution operator, which, for the ket-state $|\Psi(t)\rangle$, becomes

$$i\hbar\frac{\partial|\Psi(t)\rangle}{\partial t}=H|\Psi(t)\rangle. \tag{1.9}$$

If the Hamiltonian operator does not depend on time, the solution of Eq. (1.8) such that $U(0)=\mathbb{1}$ is simply

$$U(t)=\exp\left\{-\frac{i}{\hbar}Ht\right\}. \tag{1.10}$$

If the Hamiltonian depends on time but possesses the property to commute with itself at different times: $[H(t'),H(t)]=0$, for any t and t', the solution of Eq. (1.8) may be formally written

$$U(t)=\exp\left\{-\frac{i}{\hbar}\int_0^t Hdt\right\}. \tag{1.11}$$

Finally, if the Hamiltonian do not commute at different times, that is, $[H(t'),H(t)]\neq 0$, the formal solution of the Schrödinger equation has been formally written by Dyson as

$$U(t)=1+\sum_{n=1}^{\infty}\left(-\frac{i}{\hbar}\right)^n\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_nH(t_1)H(t_2)\cdots H(t_n), \tag{1.12}$$

where $t_1>t_2>t_3>\cdots>t_n$.

In this thesis we deal with physical systems represented by time-dependent Hamiltonians non-commuting at different times. In the following chapter we start by investigating the dynamical problem of a single two-level system. We present specific solvability conditions for the dynamical problem as well as the related exact expression of the time evolution operator. In the other chapters we show how to exploit the knowledge of analytically treatable single qubit scenarios to get exact solutions for the dynamical problems related to larger and more complex physical systems. Our attention is mainly focused on interacting spin systems subjected to classical time-dependent fields. A careful analysis of the symmetries possessed by the Hamiltonian is the key to get both the formal expression of the time evolution operator and its explicit analytical form for specific exactly solvable scenarios of remarkable physical interest.

1.2 The two-level dynamical problem and exact solutions

Searching new exactly solvable time-dependent scenarios of a single spin-1/2 could be very interesting and worth both from basic and applicative points of view, especially in the quantum control context. To pursue this target, over the last years, new methods have been developed to face the problem with an original strategy [35, 36, 37].

1.2.1 Parametric solutions

The su(2) Hamiltonian of a single spin-1/2 subjected to a generic time-dependent magnetic field $\mathbf{B}(t) \equiv [B_x(t), B_y(t), B_z(t)]$ may be cast as follows

$$H(t) = \Omega(t)\hat{\sigma}^z + \omega_x(t)\hat{\sigma}^x + \omega_y(t)\hat{\sigma}^y = \begin{pmatrix} \Omega(t) & \omega(t) \\ \omega^*(t) & -\Omega(t) \end{pmatrix}, \tag{1.13}$$

with

$$\Omega(t) = \frac{\mu_B g}{2} B_z(t), \tag{1.14a}$$

$$\omega(t) = \omega_x - i\omega_y = \frac{\mu_B g}{2}[B_x(t) - iB_y(t)] \equiv |\omega(t)| e^{i\phi_\omega(t)}. \tag{1.14b}$$

Here $\hat{\sigma}^k$ $(k = x, y, z)$ are the spin-1/2 Pauli matrices and $\mu_B g$ is the magnetic moment associated to the spin-1/2, g and μ_B being the appropriate Landé factor and the Bohr magneton, respectively.

We point out that the Hamiltonian in Eq. (1.13) is the general time-dependent operator of any two-level system, unless an additive time-dependent term proportional to the identity operator $\mathbb{1}$, which would not influence the spin-1/2 dynamics since it would trivially result in a time-dependent overall phase factor in the evolution operator. Of course, depending on the physical system under scrutiny the physical meaning of the Hamiltonian parameter changes accordingly. Thus, hereafter, the words spin-1/2 and field are used with general meaning standing for two-level system and controllable external parameter, respectively.

The entries $a(t) \equiv |a(t)| \exp\{i\phi_a(t)\}$ and $b(t) \equiv |b(t)| \exp\{i\phi_b(t)\}$ of the unitary time evolution operator

$$U(t) = \begin{pmatrix} a(t) & b(t) \\ -b^*(t) & a^*(t) \end{pmatrix}, \quad |a(t)|^2 + |b(t)|^2 = 1, \tag{1.15}$$

generated by $H(t)$, must satisfy the problem $i\hbar\dot{U}(t) = H(t)U(t)$, $U(0) = \mathbb{1}$, which originates the following system of linear differential equations

$$\begin{cases} i\hbar\, \dot{a}(t) = \Omega\, a(t) - \omega\, b^*(t), \\ i\hbar\, \dot{b}(t) = \omega\, a^*(t) + \Omega\, b(t), \\ a(0) = 1, \quad b(0) = 0. \end{cases} \tag{1.16}$$

It is possible to demonstrate [38] (see Appendix A.1) that if $\Theta(t)$ is a complex-valued C^1 (continuously differentiable) function of t satisfying the nonlinear integral-differential Cauchy problem

$$\frac{1}{2}\dot{\Theta}(t) + \frac{|\omega(t)|}{\hbar}\sin\Theta(t)\cot\Big[\frac{2}{\hbar}\int_0^t |\omega(t')|\cos\Theta(t')dt'\Big] = \frac{\Omega(t)}{\hbar} + \frac{\dot{\phi}_\omega(t)}{2}, \qquad \Theta(0) = 0, \tag{1.17}$$

then the solutions of the Cauchy problem (1.16) can be represented as follows

$$a(t) = \cos\left[\frac{1}{\hbar}\int_0^t |\omega(t')|\cos\left[\Theta(t')\right]dt'\right] \times \exp\left\{i\left(\frac{\phi_\omega(t)-\phi_\omega(0)}{2} - \frac{\Theta(t)}{2} - \mathscr{R}(t)\right)\right\}, \tag{1.18a}$$

$$b(t) = \sin\left[\frac{1}{\hbar}\int_0^t |\omega(t')|\cos\left[\Theta(t')\right]dt'\right] \times \exp\left\{i\left(\frac{\phi_\omega(t)+\phi_\omega(0)}{2} - \frac{\Theta(t)}{2} + \mathscr{R}(t) - \frac{\pi}{2}\right)\right\}, \tag{1.18b}$$

with

$$\mathscr{R}(t) = \int_0^t \frac{|\omega(t')|\sin[\Theta(t')]}{\sin\left[2\int_0^{t'}|\omega(t'')|\cos[\Theta(t'')]dt''\right]}dt'. \tag{1.19}$$

Vice versa, if $a(t)$ and $b(t)$ are solutions of the Cauchy problem (1.16), then the representations given in Eqs. (1.18) are still valid and $\Theta(t)$ satisfies Eqs. (1.17).

Generally speaking, solving Eq. (1.17) is a difficult task. This equation however may be exploited in a different way, giving rise to a strategy [38] aimed at singling out exactly solvable dynamical problems represented by Eq. (1.16). Fixing, indeed, at will the function $\Theta(t)$ in Eqs. (1.17), that is, $\Theta(t)$ regarded now as a parameter (function) rather than an unknown, determines a link between $\Omega(t)$ and $\omega(t)$ under which the corresponding dynamical problem may be exactly solved in view of Eqs. (1.18). It is important to underline, however, that, depending on the choice of the function $\Theta(t)$, it could be very difficult (if not impossible) sometimes to find analytical expressions of all quantities we need, in particular for $\phi_a(t)$ and $\phi_b(t)$ because of the presence of the integral $\mathscr{R}$ in Eq. (1.19). We emphasize that if we knew the solution of the Cauchy-like problem given in Eq. (1.17), whatever $\Omega(t)$, $|\omega(t)|$ and $\dot{\phi}_\omega(t)$ are, then we would be in condition to solve in general the corresponding Cauchy dynamical problem expressed by Eqs. (1.16). It is worth noticing however that in the special physical scenario in which the driving term $\Omega(t)+\hbar\dot{\phi}_\omega(t)/2$ vanishes, the corresponding Cauchy problem admits the solution $\Theta(t)=0$, by direct inspection. The physical implication of such a solution will be considered in detail in the next section.

Another useful way of parametrizing the expressions of $a(t)$ and $b(t)$ is [38]

$$a(t) = \left(\cos[\Phi(t)] - i\frac{\beta}{\sqrt{1+\beta^2}}\sin[\Phi(t)]\right)\exp\left\{i\frac{\phi_\omega(t)}{2}\right\}, \tag{1.20a}$$

$$b(t) = \frac{1}{\sqrt{1+\beta^2}}\sin[\Phi(t)]\exp\left\{i\left(\frac{\phi_\omega(t)}{2} - \frac{\pi}{2}\right)\right\}, \tag{1.20b}$$

with

$$\Phi(t) = \sqrt{1+\beta^2}\int_0^t \frac{|\omega(t')|}{\hbar}dt', \tag{1.21}$$

β being an arbitrary real constant number and having put, without loss of generality, $\phi_\omega(0)=0$. In this case, it is possible to check that they solve the system (1.16) if the following condition holds

$$\frac{\Omega(t)}{\hbar} + \frac{\dot{\phi}_\omega(t)}{2} = \beta\frac{|\omega(t)|}{\hbar}. \tag{1.22}$$

It is stressed that this last equation does only express the condition under which, whatever β is, the representations (1.20) satisfy the Cauchy problem (1.16). This means that the real number β plays in this case the role of parameter. When Eq. (1.22) cannot be satisfied for any β, of course the solution of the dynamical problem exists but cannot be represented using Eqs. (1.20). In this case there certainly exists a function $\Theta(t)$ enabling the representation of the solutions by using Eqs. (1.18). Finally, it is interesting to underline that Eq. (1.17) turns into the simpler condition (1.22) on $\mathbf{B}(t)$ under an appropriate choice of the parameter function $\Theta(t)$ [38].

1.2.2 New exactly solvable single qubit time dependent scenarios

Before to start it is useful to report the expression of the two time-dependent parameter functions $a(t)$ and $b(t)$ when the magnetic fields acting upon the spin 1/2 are constant, that is, $\omega = const.$ and $\Omega = const.$. We have namely

$$\begin{aligned} a(t) &= \left[\cos(\nu t/\hbar) - i\frac{\Omega}{\nu}\sin(\nu t/\hbar)\right] e^{i\Phi t} \\ b(t) &= -i\frac{|\omega|}{\nu}\sin(\nu t/\hbar) e^{i\Phi t}, \end{aligned} \tag{1.23}$$

with $\nu \equiv \sqrt{\Omega^2 + |\omega|^2}$ and $\Phi = -\arctan[\omega_y/\omega_x]$.

In the following, instead, we show the practical potentiality of the method proposed in Ref. [38] and reported in appendix A.1 whose results about the parametric solutions of the two-level dynamical problem have been previously showed. We report and discuss two novel time-dependent physical scenarios of the quantum dynamics of a spin-1/2 and the exact solutions of the related dynamical problem.

First example

Firstly, let us put

$$\int_0^t \frac{|\omega(t')|}{\hbar}\cos\Theta(t')dt' = \frac{1}{2}\arcsin\left[\tanh(2\tau)\right], \qquad \tau(t) = \int_0^t \frac{|\omega(t')|}{\hbar}dt' \tag{1.24}$$

With this choice we have

$$\cos\Theta(\tau) = \frac{1}{\cosh(2\tau)}, \qquad \sin\Theta(\tau) = \tanh(2\tau) \tag{1.25}$$

from which we derive

$$\dot{\Theta}(\tau) = 2\frac{\gamma}{\cosh(2\tau)}, \qquad \Theta(\tau) = 2\arctan[\tanh(\tau)] \tag{1.26}$$

and the integral $\mathscr{R}$ is trivially integrated to yield

$$\mathscr{R}(\tau) = \tau. \tag{1.27}$$

In this instance, the two parameter time-functions a and b result

$$a(t) = \sqrt{\frac{\cosh[2\tau(t)]+1}{2\cosh[2\tau(t)]}} \times \exp\left\{\frac{\phi_\omega(t)-\phi_\omega(0)}{2} - \arctan[\tanh[\tau(t)]] - \tau(t)\right\} \tag{1.28a}$$

$$b(t) = \sqrt{\frac{\cosh[2\tau(t)]-1}{2\cosh[2\tau(t)]}} \times \exp\left\{\frac{\phi_\omega(t)+\phi_\omega(0)}{2} - \arctan[\tanh[\tau(t)]] + \tau(t) - \frac{\pi}{2}\right\}. \tag{1.28b}$$

Finally, the condition linking the Hamiltonian parameters reads

$$\Omega(t)+\frac{\dot{\phi}_\omega(t)}{2}=\frac{2|\omega(t)|}{\cosh[2\tau(t)]}. \tag{1.29}$$

Considering $|\omega(t)|=const.$ and $\dot{\phi}_\omega=0$, the plot of $\Omega/|\omega|$ is shown in Fig. 1.1a as a function of $\tau=|\omega|t/\hbar$.

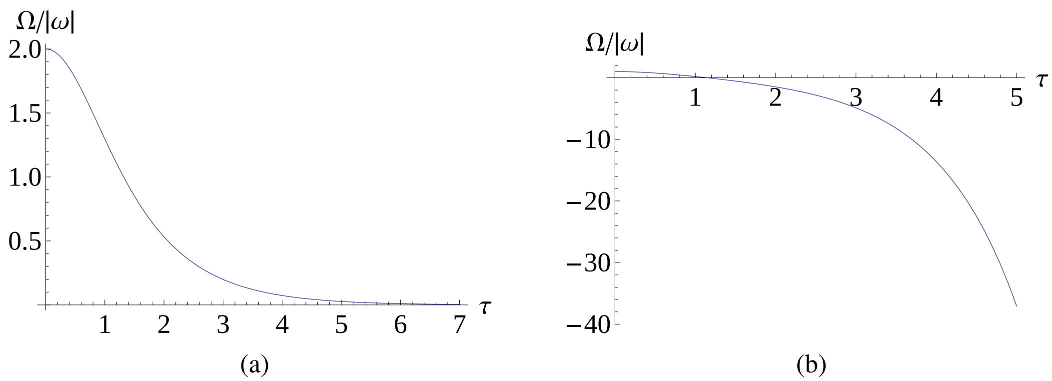


Figure 1.1: Plot of $\Omega(\tau)/|\omega|$ according to a) Eq. (1.29) and b) Eq. (1.33).

Second example

Let us consider now

$$\int_0^t \frac{|\omega(t')|}{\hbar}\cos\Theta(t')dt'=\arcsin\big[\tanh(\tau)\big]. \tag{1.30}$$

It is easy to verify that the expressions of $\cos\Theta$ and $\sin\Theta$ are the same as those in (1.25) of the previous case and so also $\dot{\Theta}$ and Θ have the same expressions as those given in (1.26); the only difference lies in the fact that the mathematical functions depends now on τ instead of on 2τ. What is different is the value of integral $\mathscr{R}$ which, in this case, results

$$\mathscr{R}(\tau)=\frac{\sinh(\tau)}{2} \tag{1.31}$$

This time the two functions defining the time evolution operator read

$$a(t)=\frac{1}{\cosh[\tau(t)]}\times\exp\left\{\frac{\phi_\omega(t)-\phi_\omega(0)}{2}-\arctan[\tanh(\tau(t)/2)]-\frac{\sinh[\tau(t)]}{2}\right\}, \tag{1.32a}$$

$$b(t)=\tanh[\tau(t)]\times\exp\left\{\frac{\phi_\omega(t)+\phi_\omega(0)}{2}-\arctan[\tanh(\tau(t)/2)]+\frac{\sinh[\tau(t)]}{2}-\frac{\pi}{2}\right\}. \tag{1.32b}$$

and the relation in Eq. (1.17) becomes

$$\Omega(t)+\frac{\dot{\phi}_\omega(t)}{2}=\frac{|\omega(t)|}{2}\left[\frac{3}{\cosh[\tau(t)]}-\cosh[\tau(t)]\right]. \tag{1.33}$$

Figure 1.1b shows the behaviour of $\Omega/|\omega|$ against τ when $|\omega(t)| = const.$ and $\dot{\phi}_\omega = 0$.

It is important to point out that the factor 1/2 (1) multiplying the function $\arcsin\big[\tanh(\tau)\big]$ in Eq. (1.24) [(1.30)] is crucial for the possibility of exactly getting the integral $\mathscr{R}$. Furthermore, such a factor has a remarkable consequence in the time-dependence of $|a|$, $|b|$ and Ω in the first and second scenarios. We saw, indeed, that the asymptotic ($t \to \infty$) values of $|a|$ and $|b|$ are very different in the two cases determining a completely different dynamical evolution in time. Finally, as we can see from Fig. 1.1b, the multiplying factor significantly determines the time trend of the longitudinal component of magnetic field which must be engineered appropriately to have the exact dynamics we are studying.

1.3 Physical meaning of the solvability conditions

The aim of this subsection is to furnish a direct physical meaning of the solvability conditions in Eqs. (1.17) and (1.22). We find the physical reason of such conditions within the so called Generalized Rabi Systems (GRSs) framework. Our investigation introduces, in a very natural way, three different classes of GRSs wherein Rabi oscillations of maximum amplitude still survive. We succeed indeed in identifying generalized resonance conditions which are, as in the Rabi scenario, at the origin of the complete population transfer between the two Zeeman levels of the spin. We bring to light that, even at resonance, these oscillations might loose its periodic character, significantly differing, thus, from the sinusoidal behaviour occurring in the Rabi scenario. We also consider time-dependent magnetic fields giving rise to exactly solvable models not satisfying the resonance condition. In this way, we are able to write down a special link among all the time-dependent parameters appearing in the Hamiltonian of the system even if the resonance condition is not met, for which, however, the dynamical problem is exactly solvable. The departure of the time evolution of the transition probability out of generalized resonance from the corresponding behaviour in the Rabi scenario is illustrated with the help of two exemplary cases.

1.3.1 Generalized resonance condition and out of resonance cases

Rabi [31, 32] and Schwinger [44] exactly solved the quantum dynamics of a spin-1/2 in the now called Rabi scenario, that is, subjected to a static magnetic field B_0 along the z-axis and an r.f. magnetic field rotating in the x-y plane with frequency ω_{xy}, namely

$$\mathbf{B}_R(t) = B_\perp[\cos(\omega_{xy}t)\mathbf{c}_1 - \sin(\omega_{xy}t)\mathbf{c}_2] + B_0\mathbf{c}_3, \tag{1.34}$$

$\mathbf{c}_1, \mathbf{c}_2$ and $\mathbf{c}_3$ being fixed unit vectors in the laboratory frame. Their seminal papers show that the probability of transition between the two Zeeman states generated by $\mathbf{B}_z = B_0\mathbf{c}_3$ is dominated by periodic oscillations reaching maximum amplitude under the so-called resonance condition $\Delta = \omega_{xy} - \omega_L = 0$. Here ω_L is the spin Larmor frequency. The exact treatment of this basic problem provides the robust platform for the NMR technology implementation [45]. The recently published special issue on semiclassical and quantum Rabi models [46] witnesses the evergreen attractiveness of this problem.

The experimental set-up considered by Rabi, as described before, leads to the Hamiltonian model

(1.13) where

$$\Omega(t) = \frac{\mu_B g}{2} B_0 \equiv \Omega_0, \tag{1.35a}$$

$$|\omega(t)| = \frac{\mu_B g}{2}\sqrt{B_x^2(t)+B_y^2(t)} = \frac{\mu_B g}{2} B_\perp \equiv |\omega_0|, \tag{1.35b}$$

$$\phi_\omega(t) = \nu_0 t \equiv \dot{\phi}_0 t. \tag{1.35c}$$

Then it is characterized by the three time-independent parameters: Ω_0, $|\omega_0|$ and $\dot{\phi}_0$. We generalize this Rabi scenario by making some out of or all these parameters time-dependent: $\Omega \to \Omega(t)$, $|\omega| \to |\omega(t)|$ and $\dot{\phi}_0 t \to \phi_\omega(t)$.

Generalizing the approach in Ref. [32], we pass from the laboratory frame to the time-dependent one tuned with $\phi_\omega(t)$, where the time-dependent Schrödinger equation for the transformed state,

$$|\psi(t)\rangle = \exp\{i\phi_\omega(t)\hat{\sigma}^z/2\}|\tilde{\psi}(t)\rangle, \tag{1.36}$$

is governed by the following effective time-dependent transformed Hamiltonian

$$H_{GR}(t) = \left(\Omega(t) + \frac{\hbar}{2}\dot{\phi}_\omega(t)\right)\hat{\sigma}^z + |\omega(t)|\hat{\sigma}^x. \tag{1.37}$$

We point out that, on the basis of the transformation defined in Eq. (1.36), such a new transformed Hamiltonian presents a simpler transverse field which depends only on the modulus $[|\omega(t)|]$ of the original one defined in Eq. (1.35). However, the dynamical effects due to the time-dependent phase $[\phi_\omega(t)]$ of the original transverse field, stem now from the new redefined longitudinal field.

It is worth noticing its strict similarity with the analogous one got in Ref. [32] where the unitary transformation is indeed a uniform rotation around the z-axis. In fact, it is enough to make $\Omega(t)$, $|\omega(t)|$ and $\dot{\phi}_\omega(t)$ time-independent in H_{GR} (GR stands for Generalized Rabi) to immediately recover the transformed Hamiltonian got by Rabi [32]. On the basis of this observation it then appears natural to refer to the following condition

$$\Omega(t) + \frac{\hbar}{2}\dot{\phi}_\omega(t) = 0, \tag{1.38}$$

as a generalized resonance condition, in accordance with the corresponding static resonance condition $\Omega_0 + \hbar\dot{\phi}_0/2 = 0$ brought to light by Rabi in Ref. [31]. We underline that the generalized resonance condition does not lead to a time-independent transformed dynamical problem (as it happens in the Rabi scenario), but, whatever H is, it easily enables the explicit construction of the time evolution operator describing the quantum motion of the spin in the laboratory frame. In view of Eq. (1.22), the entries of such an operator are indeed exactly given by Eqs. (1.20) in the limit $\beta \to 0$, namely

$$a(t) = \cos\left[\int_0^t \frac{|\omega|}{\hbar}dt'\right]\exp\left\{i\frac{\phi_\omega(t)}{2}\right\}, \tag{1.39a}$$

$$b(t) = \sin\left[\int_0^t \frac{|\omega|}{\hbar}dt'\right]\exp\left\{i\frac{\phi_\omega(t)}{2} - i\frac{\pi}{2}\right\}. \tag{1.39b}$$

By definition, we say to be in generalized out of resonance when the left hand side of Eq. (1.38) is non-vanishing, namely

$$\frac{\Omega(t)}{\hbar} + \frac{\dot{\phi}_\omega(t)}{2} = \Delta(t) \neq 0, \tag{1.40}$$

where $\Delta(t)$ is an arbitrary frequency-dimensioned well-behaved function of time. Let us observe that, on the basis of the structure of H_{GR} in Eq. (1.37), when $\Delta(t)$ is proportional to $|\omega(t)|$, the dynamical problem may be exactly solved. Indeed, this condition coincides with that expressed by Eq. (1.22) which in turn enables one to write down exact solutions of the Cauchy problem (1.16) in the form given by Eqs. (1.20). In the following we report the exact solutions of special non-trivial dynamical problems both in and out of resonance. Our aim is to illustrate the occurrence of analogies and differences in the time behaviour of the Rabi transition probability

$$P_+^-(t) = |\langle -|U(t)|+\rangle|^2 = |b(t)|^2, \tag{1.41}$$

($\hat{\sigma}^z|\pm\rangle = \pm|\pm\rangle$), when the time evolution of the magnetic field acting upon the spin cannot be described as a perfect precession around the z-axis.

1.3.2 Examples of generalized Rabi models

This subsection is aimed at showing that the Rabi transition probability $P_+^-(t) = |\langle -|U(t)|+\rangle|^2$ exhibits a remarkable sensitivity to possible different choices of the time-dependent magnetic fields under general conditions. The following examples are reported to illustrate such behaviour.

Examples of GRSs dynamics under generalized resonance condition

Let us consider, firstly, the generalized resonance condition in Eq. (1.38). We know that, under such a condition, the time evolution operator is characterized by the time behaviour of its two entries given in Eq. (1.39), so that the transition probability reads

$$P_+^-(t) = \sin^2\left[\int_0^t \frac{|\omega|}{\hbar} dt'\right]. \tag{1.42}$$

It is immediately evident that $P_+^-(t)$ exhibits different behaviours: it may be periodic or asymptotic, for example. Indeed, e.g., setting $|\omega(t)| = |\omega_0|\,\mathrm{sech}(|\omega_0|t/\hbar)$, obtainable by an x-y magnetic field varying over time as (we remind that $\dot{\phi}_0$ is a constant real parameter)

$$\mathbf{B}_{\mathrm{tr}} = B_x(t)\mathbf{c}_1 + B_y(t)\mathbf{c}_2 = B_\perp\,\mathrm{sech}(|\omega_0|t/\hbar)[\cos\left(\dot{\phi}_0 t\right)\mathbf{c}_1 - \sin\left(\dot{\phi}_0 t\right)\mathbf{c}_2], \tag{1.43}$$

we get

$$P_+^-(t) = \tanh^2(|\omega_0|t/\hbar), \tag{1.44}$$

resulting in a Landau-Zener-like transition, that is an asymptotic aperiodic inversion of population. Figures 1.2a and 1.2d represent the transverse magnetic field in Eq. (1.43) and the resulting transition probability in Eq. (1.44), respectively, plotted against the dimensionless time $\tau' = |\omega_0|t/\hbar$ with $\hbar\dot{\phi}_0/|\omega_0| = 10$.

However, of course, it is easy to understand that it is possible to make choices either resulting in a oscillating but not periodic transition probability or exhibiting a periodic behaviour, even if not coincident with that characterizing the Rabi scenario. If we consider, for example,

$$|\omega(t)| = |\omega_0|e^{-\gamma t}, \tag{1.45}$$

reproducible by engineering the transverse magnetic field as

$$\mathbf{B}_{\mathrm{tr}} = B_x(t)\mathbf{c}_1 + B_y(t)\mathbf{c}_2 = B_\perp e^{-\gamma t}[\cos\left(\dot{\phi}_0 t\right)\mathbf{c}_1 - \sin\left(\dot{\phi}_0 t\right)\mathbf{c}_2], \tag{1.46}$$

the resulting transition probability yields

$$P_+^-(t) = \sin^2[\alpha(1-e^{-\gamma t})], \tag{1.47}$$

with $\alpha = |\omega_0|/\hbar\gamma$. We point out that, for the sake of simplicity, in Eqs. (1.43) and (1.46) we have put $\phi_\omega(t) = \dot{\phi}_0 t$, even if, in general, the expression of the probability in Eq. (1.47) holds whatever $\phi_\omega(t)$ is, provided that Eq. (1.38) is satisfied. Figure 1.2b shows the time behaviour of the magnetic field in the x-y plane, against the dimensionless parameter γt, when $\alpha = 9\pi/2$ and $\dot{\phi}_0/\gamma = 10$.

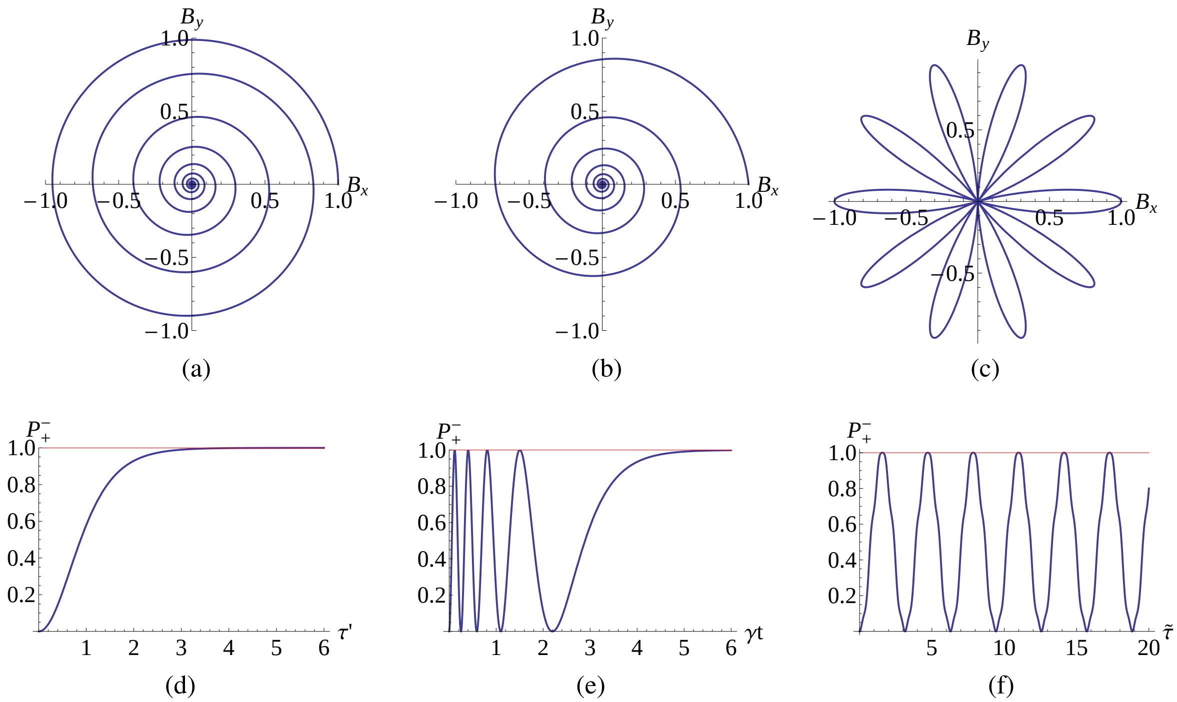


Figure 1.2: (Color online) a) The normalized magnetic field in Eq. (1.43), parametrically represented in the x-y plane and the related d) transition probability in Eq. (1.44) as a function of the dimensionless parameter $\tau' = |\omega_0|t/\hbar$ with $\dot{\phi}_0/\hbar|\omega_0| = 10$; b) the normalized magnetic field in Eq. (1.46), parametrically represented in the x-y plane and the related e) transition probability in Eq. (1.47) against the dimensionless parameter γt with $|\omega_0|/\hbar\gamma = 9\pi/2$ and $\dot{\phi}_0/\gamma = 10$; c) the normalized transverse magnetic field in Eq. (1.51), parametrically represented in the x-y plane in terms of $\tilde{\tau} = \dot{\phi}_0 t$ with $A'/B_\perp = 1$ and $\lambda = 10\dot{\phi}_0$ and the related f) transition probability in Eq. (1.52) for $k = 1$ and $n = 10$ and $C = 1$.

The time behaviour of $P_+^-(t)$ as given in Eq. (1.47) is reported in Fig. 1.2e for $\alpha = 9\pi/2$. We recognize the existence of a transient wherein $P_+^-(t)$ exhibits aperiodic oscillations of maximum amplitude which, after a finite interval of time, turn into a monotonic increase that asymptotically approaches 1. We emphasize that the number of complete oscillations, preceding the asymptotic behaviour of $P_+^-(t)$ as well as $P_+^-(\infty)$ itself, are α-dependent. Equation (1.47), indeed, predicts

$$P_+^-(\infty) = \sin^2(\alpha), \tag{1.48}$$

which immediately leads to

$$\begin{cases} P_+^-(\infty) = 0, & \alpha = n\pi, \\ P_+^-(\infty) = 1, & \alpha = \dfrac{2n+1}{2}\pi, \\ P_+^-(\infty) = \sin^2(\alpha), & \text{otherwise.} \end{cases} \tag{1.49}$$

As our third example, we consider the following modulation of $|\omega(t)|$

$$|\omega(t)| = |\omega_0| + A\cos(\lambda t), \quad 0 < A < |\omega_0|, \tag{1.50}$$

realizable by engineering the transverse magnetic field as

$$\begin{aligned} B_x(t) &= [B_\perp + A' \cos(\lambda t)]\cos(\dot{\phi}_0 t), \\ B_y(t) &= -[B_\perp + A' \cos(\lambda t)]\sin(\dot{\phi}_0 t). \end{aligned} \tag{1.51}$$

Here $\lambda = n\dot{\phi}_0$ with $n \in \mathbb{N}^*$, $A = \mu_B g A'/2$ and then $0 < A' < B_\perp$, in view of Eqs. (1.50) and (1.35). The transverse field is represented in Fig. 1.2c as a function of the dimensionless time parameter $\tilde{\tau} = \dot{\phi}_0 t$, once more supposing for simplicity $\phi_\omega(t) = \dot{\phi}_0 t$.

In this case, the Rabi's transition probability results

$$P_+^-(t) = \sin^2\Big[C\Big(\tilde{\tau} + \frac{k}{n}\sin(n\tilde{\tau})\Big)\Big], \tag{1.52}$$

with

$$C = \frac{|\omega_0|}{\hbar\dot{\phi}_0}, \quad k = \frac{A'}{B_\perp}, \quad \tilde{\tau} = \dot{\phi}_0 t, \quad n = \frac{\lambda}{\dot{\phi}_0}. \tag{1.53}$$

The behaviour of $P_+^-(t)$ in Eq. (1.52) is shown in Fig. 1.2f, having put $k = 1$, $n = 10$ and $C = 1$. Differently from the previous example, we see that, in this case, the characteristic sinusoidal behaviour of the Rabi transition probability turns into a periodic population transfer, still of maximum amplitude, between the two energy levels of the spin. We emphasize that, in view of Eq. (1.42), different time evolutions of $P_+^-(t)$ require different choices of $|\omega(t)|$ only, regardless, then, of $\Omega(t)$ and $\phi_\omega(t)$ provided they are constrained by the generalized resonance condition (1.38). For this reason, in the plots in Fig. 1, we have chosen the simplest case $\dot{\phi}_\omega(t) = \dot{\phi}_0$ [implying $\Omega(t) = -\hbar\dot{\phi}_0/2$ by Eq. (1.38)]. Our choices for $|\omega(t)|$, indeed, are not merely mathematical choices, but they aim at furnishing physical scenarios in the grasp of the experimentalists. To this end, it is hence important to take care of $\Omega(t)$ and $\phi_\omega(t)$ too. This consideration explains why we have chosen $\Omega(t) = -\hbar\dot{\phi}_0/2$ and $\phi_\omega(t) = \dot{\phi}_0 t$. It is worth noticing that had we selected different, more complex or also very exotic time-dependences for such parameters, we would get just a mere mathematical speculation since no physical effects would be present in the physical quantity under scrutiny, $P_+^-(t)$; for the latter, indeed, under the generalized resonance condition in Eq. (1.38), $\Omega(t)$ and $\dot{\phi}_\omega(t)$ are not relevant physical parameters. We stress, however, that distinct realizations of the resonance condition, keeping the same $|\omega(t)|$, introduce significant changes in the dynamical behaviour of the GRS with respect to the Rabi system. It is enough to consider, for example, that

$$\langle +|U^\dagger(t)\hat{\sigma}^{x/y}U(t)|+\rangle = \mp 2\hbar|a(t)||b(t)|\cos[\phi_a(t) + \phi_b(t)], \tag{1.54}$$

depend on both $\phi_\omega(t)$ and $|\omega(t)|$, in view of Eqs. (1.39).

Examples of GRSs dynamics in generalized out of resonance cases

We analyse now the generalized out of resonance case, defined in Eq. (1.40). Since it appears hopeless to have an exact closed treatment of the problem in Eq. (1.17) with an arbitrary $\Delta(t)$, we confine ourselves to the following specific forms

$$\hbar\Delta(t) = \begin{cases} \beta_0|\omega(t)|, \\ \beta(t)|\omega(t)|. \end{cases} \tag{1.55}$$

In the former case, the solutions $a(t)$ and $b(t)$ of the system in Eq. (1.16) may be cast as reported in Eqs. (1.20) so that

$$P_+^-(t) = \frac{1}{1+\beta_0^2}\sin^2\left[\sqrt{1+\beta_0^2}\int_0^t \frac{|\omega(t')|}{\hbar}dt'\right]. \tag{1.56}$$

In the limit $\beta_0 \to 0$ we recover Eq. (1.42) from this equation. Thus, we may compare $P_+^-(t)$ in the resonant and this non-resonant cases when $|\omega(t)|$ is fixed in the same way. It is easy to convince oneself that the main effect of a positive value of the parameter β_0 on $P_+^-(t)$ is nothing but a scale effect determined by the ratio $1/(1+\beta_0^2)$.

We wish now to discuss some exactly solvable scenarios of generalized, out of resonance, Rabi problems wherein $\hbar\Delta(t) = \beta(t)|\omega(t)|$. The form of $\Delta(t)$ written before naturally emerges when the independent variable t in Eq. (1.17) is substituted by

$$\tau(t) = \int_0^t \frac{|\omega(t')|}{\hbar}dt', \tag{1.57}$$

and a choice at will of $\Theta[\tau(t)]$ is performed in accordance with Ref. [38]. We stress however that the corresponding function $\beta(t)$ would be functionally dependent on $|\omega(t)|$, that is, we determine $\beta[\tau(t)]$ once we have chosen $|\omega(t)|$. We emphasize that the fact that the function $\beta(t)$ is not independent of the function $|\omega(t)|$ does not spoil of interest such a particular procedure. In the following examples we indeed report two applications of the general strategy here exposed.

Case 1

In this subsection and the following one we will omit the t-dependence in $\tau(t)$ to save some writing. To illustrate the applicability of our parametrization given in Eqs. (1.18) and (1.19), we cannot simply confine ourselves to assign at will $\Theta(\tau)$. Indeed, we must overcome the unique analytical difficulty related to the calculation of $\mathscr{R}(t)$ in Eq. (1.19) as previously underlined. In practice, then, what is demanded is to search specific choices of $\Theta(\tau)$ such that the integral expressing $\mathscr{R}(t)$ becomes evaluable. The following two examples provide a successful application of such a strategy.

Assuming the solution of the problem (1.17) as

$$\Theta(t) = 2\tan^{-1}\left(\frac{2\tau}{\sqrt{2+4\tau^2}}\right), \tag{1.58}$$

it is straightforward to show that

$$\int_0^t \frac{|\omega(t')|}{\hbar}\cos[\Theta(t')]dt' = \frac{1}{2}\tan^{-1}(2\tau). \tag{1.59}$$

Equation (1.17) immediately yields

$$\Delta(t) = \frac{4(1+\tau^2)}{(1+4\tau^2)\sqrt{2+4\tau^2}}\frac{|\omega(t)|}{\hbar} \equiv \tilde{\beta}(\tau)|\omega(t)| = \beta(t)|\omega(t)|. \tag{1.60}$$

Within such a scenario, the specialized expressions of Eqs. (1.18) result

$$|a(t)| = \sqrt{\frac{\sqrt{1+4\tau^2}+1}{2\sqrt{1+4\tau^2}}}, \quad |b(t)| = \sqrt{\frac{\sqrt{1+4\tau^2}-1}{2\sqrt{1+4\tau^2}}}, \tag{1.61}$$

and

$$\phi_a(t) = \frac{\phi_\omega(t)-\phi_\omega(0)}{2} - \tan^{-1}\left(\frac{2\tau}{\sqrt{2+4\tau^2}}\right) + \frac{i}{\sqrt{2}}\text{EllipticE}[i\sinh^{-1}(2\tau),1/2], \tag{1.62a}$$

$$\phi_b(t) = \frac{\phi_\omega(t)+\phi_\omega(0)}{2} - \tan^{-1}\left(\frac{2\tau}{\sqrt{2+4\tau^2}}\right) - \frac{i}{\sqrt{2}}\text{EllipticE}[i\sinh^{-1}(2\tau),1/2] - \frac{\pi}{2}, \tag{1.62b}$$

with $\text{EllipticE}(\phi,m) = \int_0^\phi [1-m\sin^2(\theta)]^{1/2}d\theta$. It is interesting to consider a simple case in which $|\omega(t)| = const. = |\omega_0|$. In this instance we have such a situation that $P_+^-(t) = |b(t)|^2$ ($P_+^+(t) = |a(t)|^2$) goes from 0 (1), at $t = 0$, to $1/2$ $\big(1/2\big)$, when $t \to \infty$, as it is seen in Fig. 1.3b: full blue and dashed red lines, respectively. In Fig. 1.3a we may appreciate the time behaviour of $\hbar\Delta(t)/|\omega_0|$ related to this specific physical scenario. This specific out of resonance time-dependent scenario, then, asymptotically evolves the initial state $|+\rangle$ towards an equal-weighted superposition of the two eigenstates of $\hat{\sigma}^z$.

Case 2

The second scenario is based on the following assumption

$$\Theta(t) = 2\tan^{-1}\left(\frac{\tau}{\sqrt{2+\tau^2}}\right), \tag{1.63}$$

which, notwithstanding its apparent similarity with the previous case given in Eq. (1.58), leads however to a remarkable different temporal behaviour of the correspondent generalized Rabi system. This time it results in

$$\int_0^t \frac{|\omega(t')|}{\hbar}\cos[\Theta(t')]dt' = \tan^{-1}(\tau), \tag{1.64}$$

so that the solutions of (1.16) read

$$|a(t)| = \frac{1}{\sqrt{1+\tau^2}}, \qquad |b(t)| = \frac{\tau}{\sqrt{1+\tau^2}} = |a(t)|\tau, \tag{1.65}$$

and

$$\phi_a(t) = \frac{\phi_\omega(t)-\phi_\omega(0)}{2} - \tan^{-1}\left(\frac{\tau}{\sqrt{2+\tau^2}}\right) - \frac{1}{2}\left[\frac{\tau\sqrt{2+\tau^2}}{2} + \sinh^{-1}\left(\frac{\tau}{\sqrt{2}}\right)\right], \tag{1.66a}$$

$$\phi_b(t) = \frac{\phi_\omega(t)+\phi_\omega(0)}{2} - \tan^{-1}\left(\frac{\tau}{\sqrt{2+\tau^2}}\right) + \frac{1}{2}\left[\frac{\tau\sqrt{2+\tau^2}}{2} + \sinh^{-1}\left(\frac{\tau}{\sqrt{2}}\right)\right] - \frac{\pi}{2}. \tag{1.66b}$$

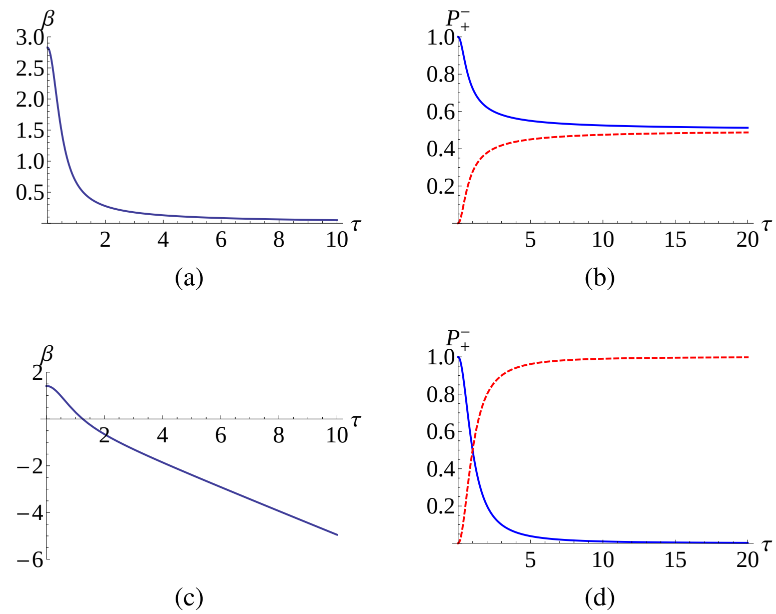


Figure 1.3: (Color online) Time-dependence of the "detuning" $\hbar\Delta(\tau)/|\omega_0| = \beta(\tau)$ against the dimensionless time $\tau = |\omega_0|t/\hbar$ related to the example a) 1.60 and c) 1.67; transition probabilities $P_+^+(\tau) = |a(\tau)|^2$ (full blue) and $P_+^-(\tau) = |b(\tau)|^2$ (dashed red) for the time-dependent scenario b) 1.61 and d) 1.65.

Finally, the special form of $\Delta(t)$ underlying this specific scenario is

$$\Delta(t) = \left[\frac{2+(1-\tau^2)(2+\tau^2)}{2(1+\tau^2)\sqrt{2+\tau^2}}\right]\frac{|\omega(t)|}{\hbar}. \tag{1.67}$$

In this case, it is easy to see that if $|\omega| = const. = |\omega_0|$, $P_+^-(t) = |b(t)|^2$ ($P_+^+(t) = |a(t)|^2$) goes from 0 (1) to 1 (0) asymptotically. These behaviours, reproducing the transition probabilities in the Landau-Zener-like transition probabilities, are illustrated by full blue and dashed red lines, respectively, in Fig. 1.3d.

In this case, the time behaviour of the "detuning" $\hbar\Delta(t)/|\omega_0|$ is characterized by an asymptotic linear dependence on t, as shown in Fig. 1.3c. As in the resonant scenario, even here different time-dependences of the magnetic field may give rise to qualitatively different time evolutions of $P_+^-(t)$ with respect to the Rabi scenario. We emphasize that the scenarios and the formulas reported for the two examples are valid whatever the time-dependence of $|\omega(t)|$ is. Of course, depending on the choice of $|\omega(t)|$ the expressions for $\beta(t)$ and for $a(t)$ and $b(t)$ could become complicated functions of time, but the related time-dependent scenario keeps the property to be an exactly solvable case for the spin-1/2 dynamical problem.

As a final remark we want to emphasize that it could be very hard to get analytical expressions for $a(t)$ and $b(t)$, in Eq. (1.18a) and (1.18b), respectively, depending on the choice of $\Theta(t)$ and two of the three Hamiltonian parameters. Nevertheless, such a bottleneck does not influence our capability to predict the Rabi transition probability and the expression of $\Delta(t)$ in order to know how to engineer the magnetic fields to get the desired time evolution. Indeed, we would be always able to find accordingly the expressions of $|a(t)|$ and $|b(t)|$. Thus, as a consequence, given $|\omega(t)|$, when $\Omega(t)$ and $\dot{\phi}_\omega(t)$ are chosen in such a way to generate the same detuning $\Delta(t)$, the related different physical scenarios share the same analytical expressions of $|a(t)|$ and $|b(t)|$. Then all physical observables depending only on these quantities share the same expressions as well, e.g. $P_+^-(t)$ or

$$\langle \pm|U^\dagger(t)\hat{\sigma}^z U(t)|\pm\rangle = \pm\hbar(|a(t)|^2 - |b(t)|^2). \tag{1.68}$$

1.4 Summary and remarks

The Rabi scenario consists in a spin-1/2 subjected to a time-dependent magnetic field precessing around the quantization axis ($\hat{z}$) [31] and is characterized by three time-independent parameters: Ω_0, $|\omega_0|$ and $\dot{\phi}_0$. Rabi shows that when $\Omega_0 + \hbar\dot{\phi}_0/2 = \Delta = 0$ the transverse magnetic field acts as a probe of the energy separation $2\Omega_0$ due to the longitudinal field alone. The measurable physical quantity revealing Ω_0 is the transition probability $P_+^-(t) = \langle -|U(t)|+\rangle$ which, at resonance, oscillates between 0 and 1 with frequency now referred to as Rabi frequency.

We have generalized this Rabi scenario by assuming an su(2) general time-dependent Hamiltonian model where then Ω_0, $|\omega_0|$ and $\dot{\phi}_0$ are now replaced with time-dependent counterparts. Along the lines of the Rabi approach [32], we firstly show that, in the frame moving with the time-dependent angular frequency $\dot{\phi}_\omega(t)$, the condition $\Omega(t) + \hbar\dot{\phi}_\omega(t)/2 = \Delta(t) = 0$ plays the same role of the Rabi resonance condition in the Rabi scenario. Such an occurrence makes of basic interest a direct comparison between the Rabi scenario and its generalized version on both time-dependent resonance and out of resonance ($\Delta(t) \neq 0$) cases. To bring to light the occurrence of analogies and differences, we have focussed our attention on the study of the transition probability $P_+^-(t)$ between the two eigenstates of $\hat{S}^z$.

We have shown that, on resonance, $P_+^-(t)$ depends only on the integral of $|\omega(t)|$. Our examples illustrate that this circumstance determines a transition probability characterized by three possible different regimes: oscillatory (the only one dominating the Rabi scenario), monotonic and mixed which means an initial oscillatory transient followed by an asymptotic monotonic behaviour.

To capture significant dynamical consequences stemming from the detuning time dependence, we have constructed exactly solvable problems and analysed the corresponding quantum dynamics of the spin-1/2. We have thus highlighted that when $\Delta(t)$ is proportional to $|\omega(t)|$, the main effect emerging in the time behaviour of $P_+^-(t)$ is a scale effect both in amplitude and in frequency (like in the Rabi scenario).

We have further investigated two specific exactly solvable scenarios of experimental interest for which $\Delta(t)/|\omega(t)|$ varies over time. One of them predicts a Landau-Zener transition, while the other an equal weighted coherent superposition of the two states of the system. It is important to underline that our examples illustrate exactly solvable cases where all the dynamical aspects and features of the spin-1/2 system under scrutiny may be brought to light. We pointed out, however, that when one is interested in the Rabi transition probability $P_+^-(t)$ only, the knowledge of $|a(t)|$ and $|b(t)|$ is enough. We emphasize that this circumstance leads us to wider and richer classes of physical scenarios, since we need not to worry about possible analytical difficulties stemming from Eq. (1.19).

We underline that the knowledge of the exactly solvable problems previously reported provide stimulating ideas for technological applications with single qubit devices. In addition it furnishes ready-to-use tools for interacting qudits systems (as shown in the following chapters of this thesis), being of relevance in several fields from condensed matter physics [4, 47] to quantum information and quantum computing [21, 48, 49, 50].

The results reported in this chapter have been published in Ref. [51].

Chapter 2

Two Interacting Spin-Qubit Systems

In this chapter we demonstrate how the knowledge of exactly solvable single qubit dynamical problems can result fundamental to solve more complex dynamical physical situations involving more spins interacting each other. We start by analysing a system of two interacting spin-qubits subjected to local time-dependent fields. We show that the exact solution of the LMSZ scenario [33] leads us to disclose new intriguing physical effects exploitable for interesting experimental physical applications.

2.1 Physical systems

A rigid and localized dimeric structure (simply dimer) consists of a pair of independent distinguishable quantum subsystems living, by definition, in finite-dimensional Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ and, therefore, hereafter referred to as spins $\hat{\mathbf{S}}_1 \equiv (\hat{S}_1^x, \hat{S}_1^y, \hat{S}_1^z)$ and $\hat{\mathbf{S}}_2 \equiv (\hat{S}_2^x, \hat{S}_2^y, \hat{S}_2^z)$ respectively, $\hat{S}_i^a$ ($i = 1,2$; $a = x,y,z$) being the operator for the a-cartesian component of $\hat{\mathbf{S}}_i$ in the laboratory reference frame. The dimension of the Hilbert space $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$ of the dimer is $(2S_1+1)(2S_2+1)$, indeed postulating the absence in the two subsystems as well as in the compound system of classical degrees of freedom (situation previously described using the adjectives 'rigid' and 'localized'). The physical nature of $\hat{\mathbf{S}}_i$ depends on the particular scenario under scrutiny: it may be the spin of an electron or a nucleus, the angular momentum of an atom in its ground state or an effective representation of a few-level system dynamical variable. The Hamiltonian H of the dimer is then a true or effective spin Hamiltonian where the terms linear in $\hat{S}_i^a$ may (even fictitiously) be interpreted as Zeeman coupling of each of the two spins with classical, external, generally different and time-dependent effective magnetic fields $\mathbf{B}_1(t)$ and $\mathbf{B}_2(t)$ while the bilinear contributions may be thought of as stemming from the spin-spin interaction [3].

Over the last two decades a great deal of theoretical, experimental and applicative attention has been devoted to the field of Molecular Magnetic Materials, in particular after the discovery of the so-called Single Magnet Molecule (SSM), that is a single molecule behaving like a nanosized magnet associated to an unusual high value (even $S = 10$ [52]) of the spin in the ground state of the molecule. It is a matter of fact that as a result of a successful, extraordinary and synergically interdisciplinary effort aimed at searching and producing SMM in laboratory, in the last few years we have witnessed a very fast growing of efficient protocols for synthesizing a variety of such molecular magnets with the added value of possessing a number of constituent paramagnetic ions embodied in the molecule running from 2 to 10 in different samples [53]. Such important technological advances on the one hand open very good applicative perspectives in many directions, from the realization of an experimental set up for testing theoretical prediction concerning qudits-based single purpose quantum computers to the availability of new materials with magnetic properties tailored on demand to meet specific tasks. On the other hand the production of crystalline or powder samples made up of molecular magnetic units, provides an ideal platform to investigate and reveal the emergence of nonclassical signatures in the quantum dynamics of two or few interacting spins.

The simplest coupled spin system we may conceive consists, of course, of two interacting spin-1/2's only in a dimer, isolated from its environment (rest of the sample) degrees of freedom. Some binuclear copper(II) compounds, e.g. [4, 54], provide a possible scenario of this kind and in the previous references the values of the parameters characterizing the spin-spin interaction in such a molecule have been experimentally determined exploiting electron-paramagnetic resonance techniques. Motivations to investigate the emergence of quantum signatures in the behaviour of two coupled spins ($\geq 1/2$) go beyond the area of magnetic materials. Two spin-1/2 Hamiltonians provide indeed experimentally implementable powerful effective models to capture quantum properties of such systems like two coupled semiconductor quantum dots [55] or a pair of two neutral cold atoms each nested into two adjacent sites of an optical lattice made up of an isolated double wells [21]. Spin models provide a successful language to investigate possible manipulations of the qubits aimed at quantum computing purposes and quantum information transfer between two spin-qubits [56], encompassing rather different physical contents like, for example, cavity QED [57, 58], superconductors [59, 60] and trapped ions [61, 62].

2.2 The Hamiltonian model and its formal solution

The most general Hamiltonian model of an isolated dimer hosting two spin-1/2's may be written as a bilinear form involving the two sets of operators $\{\hat{S}_1^x, \hat{S}_1^y, \hat{S}_1^z, \hat{S}_1^0\}$ and $\{\hat{S}_2^x, \hat{S}_2^y, \hat{S}_2^z, \hat{S}_2^0\}$, that is,

$$H = \sum_{(i,j)\neq(0,0)} \gamma_{ij} \hat{S}_1^i \otimes \hat{S}_2^j \tag{2.1}$$

where i and j run in the set $(x, y, z, 0)$ and the operator $\hat{S}_i^0$ $(i = 1, 2)$ is the identity operator $\mathbb{1}_i$ in $\mathscr{H}_i$. The six real parameters γ_{i0} and γ_{0j} $(i \cdot j \neq 0)$ are assumed to be generally time dependent while all the other parameters characterizing the spin-spin coupling are real and time independent. Without further specific constraints on the 15 parameters γ_{ij} $(i, j = x, y, z, 0)$, the Hamiltonian possesses no symmetries and in particular it does not commute with the collective angular momentum operators $\hat{\mathbf{S}}^2 = (\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2)^2$ and/or $\hat{S}^z = \hat{S}_1^z + \hat{S}_2^z$. In such a case, even if H is time independent, the four roots of the relative secular equation, albeit determinable, are rather involved functions of all the 15 parameters and then are practically not exploitable for extracting physical prediction on the physical system under scrutiny. Thus, either legitimated by investigations on specific physical situations or motivated by the interest in studying models possessing, by construction, constants of motion, some constraints on the parameters γ_{ij} have been introduced in the literature, making the Hamiltonian (2.1) less general and at the same time non trivial and of physical interest. It is enough to quote the main declinations of the three-dimensional quantum Heisenberg models or the Dzyaloshinskii-Moriya (DM) models [63, 64, 65, 66] in conjugation or not with simplified contributions to terms describing anisotropy effects in the Hamiltonian.

The Hamiltonian model (2.1), including all contributions stemming from internal or external couplings of our two spin-1/2 system, may be cast in the following form

$$H' = \mu_B(\mathbf{B}_1 \cdot \mathbf{g}_1 \cdot \mathbf{S}_1 + \mathbf{B}_2 \cdot \mathbf{g}_2 \cdot \mathbf{S}_2) + \mathbf{S}_1 \cdot \mathbf{\Gamma}_{12} \cdot \mathbf{S}_2, \tag{2.2}$$

where $\mathbf{g}_1$, $\mathbf{g}_2$ and $\mathbf{\Gamma}_{12}$ are appropriate second-order cartesian tensors whose entries are related to the 15 parameters appearing in Eq. (2.1) and μ_B denotes the Bohr magneton. Equation (2.2) mimics the usual way of representing the Hamiltonian used in a molecular or nuclear context to describe the coupling of two true spin-1/2's. In general we may claim that $\mathbf{g}_1$ and $\mathbf{g}_2$ include possible corrections to the coupling terms between each spin and its local time-dependent external magnetic field, while the other term includes contact term-like couplings as well as anisotropic-like spin-spin couplings.

The model we are going to propose assumes in the laboratory frame that $\mathbf{B}_i(t) \equiv (0, 0, B_i^z(t))$, and

$$\mathbf{\Gamma}_{12} = \begin{pmatrix} \gamma_{xx} & \gamma_{xy} & 0 \\ \gamma_{yx} & \gamma_{yy} & 0 \\ 0 & 0 & \gamma_{zz} \end{pmatrix}, \quad \mathbf{g}_i = \begin{pmatrix} g_i^{xx} & g_i^{xy} & 0 \\ g_i^{yx} & g_i^{yy} & 0 \\ 0 & 0 & g_i^{zz} \end{pmatrix} \tag{2.3}$$

with $(i = 1, 2)$. The structure of $\mathbf{g}_i$ is, for example, appropriate when the dimer is a binuclear unit characterized by a C_2-symmetry with respect to the $\hat{z}$ axis [4].

In accordance with our previous assumptions, we investigate the quantum dynamics of the following time-dependent two spin Hamiltonian model

$$H = \hbar\omega_1 \hat{\sigma}_1^z + \hbar\omega_2 \hat{\sigma}_2^z + \gamma_{xx}\hat{\sigma}_1^x\hat{\sigma}_2^x + \gamma_{yy}\hat{\sigma}_1^y\hat{\sigma}_2^y + \gamma_{zz}\hat{\sigma}_1^z\hat{\sigma}_2^z + \gamma_{xy}\hat{\sigma}_1^x\hat{\sigma}_2^y + \gamma_{yx}\hat{\sigma}_1^y\hat{\sigma}_2^x, \tag{2.4}$$

where $\hat{\sigma}_i^x$, $\hat{\sigma}_i^y$ and $\hat{\sigma}_i^z$ $(i = 1, 2)$ are the Pauli matrices related to the respective components of the spin operator $\hat{\mathbf{S}}_i$ as

$$\hat{\mathbf{S}}_i = \frac{\hbar}{2}\hat{\boldsymbol{\sigma}}_i \tag{2.5}$$

with $\hat{\boldsymbol{\sigma}}_i \equiv (\hat{\sigma}_i^x, \hat{\sigma}_i^y, \hat{\sigma}_i^z)$, while

$$\omega_i(t) = \frac{\mu_B g_i^{zz} B_i^z(t)}{2}. \tag{2.6}$$

Note that the identity operators $\mathbb{1}_i$ are and will mostly be suppressed for notational simplicity.

2.2.1 Symmetry-based dynamical decomposition

Our Hamiltonian does not commute with $\hat{\mathbf{S}}^2$ and $\hat{S}^z$ but, by construction, it exhibits the following canonical and symmetry transformation

$$\hat{\sigma}_i^x \to -\hat{\sigma}_i^x, \quad \hat{\sigma}_i^y \to -\hat{\sigma}_i^y, \quad \hat{\sigma}_i^z \to \hat{\sigma}_i^z, \quad i = 1, 2. \tag{2.7}$$

This fact implies the existence of a unitary time-independent operator accomplishing the transformation (2.7), which is by construction a constant of motion. This unitary operator is given by $\pm\hat{\sigma}_1^z\hat{\sigma}_2^z$, being the transformation (2.7) nothing but the rotations of π around the $\hat{z}$ axis with respect to each spin. The unitary operator accomplishing this transformation is

$$e^{i\pi\hat{S}_1^z/\hbar} \otimes e^{i\pi\hat{S}_2^z/\hbar} = -\hat{\sigma}_1^z\hat{\sigma}_2^z = \cos\left(\frac{\pi}{2}\hat{\Sigma}_z\right), \tag{2.8}$$

where $\hat{\Sigma}_z \equiv \hat{\sigma}_1^z + \sigma_2^z$. Equation (3.7) shows that the constant of motion $\hat{\sigma}_1^z \hat{\sigma}_2^z$ is indeed a $\hat{\Sigma}_z$-based parity operator since in correspondence to its integer eigenvalues $M = 0, \pm 2$, $\hat{\sigma}_1^z \hat{\sigma}_2^z$ has eigenvalues $+1$ and -1 respectively.

The existence of this constant of motion implies the existence of two sub-dynamics related to the two eigenvalues of $\hat{\sigma}_1^z \hat{\sigma}_2^z$. We can extract these two sub-dynamics by considering that the operator $\hat{\sigma}_1^z \hat{\sigma}_2^z$ has the same spectrum of $\hat{\sigma}_2^z$, i.e., the same eigenvalues (± 1) with the same twofold degeneracy. Therefore, there exists a unitary time-independent operator $\mathbb{U}$ transforming $\hat{\sigma}_1^z \hat{\sigma}_2^z$ in $\hat{\sigma}_2^z$. It can be easily seen that the unitary and hermitian operator

$$T = \frac{1}{2}\left[\mathbb{1} + \hat{\sigma}_1^z + \hat{\sigma}_2^x - \hat{\sigma}_1^z \hat{\sigma}_2^x\right] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{2.9}$$

in the standard ordered basis

$$\mathscr{B} = \{|{+}{+}\rangle, |{+}{-}\rangle, |{-}{+}\rangle, |{-}{-}\rangle\}, \tag{2.10}$$

accomplishes the desired transformation:

$$T^\dagger \hat{\sigma}_1^z \hat{\sigma}_2^z T = T \hat{\sigma}_1^z \hat{\sigma}_2^z T = \hat{\sigma}_2^z. \tag{2.11}$$

Transforming H into $\tilde{H} = T^\dagger H T$, we get

$$\tilde{H} = \hbar\omega_1 \hat{\sigma}_1^z + \hbar\omega_2 \hat{\sigma}_1^z \hat{\sigma}_2^z + \gamma_{zz} \hat{\sigma}_2^z + \gamma_{xx} \hat{\sigma}_1^x - \gamma_{yy} \hat{\sigma}_1^x \hat{\sigma}_2^z + \gamma_{xy} \hat{\sigma}_1^y \hat{\sigma}_2^z + \gamma_{yx} \hat{\sigma}_1^y. \tag{2.12}$$

It is easy to check that $\hat{\sigma}_2^z$ is a constant of motion of $\tilde{H}$ and that, consequently, $\tilde{H}$ may be represented as

$$\tilde{H} = \sum_{\sigma_2^z} \tilde{H}_{\sigma_2^z} \left|\sigma_2^z\right\rangle \left\langle \sigma_2^z\right| = \tilde{H}_+ \otimes |+\rangle\langle+| + \tilde{H}_- \otimes |-\rangle\langle-| = \tilde{H}_+ \oplus \tilde{H}_- \tag{2.13}$$

where

$$\tilde{H}_\pm = \pm\gamma_{zz} + \hbar\Omega_\pm \hat{\sigma}_1^z + \gamma_\pm \hat{\sigma}_1^x + \Gamma_\pm \hat{\sigma}_1^y. \tag{2.14}$$

with

$$\Omega_\pm(t) = \omega_1(t) \pm \omega_2(t), \qquad \gamma_\pm = (\gamma_x \mp \gamma_y), \qquad \Gamma_\pm = \pm\gamma_{xy} + \gamma_{yx} \tag{2.15}$$

This implies the existence of two ($\sigma_2^z = \pm 1$) sub-dynamics related to a fictitious spin-1/2 immersed in different magnetic fields, each one possessing three components with the z one only depending on time.

2.2.2 Time evolution operator

If ω_1 and ω_2 were time independent, it would be straightforward to find the eigenstates of $\tilde{H}$ as

$$|\tilde{\psi}\rangle = |\phi_{1i}\rangle_{\sigma_2^z} \otimes |\sigma_2^z\rangle \tag{2.16}$$

($i = 1, 2$) where $|\phi_{1i}\rangle_{\pm 1}$ are the two eigenvectors of $\tilde{H}_\pm$, that is the two eigenvectors related to the sub-dynamics with $\sigma_2^z = \pm 1$. Through the relation

$$|\psi\rangle = T|\tilde{\psi}\rangle, \tag{2.17}$$

we could in turn find the eigenvectors of H and the time evolution of an arbitrary state of the two spins.

When ω_1 and ω_2 depend on time, thanks to the fact that the unitary and hermitian operator T is time independent, we succeed, in view of the structure possessed by $\tilde{H}$ as given by Eq. (2.12), in decoupling the time-dependent Schrödinger equation into two time-dependent Schrödinger equations of single spin-1/2. Therefore, we can construct the time-evolution operator of the whole dynamics of the two interacting spin-1/2's, starting from the construction of the two time evolution operators of the two sub-dynamics of single spin-1/2. Indeed, the Cauchy problem for the evolution operator U generated by H: $i\hbar\,\dot{U} = H\,U$ with $U(0) = \mathbb{1}$, is easily converted into the following two Cauchy problems related to the sub-dynamics associated to $\tilde{H}_+$ and $\tilde{H}_-$

$$i\hbar\,\dot{\tilde{U}}_\pm = \tilde{H}_\pm\,\tilde{U}_\pm, \qquad \tilde{U}_\pm(0) = \mathbb{1}, \tag{2.18}$$

where $\tilde{U} \equiv T^\dagger U T \equiv \tilde{U}_+ \otimes |+\rangle\langle+| \;+\; \tilde{U}_- \otimes |-\rangle\langle-|$. Thus, if we are able to solve these two single spin-1/2 time-dependent Schrödinger equations we then can construct

$$U = T\,\tilde{U}\,T^\dagger = T\,\tilde{U}\,T. \tag{2.19}$$

The evolution operators generated by $\tilde{H}_\pm$ can be formally cast as follows

$$\tilde{U}_\pm = e^{\mp i\gamma_{zz}t/\hbar}\begin{pmatrix} a_\pm & b_\pm \\ -b_\pm^* & a_\pm^* \end{pmatrix}, \tag{2.20}$$

with $|a_\pm|^2 + |b_\pm|^2 = 1$, The formal expression of the time evolution operator of the two interacting qubit system then reads

$$U = \begin{pmatrix} a_+e^{-i\gamma_{zz}t/\hbar} & 0 & 0 & b_+e^{-i\gamma_{zz}t/\hbar} \\ 0 & a_-e^{i\gamma_{zz}t/\hbar} & b_-e^{i\gamma_{zz}t/\hbar} & 0 \\ 0 & -b_-^*e^{i\gamma_{zz}t/\hbar} & a_-^*e^{i\gamma_{zz}t/\hbar} & 0 \\ -b_+^*e^{-i\gamma_{zz}t/\hbar} & 0 & 0 & a_+^*e^{-i\gamma_{zz}t/\hbar} \end{pmatrix}, \tag{2.21}$$

It is important to point out that if $\omega_1(t) = \omega_2(t)$

$$\tilde{H}_- = \begin{pmatrix} 0 & \Gamma_- \\ \Gamma_-^* & 0 \end{pmatrix}, \tag{2.22}$$

with $\Gamma_- = (\gamma_{xx} + \gamma_{yy}) - i(\pm\gamma_{xy} + \gamma_{yx})$, and then the related evolution operator reads

$$\tilde{U}_- = e^{i\frac{\gamma_{zz}}{\hbar}t}\begin{pmatrix} \cos(|\Gamma_-|t/\hbar) & e^{i\Phi}\sin(|\Gamma_-|t/\hbar) \\ e^{-i\Phi}\sin(|\Gamma_-|t/\hbar) & \cos(|\Gamma_-|t/\hbar) \end{pmatrix}, \tag{2.23}$$

where $\Phi = \arctan\left(\frac{\gamma_{xx}+\gamma_{yy}}{\gamma_{yx}-\gamma_{xy}}\right)$. In this instance, hence, the whole evolution operator of the initial dynamics becomes

$$U = \begin{pmatrix} a_+e^{-i\gamma_{zz}t/\hbar} & 0 & 0 & b_+e^{-i\gamma_{zz}t/\hbar} \\ 0 & e^{i\frac{\gamma_{zz}}{\hbar}t}\cos\left(\frac{|\Gamma_-|}{\hbar}t\right) & e^{i(\Phi+\frac{\gamma_{zz}}{\hbar}t)}\sin\left(\frac{|\Gamma_-|}{\hbar}t\right) & 0 \\ 0 & e^{-i(\Phi-\frac{\gamma_{zz}}{\hbar}t)}\sin\left(\frac{|\Gamma_-|}{\hbar}t\right) & e^{i\frac{\gamma_{zz}}{\hbar}t}\cos\left(\frac{|\Gamma_-|}{\hbar}t\right) & 0 \\ -b_+^*e^{-i\gamma_{zz}t/\hbar} & 0 & 0 & a_+^*e^{-i\gamma_{zz}t/\hbar} \end{pmatrix}. \tag{2.24}$$

Of course, the evolution operator has the same form as that given by Eq. (2.21), where the two-by-two internal block is now completely determined regardless of the way H depends on time. This means that when $\omega_1(t) = \omega_2(t) = \omega(t)$ in the Hamiltonian model given in Eq. (2.4), the time evolutions of $|+-\rangle$ and $|-+\rangle$ (and so, of every linear combination of these states) are independent of $\omega(t)$ and are characterized by Bohr frequencies related to the coupling constants appearing in H.

It is useful to underline that the condition $\omega_1(t) = \omega_2(t)$ is not implied simply by the condition $\mathbf{B}_1(t) = \mathbf{B}_2(t)$ because in general we may have different $\mathbf{g}$-tensors (or factors) for the two spins which "rule" the coupling with the magnetic field and are responsible for the different effective local magnetic fields in the two sites, even when $\mathbf{B}_1(t) = \mathbf{B}_2(t)$. So, the more general condition implying $\omega_1(t) = \omega_2(t)$ is

$$\mathbf{B}_1(t) \cdot \mathbf{g}_1 = \mathbf{B}_2(t) \cdot \mathbf{g}_2. \tag{2.25}$$

To conclude, we emphasize that our analysis shows that, in this context, the knowledge of exactly solvable dynamical problems of single spin-1/2 plays a crucial role in identifying exactly solvable scenarios for the two-spin-qubit system. If we determine, indeed, the time-dependence of the effective fields in the two subdynamics, $\Omega_\pm(t)$, for which we are able to solve the related two-level dynamical problems, we get easily the time dependence of ω_1 and ω_2 [and that of B_1^z and B_2^z through the relation (2.6)] for which the two-spin-1/2 dynamical problem can be analytically solved, namely

$$\omega_1 = \frac{\Omega_+ + \Omega_-}{2\hbar}, \qquad \omega_2 = \frac{\Omega_+ - \Omega_-}{2\hbar}. \tag{2.26}$$

2.3 Landau-Majorana-Stückelberg-Zener scenario

The Landau-Majorana-Stückelberg-Zener (LMSZ) scenario [33] and the Rabi one [31] represent two milestones among exactly solvable time-dependent semi-classical models for two-level systems. A common fundamental property of these two models is the possibility of realizing a full population inversion in a two-state quantum system. In the former case through an adiabatic passage via a level crossing, in the second case thanks to the application of a resonant π-pulse.

It is important to underline that the LMSZ scenario, differently from the Rabi case, is an ideal model. The word "ideal" refers to the fact that it consists in a process characterized by an infinite time duration resulting, then, practically unrealisable. This fact leads, indeed, to not physical properties such as, for example, the fact that the energies of the adiabatic states diverge at initial ($-\infty$) and final instant ($+\infty$). As a consequence, both mathematical and physical problems arise when amplitudes and not only probabilities are necessary, e.g. when initial states present coherences [68, 67]. In such cases one can alternatively use either the exact solutions of the finite LMSZ scenario [69] or the Allen-Eberly-Hioe model [70], the Demkov-Kunike model [71] or other models [72, 73], where no divergency problems arise and the transition probability is rather simple.

However, despite this circumstance, it is a matter of fact that the LMSZ grasps peculiar dynamical aspects of a lot of physical systems [74]. This relevant aspect has increased the popularity of the LMSZ model and several efforts have been done towards its generalization to the case of N-level quantum systems [67, 75, 76] and total crossing of bare energies [77]. Moreover, its experimental feasibility gave it a basic role in the area of quantum technology thanks also to the several sophisticated techniques developed for a precise local manipulation of the state and the dynamics of a single qubit in a chain [78, 79, 80, 81, 82, 83].

In such an applicative scenario, as we know, several sources of incoherences can be present [21, 48, 49, 50]: incoherent (mixed) states, relaxation processes (e.g., spontaneous emission) or interaction with a surrounding environment (e.g., nuclear spin bath). They generate incoherent excitation leading to departure from a perfect (ideal) population transfer. Therefore, more realistic descriptions of quantum systems subjected to LMSZ scenario comprising such effects have been proposed [84, 85, 86, 87, 88, 89].

In this respect, the most relevant influence in the dynamics of a spin-qubit primarily stems from the coupling with its nearest neighbours. Recently the attention has been focused on double interacting spin-qubit systems subjected to LMSZ scenario [90, 91, 92, 93, 94, 95]. These papers investigate the coupling effects in the two-spin system dynamics in view of possible experimental techniques and protocols. Moreover, such systems, under specific conditions, behave effectively as a two-level system with relevant applicability in quantum information and computation sciences [96]. In the references cited before, indeed, generation of entangled states [90] or the singlet-triplet transition [79, 91, 92] in the two-qubit system under the LMSZ scenario have been studied.

With the same objective in mind, that is to characterize physical effects stemming from the coupling between two spin-qubits subjected to a LMSZ scenario, in this section we study a special case of the two-spin-1/2 C_2-symmetry Hamiltonian model in Eq. (2.4). We consider only the 'diagonal' coupling terms consisting in isotropic or anisotropic exchange interaction. The two spin-1/2's are moreover subjected only to a LMSZ ramp with no transverse static field. We show that LMSZ transitions for the two spin-qubits are still possible thanks to the presence of the coupling, playing the role of an effective transverse field. Such an effect, we call *coupling-assisted LMSZ transition*, deserves particular attention for two reasons. First, it can be exploited to estimate the presence and the relative weight of different coupling parameters determining the symmetry of the Hamiltonian and then the dynamics of the two spins. Second, through such an estimation, it is possible to set the slope of the field ramp in such a way to generate asymptotic maximally entangled states of the two qubits.

We investigate the case in which a LMSZ ramp is applied on either just one or both the spins. Our following theoretical analysis is based on the possibility of experimentally addressing at will the spin systems exploiting, for example, the Scanning Tunneling Microscopy (STM). It appears hence appropriate to furnish a sketch of such a technique. STM proved to be an excellent experimental technique in controlling the dynamics of spin-qudit systems for two main reasons: 1) the possibility of building atom by atom atomic-scale structures [97], such as spin chains and nano-magnets [98]; 2) the possibility of controlling the whole system by addressing a single element (qudit) while it interacts with the others [98, 99, 100], succeeding in realizing, for example, logic operations [97]. The manipulation of a single qudit dynamics is performed through the exchange interaction between the atom on the tip of the scanning tunneling microscope and the target atom in the chain. It is possible to show that such an interaction is equivalent to a magnetic field applied on the atom we want to manipulate [82, 98]. In this way, it is easy to guess that a time-dependent distance between the tip and the target atom generates a time-dependent exchange coupling, giving rise, in turn, to a time-dependent effective magnetic field on the atom of the chain, as analysed in Ref. [82]. Basing on such an observation, in Ref. [83] the authors study the spin dynamics and entanglement generation in a spin chain of Co atoms on a surface of Cu_3N/Cu(110). Precisely, they consider a LMSZ ramp along the z direction produced in a time window of $20ps$ and a short Gaussian pulse in the x direction (half-width: $10ps$).

2.3.1 Coupling-based collective LMSZ transitions

At the light of the STM experimental scenario, we take into account firstly the case of a LMSZ ramp applied on the first spin such that

$$\hbar\omega_1(t) = \alpha t/2, \quad \hbar\omega_2(t) = 0, \qquad t \in (-\infty, \infty), \tag{2.27}$$

where α is related to the velocity of variation of the field, $\dot{B}_z \propto \alpha$, and it is considered a positive real number without loss of generality. Moreover, in this section devoted to the study of the LMSZ scenario, we put for convenience (but without loss of generality) the two parameters related to the DM interaction equal to zero, namely $\gamma_{xy} = \gamma_{yx} = 0$. Let us consider the two spins initialized in the state $|--\rangle$. In this instance, the subdynamics governed by H_+ [see Eq. (2.13)] is characterized by a LMSZ scenario where the longitudinal (z) magnetic field produces the standard LMSZ ramp $\hbar\Omega_+(t) = \hbar\omega_1(t) = \alpha t/2$ and the transverse effective magnetic field along the x-direction is given by γ_+. It is well-known that the dynamical problem for such a time-dependent scenario can be analytically solved. The transition probability of finding the two-spin system in the state $|++\rangle$ coincides with the probability of finding the fictitious spin-1/2 subjected to H_+ in its state $|+\rangle$ starting from $|-\rangle$ and reads [33]

$$P_+ = |\langle ++|U_+(\infty)|--\rangle|^2 = 1 - \exp\{-2\pi\gamma_+^2/\hbar\alpha\}. \tag{2.28}$$

If we now, instead, consider the two spins initially prepared in $|-+\rangle$, the probability of finding the two-spin system in the state $|+-\rangle$, results

$$P_- = |\langle +-|U_-(\infty)|-+\rangle|^2 = 1 - \exp\{-2\pi\gamma_-^2/\hbar\alpha\}. \tag{2.29}$$

This time the transition probability is governed by the fictitious magnetic field given by γ_-. The effective longitudinal magnetic field, instead, is the same, namely $\hbar\Omega_-(t) = \hbar\omega_1(t) = \alpha t/2$. We see that in both cases, although a constant transverse magnetic field is absent, LMSZ transitions of the two-spin system are possible thanks to the presence of the coupling between them. It is important to stress that, for the cases considered before, if $\gamma_x = \gamma_y$ (isotropic exchange interaction case) we cannot have transition in the first case, that is in the subdynamics involving $|++\rangle$ and $|--\rangle$. In this instance, indeed, P_+ happens to be 0 at any time.

Isotropy effects: local LMSZ transition by nonlocal control and state transfer

The symmetry-based dynamical decomposition and the application of the STM LMSZ scenario in each subdynamics allow us to bring to light peculiar evolutions of physical interest. For example, if we consider $\gamma_x \neq \gamma_y$ and the following initial condition

$$|+\rangle \otimes \frac{|+\rangle + |-\rangle}{\sqrt{2}}, \tag{2.30}$$

the two states $|++\rangle$ and $|+-\rangle$ evolve independently and applying the LMSZ ramp we have the probability $P = P_+P_-$ to find asimptotically the two-spin system in the state

$$|-\rangle \otimes \frac{|+\rangle + |-\rangle}{\sqrt{2}}. \tag{2.31}$$

We see, that such a dynamics leaves unaffected the second spin, while it produces a LMSZ transition only on the first spin. It is also interesting to put in evidence the dynamical evolution of the symmetric initial condition

$$\frac{|+\rangle + |-\rangle}{\sqrt{2}} \otimes |+\rangle. \tag{2.32}$$

This time, we get the same probability $P = P_+ P_-$ of finding asymptotically the two-spin system in

$$\frac{|+\rangle + |-\rangle}{\sqrt{2}} \otimes |-\rangle. \tag{2.33}$$

This case results less intuitive even though we are reproducing the same dynamics but with interchanged roles of the two spins. In this instance, in fact, we generate a LMSZ transition only on the second spin by locally applying the field on the first one. This shows that the coupling between the two spins plays a key role to achieve a non-local control of the second spin by locally manipulating the first ancilla qubit.

If we consider, instead, $\gamma_x = \gamma_y = \gamma/2$ we know that the transition $|--\rangle \leftrightarrow |++\rangle$ is suppressed. This means that if we consider as initial conditions the states in Eqs. (2.30) and (2.32), we get asymptotically, this time, the states

$$\frac{|+\rangle + |-\rangle}{\sqrt{2}} \otimes |+\rangle, \tag{2.34a}$$

$$|+\rangle \otimes \frac{|+\rangle + |-\rangle}{\sqrt{2}}, \tag{2.34b}$$

respectively, with probability $P = 1 - \exp\{-2\pi\gamma^2/\hbar\alpha\}$. We see that the isotropy properties of the exchange interaction consistently change the dynamics of the system. When the exchange interaction is isotropic, indeed, the asymptotic states reached by the initial conditions (2.30) and (2.32) radically change. In these cases, the resulting physical effect is a state transfer or a state exchange between the two spin-qubits. Therefore, the different state transitions from the state (2.30) [(2.32)] to the state (2.31) or (2.34a) [(2.33) or (2.34b)] (different responses of the system under LMSZ ramp) can reveal the level of isotropy of the exchange interaction.

Coupling parameter estimation

It is interesting noticing that the coupling-based LMSZ transition could be used to estimate the coupling parameters. By measuring P_+ and P_- (Eqs. (2.28) and (2.29), respectively) we get an estimation of γ_+ and γ_- and then of the two coupling parameters γ_x and γ_y. Supposing to know P_+ and P_-, we have indeed

$$\begin{aligned} \gamma_x &= \frac{1}{2}\sqrt{\frac{\hbar\alpha}{2\pi}}\left[\sqrt{\log\left(\frac{1}{1-P_-}\right)} + \sqrt{\log\left(\frac{1}{1-P_+}\right)}\right], \\ \gamma_y &= \frac{1}{2}\sqrt{\frac{\hbar\alpha}{2\pi}}\left[\sqrt{\log\left(\frac{1}{1-P_-}\right)} - \sqrt{\log\left(\frac{1}{1-P_+}\right)}\right]. \end{aligned} \tag{2.35}$$

We wish to emphasize that we may estimate the coupling parameters also through the Rabi oscillations occurring in the two subspaces. Applying, indeed, a constant field ω_1 on the first spin, the two probabilities P_+ and P_- become

$$
\begin{aligned}
P_+ &= \frac{\gamma_+^2}{\hbar^2\omega_1^2+\gamma_+^2}\sin^2\left(\sqrt{\omega_1^2+\gamma_+^2/\hbar^2}\;t\right),\\
P_- &= \frac{\gamma_-^2}{\hbar^2\omega_1^2+\gamma_-^2}\sin^2\left(\sqrt{\omega_1^2+\gamma_-^2/\hbar^2}\;t\right).
\end{aligned}
\tag{2.36}
$$

So, by measuring the frequency and the amplitude of the oscillations in the two cases we may get information about the the relative weights of the coupling parameters.

2.3.2 Entanglement generation through *non*-adiabatic process

A precise estimation of the coupling parameters is useful also to generate entangled states of the two spins. By the knowledge of them, indeed, we may set the parameter α in order to get asymptotically $P_\pm = 1/2$, generating so an entangled state. If the two spins start from state $|--\rangle$ or $|-+\rangle$, they reach asymptotically the pure state $(|++\rangle + e^{i\phi}|--\rangle)/\sqrt{2}$ in the first case and $(|+-\rangle + e^{i\phi}|-+\rangle)/\sqrt{2}$ in the second case, which are maximally entangled states. The asymptotic curves of the concurrence (the entanglemnt measure for two spin-1/2's introduced in Ref. [101]), in fact, when the two-spin system is initialized in $|--\rangle$ or $|-+\rangle$, read respectively

$$C = 2|c_{++}c_{--}| = 2\sqrt{P_+(1-P_+)} = 2\sqrt{(1-e^{-2\pi\beta_+})e^{-2\pi\beta_+}} \tag{2.37a}$$

$$C = 2|c_{+-}c_{-+}| = 2\sqrt{P_-(1-P_-)} = 2\sqrt{(1-e^{-2\pi\beta_-})e^{-2\pi\beta_-}} \tag{2.37b}$$

and they exhibit a maximum for $\beta_+ = \beta_- = \log(2)/2\pi \approx 0.11$. In the previous expressions we put $\beta_+ = \gamma_+^2/\hbar\alpha$ and $\beta_- = \gamma_-^2/\hbar\alpha$, while c_{++} and c_{--} (c_{+-} and c_{-+}) are the asymptotic amplitudes of the states $|++\rangle$ and $|--\rangle$ ($|+-\rangle$ and $|-+\rangle$), respectively. Therefore, $\log(2)/2\pi$ is exactly the value the LMSZ parameters β_+ ad β_- must have to realize the generation of the entangle states $(|++\rangle + e^{i\phi}|--\rangle)/\sqrt{2}$ and $(|+-\rangle + e^{i\phi}|-+\rangle)/\sqrt{2}$ when the two spins start from $|--\rangle$ or $|-+\rangle$, respectively. Figure 2.1a reports the two curves for $\beta_-/2 = \beta_+ = \beta$.

We may verify this fact by investigating the behaviour of the concurrence in time. To this end, the exact solutions of the two time-dependent parameters determining the two time evolution operators U_+ and U_- in Eq. (2.20), related to each subdynamics, are necessary and they reads, namely [69]

$$
\begin{aligned}
a_\pm =& \frac{\Gamma_f(1-i\beta_\pm)}{\sqrt{2\pi}}\times\\
&[D_{i\beta_\pm}(\sqrt{2}e^{-i\pi/4}\tau)\,D_{-1+i\beta_\pm}(\sqrt{2}e^{i3\pi/4}\tau_i) + D_{i\beta_\pm}(\sqrt{2}e^{i3\pi/4}\tau)\,D_{-1+i\beta_\pm}(\sqrt{2}e^{-i\pi/4}\tau_i)],\\
b_\pm =& \frac{\Gamma_f(1-i\beta_\pm)}{\sqrt{2\pi\beta}}e^{i\pi/4}\times\\
&[-D_{i\beta_\pm}(\sqrt{2}e^{-i\pi/4}\tau)\,D_{-1+i\beta_\pm}(\sqrt{2}e^{i3\pi/4}\tau_i) + D_{i\beta_\pm}(\sqrt{2}e^{i3\pi/4}\tau)\,D_{-1+i\beta_\pm}(\sqrt{2}e^{-i\pi/4}\tau_i)].
\end{aligned}
\tag{2.38}
$$

Γ_f is the Gamma function, while $D_\nu(z)$ are the parabolic cylinder functions [102] and $\tau = \sqrt{\alpha/\hbar}\;t$ is a time dimensionless parameter; τ_i identify the initial time instant. If the system starts, e.g., from the

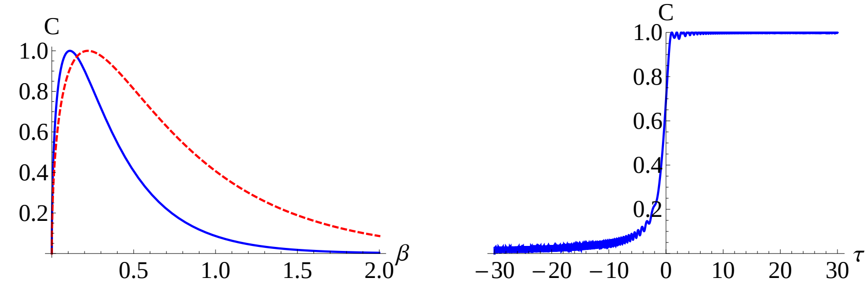


Figure 2.1: (Color online) a) The two curves of the concurrence in Eq. (2.37a) (full blue line) and Eq. (2.37b) (red dashed line) for $\beta_-/2 = \beta_+ = \beta$; b) Time behaviour of concurrence for the initial condition $|--\rangle$ and $\beta_+ = 0.1$ plotted against the dimensionless time $\tau = \sqrt{\alpha/\hbar}\, t$.

state $|--\rangle$ the amplitudes result

$$c_{++} = b_+, \quad c_{--} = a_+^*, \quad c_{+-} = c_{-+} = 0, \tag{2.39}$$

and the related time-behaviour of the concurrence $C = 2|b_+||a_+|$ for $\beta_+ = 0.1$ is reported in Fig. 2.1b. We see, as expected, that such a choice of the LMSZ parameter generate a maximally entangled state of the two spin-qubits. It is important to point out that, on the basis of Eqs. (2.38), the parameter β_+ determines not only the asymptotic value of the concurrence but also its time behaviour. This fact is confirmed and can be appreciated by Figs. 2.2a and 2.2b reporting the concurrence against the dimensionless parameter τ for $\beta_+ = 1/2$ and $\beta_+ = 2$, respectively. The physical meaning of the asymptotic vanishing of C in Fig. 2.2b is that for the specific value of β_+ the system evolves quite adiabatically towards the factorized states $|++\rangle$. On the contrary, in Figs. 2.2a the slope of the ramp induces a non adiabatic evolution towards a coherent not factorizable superposition of $|++\rangle$ and $|--\rangle$. We emphasize, then, that the generation of a maximally entangled coherent superposition state of the two spin-qubits requires a non-adiabatic process. This circumstance turns out to be reasonable at the light of the fact that, to this end, we have to generate asymptotically a LMSZ probability equal to 1/2; an adiabatic LMSZ dynamics, instead, would produce a transition probability reaching asymptotically the maximum value, 1.

We would get analogous results by studying the LMSZ process when the two spin-qubits start from the state $|-+\rangle$. In this case, only the states $|-+\rangle$ and $|+-\rangle$ would be involved and the LMSZ parameter determining the different concurrence regimes would be β_-. For such initial conditions, then, the ratio β_+/β_-, imposing precise relationships between the coupling parameters γ_x and γ_y, does not matter.

Such a ratio, conversely, results determinant for other initial conditions, e.g. the one considered in Eq. (2.31). In this case the amplitudes read

$$c_{++} = a_+, \quad c_{--} = -b_+^*, \quad c_{+-} = a_-, \quad c_{-+} = -b_-^*. \tag{2.40}$$

In Figs. 2.3a-2.3f we may appreciate the influence of both the ratio β_-/β_+ and the free parameter β_+; the former influences only qualitatively the behaviour of the concurrence, while the latter both qualitatively and quantitatively. This time the concurrence vanishes for high values of β_+ too, witnessing an

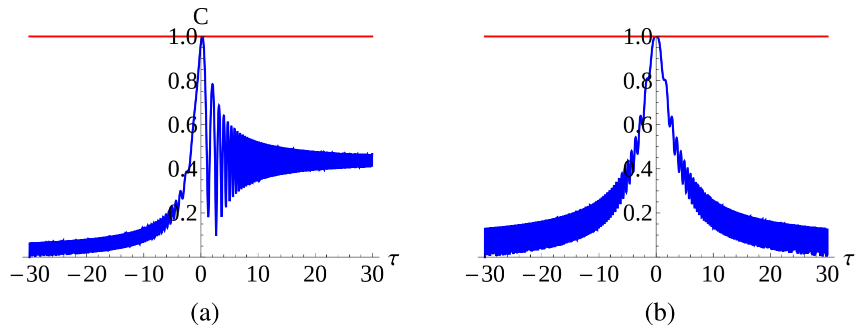


Figure 2.2: (Color online) Time behaviour of the concurrence against the dimensionless parameter $\tau = \sqrt{\alpha/\hbar}\, t$ during a LMSZ process when the system starts from the state $|--\rangle$ for a) $\beta_+ = 1/2$ and b) $\beta_+ = 2$. The upper straight curve corresponds to $C(\tau) = 1$.

asymptotic factorized state. For small values of β_+, instead, positive values of entanglement even for large times indicate a superposition of the four standard basis states.

To conclude, we note that the parameters of the applied magnetic field, including the magnetic field gradient, can be controlled in very wide ranges. For example, the magnetic field gradient can reach values as large as 150-200 T/m in a microfabricated ion trap [103], which is far beyond what is needed here. The most important parameter for the feasibility of our scheme is the spin-spin coupling constant γ. In nuclear magnetic resonance, its values typically vary from 10 Hz to 300 Hz depending on the molecule [104], which implies that entanglement can be created on the millisecond scale. A very interesting physical platform, which allows the tuning of the spin-spin coupling in a broad range, is provided by microwave-driven trapped ions in the presence of a static magnetic-field gradient [103, 105]. The effective spin-spin coupling is proportional to the magnetic-field gradient and can reach the kHz range. A third example is provided by Rydberg atoms and ions where, due to the huge electric-dipole moments of the Rydberg states, the effective spin-spin coupling can reach a few MHz [106, 107, 108]. This implies entanglement creation on the sub-microsecond scale.

2.3.3 Effects of classical noise

In experimental physical contexts involving atoms, ions and molecules investigated and manipulated by application of lasers and fields, the presence of noise in the system stemming from the coupling with a surrounding environment is unavoidable. Although a lot of technological progresses and experimental expedients have been developed, it is necessary to introduce such decoherence effects in the theoretical models for a better understanding and closer description of the experimental scenarios. There exist different approaches to treat the influence of a thermal bath; one is to consider the presence of classical noisy fields [86, 109] stemming, e.g., from the presence and the influence of a surrounding nuclear spin bath [86].

In the last reference the authors study a noisy LMSZ scenario for a N-level system. They take into account a time-dependent magnetic field $\eta(t)$ only in the z-direction and characterized by the following time correlation function $\langle \eta(t)\eta(t')\rangle = 2G\delta(t-t')$. The authors show that the LMSZ transition probability P_-^+ for a spin-1/2 to be found in the state $|+\rangle$ starting from $|-\rangle$, in case of large values of

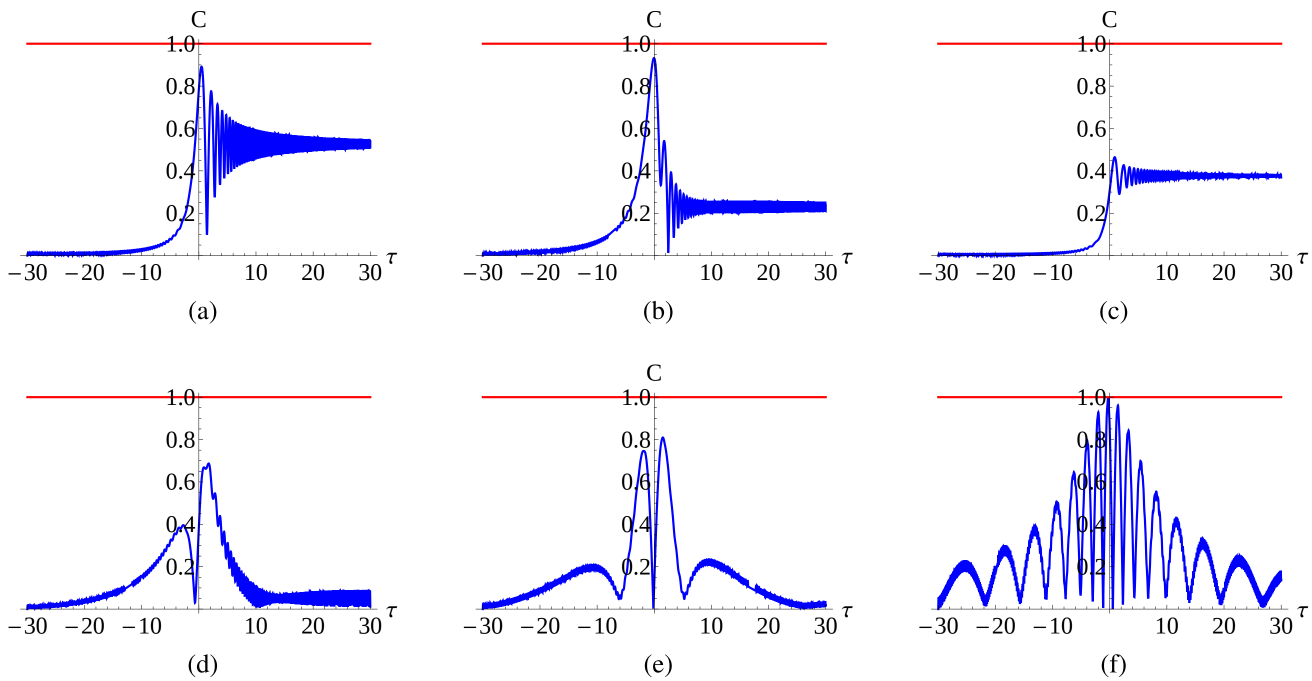


Figure 2.3: (Color online) Time behaviour of the concurrence against the dimensionless parameter $\tau = \sqrt{\alpha/\hbar}\, t$ during a LMSZ process when the system starts from the state $(|{+}{+}\rangle + |{+}{-}\rangle)/\sqrt{2}$ for $\beta_-/\beta_+ = 1/2$ and a) $\beta_+ = 1/2$, d) $\beta_+ = 2$; $\beta_-/\beta_+ = 2$ and b) $\beta_+ = 0.5$, e) $\beta_+ = 2$; $\beta_-/\beta_+ = 2$ and c) $\beta_+ = 0.1$, f) $\beta_+ = 10$. The upper straight curve corresponds to $C(\tau) = 1$.

G, changes as

$$P_{-}^{+} = \frac{1 - \exp\{-2\pi g^2/\hbar\alpha\}}{2}, \tag{2.41}$$

where g is the energy contribution due to the coupling of the spin-1/2 with the constant transverse magnetic field and α is the slope of the longitudinal magnetic field. We see that the value of G, provided that it is large, does not influence the transition probability. The unique effect of the noisy component is the loss of coherence. The field indeed cannot generate transitions between the two diabatic states, being only in the same direction of the quantization axis. In this way the transition probability, as reasonable, results hindered by the presence of the noise, since, for $g^2/\alpha \gg 1$, the system reaches at most the maximally mixed state.

This result is of particular interest in our case since the addition of the noisy component $\eta(t)$ leaves completely unaffected the symmetry-based Hamiltonian transformation and the validity of the dynamics-decoupling procedure. Thus, also in this case, the dynamical problem of the two-qubit system may be converted into two independent spin-1/2 problems affected by a random fluctuating z-field. Thus, we may write easily the transition probabilities when the two spins are subjected to a unique homogeneous field influenced by the noisy component considered before. We have precisely

$$P_{+} = \frac{1 - \exp\{-2\pi \gamma_{+}^2/\hbar\alpha\}}{2}, \quad \omega_1(t) = \omega_2(t) = [\alpha t + \eta(t)]/4, \tag{2.42}$$

We underline that the transition probability P_{-} vanishes in case of an unique homogeneous magnetic field. In the related subdynamics, indeed, the effective field ruling the two-spin dynamics is zero, namely $\Omega_{-}(t) = 0$. Moreover, for $\gamma_x = \gamma_y$ we would have no physical effects, since, in such a case, also P_{+} would result zero.

Another way to face with the problem of open quantum systems is to use non-Hermitian Hamiltonians effectively incorporating the information of the fact that the system they describe is interacting with a surrounding environment [24, 110, 111, 112, 113]. We may suppose, for example, that the spontaneous emission from the up-state to the down-one is negligible and that some mechanism makes the up-state $|+\rangle$ irreversibly decaying out of the system with rate ξ and ξ' for the first and second spin-1/2, respectively. It is well known that we can phenomenologically describe such a scenario by introducing the non-Hermitian terms $i\xi\hat{\sigma}_1^z/2$ and $i\xi'\hat{\sigma}_2^z/2$ in our Hamiltonian model. Analogously to the case of a noisy field component, also the introduction of these terms does not alter the symmetry of the Hamiltonian model. The symmetry-based transformation leads us to two independent non-Hermitian two-level models. In the same way we may exploit the results got for a single qubit with a decaying state subjected to the LMSZ scenario [84, 85, 87] and reread them in terms of the two-spin-1/2 language. We know that the decaying rate affects only the time-history of the transition probability but not, surprisingly, its asymptotic value [84]. However, this result is valid for the ideal LMSZ scenario; considering the more realistic case of a limited time window, it has been demonstrated, indeed, that a decaying rate-dependence for the population of the up-state arises [85].

2.4 X-States and quantum correlations

Quantum correlations became in the last two decades a field of large interest, due to their crucial role played in the quantum information science [9, 114]. Much effort and work are presently devoted to characterize and quantify the quantum correlations, like entanglement, steering and discord existing in multipartite quantum states [115]. These quantum correlations are considered useful physical quantum resources with promising applications in quantum information processing and transmission tasks and protocols. It is well known that quantum entanglement does not describe all the properties of non-classical nature of the quantum correlations. In this respect, quantum discord has been proposed as a measure of quantum correlations, beyond entanglement [116, 117], which can also exist in separable mixed states. The physical understanding of quantum discord constantly advanced and several operational interpretations to discord have been proposed [118].

It is nowadays possible to realize physical scenarios where quantum coherence turns out to be robustly protected against detrimental classical and quantum uncontrollable sources. This circumstance has spurred a growing interest toward the quantum dynamics of closed bipartite physical systems subjected to controllable time dependent external classical fields. When the corresponding Hamiltonian model is both non trivial and exactly solvable, one might indeed undertake a systematic, hopefully exact, study of the unitary time evolution of correlations get established in the closed system, not traceable back to classical physics.To follow and to interpret, for example, the appearance, time variation and death at finite time instants of entanglement and quantum discord is of relevance from both a theoretical and applicative point of view. On the one hand, such a knowledge may significantly contribute to highlight the meaning of crucial concepts like non locality and dechoerence and to capture their connection with properties experimentally exhibited by the system. On the other hand, due to such an interpretative potentiality, it provides the key to clarify the role of quantum correlations as resources for quantum technologies.

Quantum discord is defined in Ref. [116] as the difference between the quantum generalizations of two classical expressions of the mutual information. One knows that any entanglement measure vanishes for a separable state. This is not the case with quantum discord, which may have non-zero values for separable states. In Ref. [119] it was shown that states with large amount of discord, and at the same time separable, are useful in precesses in quantum technology. Therefore, the study of such states, i.e. separable ones, characterized by non-zero discord, is of great interest in quantum information theory. In general the analytical formula of quantum discord is difficult to be obtained, since it requires an extremization procedure. The reason making computing quantum discord so difficult stems from the fact that the time required for such a target becomes exponentially larger and larger as a function of the dimension of the Hilbert space of the bipartite system under scrutiny [120]. However, in the case of continuous variable systems, for example Gaussian states, an explicit formula of quantum discord was found, if one restricts the set of all quantum measurements to Gaussian ones [121]. On the other hand, for discrete quantum systems such as for two qubits, the characterization of quantum discord is more difficult to be made in the general case. For the particular situation of the so-called class of two-qubit X-states, the quantum discord was evaluated firstly numerically [122], and secondly analytically [123, 124].

2.4.1 Concurrence and quantum discord for the two-spin model

It is possible to convince oneself that an initial state characterized by an X-structure $\rho(0) = \rho_X$ [see Appendix C], under the action of the Hamiltonian (2.4), evolves keeping the X-structure at any time instant. Thus, preparing our two-spin system in a general X-state, at any subsequent time instant $\rho(t) = U(t)\rho(0)U^\dagger(t)$ is still an X-state, $U(t)$ being the operator defined in Eq. (2.21). Such a property exhibited by the X-states $\rho_X(t)$ is due to the special structure of the time evolution operator which, in turn, is determined by the symmetry properties of the Hamiltonian. The C_2-symmetry with respect to the z-direction, possessed by the Hamiltonian, indeed, causes the existence of two dynamically invariant Hilbert subspaces related to the two eigenvalues of the constant of motion $\hat{\sigma}_1^z\hat{\sigma}_2^z$. Thus, every state which does not mix the two subspaces at the initial time instant, like the X-states, will keep this property at any following time instant. This fact has relevant implications from a physical point of view: since we know how to calculate analytically the quantum discord for a generic X-state [see Appendix C], we are able then to calculate exactly its general time-dependent expression for our model. Such a general analytical expression will depend on the four parameters $a_\pm(t)$ and $b_\pm(t)$. In case of exactly scenarios, it means that we have the analytical form of these two parameters, so that we would get an explicit time-dependent expression of the quantum discord.

The explicit expressions of $a_\pm(t)$ and $b_\pm(t)$ depend on the specific time-dependences of the Hamiltonian parameters. Although, as shown before, the dynamical problem of the two spins may be converted into two independent problems of single spin-1/2, we know that we are not able to find the analytical solution of the time-dependent Schrödinger equation for a spin-1/2 subjected to a generic time-dependent Hamiltonian (that is for generic time-dependences of the Hamiltonian parameters). Therefore, the knowledge of specific exactly solvable time-dependent scenarios for a single spin-1/2 becomes crucial. In appendix B, on the basis of the single qubit exactly solvable scenarios (1.29) and (1.33), we report two exactly solvable time-dependent scenarios for the two spin-1/2's with the related analytical expressions of the parameters $a_\pm(t)$ and $b_\pm(t)$. Thus, for such exactly solvable time-dependent models of the two-spin system we are able to calculate the explicit form of a generic state and, in particular for an X-state, the explicit time evolution of the quantum discord.

In the folwing we analyse the time-dependendence both of the concurrence and the quantum discord related to the specific class of initial X-states known as Werner states [125], namely:

$$\rho_W^{(\alpha)} = \frac{1-\alpha}{4} I \otimes I + \alpha\, |\Psi^-\rangle\langle\Psi^-|, \tag{2.43}$$

where $|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$ is the singlet state and $\alpha \in [-\frac{1}{3}, 1]$ is a real parameter. In order to investigate the dynamics of the correlations, we use as a measure of entanglement of the two qubits the concurrence, which was introduced by Wootters [101]. The analytical calculation of the quantum discord is performed accordingly to the procedure exposed in Appendix C.

If ρ is the density operator of a two-qubit system, then its spin-flipped state is given by $\rho' = (\sigma_2 \otimes \sigma_2)\,\rho^*\,(\sigma_2 \otimes \sigma_2)$, where ρ^* is the complex conjugate of ρ. Let us denote the eigenvalues of the non-Hermitian matrix $\rho\,\rho'$ by ν_1, ν_2, ν_3, and ν_4, in decreasing order. The concurrence is defined by

$$C(\rho) = \max\{\sqrt{\nu_1} - \sqrt{\nu_2} - \sqrt{\nu_3} - \sqrt{\nu_4}, 0\}. \tag{2.44}$$

The expression of the concurrence of the Werner state (2.43) is [123]:

$$C(\rho_W) = \max\left\{\frac{3\,\alpha - 1}{2}, 0\right\}. \tag{2.45}$$

For $\alpha \in (\frac{1}{3}, 1]$ the concurrence is greater than zero, which means that the Werner state (2.43) is inseparable. The expression of the concurrence of an arbitrary X state was found in Ref. [126]:

$$C(\rho_{\mathrm{X}}) = 2\,\max\left\{0, |\rho_{23}| - \sqrt{\rho_{11}\,\rho_{44}}, |\rho_{14}| - \sqrt{\rho_{22}\,\rho_{33}}\right\}. \tag{2.46}$$

Let us define now the state $|\xi\rangle$ of two qubits as follows:

$$|\xi\rangle = \mu|+-\rangle + \nu|-+\rangle, \tag{2.47}$$

with μ and ν complex parameters satisfying $|\mu|^2+|\nu|^2=1$. We construct a special class of two-qubit mixed states, which generalizes the Werner state, as follows:

$$\eta_{\mu,\nu}^{(\alpha)} = \frac{1-\alpha}{4} I\otimes I + \alpha|\xi\rangle\langle\xi|, \tag{2.48}$$

where $\alpha \in [-\frac{1}{3},1]$. For $\mu = 1/\sqrt{2}$ and $\nu = -1/\sqrt{2}$, the state $\eta_{\mu,\nu}^{(\alpha)}$ becomes the Werner state (2.43). By using the expression of the concurrence of an X state given by Eq. (2.46), one obtains:

$$C(\eta_{\mu,\nu}^{(\alpha)}) = \max\left\{0, g\left(\alpha,\mu\right)\right\}, \tag{2.49}$$

with

$$g(\alpha,\mu) = 2|\alpha||\mu|\sqrt{1-|\mu|^2} - \frac{1-\alpha}{2}. \tag{2.50}$$

It is possible to verify that for $\alpha \in [-\frac{1}{3},\frac{1}{3}]$, one gets $g(\alpha,\mu) \leq 0$ for any $|\mu| \in [0,1]$, implying a vanishing concurrence $C(\eta_{\mu,\nu}^{(\alpha)}) = 0$. For $\alpha \in \left(\frac{1}{3},1\right]$, instead, the equation $g(\alpha,\mu)=0$ may be cast in he following form:

$$\alpha = \frac{1}{1+4|\mu|\sqrt{1-|\mu|^2}}. \tag{2.51}$$

If we represent Eq. (2.51) in the α-$|\mu|$-plane, the curve $\alpha(|\mu|)$ distinguishes the region wherein the Concurrence vanishes from the one where the Concurrence is positive. In other words, Eq. (2.51) defines in the α-$|\mu|$ plane the border between appearance and disappearance of Entanglement between the two spins within the class of the generalized Werner states $\eta_{\mu,\nu}^{\alpha}$. In particular when $\alpha \leq 1/3$ the Concurrence is zero whatever μ is. When, instead, $\alpha > 1/3$ there always exists an α-dependent interval $[|\mu_1|,|\mu_2|]$ within which the Concurrence is different from zero. In Fig. 2.4 we plot α in terms of $|\mu|$ by using Eq. (2.51) for which the concurrence of the state $\eta_{\mu,\nu}^{(\alpha)}$ is equal to zero.

We obtain the expression of the concurrence of the state $\eta_{\mu,\nu}^{(\alpha)}$:

$$C(\eta_{\mu,\nu}^{(\alpha)}) = \begin{cases} 0 & \text{for} \quad |\mu| \in \left[0, \frac{1}{2} - \frac{\sqrt{3\alpha^2+2\alpha-1}}{4\alpha}\right] \\ 2\alpha\,|\mu|\sqrt{1-|\mu|^2} - \frac{1-\alpha}{2} & \text{for} \quad |\mu| \in \left(\frac{1}{2} - \frac{\sqrt{3\alpha^2+2\alpha-1}}{4\alpha}, \frac{1}{2} + \frac{\sqrt{3\alpha^2+2\alpha-1}}{4\alpha}\right) \\ 0 & \text{for} \quad |\mu| \in \left[\frac{1}{2} + \frac{\sqrt{3\alpha^2+2\alpha-1}}{4\alpha}, 1\right] \end{cases} \tag{2.52}$$

It is easy to persuade oneself that $C(\eta_{\mu,\nu}^{(\alpha)}) = C(\rho_W) = (3\alpha-1)/2$ under the condition $|\mu| = 1/\sqrt{2}$. This implies, in particular, the invariance of the concurrence of ρ_W [Eq. (2.43)] when one turns $|\Psi^-\rangle$ into $|\Psi^+\rangle$.

It is worth noticing, in addition, that

$$\eta_{\mu,\nu}^{(\alpha)}(t) = \frac{1-\alpha}{4} I\otimes I + \alpha|\xi(t)\rangle\langle\xi(t)| \tag{2.53}$$

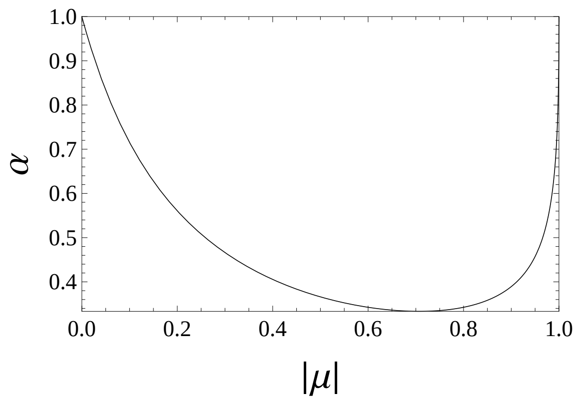


Figure 2.4: The plot of α in terms of $|\mu|$ by using Eq. (2.51) for which the generalized Werner state $\eta_{\mu,\nu}^{(\alpha)}$ is characterized by vanishing concurrence.

where

$$|\xi(t)\rangle = U(t)|\Psi^-\rangle = c_{+-}(t)|01\rangle + c_{-+}(t)|10\rangle, \tag{2.54}$$

with

$$c_{+-}(t) = \frac{1}{\sqrt{2}} \exp\left(i\frac{\gamma_{33}}{\hbar}t\right) (a_- - b_-); \tag{2.55}$$

$$c_{1-+}(t) = -\frac{1}{\sqrt{2}} \exp\left(i\frac{\gamma_{33}}{\hbar}t\right) \left(a_-^* + b_-^*\right). \tag{2.56}$$

This fact means that the generalized Werner states $\eta_{\mu,\nu}^{(\alpha)}$ evolve keeping their α-dependent structure. Hence the time evolution of a generalized Werner state characterized by a particular value of α generates only “horizontal movements” in the α-$|\mu|$ plane in Fig. 2.4. This circumstance implies that, during its time evolution, a generalized Werner state may enter into or go out the non-zero-Concurrence region identified in Fig. 2.4. For example, if we consider as initial condition the entangled generalized Werner state defined by $\alpha = \mu = 0.5$, it may happen that, at a certain time instant, μ becomes less than ≈ 0.25. In this case, then, a sudden death of Entanglement is exhibited. Of course, if $|\mu|$ comes back to its original value in a finite interval of time, a re-birth of Entanglement would follow a plateaux of zero-Concurrence. Such a possibility is confirmed by the plots reported in the following where we compare the Concurrence and the Quantum Discord in time for our two-spin system under the two exactly solvable time-dependent scenarios (1.3.2) and (1.3.2).

When the applied magnetic fields have the expressions in Eq. (1.3.2), we get

$$C(\rho(t)) = \max\left\{0, |\alpha|\sqrt{1-\tanh^2(2\tau_-)\sin^2(2\tau_-)} - \frac{1-\alpha}{2}\right\}. \tag{2.57}$$

The analytical expression of the concurrence when the magnetic fields vary over time as in Eq. (1.3.2), becomes instead

$$C(\rho(t)) = \max\left\{0, |\alpha|\sqrt{1-4\frac{\tanh^2(\tau_-)}{\cosh^2(\tau_-)}\sin^2[\sinh(\tau_-)]} - \frac{1-\alpha}{2}\right\}. \tag{2.58}$$

In both cases we see that for $\alpha \in \left[-\frac{1}{3}, \frac{1}{3}\right]$, the concurrence is equal to zero, while the quantum discord is non-vanishing [see Figs. 2.5 a) and 2.6 a)]. For $\alpha \in \left(\frac{1}{3}, 1\right)$, instead, the phenomenon of sudden death of entanglement followed by revival of entanglement occurs as previously predicted [see Figs. 2.5 b) and 2.6 b)].

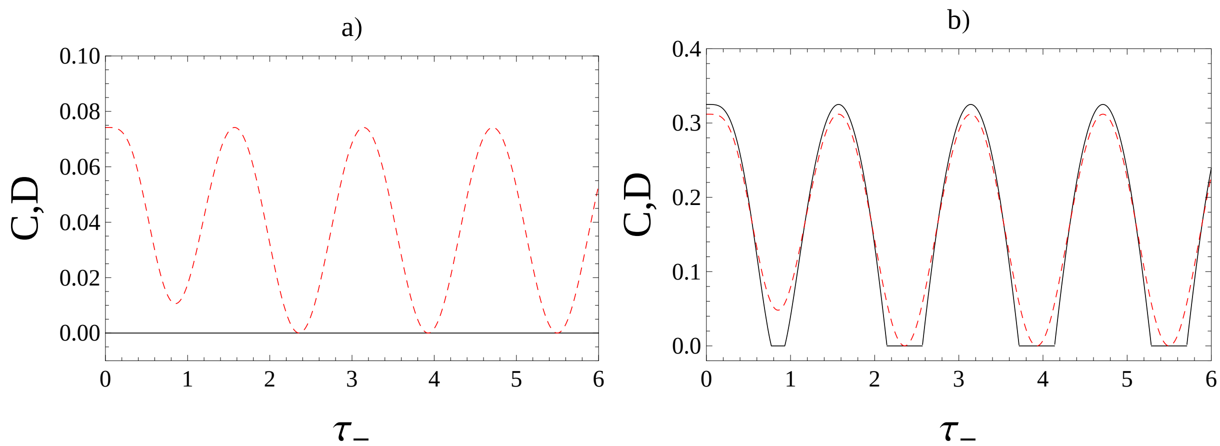


Figure 2.5: The concurrence (black, solid) and quantum discord (red, dashed) when the state at $t = 0$ is the Werner state (2.43) in the time-dependent scenario identified by Eq. (B.1) in terms of $\tau_- = \frac{|\Gamma_-|}{\hbar} t$ for a) $\alpha = 0.25$ and b) $\alpha = 0.55$.

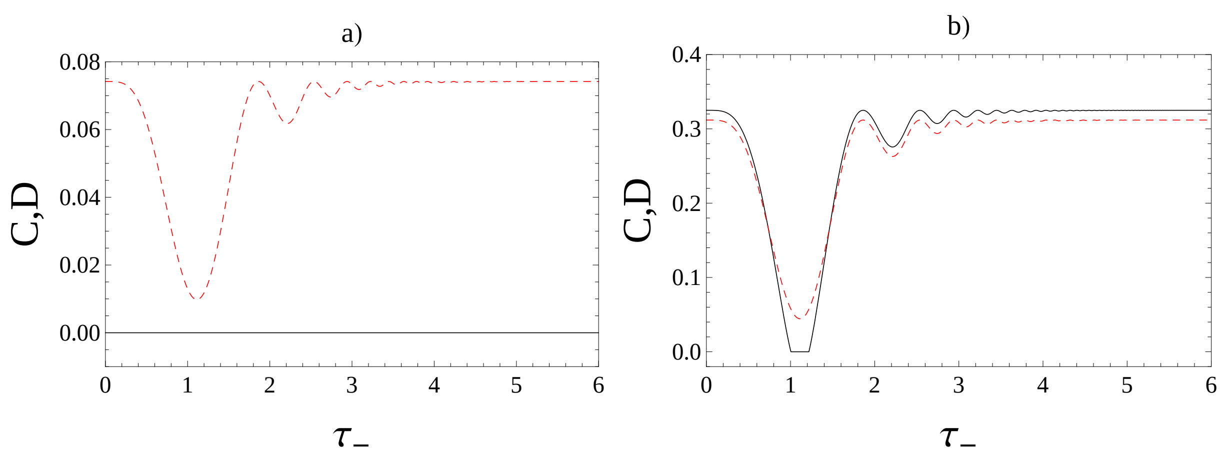


Figure 2.6: The concurrence (black, solid) and quantum discord (red, dashed) when the state at $t = 0$ is the Werner state (2.43) in the time-dependent scenario identified by Eq. (B.3) in terms of $\tau_- = \frac{|\Gamma_-|}{\hbar} t$ for a) $\alpha = 0.25$ and b) $\alpha = 0.55$.

2.5 Summary and remarks

We have considered a physical system of two interacting spin-1/2's whose coupling comprises the terms stemming from the anisotropic exchange interaction and the anisotropic Dzyaloshinskii-Moryia

[63, 64] interaction. Moreover, each of them is subjected to a local time-dependent field. The C_2-symmetry (with respect to the quantization axis $\hat{z}$) possessed by the Hamiltonian allowed us to identify two independent single spin-1/2 sub-problems nested in the quantum dynamics of the two spin-qubits. This fact gave us the possibility of decomposing the dynamical problem of the two spin-1/2's into two independent problems of single spin-1/2. We underline, in addition, that the dynamical reduction exposed in Sec. 2.2.1 is independent of the time-dependence of the fields.

In this way, considering a LMSZ ramp applied on the two spin-qubits, our system can be regarded as a four-level system presenting an avoided crossing for each pair of instantaneous eigenenergies related to the two dynamically invariant subspaces. This aspect turned out to be the key to solve easily and exactly the dynamical problem, bringing to light several physically relevant aspects.

We have shown that, although a transverse chirp [90] or a constant field is absent, LMSZ transitions are still possible, precisely from $|--\rangle$ to $|++\rangle$ and from $|-+\rangle$ to $|+-\rangle$ (the two couples of states spanning the two dynamically invariant Hilbert spaces related to the symmetry Hamiltonian). Such transitions occur thanks to the presence of the coupling between the spins which plays as effective static transverse field in each subdynamics.

It is worth noticing that, in our model, the two LMSZ sub-dynamics are ruled either by different combinations of the externally applied fields (when the local fields are different) or by the same field (under the STM scenario, that is when one local field is applied on just one spin). In the latter case we showed the possibility of 1) a non-local control, that is to manipulate the dynamics of one spin by applying the field on the other one and 2) a state exchange/transfer between the two spins. We brought to light how such effects are two different replies of the system depending on the isotropy properties of the exchange interaction.

Concerning the interaction terms, each subdynamics is characterized by different combinations of the coupling parameters. This aspect has relevant physical consequences since, as showed, by studying the LMSZ transition probability in the two subspaces, it is possible both to evaluate the presence of different interaction terms and to estimate their weights in ruling the dynamics of the two-spin system. We have brought to light how the estimation of the coupling parameters could be of relevant interest since, through this knowledge, we may set the slope of variation of the LMSZ ramp as to generate asymptotically entangled states of the two spin-1/2's. Moreover, we reported the exact time-behaviour of the entanglement for different initial conditions and we analysed how the coupling parameters can determine different entanglement regimes and asymptotic values.

In this respect, we underline that in Ref. [90] the authors considered a system of two spin-1/2's interacting only through the term $\hat{\sigma}_1^z\hat{\sigma}_2^z$ and subjected to the same magnetic field consisting in a Gaussian pulse uniformly rotating in the $x-y$ plane and a LMSZ ramp in the z direction. They showed that the coupling between the two spins enhances significantly the probability to drive adiabatically the two-spin system from the separate state $|--\rangle$ to the entangled state $(|+-\rangle+|-+\rangle)/\sqrt{2}$. In this case the procedure to generate an entangled state is different from the scenario considered here because of the different symmetries of the Hamiltonians ruling the two-spin dynamics. Indeed, in Ref. [90] the Hamiltonian commutes with $\hat{S}^2$ and consequently two dynamically invariant Hilbert subspaces exist: one of dimension three and the other of dimension one. The three-dimensional subspace is spanned by the states $|++\rangle$, $(|+-\rangle+|-+\rangle)\sqrt{2}$ and $|--\rangle$, making possible the preparation of the entangled state of the two spin-1/2's by an adiabatic passage when they start from the separate state $|--\rangle$. In our case, instead, $\hat{S}^2$ is not constant while the integral of motion is $\hat{s}_1^z\hat{s}_2^z$. The symmetries of the Hamiltonian, thus, generate two two-dimensional dynamically invariant Hilbert subspaces: one spanned by

$|++\rangle$ and $|--\rangle$ and the other by $|+-\rangle$ and $|-+\rangle$. Then, in our case, the transition between the states considered in the other work is impossible since such states belong to different invariant subspaces.

We have emphasized how our symmetry-based analysis has proved to be useful also to get exact results when a classical random field component or non-Hermitian terms are considered to take into account the presence of a surrounding environment interacting with the system. In this case, the dynamics decomposition is unaffected by the presence of the noise or the dephasing terms and then we may apply the results previously reported for a two-level system [84, 85, 86] and reread them in terms of the two spin-1/2's.

Finally, we have brought to light the fact that the symmetry properties of the Hamiltonian model guarantee that an X-density matrix evolves keeping such a structure and, on the other hand, the quantum discord of such a state can be analytically determined. This is why we choose an X-state as initial condition and in this class we concentrate on generic α-parametric Werner states. Our analysis predicts in two time-dependent scenarios the presence of sudden death-sudden revival phenomena in the concurrence when a non-vanishing quantum discord is present.

As conclusive remarks, we wish to stress that the results concerning the LMSZ transitions are valid not only within the STM scenario, but they are applicable to other physical platforms. Indeed, the local LMSZ model for a spin-qubit interacting with another neighbouring spin-qubit may be reproduced also in laser-driven cold atoms in optical lattices where highly-selective individual addressing has been experimentally demonstrated [127]. Another prominent example is laser-driven ions in a Paul trap where spatial individual addressing of single ions in an ion chain has been routinely used for many years [128, 129]. Yet another example is microwave-driven trapped ions in a magnetic-field gradient where individual addressing with extremely small cross-talk has been achieved in frequency space [130, 131].

We point out, in addition, that our results concerning the LMSZ dynamics of the two spin-1/2's are deeply different from the ones reported in other Refs. [79, 91, 92] where systems of two spin-1/2's in a LMSZ framework have been investigated on the basis of an approximate treatment. In these papers, indeed, the two spin-qubits are not directly coupled, but they interact through a common nuclear spin bath which they are coupled to. Such a composite system behaves as a two-level system under several assumptions and to derive the effective single spin-1/2 Hamiltonian requires several approximations. In Ref. [92], in particular, the effective Hamiltonian describes the coupling between the two-level system and a longitudinal time-dependent field which is not a pure LMSZ ramp, presenting a complicated functional dependence on the original Hamiltonian parameters. There is, in addition, a time-dependent effective interaction between the two states possessing a complicated functional dependence on the confinement energy as well as the tunneling and Coulomb energies. Although such an effective Hamiltonian goes beyond the standard LMSZ scenario, it may be considered similar to the LMSZ one since both Hamiltonians describe an adiabatic passage through an anticrossing. In our case, instead, the two spin-1/2's are directly coupled, besides being subjected to a random field stemming from the presence of a spin bath. Furthermore, the effective two-state Hamiltonians governing the two-qubit dynamics in the two invariant subspaces are easily got without involving any assumption and/or approximation. The two two-level Hamiltonians, indeed, are derived only on the basis of a transparent mathematical mapping between the two-qubit states in each subspace and the states of a fictitious spin-1/2. Moreover, they describe exactly a LMSZ scenario with a standard avoided crossing where the transverse constant field is effectively reproduced by the coupling existing between the two qubits. The treatment at the basis of our analysis remarkably has enabled us to explore peculiar dynamical aspects of the system under scrutiny, leading, for example, to the exact evolution of the entanglement get established between the two spins.

We underline, moreover, that our study is not a special case of the one considered in Ref. [94], where a Lipkin-Meskow-Glick (LMG) interaction model for N spin-qubits subjected to a LMSZ ramp is considered. The numerical results reported in Ref. [94] are, indeed, based on the mean field approximation. In addition, there is no possibility of considering in the LMG model effects stemming from the anisotropy between x and y interaction terms.

The results discussed in this chapter have been reported in Refs. [132, 133, 134, 135].

Chapter 3

Interacting Qudits and Qubit Chains

Spin chains offer a priviliged experimental scenario for quantum technology applications thanks to the possibility of entanglement generation [136, 137, 138] also over long distances [139]. Entanglement, indeed, is the key resource for quantum information tasks [140] and its manipulation by field application [60] is of course of fundamental importance. In this respect, the possibility of realizing a local application of fields on a single qudit while it interacts with other ones is of basic interest to generate physical effects in the spin chain by manipulating the single spin dynamics. Through the Scanning Tunneling Microscopy (STM), for example, it is possible to construct atom by atom a chain of interacting nanomagnets and to manipulate the state of a single spin by applying a local magnetic field on atomic scale with a STM tip [82, 83, 97, 98, 99, 100, 141].

In this context, a growing interest in qudits - N-level quantum systems - should be emphasized. Interacting spin systems with $s > 1/2$ reveal a rich variety of phenomena in condensed matter and atomic physics. For example, spin models with higher spin length may exhibit novel topological phases described by a hidden order parameter [142]. Moreover, various strongly interacting spin-boson systems can be mapped onto coupled spin models [143, 144, 145]. Apart from the methods used to solve analytically various spin-1/2 systems, in general, models with $s > 1/2$ are highly complex and do not permit analytical treatment.

Qutrits, and qudits in general, offer numerous advantages over qubits beyond the obvious exponential increase of their Hilbert space. For example, qutrits allow the construction of new types of quantum protocols [146, 147] and entanglement [148], Bell inequalities resistant to noise [149], larger violations of nonlocality [150], more secure quantum communication [151, 152], optimization of the Hilbert space dimensionality vs. control complexity [153], and others. To this end, efficient recipes for manipulation of qutrits [154, 155] and qudits [156] have been proposed.

In this chapter we show how our approach based on the Hamiltonian symmetry analysis, leading us to a reduction of the main dynamical problem into easier sub-problems, turns out to be an useful mathematical tool to treat and solve also more complex interacting spin-qudit systems. Here we report the results obtained for two interacting qutrits, two interacting qudits and for a large system of N qubits coupled through unconventional high order interaction terms. Our scope is to furnish detailed information about the exact dynamics of such spin-chain systems when they are subjected to classical external time-dependent field and to exploit such a knowledge in order to bring to light relevant physical applications.

3.1 Two qutrits

3.1.1 The model and its symmetry-based analysis

Analogously to the case of two interacting qubits considered in the previous chapter, here we consider a system of two interacting spin-1 systems denoted by $\mathbf{S}_1$ and $\mathbf{S}_2$, respectively living in the Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, in a physical model described by the Hamiltonian

$$H = \mu_B(g_1 B_1^z \hat{S}_1^z + g_2 B_2^z \hat{S}_2^z) + J_0 \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2 + \hat{\mathbf{S}}_1 \cdot \mathbf{D}_{12} \cdot \hat{\mathbf{S}}_2. \tag{3.1}$$

The first two terms in Eq. (3.1) describe the interaction of the two spins with two (generally different) parallel local magnetic fields oriented along the $\hat{z}$-axis, B_1^z and B_2^z, with the assumption of scalar g-factors, g_1 and g_2. The third term represents the Heisenberg isotropic exchange interaction of coupling strength J_0, while the last term, through the second-order traceless Cartesian tensor $\mathbf{D}_{12}$, accounts for symmetric spin-spin anisotropic couplings stemming from the dipole-dipole (d-d) interaction and anisotropic exchange interaction.

Although we only consider two interacting spin-1 systems, our model covers a broad range of physical situations. For example, in solid state physics the coupling between two molecules, which in their ground state possess a total angular momentum (effective spin) $\mathbf{S} = 1$, is described using the Hamiltonian model (3.1), with the proviso that spin-orbit effect can be neglected [3]. An optical lattice of two wells, each containing a single atom of spin 1, provides another possible physical scenario wherein manipulation of the atom-atom coupling constants is within experimental reach [157]. In addition, the interaction between nanomagnets with a total spin of 1, which is of great interest in quantum computing, is described by the Hamiltonian model (3.1) [158]. Recently, it was shown that the interaction between two separated nitrogen-vacancy centres in diamond can be described by a Heisenberg spin-1 model [159]. Moreover, spin-1 models can be realized in a linear ion crystal by using atomic species with three metastable levels driven by laser fields [160, 161].

Now, we suppose that our system possesses C_2-symmetry with respect to the $\hat{z}$ direction. In this case the $\mathbf{D}_{12}$ matrix takes the form

$$\mathbf{D}_{12} = \begin{pmatrix} d_{xx} & d_{xy} & 0 \\ d_{yx} & d_{yy} & 0 \\ 0 & 0 & d_{zz} \end{pmatrix} \tag{3.2}$$

and the Hamiltonian (3.1) may be written as

$$H = \hbar\omega_1 \hat{\Sigma}_1^z + \hbar\omega_2 \hat{\Sigma}_2^z + \gamma_x \hat{\Sigma}_1^x \hat{\Sigma}_2^x + \gamma_y \hat{\Sigma}_1^y \hat{\Sigma}_2^y + \gamma_z \hat{\Sigma}_1^z \hat{\Sigma}_2^z + \gamma_{xy} \hat{\Sigma}_1^x \hat{\Sigma}_2^y + \gamma_{yx} \hat{\Sigma}_1^y \hat{\Sigma}_2^x, \tag{3.3}$$

where the Pauli operators $\hat{\Sigma}_i^k$ ($i = 1,2$; $k = x,y,z$) for a spin-1 system are related with the spin-1 operator components as

$$\hat{S}_i^x = \frac{\hbar}{\sqrt{2}} \hat{\Sigma}_i^x, \quad \hat{S}_i^y = \frac{\hbar}{\sqrt{2}} \hat{\Sigma}_i^y, \quad \hat{S}_i^z = \hbar \hat{\Sigma}_i^z. \tag{3.4}$$

The seven real parameters appearing in Eq. (3.3) are given by

$$\begin{aligned}
&\omega_1 = \mu_B g_1 B_1^z, \quad \omega_2 = \mu_B g_2 B_2^z,\\
&\gamma_x = \frac{\hbar^2}{2}(J_0 + d_{xx}), \quad \gamma_y = \frac{\hbar^2}{2}(J_0 + d_{yy}), \quad \gamma_z = \hbar^2(J_0 + d_{zz}),\\
&\gamma_{xy} = \frac{\hbar^2}{2} d_{xy}, \quad \gamma_{yx} = \frac{\hbar^2}{2} d_{yx}.
\end{aligned} \tag{3.5}$$

We keep the considerations as general as possible, without the restrictions of a specific physical situation. Hence hereafter we do not attribute any specific symmetry constraints to the real parameters appearing in the Hamiltonian model (3.3). In this manner, our model includes several models in the literature as special cases. These include the *XXX* ($\gamma_x = \gamma_y = \gamma_z$), *XXZ* ($\gamma_x = \gamma_y$) and *XYZ* models for two qutrits subjected to an inhomogeneous magnetic field, generalized with the inclusion of the Dzyaloshinskii-Moriya (DM) interaction ($\gamma_{yx} = -\gamma_{xy}$) [63, 64, 65]. In addition, from our Hamiltonian model one may easily recover a lot of other models, e.g., the *XX* and *XY* models ($\gamma_z = 0$) with (or not) the contribution derived by the DM interaction and (or not) the presence of a homogeneous or inhomogeneous magnetic field, recently taken as starting point for investigating the appearance of thermal entanglement in the system of two interacting qutrits [162, 163, 164].

The following symmetry transformation of H

$$\begin{cases}
\hat{\hat{\Sigma}}_1^x = -\hat{\Sigma}_1^x, \quad \hat{\hat{\Sigma}}_1^y = -\hat{\Sigma}_1^y, \quad \hat{\hat{\Sigma}}_1^z = \hat{\Sigma}_1^z,\\
\hat{\hat{\Sigma}}_2^x = -\hat{\Sigma}_2^x, \quad \hat{\hat{\Sigma}}_2^y = -\hat{\Sigma}_2^y, \quad \hat{\hat{\Sigma}}_2^z = \hat{\Sigma}_2^z,
\end{cases} \tag{3.6}$$

is canonical, such that $H \to H$, which implies the existence of a unitary time-independent operator accomplishing the transformation given by Eq. (3.6) which, by construction, is a constant of motion. Because the transformation (3.6) is nothing but a rotation of π around the $\hat{z}$-axis of each spin, we can write the unitary operator accomplishing this transformation as

$$\hat{K} = e^{i\pi \hat{S}_1^z/\hbar} \otimes e^{i\pi \hat{S}_2^z/\hbar} = e^{i\pi \hat{\Sigma}_1^z} \otimes e^{i\pi \hat{\Sigma}_2^z} = 1 - 2\left[(\hat{\Sigma}_1^z)^2 + (\hat{\Sigma}_2^z)^2\right] + 4(\hat{\Sigma}_1^z)^2(\hat{\Sigma}_2^z)^2. \tag{3.7}$$

It is possible to verify that the operator $\hat{K}$ can be written as

$$\hat{K} = \cos(\pi \hat{\Sigma}_{\mathrm{tot}}^z), \tag{3.8}$$

with $\hat{\Sigma}_{\mathrm{tot}}^z = \hat{\Sigma}_1^z + \hat{\Sigma}_2^z$ being the total spin of the composed system along the z direction. Equation (3.8) shows that the constant of motion $\hat{K}$ is indeed a parity operator with respect to the collective Pauli spin variable $\hat{\Sigma}_{\mathrm{tot}}^z$, since in correspondence to its integer eigenvalues $M = 2, 1, 0, -1, -2$, $\hat{K}$ has eigenvalues +1 and -1 depending on the parity of M.

The existence of this constant of motion subdivides the 9D Hilbert space of the system into two dynamically invariant and orthogonal subspaces corresponding to the two eigenvalues +1 and -1 of $\hat{K}$. The subspace relative to $K = 1$ ($K = -1$), and then to even (odd) values of M, will be hereafter referred to as even- (odd-) parity subspace. As a consequence, there exist a unitary and Hermitian operator $\hat{T}$ (consisting in an appropriate reordering of the standard basis states) which transforms H into $\tilde{H} = \hat{T}^\dagger H \hat{T}$, whose matrix form consists of two blocks, one of dimension 4, related to the eigenvalue

-1 of the new constant of motion $\hat{\tilde{K}} \equiv \hat{T}^\dagger \hat{K} \hat{T}$, and the other of dimension five related to the eigenvalue +1 of $\hat{\tilde{K}}$, representing the two orthogonal sub-dynamics. The new Hamiltonian $\tilde{H}$ can be written as

$$\tilde{H} = \hat{P}_{-1} \tilde{H} \hat{P}_{-1} + \hat{P}_{+1} \tilde{H} \hat{P}_{+1}, \tag{3.9}$$

where we introduced the hermitian operator $\hat{P}_{-1}$ ($\hat{P}_{+1}$) projecting a generic state of the total Hilbert space $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$ into the $\tilde{K}$-invariant subspace $\mathscr{H}_-$ ($\mathscr{H}_+$) relative to its eigenvalue -1 (+1) such that $\hat{P}_{-1} \tilde{H} \hat{P}_{-1}$ ($\hat{P}_{+1} \tilde{H} \hat{P}_{+1}$) consists in the upper (lower) block of $\tilde{H}$, or better in a matrix with the same dimension (9) of $\tilde{H}$ but with non vanishing entries only in the upper (lower) four (five) dimensional block.

It is worth noticing that the arguments leading to the possibility of representing the Hamiltonian in accordance with Eq. (3.9) hold their validity even for a more general Hamiltonian model $\hat{H}_{\text{gen}}$ obtainable from $\hat{H}$ adding terms commuting with $\hat{K}$, e.g., $(\hat{\Sigma}_1^x)^2$, $\hat{\Sigma}_1^z (\hat{\Sigma}_2^y)^2$ and $\hat{\Sigma}_1^x \hat{\Sigma}_1^y \hat{\Sigma}_2^y \hat{\Sigma}_2^x$,

$$H_{\text{gen}} = H + \text{terms commuting with } \hat{K}. \tag{3.10}$$

However, we confine ourselves to the Hamiltonian model (3.3) since it is comparatively more accessible in laboratory and in addition, as we will show in the following sections, it generates interesting quantum dynamical behaviour.

Putting

$$\begin{aligned}
\Omega_+ &= \omega_1 + \omega_2, \\
\Omega_- &= \omega_1 - \omega_2, \\
\gamma_1 &= \gamma_x - \gamma_y - i(\gamma_{xy} + \gamma_{yx}), \\
\gamma_2 &= \gamma_x + \gamma_y + i(\gamma_{xy} - \gamma_{yx}),
\end{aligned} \tag{3.11}$$

the 4×4 block reads

$$\tilde{H}_- \equiv \begin{pmatrix} \hbar\omega_1 & \gamma_2 & \gamma_1 & 0 \\ \gamma_2^* & \hbar\omega_2 & 0 & \gamma_1 \\ \gamma_1^* & 0 & -\hbar\omega_2 & \gamma_2 \\ 0 & \gamma_1^* & \gamma_2^* & -\hbar\omega_1 \end{pmatrix} \tag{3.12}$$

and the four states of the original standard basis involved in such a subspace are

$$|e_1\rangle = |10\rangle, \quad |e_2\rangle = |01\rangle, \quad |e_3\rangle = |0-1\rangle, \quad |e_4\rangle = |-10\rangle. \tag{3.13}$$

The lower block of $\hat{\tilde{H}}$ is represented by the 5×5 matrix

$$\tilde{H}_+ \equiv \begin{pmatrix} \hbar\Omega_+ + \gamma_z & 0 & \gamma_1 & 0 & 0 \\ 0 & \hbar\Omega_- - \gamma_z & \gamma_2 & 0 & 0 \\ \gamma_1^* & \gamma_2^* & 0 & \gamma_2 & \gamma_1 \\ 0 & 0 & \gamma_2^* & -\hbar\Omega_- - \gamma_z & 0 \\ 0 & 0 & \gamma_1^* & 0 & -\hbar\Omega_+ + \gamma_z \end{pmatrix} \tag{3.14}$$

where the five standard basis states spanning this subspace are

$$\begin{aligned}
&|e_5\rangle = |11\rangle, \quad |e_6\rangle = |1-1\rangle, \quad |e_7\rangle = |00\rangle, \\
&|e_8\rangle = |-11\rangle, \quad |e_9\rangle = |-1-1\rangle.
\end{aligned} \tag{3.15}$$

Equation (3.9) implies that the quantum dynamics of two qutrits interacting according to the model of Eq. (3.3) factorizes into an effective spin-$\frac{3}{2}$ system and an effective spin-2 system.

We note that the mathematical steps leading from Eq. (3.3) to Eq. (3.9) reproduce analogous results even if we use, mutatis mutandis, the same Hamiltonian model where qudits systematically substitute the appearing qutrits. Of course the dimensions of the dynamically invariant subspaces existing in the qudits case strictly depends on the dimension of the qudits Hilbert space. In the next sections we will show that in the case of the qutrits a further aspect of such reducibility of the quantum dynamics of the system emerges, leading to physically transparent and far-reaching consequences.

Four-dimensional subdynamics

The eigenvectors of the Hamiltonian $\tilde{H}_-$ (3.12) may be exactly derived by solving the fourth degree relative secular equation. The corresponding eigenvalues are

$$\mathscr{E}_1 = E_1 + E_2, \quad \mathscr{E}_2 = E_1 - E_2, \quad \mathscr{E}_3 = -\mathscr{E}_2, \quad \mathscr{E}_4 = -\mathscr{E}_1, \tag{3.16}$$

where

$$E_1 = \sqrt{\frac{(\hbar\Omega_+)^2}{4} + |\gamma_1|^2}, \qquad E_2 = \sqrt{\frac{(\hbar\Omega_-)^2}{4} + |\gamma_2|^2}. \tag{3.17a}$$

The four eigenvalues of $\tilde{H}_-$, in view of Eq. (3.16), may be obtained summing elements of the two pairs $\{E_1, -E_1\}$ and $\{E_2, -E_2\}$ in all possible ways. This circumstance hints that the quantum dynamics of the two qutrits restricted to the four dimensional Hilbert subspace generated by $|e_k\rangle$ with $k = 1,2,3,4$, is traceable back to that of two effective non-interacting spin-$\frac{1}{2}$ systems, respectively described by two bi-dimensional traceless Hamiltonians H_1 and H_2 with eigenvalues $\pm E_1$ and $\pm E_2$.

To verify this intuition we search for a mapping between the two qutrits original basis states in (3.13) and the two spin-$\frac{1}{2}$ basis, that is $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$, in accordance to which the generic eigenstate $|\psi_k\rangle$ of $\tilde{H}_-$ may be represented as a tensorial product between an eigenstate of H_1 and an eigenstate of H_2. Such a mapping consists simply in

$$\begin{aligned} |10\rangle &\leftrightarrow |++\rangle, \\ |01\rangle &\leftrightarrow |+-\rangle, \\ |0-1\rangle &\leftrightarrow |-+\rangle, \\ |-10\rangle &\leftrightarrow |--\rangle, \end{aligned} \tag{3.18}$$

where we define the effective spin-$\frac{1}{2}$ states as $\sigma_i^z |\pm\rangle_i = \pm |\pm\rangle_i$ with $i = 1,2$. Indeed, it is straightforward to show that the sub-dynamics of the two spin-1 systems interacting according to (3.3), related to the $\hat{\tilde{K}}$-invariant subspace of dimension four characterized by the eigenvalue $\tilde{K} = -1$, may be reinterpreted as the dynamics of two decoupled effective spin-$\frac{1}{2}$ systems. Indeed, we can write $\tilde{H}_-$ as

$$\tilde{H}_- = H_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes H_2, \tag{3.19}$$

where we define

$$H_1 = \frac{\hbar(\omega_1 + \omega_2)}{2}\hat{\sigma}_1^z + (\gamma_x - \gamma_y)\hat{\sigma}_1^x + (\gamma_{xy} + \gamma_{yx})\hat{\sigma}_1^y, \tag{3.20a}$$

$$H_2 = \frac{\hbar(\omega_1 - \omega_2)}{2}\hat{\sigma}_2^z + (\gamma_x + \gamma_y)\hat{\sigma}_2^x - (\gamma_{xy} - \gamma_{yx})\hat{\sigma}_2^y. \tag{3.20b}$$

The physical interpretation of this sub-dynamics in terms of two spin-$\frac{1}{2}$ systems is clear and direct: H_1 (H_2) describes a fictitious spin-$\frac{1}{2}$ system immersed in an effective magnetic field $\vec{B}_1^{\text{eff}}$ ($\vec{B}_2^{\text{eff}}$) expressible as

$$\begin{aligned}\vec{B}_1^{\text{eff}} &= \Big((\gamma_x-\gamma_y),(\gamma_{xy}+\gamma_{yx}),\frac{\hbar\mu}{2}(g_1B_1^z+g_2B_2^z)\Big),\\ \vec{B}_2^{\text{eff}} &= \Big((\gamma_x+\gamma_y),(\gamma_{yx}-\gamma_{xy}),\frac{\hbar\mu}{2}(g_1B_1^z-g_2B_2^z)\Big),\end{aligned} \tag{3.21}$$

such that we have $\tilde{H}_- = \sum_{i=1}^2 \vec{\sigma}_i\cdot\vec{B}_i^{\text{eff}}$.

Since $\tilde{H}_-$ of Eq. (3.19) describes two decoupled spin-$\frac{1}{2}$ systems, the eigenvectors of $\tilde{H}_-$ may be written in the following factorized form

$$\tilde{H}_- \to \begin{cases}|\psi_{11}\rangle\otimes|\psi_{21}\rangle \to |\psi_1\rangle,\\ |\psi_{11}\rangle\otimes|\psi_{22}\rangle \to |\psi_2\rangle,\\ |\psi_{12}\rangle\otimes|\psi_{21}\rangle \to |\psi_3\rangle,\\ |\psi_{12}\rangle\otimes|\psi_{22}\rangle \to |\psi_4\rangle,\end{cases} \tag{3.22}$$

where $\{|\psi\rangle_{11},|\psi\rangle_{12}\}$ ($\{|\psi\rangle_{21},|\psi\rangle_{22}\}$) are the eigenvectors of H_1 (H_2). The corresponding eigenenergies for each state are given by Eq. (3.16).

We emphasize that the two-qutrit systems may be prepared in a state whose evolution is dominated by one admissible Bohr frequency only exactly mappable in the time evolution of a single spin-$\frac{1}{2}$ system subjected to an appropriate magnetic field [see Eq. (3.21)]. In other words, the quantum dynamics of two qutrits generated by the Hamiltonian (3.3) possesses symmetry properties leading to such a peculiar dynamical behaviour. We note finally that, since the unitary operator $\hat{T}$ transforming H into the direct sum of H_- and H_+ is independent of time, the demonstration of the fact that the quantum dynamics induced by H_- may be traced back to that of two effective spin-$\frac{1}{2}$ systems might provide significant advantages even when H is time-dependent, at least in its 4D dynamically invariant subspace.

Five-dimensional subdynamics

As previously shown, the quantum dynamics of the two coupled spin-1 systems is reducible to two quantum sub-dynamics, the first one described by Eq. (3.12) and second one by Eq. (3.14). The lucky mathematical occurrence leading us to trace back the quantum dynamics of the two spin-1 systems to that of two non interacting spin-1/2 systems in the four dimensional invariant subspace cannot emerge in the other invariant subspace essentially because its dimension 5 is a prime number. Then, in the spirit of the previous section, the only observation we may do is that in such five dimensional subspace the quantum dynamics of the two spin-1 systems may be mapped into that of a spin-2. Unfortunately the effective (through an appropriate mapping) representation of $\tilde{H}_5$ in terms of a spin-2 operators is very involved appearing strongly non linear and practically impossible to be related to a convincing physical scenario. This is why we do not proceed further along this direction confining ourselves to the consideration of particular conditions easily providing the possibility of extracting useful properties possessed by our model.

The first aspect related to the model deserving attention is that by comparing the reduced matrices given by Eqs. (3.12) and (3.14) it is possible to note that the parameter γ_z influences the sub-dynamics in the five dimensional dynamically invariant subspace of H only. As a consequence we may choose specific values of this parameter without modifying the dynamical properties of the system in the four dimensional dynamically invariant subspace. It is possible to convince oneself that for $\gamma_z = 0$ the eigensolutions of $\tilde{H}_+$ may be exactly found.

Furthermore, it is possible to verify that if we assume

$$\begin{cases} \gamma_x = \gamma_y = \gamma/2 \\ \gamma_{xy} = -\gamma_{yx} = D_z/2 \end{cases} \tag{3.23}$$

the five dimensional block is reduced into two one dimensional block and a three dimensional one as it can be appreciated from what follows

$$\tilde{H}_+ = \begin{pmatrix} \Omega_+ + \gamma_z & 0 & 0 & 0 & 0 \\ 0 & \Omega_- - \gamma_z & (\gamma + iD_z) & 0 & 0 \\ 0 & (\gamma - iD_z) & 0 & (\gamma + iD_z) & 0 \\ 0 & 0 & (\gamma - iD_z) & -\Omega_- - \gamma_z & 0 \\ 0 & 0 & 0 & 0 & -\Omega_+ + \gamma_z \end{pmatrix}. \tag{3.24}$$

The previous specific conditions (3.23) have a clear interesting physical meaning: the first condition imposes an isotropic XY-exchange interaction while the second one takes into account the antisymmetric exchange or Dzyaloshinskii-Moriya interaction $\mathbf{D} \cdot (\hat{\mathbf{S}}_1 \times \hat{\mathbf{S}}_2)$ with $\mathbf{D} \equiv (0, 0, D_z)$. This model is well known in literature and was studied in connection with the properties of thermal entanglement [65].

It is interesting to point out, moreover, that, in this instance, the three dimensional block may be described in terms of the single spin-1 Pauli operators defined in Eq. (3.4) and the related Hamiltonian precisely reads

$$\tilde{H}_3 = \gamma \hat{\Sigma}^x - D_z \hat{\Sigma}^y + \Omega_- \hat{\Sigma}^z - \gamma_z (\hat{\Sigma}^z)^2. \tag{3.25}$$

We see immediately that putting $\gamma_z = 0$ we have a SU(2) three dimensional fictitious sub-dynamics of a single spin-1 subjected to the effective external magnetic field $\mathbf{B}_1 \equiv (\gamma, D_z, \Omega_-)$, so that we may write $\tilde{H}_3 = \sum_j \mathbf{B}_1^j \cdot \hat{\Sigma}^j$. This observation is particularly significant at the light of the interplay between the new results obtained for SU(2) bidimensional time-dependent dynamics. In this way, under the conditions (3.23) and $\gamma_z = 0$, we may study analytically and know exactly the five dimensional sub-dynamics of the two spin-1 systems also in a time-dependent scenario, more precisely when the two magnetic fields are time-dependent, i.e. when we have $\omega_1(t)$ and $\omega_2(t)$ as considered in the following paragraph.

It is worth to point out, finally, that the conditions (3.23), contrary to the conditions on γ_z, modify the dynamics in the four dimensional subspace, too. In this instance, indeed, we obtain a four dimensional sub-dynamics of the two spin-1's well described in terms of two decoupled fictitious spin 1/2's in which the first spin is subjected to a magnetic field only in the z-direction while the second spin is immersed in a magnetic field having a direction depending on the coupling parameters of the model. It can be appreciated and easily verified from Eqs. (3.27) providing conditions (3.23). Therefore under conditions (3.23) both the sub-dynamics are exactly treatable or in other words the full model may be exactly solved.

3.1.2 LMSZ scenario

Now, we want to study the two interacting qutrits when they are subjected to time-dependent fields, $\omega_1(t)$ and $\omega_2(t)$. In the following we show that we are able to construct formally the time evolution operator for both four- and five-state subdynamics. In particular, we analyse the case in which the z-magnetic field is a ramp as in the LMSZ scenario [33]. We are interested in revealing intriguing dynamical effects stemming from the homogeneity or heterogeneity of both the coupling parameters and the two fields. In addition, we want to exploit our symmetry-based approach to take into account the influence of a surrounding environment by considering a random fluctuating field component.

To this end we consider the following specialized model of two interacting qutrits subjected to local time-dependent fields

$$H = \hbar\omega_1\hat{\Sigma}_1^z + \hbar\omega_2\hat{\Sigma}_2^z + \gamma_x\hat{\Sigma}_1^x\hat{\Sigma}_2^x + \gamma_y\hat{\Sigma}_1^y\hat{\Sigma}_2^y + \gamma_z\hat{\Sigma}_1^z\hat{\Sigma}_2^z. \tag{3.26}$$

Our scope is to study a Landau-Majorana-Stückelberg-Zener (LMSZ) scenario for the two qutrits and analyse how the coupling between them and a noisy component of the magnetic field affect their dynamics.

In this case the Hamiltonian model of two decoupled fictitious spin-1/2's describing effectively the two qutrit dynamics in the four-dimensional Hilbert subspace reads $H_- = H_1 \otimes \hat{\mathbb{1}}_2 + \hat{\mathbb{1}}_1 \otimes H_2$, with

$$H_1 = \frac{\hbar\Omega_+}{2}\hat{\sigma}_1^z + \gamma_-\hat{\sigma}_1^x, \qquad H_2 = \frac{\hbar\Omega_-}{2}\hat{\sigma}_2^z + \gamma_+\hat{\sigma}_2^x. \tag{3.27}$$

Under the following conditions $\gamma_z = 0$ and $\gamma_x = \gamma_y = \gamma/2$ the five-dimensional block, instead, is decomposed in two one-dimensional blocks and a three-dimensional one. In this instance the latter possesses an su(2) structure and can be written in terms of spin variables of a fictitious spin-1, namely

$$H_3 = \gamma\hat{\Sigma}^x + \hbar\Omega_-\hat{\Sigma}^z. \tag{3.28}$$

We emphasize that the choice $\gamma_z = 0$ is necessary to get an su(2)-symmetry structure of the matrix within the three-dimensional subspace. This choice, however, does not alter the four-dimensional subdynamics since H_1 and H_2 in Eq. (3.27) do not depend on γ_z.

Four-dimensional subdynamics

General solution

We may formally write the time evolution operator U_j ($j = 1,2$) related to H_j, solution of the Schrödinger equation $i\hbar\dot{U}_j = H_j U_j$, as follows

$$U_j = \begin{pmatrix} a_j & b_j \\ -b_j^* & a_j^* \end{pmatrix}, \tag{3.29}$$

where a_j and b_j are time-dependent Cayley-Klein parameters satisfying $|a_j|^2 + |b_j|^2 = 1$. The time evolution operator U_-, satisfying the Schrödinger equation $i\hbar\dot{U}_- = H_- U_-$, then reads

$$U_- = U_1 \otimes U_2 = \begin{pmatrix} a_1a_2 & a_1b_2 & b_1a_2 & b_1b_2 \\ -a_1b_2^* & a_1a_2^* & -b_1b_2^* & b_1a_2^* \\ -b_1^*a_2 & -b_1^*b_2 & a_1^*a_2 & a_1^*b_2 \\ b_1^*b_2^* & -b_1^*a_2^* & -a_1^*b_2^* & a_1^*a_2^* \end{pmatrix}. \tag{3.30}$$

The mathematical expressions of $a_j(t)$ and $b_j(t)$ depend on the time-dependence of the two local magnetic fields $\omega_1(t)$ and $\omega_2(t)$.

STM dynamics: non-local control and state transfer

We firstly analyse the case of a single local z-magnetic field $B_z(t)$ applied on the first spin consisting in a LMSZ ramp, such that

$$\hbar\omega_1(t) = \alpha t, \quad t \in (-\infty, \infty), \tag{3.31}$$

where α is considered a positive real number and rules the adiabaticity of the process since $\dot{B}_z \propto \alpha$. Let us consider the case of an excitation present in the system and localized in one of the two qutrits, say the second spin; in this case the initial state of the two qutrits (fictitious qubits) is $|-10\rangle$ ($|--\rangle$). In this instance, each fictitious spin-1/2 is subjected to a LMSZ scenario with $\omega_1(t)$ as longitudinal magnetic field and a constant (effective) transverse magnetic field determined by the coupling parameters [see Eq. (3.27)]. In this way, the first and second fictitious spin-1/2 have the probability to make the transition to the up-state, respectively

$$P_1 = 1 - \exp\{-2\pi\beta_-\}, \tag{3.32}$$

and

$$P_2 = 1 - \exp\{-2\pi\beta_+\}, \tag{3.33}$$

with $\beta_\pm = \gamma_\pm^2/\hbar\alpha$. Thus, the joint probability for the two fictitious spin-1/2's to be found in the state $|++\rangle$, $|+-\rangle$ and $|-+\rangle$, starting from $|--\rangle$, are respectively

$$P_1P_2, \qquad P_1(1-P_2), \qquad (1-P_1)P_2, \tag{3.34}$$

being nothing but the probability of finding the two qutrits in the state $|10\rangle$, $|01\rangle$ and $|0-1\rangle$, respectively. We know that in the standard LMSZ scenario applied on a single spin-qubit, the transverse field couples the two levels and is then responsible of the avoided crossing. It is worth noticing that, in our case, the transverse field role is played by the coupling existing between the two qutrits, as it is clear by the two Hamiltonians in Eq. (3.27). Hence, we may reproduce adiabatic conditions by appropriately setting the ratio between the longitudinal fields and the coupling parameters in order to have a full LMSZ transition of the two fictitious spin-1/2's. The three probabilities in Eq. (3.34) are reported in Fig. 3.1 against the parameter $\beta = \beta_+$ for $\beta_+/\beta_- = 2$. In this case we are realizing a local control of the dynamics of the first qutrit, leaving the other one unaltered. For a complete LMSZ transition, indeed, the first qutrit accomplishes the LMSZ transition $|-1\rangle \to |1\rangle$, while the second qutrit's state does not change.

Analogously, we may consider the excitation initially localized in the first spin-1, so that the two qutrits start from the state $|0-1\rangle$. In this instance the two-qutrit system is asymptotically driven to the state $|01\rangle$ and the probability of the related transition acquires the same expression as the previous one in Eq. (3.34). It is worth noticing that in this case we generate a LMSZ transition from $|-1\rangle$ to $|1\rangle$ in the second spin, by applying a local magnetic field only on the first qutrit which, instead, remains in its initial state. Such a circumstance, thus, may be identified as the achievement of a non-local control of the second qutrit.

Another interesting effect to be highlighted is the possibility of realizing a state transfer between the two qutrits. Indeed, if the two qutrits (fictitious qubits) are initialized in the state $|-10\rangle$ ($|--\rangle$) and we assume $\gamma_x = \gamma_y$, the transition probability of the first fictitious spin-1/2 is forbidden, while the second one passes to $|+\rangle$ with probability $P = P_2$. In this way, the two qutrits (fictitious qubits) reach the state $|0-1\rangle$ ($|-+\rangle$) having interchanged their initial state. The same effect is present if the two qutrits

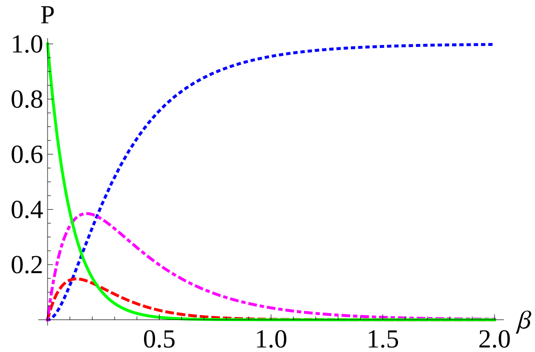


Figure 3.1: (Color online) a) Asymptotic LMSZ probabilities [Eq. (3.34)] of finding the two qutrits in the state $|10\rangle$ (blue dotted line), $|01\rangle$ (magenta dot-dashed line), $|0-1\rangle$ (red dashed line) and $|-10\rangle$ (green full line), when they start from the state $|-10\rangle$ for $\gamma_x \neq \gamma_y$, $\beta = \beta_+$ and $\beta_+/\beta_- = 2$.

are initially prepared in $|10\rangle$ passing to $|01\rangle$. In such a case, the transitions between the states of the two qutrit system in the four dimensional subspace are different since the condition $\gamma_x = \gamma_y$ introduces a further symmetry in the model related to the commutation of H with $\hat{\Sigma}^z_{\text{tot}}$. This fact generates, in the subspace under scrutiny, the existence of other two dynamically invariant subspaces related to the eigenvalues of $\hat{\Sigma}^z_{\text{tot}}$. It is easy to verify that, this time the two qutrits starting from $|-10\rangle$ ($|10\rangle$) can be asymptotically found only in the state $|0-1\rangle$ ($|01\rangle$).

At the light of the STM scenario, the physical effects previously discussed and analytically derived are of relevant interest. They show, indeed, that the presence of the coupling between the two qutrits allows us to manipulate the dynamics of the whole two-qutrits chain by the application of a single local magnetic field on one of the two spins, being exactly one of the task of the application of the STM technique. Moreover, the previous examples brought to light that, by studying the kind of transitions occurring in the two-qutrit system, we may get information about the coupling parameters determining the symmetries of the Hamiltonian.

Effects of environment

We wish to show now that the mapping of the two-qutrit dynamics into that of two decoupled spin-1/2's in the four-dimensional subspace is useful not only to solve exactly the problem in ideal conditions, but also to take into account possible external influences due to the action of a surrounding environment, such as nuclear spin bath. In Ref. [80], for example, it is experimentally demonstrated that decoherence effects in the dynamics of a NV center in diamond (consisting in a three-level system), subjected to a LSZ interferometer, comes from the dipolar interaction of the system with the surrounding ^{13}C nuclear spins random fluctuating at room temperature. Analogously to the procedure we adopted for two interacting qubits, such external influences may be theoretically regarded, for example, as noise in the magnetic field component. In that case we consider the results reported in Ref. [86] where the authors study the dynamics of a spin S subjected to a noisy LMSZ scenario. The noisy time-dependent magnetic field $\eta(t)$ is considered only in the z direction and characterized by a time correlation function of the form $\langle \eta(t)\eta(t') \rangle = 2\Gamma\delta(t-t')$. Reference [80] experimentally legitimates such an assumption; in that case, indeed, the authors shows as the transverse fluctuations can be neglected. In such a way the noisy component cannot generates transitions between the different states but it leads only to loss of coherence. In Ref. [86], the authors show how the LMSZ transition probability is affected by the presence of such a noisy magnetic field in the case of a spin-1/2, a spin-1 and a spin-3/2. For a spin-1/2 and for large values of Γ we have asymptotically

$$P_{-}^{+} = \frac{1 - \exp\{-2\pi g^2/\hbar\alpha\}}{2}, \tag{3.35}$$

where g is the energy contribution due to the coupling of the spin-1/2 with the constant transverse magnetic field. We see that the transition probability does not depend on the specific value of Γ, provided that Γ is large. Moreover, it is important to note that the effect of the noise is to hinder the transition. Indeed, in the most convenient case, that is for $g^2/\hbar\alpha \gg 1$, the system reaches at most an equally populated condition of the two states. This is of particular interest for us since we have shown that the transition of the two qutrits studied before can be reduced to the LMSZ transition of a spin-1/2. Then, it means that the result previously reported can be exploited in our case to find the corrected LMSZ transition probability for the two qutrits when the field is affected by a noisy component. For example, if $\gamma_x \neq \gamma_y$, the probability in Eq. (3.34) becomes $P_1 P_2/4$, reasonably meaning that, under the effect of noise, we reach an equally populated condition of the four states involved in the subdynamics under scrutiny. Analogously, if $\gamma_x = \gamma_y$, had the two qutrits started form $|-10\rangle$ we get the probability $P_2/2$ of transition to the state $|0-1\rangle$, reaching this time an equally populated condition between these two states.

Such observation is based on the fact that, adding the noisy component $\eta(t)$ to the field applied to the first qutrit, nothing changes in the dynamics-decoupling procedure. The Hamiltonian transformation is completely unaffected since the only difference consists in a redefinition of the longitudinal field. In this way, what we obtain is an effective z-field for the two fictitious spin-1/2's supplemented by a random field component. Thus, also in this case, we may reduce the two-qutrit dynamical problem into the analysis of the quantum dynamics of two decoupled spin-1/2's.

In this respect, it is worth pointing out that the argument previously exposed continues to be valid also when we consider the possibility that the exited states $|0\rangle$ and $|1\rangle$ of the two qutrits decay irreversibly out of the system by some mechanism. Let us suppose that the spontaneous emission from the exited states to the ground one is negligible and that the two decay rates for the state $|0\rangle$ and $|1\rangle$ are $\tilde{\Gamma}$ ($\tilde{\Gamma}'$) and $2\tilde{\Gamma}$ ($2\tilde{\Gamma}'$), respectively, for the first (second) qutrit. It is easy to see that the analysis of such a scenario is equivalent, up to add a constant imaginary term, to phenomenologically introduce the non-Hermitian terms $i\tilde{\Gamma}\hat{\Sigma}_1^z$ and $i\tilde{\Gamma}'\hat{\Sigma}_2^z$ in our Hamiltonian model. Also this time we have a simple redefinition of the parameters in front of the operators $\hat{\Sigma}_1^z$ and $\hat{\Sigma}_2^z$ without altering the symmetries possessed by the Hamiltonian H. Therefore, in such a case, within the four-dimensional subspace the two-qutrit dynamics may be described in terms of two decoupled two-level systems subjected to effective external fields and characterized by decaying states. Several results have been reported for a single qubit with a decaying state subjected to the LMSZ scenario [84, 85, 87]. Precisely, it has been proved that, on the one hand, in the standard (ideal) LMSZ scenario, the decay rate influences only its the time-history of the transition probality but not its asymptotic value [84]; on the other hand, in the more realistic LMSZ scenario characterized by a limited time-window, the exited state population exhibits a dependence on the decay rate [85]. We emphasize that even such results allow to make quantitative predictions on the LMSZ transition probabilities for the system under scrutiny.

Determination of γs

Now, we want to discuss the possibility of applying local fields on both the qutrits. Let us consider, firstly, the case

$$\omega_1(t) = \omega_2(t) = \alpha t/2, \tag{3.36}$$

with t going from $-\infty$ to $+\infty$.

In this case, the Hamiltonians of the two fictitious spin-1/2's, through which we describe effectively the dynamics of the two qutrits in the four dimensional subspace, read

$$H_1 = \hbar\Omega_+(t)\hat{\sigma}_1^z + \gamma_-\hat{\sigma}_1^x, \qquad H_2 = \gamma_+\hat{\sigma}_2^x, \tag{3.37}$$

with $\Omega_+(t) = \alpha t$. We see that the second fictitious spin-1/2 is subjected only to a magnetic field in the x-direction, while the first one is subjected to standard Landau-Zener scenario. As before, the role of the external transverse constant field is effectively played by the coupling existing between the two spins.

We study now the instance in which only one excitation is present in the system, equally shared by the two qutrits. We consider, then, the entangled state $(|-10\rangle + |0-1\rangle)/\sqrt{2}$ as initial condition. By the mapping in Eq. (3.18), such a state, rewritten in terms of the two spin-1/2 states, acquires the form

$$|-\rangle \otimes \frac{|+\rangle + |-\rangle}{\sqrt{2}}. \tag{3.38}$$

It is easy to see that the second spin does not change its state in time since the latter is an eigenvalue of H_2. The first spin, instead, evolves according to the LMSZ dynamics, so that the probability to find it in the opposite state $|+\rangle$ at very large time instants ($t \to \infty$) is P_1. Of course, it expresses too the probability of the two spin-1/2's to be found in the state $|+\rangle \otimes \frac{|+\rangle+|-\rangle}{\sqrt{2}}$. The relevant point is that, in view of Eq. (3.18), it provides the probability for the two qutrits of reaching the state

$$\frac{|10\rangle + |01\rangle}{\sqrt{2}}. \tag{3.39}$$

Thus, if $\beta_- \gg 1$, through the linear ramp we have created an excitation in the system. It is important to underline that such a transition depends strongly on the coupling parameters between the two qutrits, since their difference constitute the effective transverse magnetic field entering in the expression of the LMSZ parameter β_-. Indeed, if the two parameters are equal or very close, the transition is forbidden, while, if they are opposite, the transition probability reaches its maximum efficiency. This suggests us that, choosing at will α and studying the characteristic time of the transition, we may get information about the value of γ_-.

If we now consider

$$\omega_1(t) = -\omega_2(t) = \alpha t/2 \tag{3.40}$$

and the two qutrits initially prepared in the state $(|-10\rangle + |0-1\rangle)/\sqrt{2}$, we get a specular dynamics. That is, the first fictitious spin-1/2, subjected only to a static x-magnetic field ($H_1 = \gamma_-\hat{\sigma}_1^x$), does not evolve, while the second fictitious spin-1/2 makes a transition from $|-\rangle$ to $|+\rangle$ (being $H_2 = \hbar\alpha t\hat{\sigma}_2^z + \gamma_+\hat{\sigma}_2^x$). Studying such a transition, this time, we get information about γ_+ since it rules the characteristic time of such a transition. Finally, by comparing the two values of γ_+ and γ_- we may estimate the original coupling parameters of the two qutrits γ_x and γ_y.

Dark states

We emphasize that, under the conditions $\gamma_x = \gamma_y = \gamma/2$ and $\omega_1(t) = \omega_2(t) = \omega(t)/2$ (unique homogeneous magnetic field), the following four states

$$|\psi^0_{1/2}\rangle = \frac{|10\rangle \pm |01\rangle}{\sqrt{2}}, \qquad |\psi^0_{3/4}\rangle = \frac{|-10\rangle \pm |0-1\rangle}{\sqrt{2}} \tag{3.41}$$

result steady states independently of the time dependences of the magnetic field. This may be easily understood in terms of the two spin-1/2's. Indeed, the second spin-1/2 is in an eigenstate $[(|+\rangle \pm |-\rangle)/\sqrt{2}]$ of its constant Hamiltonian $H_2 = \gamma\hat{\sigma}^x$ and evolves trivially, only acquiring the phase factor $\exp\{-i\gamma t/\hbar\}$; the first fictitious spin-1/2, instead, (being in the state $|\pm\rangle$) keeps only the phase factor $\exp\{-i\int_0^t \omega(t)dt\}$ since its Hamiltonian $H_1 = \hbar\omega(t)\hat{\sigma}^z_1$ does not mix the two standard basis states. This means that for these four states we have ($j = 1\ldots 4$)

$$\begin{aligned} &H(t)|\psi^0_j\rangle = E_j(t)|\psi^0_j\rangle, \\ &E_{1/2}(t) = \omega(t) \pm \gamma, \quad E_{3/4}(t) = -E_{2/1}(t) \end{aligned} \tag{3.42}$$

implying

$$|\psi_j(t)\rangle = \exp\left\{-i\int_0^t dt' E(t')/\hbar\right\}|\psi^0_j\rangle. \tag{3.43}$$

It is easy to see that, considering the time-independent case, such states result to be the eigenstates of the Hamiltonian (3.3). So, this model, in this specific case, presents a peculiar characteristic consisting in maintaining its steady states also when the Hamiltonian parameters are time-dependent. A remarkable consequence of this circumstance is that the following class of states $\rho_0 = \sum_j p_j|\psi^0_j\rangle\langle\psi^0_j|$ ($\sum_j p_j = 1$), comprising e.g. the thermal state ($p_j = \exp\{-E_j/k_BT\}$, k_B and T being the Boltzman constant and the Temperature, respectively), do not evolve in time, that is

$$\rho(t) = \sum_j p_j|\psi_j(t)\rangle\langle\psi_j(t)| = \sum_j p_j|\psi^0_j\rangle\langle\psi^0_j| = \rho_0. \tag{3.44}$$

Therefore, any physical observable calculated for such class of states exhibit a constant value in time. We can call such states 'dark states' since, under the conditions written before, they are unaffected by both the coupling and the longitudinal time-dependent field, also when the latter presents a random fluctuating behaviour.

Analogously, if we have $\gamma_x = -\gamma_y$ and $\omega_1(t) = -\omega_2(t)$ the four dark states are

$$\frac{|10\rangle \pm |0-1\rangle}{\sqrt{2}}, \qquad \frac{|01\rangle \pm |-10\rangle}{\sqrt{2}}. \tag{3.45}$$

Finally, we emphasize that the previous results are not restricted to the LMSZ scenario, but they are valid whatever the time-dependence of the field is.

Entanglement

The negativity, introduced by G. Vidal and R. F. Werner in [165], of a two-qutrit system described by the density matrix ρ reads [166]

$$\mathcal{N}_\rho = \frac{||\rho^{T_B}||_1 - 1}{2}, \tag{3.46}$$

where ρ^{T_B} is the partial transpose of the matrix ρ with respect to the subsystem B. The symbol $||\cdot||_1$ is the trace norm which, for a Hermitian matrix, results in the sum of the absolute values of the negative eigenvalues of ρ^{T_B} which is Hermitian and such that $\mathrm{Tr}\{\rho^{T_B}\} = 1$. The range of values of $\mathscr{N}_\rho$ is $[0,1]$ [166] and its calculation is independent of the factorized orthonormal basis in which the matrix ρ is represented as well as of the subsystem with respect to which we calculate the partial transpose, since $(\rho^{T_A})^T = \rho^{T_B}$ and $||X||_1 = ||X^T||_1$ for any operator X.

A generic pure state $\hat{\rho} = |\Psi\rangle\langle\Psi|$ belonging to $\mathscr{H}_-$ may be expanded as $|\Psi\rangle = \sum_{k=1}^4 c_k |e_k\rangle$ ($\sum_{k=1}^4 |c_k|^2 = 1$) in view of Eq. (3.13). The corresponding eigenvalues of $\hat{\rho}^{T_2}$ are

$$\Upsilon_1 = 1-x, \quad \Upsilon_2 = x, \quad \Upsilon_3 = \sqrt{x(1-x)}, \quad \Upsilon_4 = -\Upsilon_3, \tag{3.47}$$

with $x = |c_1|^2 + |c_4|^2$. Therefore, the negativity of a generic pure state can be written as

$$\mathscr{N}_{\hat{\rho}} = \sqrt{x(1-x)}, \tag{3.48}$$

which is well defined (since $x \in [0,1]$) and reaches its maximum value $\mathscr{N}_{\hat{\rho}}^{\max} = 1/2$ at $x = 1/2$. Thus in the four dimensional dynamically invariant subspace of $\hat{H}$ the negativity exhibited by the two coupled qutrits in a pure state reaches 1/2 as upper limit. Consequently, the negativity of the two qutrits, since a generic mixed state $\hat{\rho} = \sum_{r=1}^4 p_r |\psi_r\rangle\langle\psi_r|$ with $|\psi_r\rangle$ in $\mathscr{H}_-$ and ($\sum_{r=1}^4 p_r = 1, p_r \geq 0$), possesses the same upper bound 1/2 since [165]

$$\mathscr{N}_{\hat{\rho}}\left(\sum_r |\psi_r\rangle\langle\psi_r|\right) \leq \sum_r p_r \mathscr{N}_{\hat{\rho}}(|\psi_r\rangle\langle\psi_r|) \leq \frac{1}{2}. \tag{3.49}$$

The existence of such an upper limit is directly traceable back to the circumstance, easily demonstrable, that every pure state in $\mathscr{H}_-$ possesses a Schmidt decomposition with at most two non-vanishing Schmidt coefficients, namely k_1 and k_2 expressible as

$$k_1 = \sqrt{|c_2|^2 + |c_3|^2}, \quad k_2 = \sqrt{|c_1|^2 + |c_4|^2}, \quad k_3 = 0. \tag{3.50}$$

When $k_1 k_2 > 0$ the Concurrence [$C(|\psi\rangle)$] of two qutrits introduced by Cereceda [167] reaches its maximum value $\frac{\sqrt{3}}{2}$ and since in such a case [166]

$$C(|\psi\rangle) = \sqrt{3}\mathscr{N}(|\psi\rangle), \tag{3.51}$$

an upper bound for the Negativity equal to $\frac{1}{2}$ emerges in accordance with our previous conclusion. Thus no pure state in $\mathscr{H}_-$ exhibits maximum entanglement ($C(|\psi\rangle) = 1$).

It's possible to show that a generic normalized entangled state of the two qutrits in $\mathscr{H}_-$, saturating the negativity at the value $\mathscr{N}_\rho = \frac{1}{2}$, up to a global phase factor, may be parametrically represented as

$$\begin{aligned} |\Psi\rangle_{\mathscr{N}} \quad = \quad & \frac{1}{\sqrt{2}}\Big[\Big(\cos(\theta)|1\rangle + e^{i\phi}\sin(\theta)|-1\rangle\Big)_1 \otimes |0\rangle_2 \\ & +e^{i\Phi}|0\rangle_1 \otimes \Big(\cos(\theta')|1\rangle + e^{i\phi'}\sin(\theta')|-1\rangle\Big)_2\Big]. \end{aligned} \tag{3.52}$$

where θ, θ', ϕ, ϕ' and Φ freely run in $[0, 2\pi]$.

If we now consider as initial condition the two-qutrit state $|-10\rangle$, through the exact form of the time evolution operator in Eq. (3.56), it is easy to verify that

$$x(t) = |c_1(t)|^2 + |c_4(t)|^2 = |a_1|^2|a_2|^2 + |b_1|^2|b_2|^2 \tag{3.53}$$

Concerning the LMSZ scenario, at infinite time so we have

$$x(\infty) = P_1P_2 + (1-P_1)(1-P_2), \tag{3.54}$$

where the expressions of P_1 and P_2 are reported in Eq. (3.32) and (3.33), respectively. If we put the expression in Eq. (3.54) into Eq. (3.48), we get the asymptotic expression of the Negativity. In Fig. 3.2a such an expression of the negativity is reported against the LMSZ parameter $\beta = \beta_+$, for $\beta_-/\beta_+ = 1/2$. We see that two maxima are present and they correspond to the values $\log(2)/2\pi \approx 0.11$ and $\log(2)/\pi \approx 0.22$. It means that, by appropriately setting the parameter β, when the two-qutrit system start from the state $|-10\rangle$, through the LMSZ process we may generate asymptotically an entangled state of the two spin-qutrits with the maximum level of entanglement possible in such a subspace. This fact is confirmed by Fig. 3.2b where the time behaviour of the Negativity is reported against the dimensionless parameter $\tau = \sqrt{\alpha/\hbar}\, t$ for $\beta = 0.11$. In this case, we used the expression of $x(t)$ in Eq. (3.53) with the exact solution of the LMSZ dynamical problem [69] reported in Eq. (2.38). We emphasize that the parameter β, besides the asymptotic value, deeply influences the trend in time of the Negativity curve, as it can be appreciated by Figs. 3.2c and 3.2d, related to $\beta = 0.5$ and $\beta = 2$, respectively.

We stress that it is not possible to get physical information about the entanglement get established between the two qutrits by studying correlations emerging between the two fictitious qubits. Indeed, by the mapping in Eq. (3.18), it is easy to see that entangled states of the two qutrits, such as $(|10\rangle + |01\rangle)/\sqrt{2}$, correspond to separable states of the two qubits, $(|{+}{+}\rangle + |{+}{-}\rangle)/\sqrt{2}$, and, *vice versa*, separable states of the qutrits $(|10\rangle + |-10\rangle)/\sqrt{2}$ correspond to entangled states of the qubit system, $(|{+}{+}\rangle + |{-}{-}\rangle)/\sqrt{2}$. Such a feature stems from the non-locality of the mapping established between the two systems. This observation implies that, within the four-dimensional subspace, we cannot use the Concurrence, but we are obliged to consider another Entanglement measure. This is why we use Negativity to quantify the Entanglement get established between the two qutrits.

Five-dimensional subdynamics

General solution

We have seen before that the central block of H_+ has an su(2) structure and then it is interpretable as the Hamiltonian of a (fictitious) spin-1 subjected to (fictitious as well) magnetic fields [see Eq. (3.28)]. It is well known that the time evolution operator related to a 3x3 su(2) Hamiltonian may be put in the following form [40]

$$U_3 = \begin{pmatrix} a_3^2 & \sqrt{2}a_3b_3 & b_3^2 \\ -\sqrt{2}a_3b_3^* & |a_3|^2 - |b_3|^2 & \sqrt{2}a_3^*b_3 \\ b_3^{*2} & -\sqrt{2}a_3^*b_3^* & a_3^{*2} \end{pmatrix}, \tag{3.55}$$

where a_3 and b_3 are two time-dependent parameters, solution of the analogous dynamical problem for a single spin-1/2. In other words, a_3 and b_3 may be found by solving the dynamical problem of a single

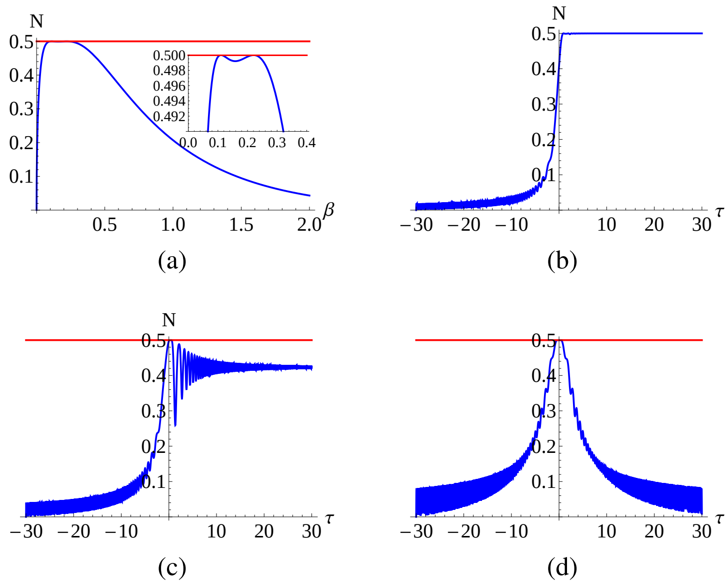


Figure 3.2: (Color online) a) β-dependence of the asymptotic Negativity of the two qutrits [Eqs. (3.48) and (3.54)] for the initial condition $|-10\rangle$. Time behaviour of the Negativity against the dimensionless parameter $\tau = \sqrt{\alpha/\hbar}\, t$ during a LMSZ process when the two-qutrit system starts from the state $|-10\rangle$ for $2\beta_- = \beta_+$ and b) $\beta_+ = 0.11$, c) $\beta_+ = 1/2$ and d) $\beta_+ = 2$. The upper straight curve represents $\mathscr{N} = 0.5$.

spin-1/2 subjected to the same magnetic field acting upon the fictitious spin-1. Thus, we may formally write the time evolution operator U_+, solution of the Schrödinger equation $i\hbar\dot{U}_+ = H_+U_+$, as follows

$$U_+ = \begin{pmatrix} e^{-\frac{i}{\hbar}\int\Omega_+} & 0 & 0 & 0 & 0 \\ 0 & a_3^2 & \sqrt{2}a_3b_3 & b_3^2 & 0 \\ 0 & -\sqrt{2}a_3b_3^* & |a_3|^2-|b_3|^2 & \sqrt{2}a_3^*b_3 & 0 \\ 0 & b_3^{*2} & -\sqrt{2}a_3^*b_3^* & a_3^{*2} & 0 \\ 0 & 0 & 0 & 0 & e^{\frac{i}{\hbar}\int\Omega_+} \end{pmatrix}. \tag{3.56}$$

Dark states

First of all, it is important to underline that also for the five-dimensional subdynamics we have dark states. Indeed, if the two qutrits are initially prepared in $|11\rangle$ or $|-1-1\rangle$, independently of the time-dependence of the z-magnetic field, the two-qutrit system remains in its initial state, also if the magnetic field component randomly fluctuates remaining along the z-direction. Moreover, if we consider the case $\omega_1(t) = \omega_2(t)$, also a generic state belonging to the three-dimensional subspace, namely

$$c_1|1-1\rangle + c_2|00\rangle + c_3|-11\rangle, \tag{3.57}$$

is completely unaffected by the presence of time-dependent magnetic fields, since in this instance $\Omega_-(t) = 0$ and the Hamiltonian governing the three-dymensional dynamics is simply $H_3 = \gamma\hat{\sigma}^x$. Such states, then, evolves only under the action of the coupling between the two qutrits. It means then that the three eigenstates of $\hat{\Sigma}^x$ rewritten in terms of two-qutrit states

$$\begin{aligned} |\psi_5^0\rangle &= \frac{|1-1\rangle + \sqrt{2}|00\rangle + |-11\rangle}{2}, \\ |\psi_6^0\rangle &= \frac{|1-1\rangle - |-11\rangle}{\sqrt{2}} \\ |\psi_7^0\rangle &= \frac{|1-1\rangle - \sqrt{2}|00\rangle + |-11\rangle}{2} \end{aligned} \tag{3.58}$$

result steady state of the two-qutrit system also when a unique homogeneous time-dependent field is applied on the two spin-1's. Consequently, every classical mixture of these three states does not evolve and every physical quantity related to this state is constant in time. Given that the states in Eq. (3.44) have the same property under the same conditions ($\omega_1(t) = \omega_2(t)$ and $\gamma_x = \gamma_y$), we may conclude that, in this scenario, the thermal state of the system and, more in general, every mixture involving the steady states $|11\rangle$, $|-1-1\rangle$ and the ones in Eqs. (3.44) and (3.58), namely

$$\rho = k_1|11\rangle\langle 11| + \sum_{j=1}^{7} p_j|\psi_j^0\rangle\langle\psi_j^0| + k_2|-1-1\rangle\langle -1-1|, \tag{3.59}$$

such that $k_1 + k_2 + \sum_j p_j = 1$, is a stationary state of the two-qutrit system.

STM scenario and LMSZ transition probabilities

We investigate now the STM experimental scenario characterized by a single local magnetic field on the first spin-1, namely $\omega_1(t) = \alpha t$, and the two qutrits initialized in the state $|1-1\rangle$. In this case the two-qutrit system behaves effectively like a three-level system (spin-1) subjected to a LMSZ ramp with an effective constant transverse magnetic field related to the coupling constant γ. For such a time-dependent scenario, the transition probabilities, from $|1-1\rangle$ to the other two states $|00\rangle$ and $|-11\rangle$, may be found analytically. Indeed, at the light of the spin-1 - spin-1/2 transition probability relationship based on the SU(2) group structure, for large time instants, we have

$$P_{-1}^{+1} = P_3^2, \quad P_{-1}^{0} = 2P_3(1-P_3), \quad P_{-1}^{-1} = (1-P_3)^2, \tag{3.60}$$

where $P_3 = (1-e^{-2\pi\beta'})$ and $\beta' = 2\gamma^2/\hbar\alpha$. Also in this case, we appreciate how the coupling between the two qutrits is responsible of an avoided crossing and a consequent full adiabatic LMSZ transition for the fictitious spin-1. In the previous expressions we have labelled with -1, 0 and 1 the states $|1-1\rangle$, $|00\rangle$ and $|-11\rangle$, respectively. The plots of the asymptotic probabilities are reported in Fig. 3.3 against the coupling-dependent LMSZ parameter β'. We see that the interplay between the coupling parameter

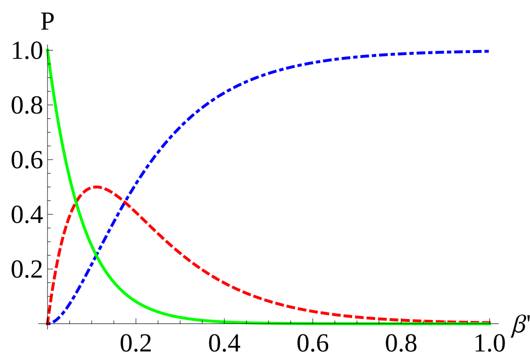


Figure 3.3: (Color online) Asymptotic LMSZ probabilities [Eq. (3.60)] of finding the two qutrits in the state $|1-1\rangle$ (blue dot-dashed line), $|00\rangle$ (red dashed line) and $|-11\rangle$ (green full line), when they start from the state $|-11\rangle$ for $\gamma_x = \gamma_y$.

γ and the ramp of the magnetic field α, defining β', deeply influences the transition probability. For high values of the parameter β' we get a complete LMSZ transition of both the spins, getting, also this time, a state transfer between the two qutrits. This means that, measuring the state of the system and varying the ramp α, we may estimate the parameter γ determining the strength of coupling between the two qutrits.

Noise effects

We consider now the field along the z axis affected by the random fluctuating contribution as done before. We may exploit again the results reported in Ref. [86] where the authors solved the dynamical problem of a noisy ramp in a LMSZ scenario also for a spin-1. In such a case, the transition probabilities affected by a noisy field component along the z-axis and characterized by the following time-correlation function $\langle \eta(t)\eta(t')\rangle = 2\Gamma\delta(t-t')$, become

$$
\begin{aligned}
P_{-1}^{+1} &= \frac{1}{6}(2+e^{-3\pi\beta'}-3e^{-\pi\beta'}),\\
P_{-1}^{0} &= \frac{1}{3}(1-e^{-3\pi\beta'}),\\
P_{-1}^{-1} &= \frac{1}{6}(2+e^{-3\pi\beta'}+3e^{-\pi\beta'}).
\end{aligned}
\tag{3.61}
$$

Also these expressions, valid for large values of Γ, are independent of the value of the same Γ. We see that, also this time, the main effect of the noise is to hinder the transition generating at most equally populated states when $\beta' \gg 1$. In this way, we brought to light how the symmetry-based analysis of the model reported in the second sections plays a key role for disclosing the exact quantum dynamics of the two interacting qutrits subjected to time-dependent magnetic fields, both in ideal and more realistic conditions.

Entanglement

In the three-dimensional subspace the Negativity for the general state in Eq. (3.57) reads

$$\mathscr{N} = |c_1||c_2| + |c_2||c_3| + |c_1||c_3|. \tag{3.62}$$

Its time evolution related to the initial condition $|-11\rangle$ results

$$\mathscr{N}(t) = |a_3||b_3|[\sqrt{2} + |a_3||b_3|], \tag{3.63}$$

and then asymptotically we get

$$\mathscr{N}(\infty) = P_3(1-P_3) + \sqrt{2P_3(1-P_3)}, \tag{3.64}$$

where P_3 is defined after Eqs. (3.60). This quantity reaches its maximum value for $P_3 = 1/2$ and then for $\beta' = \log(2)/2\pi \approx 0.11$ (see Fig. 3.4a). This means that, for such a value of the parameter β', the LMSZ process generates asymptotically an entangled state of the two qutrits with the maximum available value of Negativity for the initial condition under scrutiny, as confirmed by Fig. 3.4b. We got the latter figure by putting in Eq. (3.63) the expressions of a_+ and b_+ (or, equivalently, a_- and b_-) in Eqs. (2.38), replacing β_+ (β_-) with β'. In the same way we have analysed the time behaviour of the Negativity for the same initial condition for other two values of the parameter β', namely $\beta' = 1/2$ (Fig. 3.4c) and $\beta' = 2$ (Fig. 3.4d). Also this time we find that the LMSZ parameter deeply influences not only the asymptotic value but also the trend in time of the Negativity.

3.1.3 Summary and remarks

In this paragraph we have reported the study of the quantum dynamics of two interacting qutrits subjected to local time-dependent fields. We have taken into account the anisotropic as well as isotropic Heisenberg interaction. The field applied on just one of the two qutrits or on both the two spin-1's has been considered linearly varying on time (LMSZ ramp) along the quantization z-axis. Atomic species

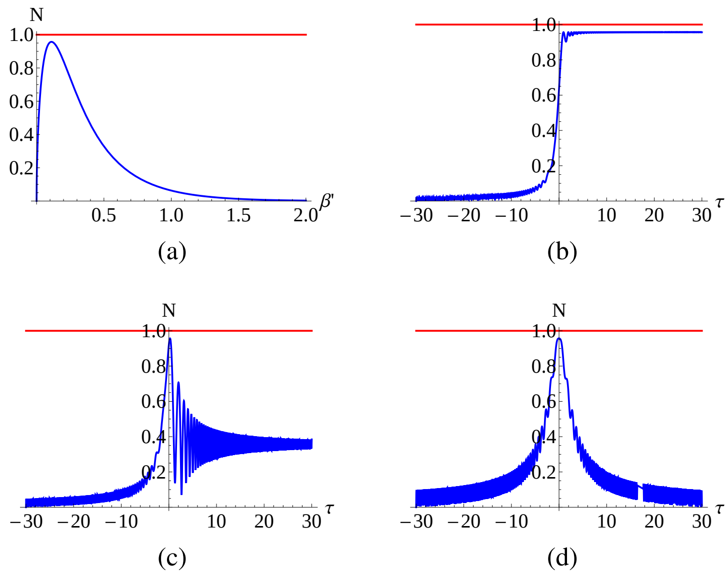


Figure 3.4: (Color online) a) β'-dependence of the asymptotic Negativity of the two qutrits [Eqs. (3.63)] for the initial condition $|-11\rangle$. Time behaviour of the Negativity against the dimensionless parameter $\tau = \sqrt{\alpha/\hbar}\, t$ during a LMSZ process when the two-qutrit system starts from the state $|-11\rangle$ for b) $\beta' = 0.11$, c) $\beta' = 1/2$ and d) $\beta' = 2$. The upper straight curve represents $\mathscr{N} = 0.5$.

with three metastable levels may be used in a linear ion crystal to realize the interacting spin-1 model under scrutiny through the application of laser fields [160, 161]. Moreover, a broad range of physical situations may be covered by such a model: two spin-1's in a double well optical lattice [157], interacting spin-1 nanomagnets [158] and effective interaction between two separated nitrogen-vacancy centres in diamond [159].

The dynamical problem has been solved thank to the reduction to two easier problems: one of two non-interacting fictitious spin-1/2's and the other of a fictitious three-level system. Such a reduction relies on the symmetry-based analysis of the Hamiltonian model which is unaffected by the time-dependences of the applied fields and, more generally, by the time-dependences of all Hamiltonian parameters. This means that the same analysis may be developed considering other possible time-dependences of the field leading to exactly solvable problems [35, 36, 37, 38].

The main result is the physical effect we called coupling-driven LMSZ transition. It consists in the fact that, although a transverse constant field is absent, LMSZ transitions between two-qutrit states are still possible thanks to the presence of the coupling between the two spin-1's. Indeed, the fictitious dynamics of the two decoupled qubits and the one of a fictitious spin-1 are characterized by a LMSZ longitudinal field and a fictitious constant transverse field stemming from the coupling existing between the spin-qutrits. This fact implies that, avoided crossings in the two qutrit system are possible thanks to the presence of such an interaction. A remarkable consequence of this circumstance consists in the fact that an appropriate ratio between the applied fields and the coupling parameters may result favourable for performing adiabatic dynamics with consequent full LMSZ transitions of the two spin-1 system. The knowledge of such a physical effect makes it possible to have control on the dynamics of the system under scrutiny as well as to get information about the interaction characterizing the same system. We have brought to light, moreover, how the LMSZ transition probabilities change according to the (an)isotropy of the coupling terms.

We have showed that the physical relevance of the coupling-driven LMSZ transitions is twofold. First, by the knowledge of the transition probabilities we may estimate the coupling parameters of the two-qutrit model. Second, basing on such an estimation, we illustrated that an appropriate and specific choice of the slope of the LMSZ ramp can generate asymptotically entangled states of the two qutrits. We have analysed the level of entanglement by studying both the asymptotic Negativity against the LMSZ parameters and its time evolution. In the latter case, we have used the exact solutions of the LMSZ dynamical problem [69] and we have investigated the effects of the coupling determining the LMSZ parameter. We reported how such a parameter, depending on the ratio of the squared coupling and the slope of the ramp, determines not only the asymptotic value, but also the trend of the Negativity.

Finally, we have discussed also how the LMSZ transition probabilities are modified by the presence of a noisy field component stemming from the interaction of the the two-qutrit system with a surrounding environment. Such an analysis is based on the fact that the dynamical reduction is unaffected by the presence of the noise and so, also in this case, we may reduce the two-spin-1 problem to easier problems whose solutions are known in literature. Following the same philosophy, we have exposed the possibility of treating exactly the problem also by introducing the environment effects with non-Hermitian terms in the Hamiltonian model.

It is worth to emphasize that the Hamiltonian model in Eq. (3.3) keeps its symmetry also for two larger spin systems, that is, for two interacting spins $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{J}}_2$. In such a case, it is always possible to decompose the dynamical problem into two sub-problems related to the two dynamically invariant subspaces linked to the two eigenvalues (1 and -1) of the constant of motion $\cos[\pi(\hat{J}_1^z+\hat{J}_2^z)]$. However, for larger spin systems, the sub-dynamics could be very difficult to solve due to the high degeneracy of both eigenvalues. The symmetry-based existence of two dynamically invariant subspaces regardless of the values of the two spins, indeed, does not represent the successful key of our approach in its own. What has been reported here, thus, has the merit of explicitly showing that the resolution of the related dynamical problems cannot be derived simply generalizing technical aspects characterizing the analogous dynamical problem of two qubits.

The results reported in this first paragraph have been published in Ref. [190, 191]

3.2 Two qudits

An isolated dimeric unit of ions, each one exhibiting an effective spin $\hat{\mathbf{J}}$, may be regarded, as a system of two interacting spins. For some compounds of dimeric units it has been experimentally proven that neglecting the couplings between spins in neighbouring units is legitimate, implying that the quantum dynamics of the same compound may be derived from that of a single dimer [4, 54]. Experimental and theoretical investigations on biradical compounds provide a further example of a physical system describable in terms of two interacting spins. In a liquid solution a compound of biradicals can be described by a symmetric spin Hamiltonian model [3]. Research activity involving biradical compound systems run from high controllable low dimensional quantum magnets realization to the study of Bose-Einstein condensation phenomena for magnetic excitations [47]. In the area of quantum computing, finally, spin Hamiltonian models describing the quantum dynamics of two electron spins in a double quantum dot [48, 168, 169, 170, 171] or in a double quantum well [21], furnish a theoretical basis for manipulating two-electron based qubits.

3.2.1 The model and the solution of the dynamical problem

Our physical system consists of two independent, localized and distinguishable spins of different value and physical nature, in general, represented by their relative spin operators $\hat{\mathbf{j}}_1$ and $\hat{\mathbf{j}}_2$, respectively, with $\hat{\mathbf{j}}_i \equiv (\hat{j}_i^x, \hat{j}_i^y, \hat{j}_i^z)$ $(i = 1,2)$. By definition $[\hat{j}_1^\alpha, \hat{j}_2^\beta] = 0$ $(\alpha, \beta = x, y, z)$ and $\hat{\mathbf{j}}_i \wedge \hat{\mathbf{j}}_i = i\hbar \hat{\mathbf{j}}_i$. The i-th spin is subjected to the local external controllable time-dependent magnetic field

$$\mathbf{B}_i(t) = B_i^x(t)\mathbf{c}_x + B_i^y(t)\mathbf{c}_y + B_i^z(t)\mathbf{c}_z, \tag{3.65}$$

such that

$$-\gamma_1 \mathbf{B}_1(t) = -\gamma_2 \mathbf{B}_2(t) \equiv \mathbf{\Omega}(t), \tag{3.66}$$

$\gamma_i = g_i \mu_0$ being the magnetic moment associated to the i-th spin, with g_i the appropriate Landé factor, and μ_0 the appropriate Bohr magneton. We observe that $\mathbf{B}_1(t)$ and $\mathbf{B}_2(t)$ are parallel (anti-parallel) if $\gamma_1\gamma_2 > 0$ (< 0). Condition (3.66) means that the two spins exhibit the same Zeeman spitting. The possibility of such a control of the magnetic field acting individually on each spin is in the grasp of experimentalists as realized in a double quantum dot system [50, 172].

Let us suppose that the two spins are in addition coupled via a ferromagnetic or anti-ferromagnetic isotropic Heisenberg interaction of strength λ, so that the corresponding Hamiltonian model may be written down as follows (from now on we set $\hbar = 1$):

$$H(t) = H_0(t) + H_I \tag{3.67}$$

with

$$H_0(t) = \sum_{i=1}^{2} \mathbf{\Omega}(t) \cdot \hat{\mathbf{j}}_i, \quad H_I = -\lambda \hat{\mathbf{j}}_1 \cdot \hat{\mathbf{j}}_2, \tag{3.68}$$

acting upon the $(2j_1+1)(2j_2+1)$-dimensional Hilbert space $\mathscr{H}$ of the two spins. We underline that with respect to the model analysed before for two qubits and two qutrits, here we consider only the isotropic exchange interaction and all the components of the time-dependent field acting upon the two qudits.

We emphasize that recent experimental advances in the area of ^{28}Si-based solid state quantum computing makes our Hamiltonian model (3.68) of some help to represent such a physical scenario [172]. In addition biradical compounds in liquid phase provide another interesting experimental situation usefully describable making use of our noiseless model [3].

To proceed with the analysis of the model it is useful to rewrite the Hamiltonian in terms of the total spin angular momentum operator $\hat{\mathbf{J}} = \hat{\mathbf{j}}_1 + \hat{\mathbf{j}}_2$, getting

$$H(t) = \mathbf{\Omega}(t) \cdot \hat{\mathbf{J}} - \frac{\lambda}{2}\hat{\mathbf{J}}^2 + K \tag{3.69}$$

where $K \equiv \frac{\lambda}{2}(\hat{\mathbf{j}}_1^2 + \hat{\mathbf{j}}_2^2)$ is proportional to the identity operator. Equation (3.69) clearly shows that our time-dependent Hamiltonian $H(t)$ commutes with $\hat{\mathbf{J}}^2 = (\hat{\mathbf{j}}_1 + \hat{\mathbf{j}}_2)^2$ and this implies that $\mathrm{Tr}\{\rho(t)\hat{\mathbf{J}}^2\} =$

$\mathrm{Tr}\{\rho(0)\hat{\mathbf{J}}^2\}$ at any time instant. Here $\rho(t) = U(t)\rho(0)U^\dagger(t)$, $U(t)$ being the unitary time evolution operator fulfilling the Cauchy problem

$$i\dot{U}(t) = H(t)U(t) \qquad U(0) = \mathbb{1} \tag{3.70}$$

and $\rho(0)$ the initial density matrix of the two spins.

The conservation of $\hat{\mathbf{J}}^2$ leads to the existence of $(j_1 + j_2) - |j_1 - j_2| + 1$ orthogonal, dynamically invariant subspaces $\mathscr{H}^{(j)}$ such that

$$\mathscr{H} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \mathscr{H}^{(j)} \tag{3.71}$$

$\mathscr{H}^{(j)}$ denoting the invariant $(2j+1)$-dimensional subspaces of $\hat{\mathbf{J}}^2$ pertaining to its eigenvalue $j(j+1)$. The Hamiltonian operator may be written as

$$H(t) = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} H^{(j)}(t), \tag{3.72}$$

and accordingly generates the time evolution operator, solution of the Cauchy problem defined in Eq. (3.70) in the form

$$U(t) = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} U^{(j)}(t). \tag{3.73}$$

$H^{(j)}(t)$ is the effective Hamiltonian of the two spins governing their dynamics in the $(2j+1)$-dimensional dynamically invariant subspace $\mathscr{H}^{(j)}$ of $H(t)$, whereas $U^{(j)}(t)$ is the related time evolution operator, solution of the (restricted) Cauchy problem

$$i\dot{U}^{(j)}(t) = H^{(j)}(t)U^{(j)}(t), \qquad U^{(j)}(0) = \mathbb{1}^{(j)}, \tag{3.74}$$

$\mathbb{1}^{(j)}$ being the identity operator in $\mathscr{H}^{(j)}$.

Since the term $K' \equiv -\frac{\lambda}{2}\hat{\mathbf{J}}^2 + K$ is proportional to $\mathbb{1}^{(j)}$ in $\mathscr{H}^{(j)}$, whatever j is, the effective Hamiltonian $H^{(j)}(t)$ governing the dynamics in $\mathscr{H}^{(j)}$ may be written as

$$H^{(j)}(t) = \mathbf{\Omega}(t) \cdot \hat{\mathbf{j}} + K', \tag{3.75}$$

which, formally, is the Hamiltonian of a fictitious spin $\hat{\mathbf{j}}$, with spin angular momentum j and magnetic moment γ_1, subjected to the time-dependent magnetic field $\mathbf{B}_1(t)$. Of course, due to Eq. (3.66) and Eq. (3.75), in this scenario γ_1 and $\mathbf{B}_1(t)$ may be replaced by γ_2 and $\mathbf{B}_2(t)$, respectively. This means that each effective Hamiltonian $H^{(j)}(t)$ possesses an su(2)-symmetry structure and the related time evolution operator $U^{(j)}(t)$ may be expressed [39, 173] in terms of the two time-dependent complex-valued functions, $a = a(t)$ and $b = b(t)$, entries of the evolution operator

$$U^{(1/2)}(t) = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad |a|^2 + |b|^2 = 1, \tag{3.76}$$

i.e. the solution of the Liouville-Cauchy problem (3.74) with $j = 1/2$ and $H^{(1/2)} = \mathbf{\Omega}(t) \cdot \frac{\hat{\boldsymbol{\sigma}}}{2}$. The Pauli vector is defined as $\hat{\boldsymbol{\sigma}} = \sigma^x \mathbf{c}_1 + \sigma^y \mathbf{c}_2 + \sigma^z \mathbf{c}_3$, σ^x, σ^y and σ^z being the Pauli matrices. The entries of the matrix $U^{(j)}(t)$ in the standard ordered basis of the eigenstates of the third component of the fictitious spin j: $\{|m\rangle, m = j, j-1, \dots, -j\}$, may be cast as follows [39] (time dependence is suppressed)

$$U^{(j)}_{m,m'}(a,b) = e^{-iK't} \sum_{\mu} C^{(j)}_{m,m'} a^{j+m'-\mu} (a^*)^{j-m-\mu} b^{m-m'+\mu} (b^*)^{\mu}, \tag{3.77}$$

where [39]

$$C^{(j)}_{m,m'} = (-1)^{\mu} \frac{\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}{\mu!(j+m'-\mu)!(j-m-\mu)!(m-m'+\mu)!}. \tag{3.78}$$

We point out that, whatever m and m' are, the summation, formally a series generated by μ running over the positive integer set, is a finite sum, generated by all the values of μ for which the denominator is finite; negative values of μ are then excluded. It is possible to convince oneself that, defined in this manner, $\left|U^{(j)}_{m,m'}(a,b)\right|^2$ represents the probability to find the N-level system in the state with z-projection m when it is initially prepared in the state with z-projection m'. Summing up, Eqs. (3.77) and (3.78) provide the solution of the Cauchy problem defined by Eq. (3.74) which, in turn and in view of Eq. (3.73), enables us to write down the exact time evolution operator solution of our main problem as defined by Eq. (3.70).

We emphasize that the possibility of expressing $U(t)$ in terms of only two time-dependent complex-valued functions may be traced back to the existence of su(2) structures nested in the Hamiltonian model given by Eq. (3.68). Such a property is a direct consequence of the symmetries possessed by the Hamiltonian model and paves the way to the exact determination of the evolution operator $U(t)$ generated by H.

This approach may be successfully exploited when $\mathbf{\Omega}(t)$ is such to allow the construction of explicit expression for $a(t)$ and $b(t)$ in a given specific physical situation. For example, when $\mathbf{\Omega}(t)$ coincides with that considered originally by Rabi [31], we are in condition to construct the explicit form of the evolution operator [31, 32] generated by the correspondent H given in Eq. (3.68) and as a consequence to investigate any aspect of the related quantum dynamics. It is thus of relevance that recently other su(2) time-dependent scenarios have been proposed and exactly solved [35, 36, 37, 38], with application to interacting spin systems according to approach exposed in this thesis. This circumstance opens the possibility of applying the approach here reported to several possible other scenarios of experimental interest, different from the one originally considered by Rabi. The exact knowledge of how two-spin systems evolve under controllable time-dependent magnetic fields might be exploited to comply, on demand, with technological needs or experimental requests.

3.2.2 Quantum dynamics

We investigate now possible effects of the exchange interaction H_I between $\hat{\mathbf{j}}_1$ and $\hat{\mathbf{j}}_2$ on the quantum dynamics each spin would experience if λ were absent. Let us denote by $|m_i\rangle$, $m_i = j_i, j_i - 1, \dots, -j_i$, a generic eigenstate of $\hat{j}^z_i$ $(i = 1,2)$. Suppose our two-spin system prepared in the state $|j_1, j_2\rangle \equiv |j_1\rangle|j_2\rangle$ belonging to $\mathscr{H}^{(j_1+j_2)}$. The probability $P^{-j_1,-j_2}_{j_1,j_2}(t)$ of finding the compound system in the state $|-j_1,-j_2\rangle \in \mathscr{H}^{(j_1+j_2)}$ at any time instant may be expressed as

$$P^{-j_1,-j_2}_{j_1,j_2}(t) = \left|U^{(j)}_{-j,j}(a,b)\right|^2 = \left|[-b^*]^{2j}\right|^2 = |b|^{4(j_1+j_2)}, \tag{3.79}$$

in view of Eq. (3.77), with $j = j_1 + j_2$.

This result means that, preparing the two spins in the factorized state $|j_1, j_2\rangle$, the probability of finding the system in the factorized state $|-j_1, -j_2\rangle$ is equal to the probability we would get when the same $\mathbf{\Omega}(t)$ is experienced by two *non-interacting* spins $\hat{\mathbf{j}}_1$ and $\hat{\mathbf{j}}_2$. It is possible to find a physical reason at the basis of the previous result by bringing to light remarkable features characterizing the quantum dynamics of our two-spin systems by considering the reduced dynamics of the two subsystems of interest.

The Liouville-Cauchy problem governing the time evolution of a generic state $\rho(0)$ of the two-spin system is

$$i\dot{\rho} = [H, \rho]. \tag{3.80}$$

In view of Eq. (3.73), its solution $\rho(t) = U(t)\rho(0)U^\dagger(t)$ is determined after solving the following Cauchy problem for a spin 1/2

$$i\dot{U}^{(1/2)} = H^{(1/2)}U^{(1/2)}, \quad U^{(1/2)}(0) = \mathbb{1}^{(1/2)}, \tag{3.81}$$

where $H^{(1/2)} = \mathbf{\Omega}(t) \cdot \hat{\mathbf{s}}$, with $\hat{\mathbf{s}} = \frac{1}{2}\hat{\boldsymbol{\sigma}}$.

The time evolution of the reduced density matrix of the i-th spin is related to the solution $\rho(t)$ of the Liouville-Cauchy problem (3.80) as follows

$$i\dot{\rho}_i = [H_i, \rho_i] + \mathrm{Tr}_{k\neq i}\{[H_I, \rho]\}, \tag{3.82}$$

where $H_i = \mathbf{\Omega}(t) \cdot \hat{\mathbf{j}}_i$, H_I is defined in Eq. (3.26) and $\rho_i(t)$ satisfies the initial condition $\rho_i(0) = \mathrm{Tr}_{k\neq i}\{\rho(0)\}$. The symbol $\mathrm{Tr}_{k\neq i}$ means tracing with respect to "the other spin". Equation (3.82) clearly shows that in correspondence to each $\rho(t)$ such that $[H_I(t), \rho(t)] = 0$ at any time instant, the i-th reduced density operator $\rho_i(t)$ satisfies the following Cauchy problem

$$i\dot{\rho}_i(t) = [H_i, \rho_i(t)], \quad \rho_i(0) = \mathrm{Tr}_{k\neq i}\{\rho(0)\}. \tag{3.83}$$

Since both H and H_I commute with $\hat{\mathbf{J}}^2$, it is easy to convince oneself that any density matrix $\rho(0)$, at any time satisfying $[H_I(t), \rho(t)] = 0$, may be represented in the coupled basis as

$$[\rho(0)]_{CB} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \rho^{(j)} \tag{3.84}$$

$\rho^{(j)}$ being a $(2j+1)$-dimensional semi-positive definite matrix such that $\rho(0)$ is a density matrix. As a consequence, when $\rho(0)$ belongs to the class of initial conditions given by Eq. (3.84), the solution of the Cauchy problem (3.83) may be written down as follows

$$\rho_i(t) = U_i(t)\rho_i(0)U_i^\dagger(t), \tag{3.85}$$

where $U_i(t)$ is the unitary operator governing the SU(2) time evolution of the spin $\hat{\mathbf{j}}_i$ when $\lambda = 0$.

In words, the symmetries of the Hamiltonian, under the condition (3.84), guarantee that each spin subsystem evolves as if the other one were absent, that is undergoing no influence stemming from the coupling term. It is worthwhile to remark that such a property holds whatever the magnetic field time-dependence is. At the light of this result we understand better and more deeply the result in Eq. (3.79):

since the factorized initial state $|j_1, j_2\rangle$ of the compound system belongs to the class of initial conditions given in Eq. (3.84), then, the joint probability of finding the two spins in the state $|-j_1, -j_2\rangle$ is nothing but the probability $|b|^{4j_1}$ of finding the spin j_1 in its state $|-j_1\rangle$ multiplied by the probability $|b|^{4j_2}$ of finding the spin j_2 in its state $|-j_2\rangle$.

In addition we recognize that the class of initial states given by Eq. (3.84) collects states being Interaction Free Evolving (IFE) states, recently reported in literature [174, 175, 176]. By definition they are pure or mixed states of a binary system evolving in time as if the interaction between the two subsystems were absent. We point out that, of course, any generic initial condition of the compound system presenting coherence terms between the different dynamically invariant subspaces of H and $\hat{\mathbf{J}}^2$ does not manifest such a peculiar dynamical feature since the reduced dynamics of the two spins is influenced by the existing isotropic Heisenberg coupling between the two spins.

It is finally worthwhile to observe that the initial entanglement between the two spins $\hat{\mathbf{j}}_1$ and $\hat{\mathbf{j}}_2$ in an arbitrary state of the class singled out by Eq. (3.84), does not change in time, whatever the entanglement measure adopted is. The physical reason may be traced back to the dynamical quenching of the interaction term stemming, in turn, from constraints on the evolution imposed by the symmetry properties possessed by our Hamiltonian model (3.26).

3.2.3 Dynamical effects due to the interaction parameter

To bring to light effects witnessing peculiar features in the quantum dynamic of the two coupled spins, we have to consider initial states generating coherences between different dynamically invariant subspaces of H. To this end, let us consider the $(2j_1+1)(2j_2+1)$ factorized states of the standard basis $\{|m_1, m_2\rangle; -j_1 \leq m_1 \leq j_1, -j_2 \leq m_2 \leq j_2\}$, ordered as

$$\begin{aligned} \Big\{ |j_1, j_2\rangle \equiv |e_1\rangle, |j_1, j_2-1\rangle \equiv |e_2\rangle, \ldots, |j_1, -j_2\rangle \equiv |e_{2j_2+1}\rangle, \\ |j_1-1, j_2\rangle \equiv |e_{2j_2+2}\rangle, |j_1-1, j_2-1\rangle \equiv |e_{2j_2+3}\rangle, \ldots, \\ |-j_1, j_2\rangle \equiv |e_{2j_2(2j_1+1)}\rangle, \ldots, |-j_1, -j_2\rangle \equiv |e_{(2j_1+1)(2j_2+1)}\rangle \Big\}, \end{aligned} \tag{3.86}$$

The projections of the factorized state $|\psi(0)\rangle = |j_1, j_2-1\rangle$ in the two invariant subspaces of $\hat{\mathbf{J}}^2$, labelled by (j_1+j_2) and (j_1+j_2-1), do not vanish and then the evolution of $|\psi(0)\rangle$ may be expressed as

$$|\psi(t)\rangle = \sum_k U_{k2} |e_k\rangle \tag{3.87}$$

where k runs from 1 to $(2j_1+1)(2j_2+1)$ generating the entries U_{k2} in the second column of $(U)_{SB}$, in accordance with our ordered standard basis.

To reach our goal, it is also important to choose appropriately the physical observable to be investigated. If we consider, e.g., the third component of the total spin angular momentum of the system $\hat{J}^z = \hat{j}_1^z + \hat{j}_2^z$, it is easy to verify that it commutes with the isotropic Heisenberg interaction $H_I = -\lambda \hat{\mathbf{j}}_1 \cdot \hat{\mathbf{j}}_2$. Since, in addition, $[H_0, H_I] = 0$ at any time instant, then

$$\begin{aligned} \langle\psi(t)|\hat{J}^z|\psi(t)\rangle &= \langle\psi_0(t)|\hat{J}^z|\psi_0(t)\rangle = \\ &= \langle\psi_{01}(t)|\hat{j}_1^z|\psi_{01}(t)\rangle + \langle\psi_{02}(t)|\hat{j}_2^z|\psi_{02}(t)\rangle \end{aligned} \tag{3.88}$$

where $|\psi_0(t)\rangle = U_0(t)|\psi(0)\rangle$, $|\psi_{01}(t)\rangle = U_{01}|j_1\rangle$ and $|\psi_{02}(t)\rangle = U_{02}|j_2-1\rangle$, with $\hat{j}_1^z|j_1\rangle = j_1|j_1\rangle$ and $\hat{j}_2^z|j_2-1\rangle = (j_2-1)|j_2-1\rangle$. In these expressions $U_0(t)$ is the unitary evolution operator generated by $H_0(t)$, whereas $U_{0i}(t)$ is that generated by $\mathbf{\Omega}\cdot\hat{\mathbf{j}}_i$. Thus, we predict the independence of $\langle\psi(t)|\hat{J}^z|\psi(t)\rangle$ from λ, regardless of the specific magnetic field acting upon the two-spin system. It is indeed possible to convince oneself that

$$\langle\hat{J}^z(t)\rangle = 0, \qquad \text{if } j_1 = j_2 = 1/2 \tag{3.89}$$

$$\langle\hat{J}^z(t)\rangle = \frac{|a|^2-|b|^2}{2}, \qquad \text{if } j_1 = 2j_2 = 1 \tag{3.90}$$

$$\langle\hat{J}^z(t)\rangle = |a|^2-|b|^2, \qquad \text{if } j_1 = j_2 = 1, \tag{3.91}$$

In order to predict a visible effect of the coupling between the spins, we calculate the time-dependence of the mean value of $\hat{j}_1^z$, getting

$$\langle\psi(t)|\hat{j}_1^z|\psi(t)\rangle = \sum_{i=0}^{2j_1}\sum_{k=1+i(2j_2+1)}^{(i+1)(2j_2+1)}(j_1-i)|U_{k2}|^2, \tag{3.92}$$

which in the three particular cases under scrutiny, leads respectively to the following explicit expressions

$$\langle\hat{j}_1^z(t)\rangle = \frac{1}{2}(|a|^2-|b|^2)\cos(\lambda t), \tag{3.93}$$

$$\langle\hat{j}_1^z(t)\rangle = \frac{1}{9}(|a|^2-|b|^2)\Big[5+4\cos(\frac{3\lambda}{2}t)\Big], \tag{3.94}$$

$$\langle\hat{j}_1^z(t)\rangle = \frac{1}{2}\Big[|a|^2-|b|^2+\Big(1-2|a|^2|b|^2(|a|^2-2|b|^2)\Big)\cos(2\lambda t)\Big], \tag{3.95}$$

with the short notation $\langle\hat{j}_1^z(t)\rangle = \langle\psi(t)|\hat{j}_1^z|\psi(t)\rangle$.

It is remarkable that by measuring the magnetization time-dependence of any one of the two spin subsystems we may experimentally recover information about the coupling strength, regardless of the applied magnetic field. This fact, in view of Eqs. (3.93), (3.94) and (3.95), enables us to check experimentally whether a direct interaction between the two spins exists or at least plays a non-negligible role in the Hamiltonian model describing the two-spin system in a given physical scenario.

3.2.4 Summary and remarks

In this paragraph we have brought to light that the problem of a binary system constituted by two qudits interacting through isotropic Heisenberg coupling and subjected to a general time-dependent field is reducible into a set of independent problems of single (fictitious) spin. Such property, being based on the structural symmetry imposed to the Hamiltonian model, holds whatever the time dependence of the controllable magnetic field is. This reducibility property, applicable to two quantum spins $\hat{\mathbf{j}}_1$ and $\hat{\mathbf{j}}_2$ of arbitrary magnitudes j_1 and j_2, represents then a new rather general result exploitable in several different physical contexts from condensed matter to quantum information, as underlined in the introductory part.

We provide ready-to-use SU(2)-based expressions of the unitary time evolution operator in terms of the two time-dependent complex-valued functions $a(t)$ and $b(t)$. These two functions determine the joint probability transition of the two spins from an initial state to a final state. It is remarkable that under appropriate initial conditions the reduced dynamics of each spin keeps unitarity, meaning that the initial state of the compound system behaves indeed as an IFE state. The time behaviour of the total magnetization as well as of each individual spin (supposed distinguishable) has been exactly forecasted. It deserves to be emphasized that, in principle, measurements of such time-dependences allow to achieve a feedback both on the coupling mechanism and, if confirmed as Heisenberg exchange interaction, on the coupling constant strength.

We underline that our dynamical reduction procedure is immune from effects stemming from degradation of unitary evolution due to the presence of classical random fields. This circumstance suggests the possibility to investigate effects stemming from noise, e.g. by adding to the ideal Hamiltonian model a fast fluctuating Gaussian field selected as a classical field, random both in its direction and intensity [177, 178].

As last remarks, we point out first that it is straightforward to make use of the approach reported in this paper to treat successfully the quantum dynamics of the Hamiltonian model given in Eq. (3.68) when the coupling constant λ is considered time-dependent too. Physical scenarios and experimental set-ups leading to time-dependent coupling constants between two subsystems have been recently reported [179]. Secondly, we notice that our approach does not lose its interest even when the experimental set-up in conjunction with the physical system under scrutiny prevent us from invoking distinguishability of two equal and interacting spins. In this case our approach still holds its validity provided we confine ourselves to any permutationally (symmetric or antisymmetric) invariant subspace $\mathscr{H}^{(j)}$ of the Hilbert space spanned by the eigenstates of the total angular momentum.

The results reported in this paragraph have been published in Ref. [192].

3.3 N interacting-qubit system

One of the attractive aspects in the physics of trapped ions and superconducting circuits stems from their dual relationship with quantum and semi-classical spin models. On one hand, we may effectively describe the dynamics of such kind of systems in terms of the language of spin systems. On the other hand, through these highly controllable technologies [180], we may reproduce and implement several types of spin interactions. Thus, trapped ions and superconducting circuits provide examples of quantum simulators of the dynamical behaviour of other quantum systems in accordance with the original seminal idea of Feynman [181], mathematically reformulated in terms of digital operations some years later [182].

A fascinating formal aspect of these quantum simulations is the mathematical occurrence of local N-wise spin-1/2 coupling terms in the Hamiltonian. Here N-wise means that the interaction among the N spins may be represented as an N-degree homogeneous multilinear polynomial in the $3N$ dynamical variables of all the N spins. Such a kind of coupling is of course alien to physical context like nuclear, atomic, and molecular physics. However, the usefulness of such N-spin Hamiltonian models has been recently brought to light in the treatment and the study of fermion lattice models where many-body interactions are present [42]. It is possible to implement many-body interactions of higher than second-order through both trapped ions techniques [41, 183] and superconducting transmon qubit arrays [8] by exploiting collective entangling operations [184, 185]. Their physical and technological importance stems from the possibility to ease several tasks of quantum information processing. In this manner, we may drive the generic many-qubit transition $|-\rangle^{\otimes N} \rightarrow |+\rangle^{\otimes N}$ to prepare multipartite Greenberger-Horne-Zeilinger (GHZ) states with a single operation and to implement stabilizer operators [108, 186] with local qubit rotations. This will allow for the implementation of topological codes [187], among other effects. Finally, the interest of studying higher order interactions may be found also in their relevance in describing better physical features and dynamical aspects of complex systems [188].

3.3.1 The model and its symmetries

We investigate the properties of a system of N distinguishable spin-1/2's subject to different magnetic fields and interacting in accordance to the following specific uniform N-wise interactions,

$$H = \sum_{k=1}^{N} \hbar\omega_k \hat{\sigma}_k^z + \gamma_x \bigotimes_{k=1}^{N} \hat{\sigma}_k^x + \gamma_y \bigotimes_{k=1}^{N} \hat{\sigma}_k^y + \gamma_z \bigotimes_{k=1}^{N} \hat{\sigma}_k^z. \tag{3.96}$$

Here, uniform means that no term mixing different components of different spins, e.g. $\hat{\sigma}_1^x \hat{\sigma}_2^y \hat{\sigma}_3^z \dots \hat{\sigma}_N^x$, is present in the Hamiltonian where only three "diagonal" terms appear. The coupling constants γ_x, γ_y and γ_z quantitatively characterize these three terms. $\hat{\sigma}^x$, $\hat{\sigma}^y$ and $\hat{\sigma}^z$ are the standard Pauli matrices and $\hbar\omega_k$ is the energy separation induced in the k-th spin by its relative magnetic field. We are able to exactly diagonalize this model by reducing it into a set of independent problems of single spin-1/2. It is worthwhile to note that our technique may be applied even when the Hamiltonian parameters are time dependent. This circumstance provides the key to govern the dynamics of all the spins by manipulating the time-dependent magnetic field acting upon only one out of the N spins.

The Hamiltonian (3.96) may be exactly diagonalized by means of a chain T of unitary transformations after which it may be put in the following form (see Appendix D.1). In the case of an odd number of spins, it reads

$$\tilde{H} \equiv T^\dagger H T = \hbar \left[\omega_1 + \sum_{k=2}^{N} \omega_k \prod_{k'=2}^{k} \sigma_{k'}^z \right] \hat{\sigma}_1^z + \gamma_x \hat{\sigma}_1^x + \left[(-1)^{\frac{N-1}{2}} \gamma_y \prod_{k=1}^{(N-1)/2} \sigma_{2k+1}^z \right] \hat{\sigma}_1^y + \left[\gamma_z \prod_{k=1}^{(N-1)/2} \sigma_{2k+1}^z \right] \hat{\sigma}_1^z, \tag{3.97}$$

whereas for an even number of spins, it assumes the form

$$\tilde{H} \equiv T^\dagger H T = \hbar \left[\omega_1 + \sum_{k=2}^{N} \omega_k \prod_{k'=2}^{k} \sigma_{k'}^z \right] \hat{\sigma}_1^z + \gamma_x \hat{\sigma}_1^x + \left[(-1)^{\frac{N}{2}} \gamma_y \prod_{k=1}^{N/2} \sigma_{2k}^z \right] \hat{\sigma}_1^x + \gamma_z \prod_{k=1}^{N/2} \sigma_{2k}^z. \tag{3.98}$$

The total unitary operator accomplishing this chained transformations may be written as

$$T = \frac{1}{2^{N-1}} \prod_{k=0}^{N-2} \left[\mathbb{1} + \hat{\sigma}_{N-(k-1)}^z + \hat{\sigma}_{N-k}^x - \hat{\sigma}_{N-(k+1)}^z \hat{\sigma}_{N-k}^x \right]. \tag{3.99}$$

We see that the only dynamical variable representing the k-th spin with $k \neq 1$ in $\tilde{H}$ (even and odd case), is $\hat{\sigma}_k^z$ which is constant of motion for $\tilde{H}$ even if $\frac{\partial}{\partial t}\tilde{H} \neq 0$. This means that we may treat all the

$\hat{\sigma}_k^z$ ($k \neq 1$) as numbers (+1 or -1) and the string of values of these constants of motion identifies one specific subspace out of the 2^{N-1} dynamically invariant Hilbert subspaces (this is the reason why we left the hat on top the operators). Therefore, treating $\hat{\sigma}_k^z$ ($k \neq 1$) as parameters, Eqs. (3.97) and (3.98) give us 2^{N-1} effective Hamiltonians of the first spin-1/2. Exploiting the explicit expression of T, the dynamics generated by each effective Hamiltonian is turned into the dynamics of the N-spin system which of course will take place in a 2×2 still invariant subspace of H as given by Eq. (3.96).

We remark that, in each of such two-dimensional subspaces, the dynamics involves a specific state of the standard basis (s.b.), $|s.b.\rangle$, and the relative flipped state, that is the one identified by $\bigotimes_k \hat{\sigma}_k^x |s.b.\rangle$. Then, we have, for example, subdynamics involving the following couples of states: $|+\rangle^{\otimes(N-m)}|-\rangle^{\otimes m}$ and $|-\rangle^{\otimes(N-m)}|+\rangle^{\otimes m}$ [$\hat{\sigma}_i^z|\pm\rangle = \pm|\pm\rangle$]. This means, in particular, that the dynamically invariant subspace identified by the eigenvalues $\sigma_k^z = 1$ (for all possible $k \neq 1$) involves the two states $|+\rangle^{\otimes N}$ and $|-\rangle^{\otimes N}$ of the N-spin system. This implies the possibility of easily generating GHZ states of the N-spin system through this kind of interactions between the spins, as it is well known in literature [8, 108]. Moreover, the added value of this model lies in the application of appropriately engineered (time-dependent) magnetic fields, in order to govern the transition of the spin system between the two states, or to manipulate in time the generation of specific superposition states. However, to this end, a time-dependent analysis of the problem is necessary.

Up to now, we have only considered a time-independent Hamiltonian model. Nevertheless, note that the same arguments discussed above hold for a time-dependent Hamiltonian as well. The mathematical reason is that the unitary transformation operator T is independent of the Hamiltonian parameters. In this way, we are able to break down the time-dependent Schrödinger equation for our N-spin system into a set of 2^{N-1} decoupled time-dependent Schrödinger equations (see Appendix D.2). This implies that an exactly solvable time-dependent scenario of a spin-1/2 could be an exactly solvable scenario for our N-spin system dynamics restricted in one of the 2^{N-1} dynamically invariant Hilbert subspaces. Therefore, the knowledge of exactly solvable problems of a single spin-1/2 subjected to a time-dependent magnetic field [31, 33, 35, 36, 37, 38]) becomes strategic.

3.3.2 Controllable quantum dynamics

Let us now consider the following specialized model

$$H = \hbar\omega_1 \hat{\sigma}_1^z + \gamma_x \bigotimes_k \hat{\sigma}_k^x \tag{3.100}$$

and the initial condition $|\psi(0)\rangle = |+\rangle^{\otimes N}$ ($\hat{\sigma}^z|\pm\rangle = \pm|\pm\rangle$). In this instance, following our previous symmetry-based analysis, the problem is reduced to the following fictitious single-spin-1/2 problem $\tilde{H} = \hbar\omega_1 \hat{\sigma}_1^z + \gamma_x \hat{\sigma}_1^x$, regardless of the parity of N. If we now suppose that ω_1 varies over time in such a way to produce a perfect inversion of the fictitious spin or a balanced superposition between the states $|+\rangle_1$ and $|-\rangle_1$, it means that in the language of the N spins we are producing, respectively, a perfect inversion of all the spins at the same time and a GHZ state of our N-spin system. These two cases are considered in the figure below where the exact probability transition $P_+^-(t)$ of finding the N-spin system in the state $|-\rangle^{\otimes N}$ starting from $|+\rangle^{\otimes N}$ is reported for two different time-dependences of the magnetic field acting upon the first spin, against the dimensionless time $\tau = \gamma_x t/\hbar$. The expressions both of the magnetic fields (Figs. 3.5a and 3.5c) and the related transition probabilities (solid lines in Figs. 3.5b and 3.5d, rspectively) are analytically derived by exactly solving the single-spin-1/2 dynamical problem [38]. The time-dependences of the fields and the related analytical solution of the dynamical problems are the ones reported in Secs. 1.2.2 and 1.3.2. Thus, these cases represent exactly solvable time-dependent scenarios for the dynamics of the N-spin system restricted to the two-dimensional subspace involving the states $|+\rangle^{\otimes N}$ and $|-\rangle^{\otimes N}$. Analogously, we may generate Rabi oscillations between the

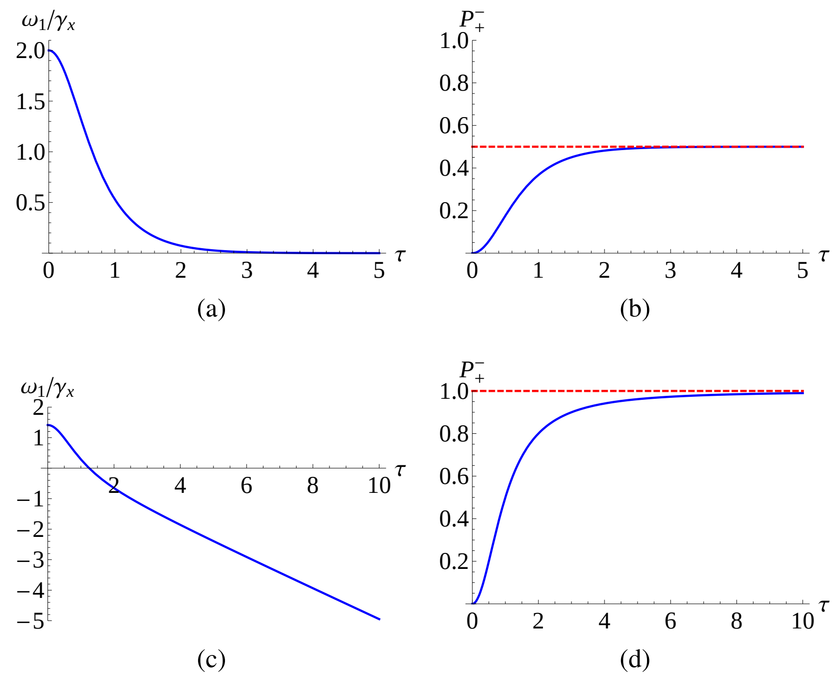


Figure 3.5: (Color online) Time dependences of the magnetic field acting upon the first spin of the chain [a) and c)] with the related time behaviour of the transition probability from $|+\rangle^{\otimes N}$ to $|-\rangle^{\otimes N}$ [b) and d), respectively]. The first case consists in generating a superposition between the two states; the second case realizes an inversion of all spins in the system. The constant dashed lines in Figs. b) and d) represent $P_+^- = 1/2$ and $P_+^- = 1$, respectively.

two states of the $N-$spin system involved in the subdynamics, by appropriately varying over time the parameter γ_x [31]. This means that, through the N-spin model under scrutiny, we may govern the dynamics of the whole N-spin system by appropriately engineering in time either the magnetic field acting upon only the first spin (ancilla qubit) or the coupling parameter or both.

3.3.3 Selective interaction-based cooling effect

Now we want to discuss a possible application of experimental interest aiming at attaining a *cooling effect* of the whole spin system. It is based on the possibility of selecting the invariant subspace wherein the N-spin-system dynamics occurs by appropriately varying over time the coupling parameter(s). This idea was presented for the first time in [189] but developed within another physical context. To this end, let us consider the following specialized model

$$H = \sum_{k=1}^{N} \hbar\omega_k \hat{\sigma}_k^z + \gamma_x(t) \bigotimes_k \hat{\sigma}_k^x + \gamma_y(t) \bigotimes_k \hat{\sigma}_k^y, \tag{3.101}$$

with $\gamma_x(t) = \gamma\cos(\nu t)$ and $\gamma_y(t) = \gamma\sin(\nu t)$ and for an odd number of spins. In this way, in each subdynamics we have Rabi oscillations between the two involved standard basis states. We know that these oscillations occur with maximum probability when the oscillation frequency of the (fictitious) transverse magnetic field (ν), is equal to the characteristic frequency between the two energy levels. Let us analyse the following conditions: 1) the characteristic frequencies of all spins are much larger than the coupling constant, $\omega_k \gg \gamma/\hbar$; 2) the oscillation frequency of the coupling constants (ν) matches the resonant condition in a specific subspace. In these instances, then, we obtain a complete oscillating behaviour in this '*selected*' subspace, while in all the other ones the system dynamics is frozen since the transition probability is negligible.

To be more explicit, let us consider for simplicity three spins and the initial condition involving all the states characterized by the first spin (ancilla qubit) in the state $|+\rangle$. In each subspace the probability of transition from the effective state $|+\rangle$ to the effective state $|-\rangle$ reads $P_+^-(t) = \frac{(\gamma/\hbar)^2}{(\gamma/\hbar)^2+\Delta_n^2} \sin\left(\frac{\omega_R}{2}t\right)$, with $\omega_R = \sqrt{\Delta_n^2 + (\gamma/\hbar)^2}$ and $\Delta_n = \left[\omega_1 + \sum_{k=2}^{3} \omega_k \prod_{k'=2}^{k} \hat{\sigma}_{k'}^z\right] + \nu_n$. Here n discriminates the different sub-dynamics and $\nu_n = \pm\nu$ depending on the sub-space, as it is clear by Eq. (3.97). It is easy to verify that if we assume now, for example, $\nu = -\sum_{k=1}^{3} \omega_k$ we have complete oscillations, that is $P_+^-(t) = \sin\left(\frac{\omega_R}{2}t\right)$, only in the subspace involving the two states $|+\rangle^{\otimes 3}$ and $|-\rangle^{\otimes 3}$. In the other subspaces, instead, providing that the condition $\omega_k \gg \gamma, \forall k$ is satisfied, the probability of transition is negligible and the dynamics is frozen, in the sense that the state remains the initial one.

This coupling-based dynamical selectivity turns out to be of particular relevance in the light of the following application of experimental interest. In the case of three spins under scrutiny, for the sake of the simplicity, let us take into account the following initial condition

$$\begin{aligned} \rho(0) =& |+\rangle\langle+|_1 \otimes [p_1|++\rangle\langle++| + \\ & + p_2|+-\rangle\langle+-| + p_3|-+\rangle\langle-+| + p_4|--\rangle\langle--|]. \end{aligned} \tag{3.102}$$

At the light of the previous discussion, considering a π-pulse, it is easy to verify that we may write the state at time t as

$$\begin{aligned} &\rho(t) \approx p_1|-\rangle\langle-|_1 \otimes |--\rangle\langle--| + \\ &+ |+\rangle\langle+|_1 \otimes [p_2|+-\rangle\langle+-| + p_3|-+\rangle\langle-+| + p_4|--\rangle\langle--|]. \end{aligned} \tag{3.103}$$

Thus, a measurement act on the first spin projecting it in the state $|-\rangle_1$, has the effect to project all other spins too into their down-states. Therefore, through an ancilla qubit and the specialized interaction model under scrutiny leading to a selective coupling, we may produce what we may call a *selective-interaction-based cooling effect* of the spin system. It is easy to understand that an analogue result may be obtained also for a greater odd number of spins.

It is important to stress that the previous procedure and result are not valid in case of an even number of spins. This is due to the fact that, as we can see in Eq. (3.98), in each subdynamics we have an effective transverse magnetic field only along the x-direction and then the Rabi scenario with the related dynamics cannot be reproduced. However, additional appropriate conditions help to circumvent the N-parity constraint giving rise even in this case to a similar result of selective-interaction-based cooling effect. Let us consider, for simplicity, only the coupling in the x-direction, that is the following further simplification of Eq. (3.96): $H(t) = \sum_{k=1}^{N} \hbar\omega_k \hat{\sigma}_k^z + \gamma_x(t) \bigotimes_k \hat{\sigma}_k^x$. It is possible to see that if ω_1 is sufficiently greater than all the other ω_k we may use the RWA [43] in each subspace and then, in this instance, we restore the presence of Rabi oscillations in each subspace. Matching the oscillation frequency of the coupling constant $\gamma_x(t)$ with the characteristic frequency of the subspace involving the states $|+\rangle^{\otimes N}$ and $|-\rangle^{\otimes N}$, namely $\nu = \sum_{k=1}^{N} \omega_k$, we obtain complete oscillations only in such a subspace. The other sub-dynamics, instead, will be characterized by a frozen dynamics, provided that $\omega_k \gg \gamma, \forall k$. Therefore, if the system starts from the analogous state written in Eq. (3.102), we achieve also for an even number of spins the 'cooling effect' thanks to the possibility to select a specific subspace in which the N-spin dynamics takes place. As a last remark it is worth to note that in this last case the result is based on the RWA, while in the different scenario for an odd number of spins, the result previously exposed is exact. It is interesting to note that this aspect can be seen also as an N-parity-dependent physical response of the system.

3.3.4 Summary and remarks

In this paragraph we have exactly solved a time-dependent model of N spin-1/2 systems comprising highly non-local interactions. First, we have shown that, thanks to non-local N-order interaction terms, it is possible to reverberate to all the spins in the system the dynamical effects generated in one of the N spins (ancilla qubit) by the application of a time-dependent magnetic field. This allows us to generate easily GHZ sates or a contemporary perfect inversion of all the spins. Second, we proposed a protocol through which we may generate a cooling effect of the whole spin system based on what we called selective interaction. The latter consists in the possibility to select a specific dynamically invariant subspace for a non-trivial dynamics of the N-spin system, by appropriately engineering the time-dependence of the coupling parameters.

The key to get such physical results lies on the possibility to solve exactly the dynamics of the N-spin system by reducing the problem into a set of independent dynamical problems of single spin-1/2's. As a final remark, it is worth noticing that this fact, identifiable as a result itself, makes possible the study of the dynamics of the system also when we consider random fluctuating components of the magnetic fields [86, 177]. This would permit to analyse possible effects on the dynamics of the N spins stemming from the coupling with an environment and to consider, then, situations closer to the experimental ones.

The results reported in this last paragraph have been published in Ref. [193].

Chapter 4

Beyond Spin Systems

The main scope of this chapter is to show new and wider horizons for the applicability of the method previously adopted to treat and solve semi-classical spin models and then physical systems living in finite-dimensional Hilbert spaces. Here, indeed, we analyse the dynamics of two interacting quantum harmonic oscillators whose Hamiltonian presents time-dependent parameters. Moreover, we compare this system with the one in which one of the two oscillators is substituted with an inverted quantum harmonic oscillator, known in literature as Glauber amplifier.

4.1 Glauber amplifier interacting with a quantum oscillator

In 1982 Glauber introduced the idea of quantum amplifier [194, 195] modelled through an inverted harmonic oscillator, that is, a system whose Hamiltonian may be represented as

$$-\hbar\omega(\hat{c}^\dagger\hat{c}+1/2), \tag{4.1}$$

where $\hat{c}$ and $\hat{c}^\dagger$ are bosonic operator satisfying the usual commutation rule $[c,c^\dagger]=1$. The eigenstates of the Hamiltonian are [196]

$$|n\rangle = \frac{(\hat{c}^\dagger)^n}{\sqrt{n!}}|0\rangle \tag{4.2}$$

where the null state is defined as $\hat{c}|0\rangle = 0$. It is important to point out that such a state is not the ground state of the system and, in particular, that the inverted oscillator does not possess a ground state. Moreover, it is worth noticing that the role of the operators $\hat{c}$ and $\hat{c}^\dagger$ is reversed with respect to that of the standard $\hat{a}$ and $\hat{a}^\dagger$ operators. Indeed, the operator $\hat{c}^\dagger$, increasing the number n of excitations, moves the inverted oscillator system towards lower energy states (and vice versa for $\hat{c}$).

In Ref. [194, 195] Glauber studies the thermodynamics of the amplifier system when it interacts with a bath of standard quantum harmonic oscillators. The same system is analysed in detail in Ref. [196]. In order to take into account interaction terms preserving the energy of the system, Glauber first considers the following interaction model: $\sum_i \hbar\omega_i\hat{a}_i^\dagger\hat{a}_i - \hbar\omega_c\hat{c}^\dagger\hat{c} + \sum_i(k_i\hat{a}_i\hat{c} + k_i^*\hat{a}_i^\dagger\hat{c}^\dagger)$, where $\hat{a}_i$ and $\hat{a}_i^\dagger$ are the bosonic operators of the i-th standard oscillator. Glauber makes evident the peculiar features of such a 'non-standard' system by making a comparison with the more familiar system comprising a 'standard' harmonic oscillator interacting with the bath.

The interest of Glauber in studying such a kind of system stems from fundamental issues concerning quantum mechanics [194, 195]. Precisely, he proposed the inverted oscillator as a toy-system through which investigate and explain irreversible processes such as wavefunction collapse and exponential dynamics, that is the exponential decaying behaviour of physical quantities. Such observed phenomena, indeed, as we know, cannot be deduced from dynamical equations and first principles at the basis of quantum mechanics. They are often described as result of phenomenological coupling with environment, even if some times this coupling is too small to correctly explain the observed decay rate. The strategy of Glauber to circumvent the phenomenological approach by introducing the inverted quantum harmonic oscillator, has the merit to introduce the irreversibility as a product of intrinsic dynamical processes, avoiding the consideration of a disturbing bath.

The idea of an inverted quantum harmonic oscillator behaving as amplifier is related to the basic issue concerning the measurement process in quantum mechanics. The measurements act, indeed, requires to be formally depicted as an amplification process, characterized by a strong irreversibility in order to avoid paradoxes such as the well-known Schrödinger's cat. The quantum nature of the Glauber amplifier, indeed, makes the amplification process both noisy and irreversible in order to prevents the occurrence of strange undesired phenomena [195].

The inverted harmonic oscillator, of course, is an ideal system impossible to be experimentally realized. However, what we are interested in are physical systems that, under specific conditions, behaves in such a way that they can be mathematically described by a quantum Glauber amplifier interacting with a bath of quantum harmonic oscillators. Two examples of such systems are: 1) a single atom with huge angular momentum J and subjected to a magnetic field; 2) a set of N two-level atoms identically coupled to the same field. When the systems start from an eigenstate of J_z (in case of N spin-1/2 the total component $J_z = \sum_i j_i^z$) with value not to far from J, their superfluorescent emission dynamics can be well approximated with that of the Glauber system. Correctly speaking, such systems are non-linear amplifiers, indeed the acceleration of the radiation rate do not continue indefinitely; however, the dynamical regime related to large values of J_z can be quite accurately described in terms of linear quantum amplifiers [194, 195].

It is relevant to highlight that the first experimental realization of a Glauber-like system has been realized in non-linear optics context through shock wave generation [197]. In this case we speak of Glauber-like amplifier since the system is properly described by the Hamiltonian of a reversed quantum harmonic oscillator rather than an inverted one: it is characterized by positive kinetic energy and a neative potential energy (in the Glauber amplifier, instead, both the energy contributions are negative). The Glauber amplifier presents thus a high potentiality both from a theoretical point of view stimulating innovative tests to fundamental physical theories and also in the applicative side in designing new devises as lasers or amplifiers.

In the following paragraphs we study the dynamics of a standard quantum oscillator coupled with a quantum Glauber amplifier when the Hamiltonian parameters are time-dependent. We are interested in the interaction terms conserving the number of excitations, rather than the energy of the system. Through the Jordan-Schwinger map, the oscillator-amplifier dynamical problem, within each dynamically invariant Hilbert space related to a precise excitation number N, is reduced into that of a single spin of value $N/2$. In this way, we are able to formally construct the time evolution operator and get the exact dynamics of the system for specific initial conditions under precise time-dependent scenarios, such as the Rabi [31, 32] and the Landau-Majorana-Stückelberg-Zener [33] ones. We calculate the mean value of the energy when the system is initially prepared in the generalized NOON state

$(\cos(\theta)|N0\rangle + e^{i\phi}\sin(\theta)|0N\rangle)/\sqrt{2}$. Furthermore, following the same spirit of the Glauber's work, we compare the dynamics of the quantum oscillator-amplifier system with that of two interacting standard quantum oscillators described by the analogous time-dependent Hamiltonian model preserving the total number of excitations. Indeed, also in this case we are able to explicitly write the formal expression of the time evolution operator and the specific one when precise time-dependent scenarios are considered. Remarkable differences between the two dynamical systems are brought to light by studying the transition probability between the states $|10\rangle$ and $|01\rangle$ and the mean value of the energy for the NOON state $(|10\rangle + |01\rangle)/\sqrt{2}$.

4.1.1 Hamiltonian model and solution of the dynamical problem

Let us consider a quantum optical system comprising a quantum oscillator interacting with a Glauber amplifier, namely:

$$H_{ho}(t) = \frac{\Omega(t)}{2}(\hat{\alpha}^\dagger\hat{\alpha} - \hat{\beta}^\dagger\hat{\beta}) + \omega(t)\hat{\alpha}^\dagger\hat{\beta} + \omega^*(t)\hat{\beta}^\dagger\hat{\alpha}, \tag{4.3}$$

It can be verified that $\hat{\mathscr{N}} = \hat{\alpha}^\dagger\hat{\alpha} + \hat{\beta}^\dagger\hat{\beta}$ is constant of motion of $H_{ho}(t)$ with integer eigenvalues $N = n_1 + n_2 = 1,2,3,\ldots$. The infinite Hilbert space, thus, can be subdivided into finite Hilbert subspaces each of which related to a precise number of collective excitations and so of different increasing dimension. It is worth pointing out that, on the basis of the Jordan-Schwinger map:

$$\hat{S}_+ = \hat{\alpha}^\dagger\hat{\beta}, \quad \hat{S}_- = \hat{\beta}^\dagger\hat{\alpha}, \quad \hat{S}_z = \frac{1}{2}(\hat{\alpha}^\dagger\hat{\alpha} - \hat{\beta}^\dagger\hat{\beta}), \tag{4.4}$$

the effective Hamiltonian governing the dynamics of the quantum optical system can be mapped in each dynamically invariant subspace into that of a spin s, namely

$$H_s(t) = \omega(t)\hat{S}_+ + \omega^*(t)\hat{S}_- + \Omega(t)\hat{S}_z, \tag{4.5}$$

where the value of the spin is linked to the number of total excitations by $s = N/2$. This is the key-point which allows us to derive the exact analytical expression of the time evolution operator of the two coupled quantum harmonic oscillator model.

We know [39, 40] that the time evolution operator $U_{1/2}$ of $H_{1/2}$ for a spin 1/2 may be written as

$$U_{1/2} = \begin{pmatrix} a(t) & b(t) \\ -b^*(t) & a^*(t) \end{pmatrix} \tag{4.6}$$

where $a \equiv a(t)$ and $b \equiv b(t)$ are two parameter time-functions, being solutions of the system in Eq. (1.16) stemming directly from the equation $i\hbar\dot{U}1/2 = H_{1/2}U_{1/2}$.

The time evolution operator U_s, solution of the equation $i\dot{U}_s = H_sU_s$, in the standard ordered basis of the eigenstates of the third component ($\hat{s}^z$) of the spin s: $\{|m\rangle, m = s, s-1, \ldots, -s\}$, may be written in terms of the same two parameter time-functions a and b [39, 40] as reported in Eqs. (3.77) and (3.78). Therefore, this means that by solving the problem for a single spin-1/2 we may derive and construct the solution for the analogous problem of a generic spin s subjected to the same time-dependent magnetic field.

We noticed before that the total Hilbert space $\mathscr{H}$ of the two quantum harmonic oscillators is divided into dynamically invariant and orthogonal Hilbert subspaces $\mathscr{H}_N$ related to the different integer eigenvalues of the integral of motion $\hat{\mathscr{N}}$, that is the different values of collective excitations of the system. We may write so

$$\mathscr{H} = \bigoplus_N \mathscr{H}_N, \tag{4.7}$$

with $N = n_1 + n_2$, and consequently the time evolution operator V of H_{ho} may be cast in the following form

$$V = \bigoplus_N V_N, \tag{4.8}$$

where V_N is a unitary operator responsible of the time evolution of the two harmonic oscillators in the subspace with N excitations. In this manner it is easy to see that V possesses the property

$$\langle n_1, n_2 | V | m_1, m_2 \rangle = \begin{cases} \neq 0, & n_1 + n_2 = m_1 + m_2 \\ 0, & n_1 + n_2 \neq m_1 + m_2 \end{cases} \tag{4.9}$$

reflecting clearly the orthogonality between the Hilbert subspaces related to different values of total excitations.

By taking into account the following equality $V_N = U_{N/2}$, on the basis of the J-S mapping, it is easy to check that the general probability amplitude in the coordinate representation, result

$$\begin{aligned} \langle x_1', x_2' | V | x_1, x_2 \rangle &= \sum_{n_1,n_2,m_1,m_2=0}^{\infty} \langle x_1', x_2' | n_1, n_2 \rangle \langle n_1, n_2 | V_N | m_1, m_2 \rangle \langle m_1, m_2 | x_1, x_2 \rangle \\ &= \sum_{N=0}^{\infty} \sum_{n,m=0}^{N} \langle x_1', x_2' | n, N-n \rangle \langle n, N-n | U_{N/2} | m, N-m \rangle \langle m, N-m | x_1, x_2 \rangle, \end{aligned} \tag{4.10}$$

where we used the completeness relation $\sum_{n_1,n_2=0}^{\infty} |n_1, n_2\rangle\langle n_1, n_2| = 1$, writeable as

$$\sum_{N=0}^{\infty} \sum_{n=0}^{N} |n, N-n\rangle\langle n, N-n| = 1, \tag{4.11}$$

with $n_1 + n_2 = N$. We see that the final expression in (4.10) is well defined since the general term $\langle n, N-n | U^{(N/2)} | m, N-m \rangle$ may be recovered by Eqs. (3.77) and (3.78), while from the basic books of quantum mechanics it is well known that [43]

$$\langle x | n \rangle = \frac{1}{\pi^{1/4}\sqrt{2^n n!}} \frac{1}{x_0^{n+1/2}} \left(x - x_0^2 \frac{d}{dx} \right)^n \exp\left\{ -\frac{1}{2} \left(\frac{x}{x_0} \right)^2 \right\}, \tag{4.12}$$

with $x_0 = \sqrt{\hbar / m\tilde{\omega}}$, where m and $\tilde{\omega}$ are the mass and the angular frequency of the classical oscillator, respectively. However, it is important to point out that, although we may write the formal expression of $\langle x_1', x_2' | V | x_1, x_2 \rangle$, such a formula cannot be practically exploited since in such a case an infinite number of invariant subspace are involved; the same happens, e.g., for coherent states.

4.1.2 Time evolution and energy mean value for NOON states

Our analysis results useful when initial conditions involving a finite number of subspaces are considered. In this respect, let us study the generalized NOON states belonging to the different subspaces, namely

$$|\Psi_N^{\theta,\phi}(0)\rangle = \cos(\theta)|N0\rangle + e^{i\phi}\sin(\theta)|0N\rangle, \qquad N=,1,2,\dots \tag{4.13}$$

On the basis of our previous analysis, it is easy to see that the evolved state of the general NOON state can be formally written as $|\Psi_N^{\theta,\phi}(t)\rangle = V_N|\Psi_N^{\theta,\phi}(0)\rangle = U_{N/2}|\Psi_N^{\theta,\phi}(0)\rangle$.

It is possible to persuade oneself that, for a general excitation number N, we have

$$\begin{aligned}
&\langle\Psi_N^{\theta,\phi}(t)|\frac{\hat{\alpha}^\dagger\hat{\alpha}-\hat{\beta}^\dagger\hat{\beta}}{2}|\Psi_N^{\theta,\phi}(t)\rangle \\
&= \frac{N}{2}(|a|^2-|b|^2)\cos(2\theta) + \mathrm{Re}[ab^*e^{-i\phi}]\sin(2\theta)\delta_{1N}, &(4.14a)\\
&\langle\Psi_N^{\theta,\phi}(t)|\hat{\alpha}\hat{\beta}^\dagger|\Psi_N^{\theta,\phi}(t)\rangle = [\langle\Psi_N^{\theta,\phi}(t)|\hat{\alpha}^\dagger\hat{\beta}|\Psi_N^{\theta,\phi}(t)\rangle]^\dagger \\
&= -Nab\cos(2\theta) + \frac{a^2e^{-i\phi}-b^2e^{i\phi}}{2}\sin(2\theta)\delta_{1N}. &(4.14b)
\end{aligned}$$

From the previous expression it is easy to check that for $N \geq 2$ and $\theta = \pi/4$ the three quantities vanish. It is worth pointing out that such a circumstance is independent of the specific time-dependence of the Hamiltonian parameters. This fact means that in the case of the initial condition $\Psi_1^{\pi/4,\phi}(0)$ the time evolution of the mean value of the energy reads

$$\langle\Psi_1^{\pi/4,\phi}(t)|H(t)|\Psi_1^{\pi/4,\phi}(t)\rangle = \Omega\mathrm{Re}[ab^*e^{-i\phi}] + \mathrm{Re}[\omega^*(a^2e^{-i\phi}-b^2e^{i\phi})], \tag{4.15}$$

while the following classes of NOON states

$$|\Psi_N^{\pi/4,\phi}(0)\rangle = \frac{|N0\rangle + e^{i\phi}|0N\rangle}{\sqrt{2}}, \qquad N \geq 2, \tag{4.16}$$

independently of the time-dependent scenario, exhibit a constant vanishing mean value of the energy in time, that is:

$$\langle\Psi_N^{\pi/4,\phi}(t)|H(t)|\Psi_N^{\pi/4,\phi}(t)\rangle = 0, \tag{4.17}$$

where $|\Psi_N^{\pi/4,\phi}(t)\rangle = V_N|\Psi_N^{\pi/4,\phi}(0)\rangle = U_{N/2}|\Psi_N^{\pi/4,\phi}(0)\rangle$. We underline that such a result is a particular case due to the coincidence of different factors: the symmetry of the states, the su(2) symmetry of the dynamics and the specific operators we have taken into account. Indeed, it is possible to verify that if we consider, in the case $N=2$, the state $(|20\rangle+|11\rangle+|02\rangle)/\sqrt{3}$, we get a non-vanishing mean value of the energy. Analogously, if consider the non-linear operators $(\alpha\beta^\dagger)^2$, $(\alpha^\dagger\beta)^2$, $(\hat{\alpha}^\dagger\hat{\alpha}-\hat{\beta}^\dagger\hat{\beta})^2/4$ and the initial state $|\Psi_2^{\theta,\phi}(0)\rangle$ we obtain

$$\begin{aligned}
&\langle\Psi_2^{\theta,\phi}(t)|\frac{(\hat{\alpha}^\dagger\hat{\alpha}-\hat{\beta}^\dagger\hat{\beta})^2}{4}|\Psi_2^{\theta,\phi}(t)\rangle = |a|^4+|b|^4+2\mathrm{Re}[(ab^*)^2e^{-i\phi}]\sin(2\theta), &(4.18a)\\
&\langle\Psi_2^{\theta,\phi}(t)|(\hat{\alpha}\hat{\beta}^\dagger)^2|\Psi_2^{\theta,\phi}(t)\rangle = [\langle\Psi_2^{\theta,\phi}(t)|(\hat{\alpha}^\dagger\hat{\beta})^2|\Psi_2^{\theta,\phi}(t)\rangle]^\dagger \\
&= \frac{(a^4e^{-i\phi}+b^4e^{i\phi})\cos(\theta)\sin(\theta)+a^2b^2}{2}, &(4.18b)
\end{aligned}$$

which are different from zero also for $\theta = \pi/4$, so that the mean value of the energy is neither vanishing nor constant in time. We stress, moreover, that such a calculation shows that correlations between the inverted and the normal quantum harmonic oscillator are present since the covariances of the operators under scrutiny result different from zero.

4.1.3 Time evolution under specific scenarios

In this section we analyse specific time-dependent scenarios to show the practical applicability of our analysis and results previously discussed.

Time-Independent Case

First of all, let us take into account the simplest case, that is when the Hamiltonian parameters are time-independent: $\Omega(t) = \Omega_0$ and $\omega(t) = \omega_0$. Moreover, let us consider, for simplicity, ω_0 a real parameter; such a choice is justified by the fact that a unitary transformation (a rotation with respect to $\hat{z}$) can be always performed in order to make ω a real parameter. In this instance the two time-function parameters a and b, solving the system in Eq. (1.16), acquire the following form

$$a(t) = \left[\cos(\tau) - i\frac{\Omega_0}{2\hbar\nu}\sin(\nu t)\right], \quad b(t) = -i\frac{\omega_0}{\hbar\nu}\sin(\nu t), \tag{4.19}$$

with $\hbar\nu \equiv \sqrt{\Omega_0^2/4 + \omega_0^2}$. In this way we can get explicit analytical expressions for all the formulas we obtained before. We can calculate, for example, the time evolution of the mean value of the energy. In Fig. 4.1 we report the θ-dependence of such a quantity when the system is initialized in the state $|\Psi_1^{\theta,0}\rangle$, whose general expression is reported in Eq. (4.15); it is easy to see that in this time scenario, as expected, the mean value energy is constant in time and depends only on the parameter θ [see Eq. (4.13)].

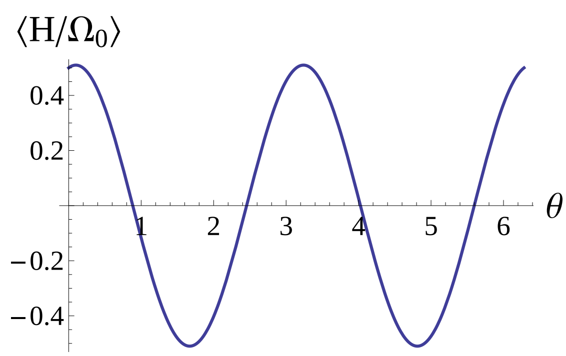


Figure 4.1: (Color online) Mean value of the energy in Eq. (4.15) scaled with respect to the parameter $\Omega(t) = \Omega_0$, with $\omega(t) = 0.1\Omega_0$ and versus the parameter θ, when the quantum oscillator-amplifier system starts from the state $|\Psi_1^{\theta,0}\rangle$ [Eq. (4.13)].

Rabi Scenario

Now, we consider the real coupling parameter oscillating in time, namely $\omega(t) = \omega_0 \cos(\nu_0 t)$, and leave the parameter $\Omega(t) = \Omega_0$ constant. Such a physical scenario, in terms of the spin language, may be reduced to the well known Rabi model [31, 32]. Precisely, under the conditions $\omega_0/\Omega_0 \ll 1$ and $\nu_0 = \Omega_0/2\hbar$ (resonance condition), only the rotating terms of the time-dependent transverse field (ω) are relevant for the dynamics of the system, so that the counter rotating ones can be disregarded. In this instance, the coupling parameter becomes $\omega = \cos(\nu_0 t) - i\sin(\nu_0 t)$. The related dynamical problem may be exactly solved and the expressions of a and b defining the time-evolution operator, solutions of (1.16) read

$$a = \cos(k\tau')e^{-i\tau'}, \quad b = -i\sin(k\tau')e^{-i\tau'}, \qquad \tau' = \nu_0 t, \quad k = \omega_0/\Omega_0. \tag{4.20}$$

The time evolution of the mean value of the energy [Eq. (4.15)] when the quantum oscillator-amplifier system starts from the state $|\Psi_1^{\pi/4,0}\rangle$ [Eq. (4.13)], is reported in Fig. 4.2a, for $\omega_0/\Omega_0 = 0.1$, with respect to the dimensionless time $\tau' = \nu_0 t$. We see the presence of the typical oscillatory regime of the Rabi scenario, being

$$\langle\Psi_1^{\pi/4,0}|U^\dagger H U|\Psi_1^{\pi/4,0}\rangle = \omega_0 \cos(k\tau'). \tag{4.21}$$

Landau-Majorana-Stückelberg-Zener Scenario

The Landau-Majorana-Stückelberg-Zener (LMSZ) scenario [33] is characterized by a linear longitudinal (in the z direction) ramp, namely, $\Omega(t) = \gamma\, t$, with $t \in (-\infty, \infty)$ and a transverse (along the x direction) constant field, $\omega = \omega^* = \omega_0$. The LMSZ scenario is an ideal model since it provides for an infinite duration of the physical procedure. To comply with more physical experimental condition, it is more appropriate to consider finite values for the initial and final time instants. In this case, the exact solution of $a(t)$ and $b(t)$ for the system in Eq. (1.16) [69] are reported in Eqs. (2.38).

The plot of the mean value of the energy for the initial condition $|\Psi_1^{\pi/4,0}\rangle$ in such a scenario is reported in Fig. 4.2b. We note that the curve is symmetric with respect to the time instant ($t = 0$) in which the avoided crossing occurs. This circumstance can be understood by writing the state of the system at a general time instant t:

$$U_{1/2}(t)|\Psi_1^{\pi/4,0}\rangle = \frac{(a+b)|10\rangle + (a^+ - b^*)|01\rangle}{\sqrt{2}}, \tag{4.22}$$

and by considering that under the LMSZ scenario $a(t)$ $[b(t)]$ goes from 1 [0] to 0 [1]. Thus, it means that the system reaches asymptotically the state $(|10\rangle - |01\rangle)/\sqrt{2}$ which differs from the initial condition $[(|10\rangle + |01\rangle)/\sqrt{2}]$ only for a relative phase factor.

4.2 Two interacting standard quantum harmonic oscillators

Let us consider now the same model for two standard quantum harmonic oscillators:

$$H'_{ho} = \frac{\Omega(t)}{2}(\hat{\alpha}^\dagger\hat{\alpha} + \hat{\beta}^\dagger\hat{\beta}) + \omega(t)\hat{\alpha}^\dagger\hat{\beta} + \omega^*(t)\hat{\beta}^\dagger\hat{\alpha}, \tag{4.23}$$

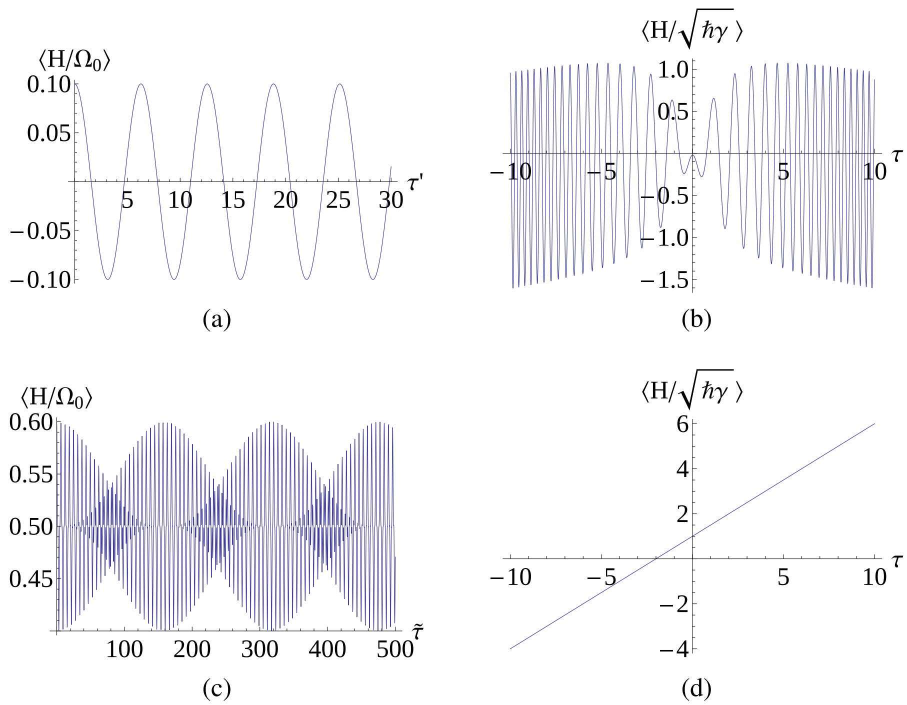


Figure 4.2: (Color online) Mean value of the energy when the oscillator-amplifier [oscillator-oscillator] system starts from $|\Psi_1^{\pi/4,0}\rangle$ for a) the Rabi scenario with $\omega_0/\Omega_0 = 0.1$ and b) the LMSZ scenario when $\omega_0^2/\hbar\gamma = 1$. Mean value of the energy when the oscillator-oscillator system starts from $|\Psi_1^{\pi/4,0}\rangle$ for c) the Rabi scenario with $\omega_0/\Omega_0 = \omega_0/2\hbar\nu_0 = 0.1$ and d) the LMSZ scenario when $\omega_0^2/\hbar\gamma = 1$.

It is easy to understand that the total excitation number $\hat{\mathscr{N}} = \hat{\alpha}^\dagger\hat{\alpha} + \hat{\beta}^\dagger\hat{\beta}$ is a constant of motion for this Hamiltonian too, $[H'(t), \mathscr{N}] = 0$. This fact implies that, also this time, we have an infinite number of dynamical invariant Hilbert subspaces related to the different eigenvalues $N = 1, 2\ldots$ of $\hat{\mathscr{N}}$. It is possible to persuade oneself that, in this case, the two oscillator dynamical problem may be mapped within the $N+1$-dimensional subspace (linked to the eigenvalue N of $\mathscr{N}$) into a spin-$N/2$ dynamical problem related to the following Hamiltonian:

$$H_N = \frac{N}{2}\Omega(t)\hat{\mathbb{1}} + \omega(t)\hat{S}^+ + \omega^*(t)\hat{S}^- \tag{4.24}$$

In this instance, the time evolution operator governing the dynamics within such a subspace can be written as

$$V_N(t) = \exp\left\{-\frac{i}{\hbar}\frac{N}{2}\int_0^t \Omega(t')dt'\right\} U_{N/2}(t) \tag{4.25}$$

with $U_{N/2}$ defined in Eq. (3.77), where $a(t)$ and $b(t)$ are the solutions of the system of differential equations originating from the spin-1/2 dynamical problem:

$$\begin{cases} \dot{a}(t) = i\omega(t)b^*(t), \\ \dot{b}(t) = -i\omega(t)a^*(t), \\ a(0) = 1, \quad b(0) = 0, \end{cases} \tag{4.26}$$

In the time-independent case, $\Omega(t) = \Omega_0$ and $\omega(t) = \omega_0$, the expressions of $a(t)$ and $b(t)$ are

$$a(t) = \cos(\omega_0\, t/\hbar), \quad b(t) = -i\sin(\omega_0\, t/\hbar). \tag{4.27}$$

In the Rabi scenario, that is, when $\omega(t) = \omega_0 e^{-i\nu_0 t}$, the two parameter time functions read instead

$$a(t) = \cos(\tilde{\tau}), \quad b(t) = -i\frac{\omega_0}{\hbar\nu_R}\sin(\tilde{\tau}), \qquad \tilde{\tau} = \nu_R\, t, \quad \nu_R = \sqrt{\nu_0^2 + \omega_0^2/\hbar^2}. \tag{4.28}$$

In the LMSZ scenario ($\Omega(t) = \gamma\, t$, $\omega = \omega^* = \omega_0$) the expressions of the two time functions are very similar to those in Eq.(4.29), namely

$$a(t) = \cos(\sqrt{\chi/2}\;\tau), \quad b(t) = -i\sin(\sqrt{\chi/2}\;\tau), \qquad \tau = \sqrt{\gamma/\hbar}\, t, \quad \chi = 2\omega_0^2/\hbar\gamma, \tag{4.29}$$

since $\Omega(t)$, in case of two interacting standard oscillators, plays no role in determining $a(t)$ and $b(t)$, as it is clear from Eq. (4.26).

The time evolution of the mean value of the energy when the two interacting quantum oscillators are initially prepared in $|\Psi_1^{\pi/4,0}\rangle$ is reported in Figs. 4.2c and 4.2d for the Rabi and the LMSZ scenario, respectively, with $\omega_0/\Omega_0 = \omega_0/2\hbar\nu_0 = 0.1$ in the first case and $\omega_0^2/\hbar\gamma = 1$ in the second case. We see that the Rabi scenario preserves, of course, its qualitative oscillatory regime, although the oscillation is consistently different presenting a beat effect, since

$$\langle\Psi_1^{\pi/4,0}|U^\dagger H U|\Psi_1^{\pi/4,0}\rangle = \frac{\Omega_0}{2} + \omega_0\cos(\nu_0 t)\left[\cos^2(\nu_R t) + \frac{\omega_0^2}{\hbar^2\nu_R^2}\sin(\nu_R t)\right] \tag{4.30}$$

A drastic change, instead, happens in the LMSZ scenario for which we have

$$\langle \Psi_1^{\pi/4,0} | U^\dagger H U | \Psi_1^{\pi/4,0} \rangle = \sqrt{\hbar\gamma}\,\tau + \omega_0. \tag{4.31}$$

This is due to the fact that the dynamics of the two-oscillator system is unaffected by the parameter $\Omega(t)$ which, in the LMSZ framework, is the main parameter driving the time evolution of the system and realizing the characteristic LMSZ dynamics as it happens for the oscillator-amplifier system.

To appreciate the difference between the dynamics of the two quantum systems even better, let us consider now the time evolution of the state $|N0\rangle$; it is easy to see that

$$P_{N0}^{0N} = \langle 0N | N0(t) \rangle \equiv \langle 0N | V_N(t) | N0 \rangle = \langle 0N | U_{N/2}(t) | N0 \rangle = |b(t)|^{2N}. \tag{4.32}$$

We note that in the time-independent case, for the two standard oscillators, $P_{N0}^{0N} = \sin^{2N}(\omega_0 t/\hbar)$ presents oscillations with maximum amplitude. In the case of an oscillator coupled with a Glauber amplifier, instead, such a transition probability, $P_{N0}^{0N} = (\omega_0/\hbar\nu)^{2N}\sin^{2N}(\nu t)$, cannot reach, in general, the maximum value $P_{N0}^{0N} = 1$, unless in the more trivial case $\Omega_0 = 0$. The opposite situation occurs in the case of the Rabi scenario. We have, indeed, $P_{N0}^{0N} = (\omega_0/\hbar\nu_R)^{2N}\sin^{2N}(\tilde{\tau})$ for two oscillators and $P_{N0}^{0N} = \sin^{2N}(k\tau')$ for the quantum oscillator-amplifier system. This circumstance can be traced back to the fact that the resonant condition cannot be satisfied in the case of two standard oscillators (Ω_0 plays no role in the dynamics). Finally, we underline that the LMSZ scenario, in case of two standard oscillators, does not

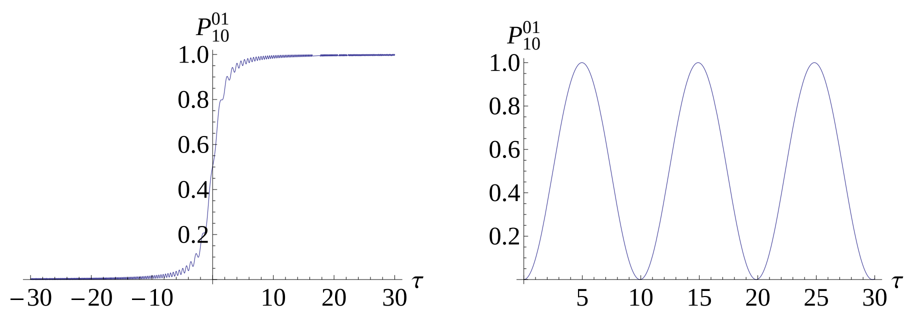


Figure 4.3: (Color online) Time evolution of the probability $P_{10}^{01} = \langle 01 | U_{1/2}(t) | 10 \rangle = |b(t)|^2$ in the LMSZ scenario with $\omega_0^2/\hbar\gamma = 1$ for a) the oscillator-amplifier system and b) the oscillator-oscillator system.

generate the typical LMSZ transition probability, but the behaviour of P_{N0}^{0N} results sinusoidal in time: $P_{N0}^{0N} = \sin^{2N}(\tilde{\tau})$. The oscillator-amplifier system, instead, exhibits an asymptotic full transition from $|N0\rangle$ to $|0N\rangle$ under adiabatic conditions, that is, when $\omega_0/\gamma \ll 1$. The two different probabilities for the LMSZ scenario are reported in Figs. 4.3a and 4.3b in case of $N = 1$ with $\omega_0^2/\hbar\gamma = 1$.

4.3 Summary and remarks

Jordan [198] and Schwinger [199] have shown that the angular momentum operators can be expressed in terms of quadratic expressions of two bosonic annihilation and creation operators $\hat{a}_1$, $\hat{a}_2$ and $\hat{a}_1^\dagger$,

$\hat{a}_2^\dagger$. Such a general statement is known as Jordan-Schwinger map (4.4). Namely, given three $N \times N$-matrices A, B and C such that $[A, B] = C$, the three operators $\hat{A} = \sum_{j,k=1}^{N} A_{jk}\hat{a}_j^\dagger \hat{a}_k$, $\hat{B} = \sum_{j,k=1}^{N} B_{jk}\hat{a}_j^\dagger \hat{a}_k$ and $\hat{C} = \sum_{j,k=1}^{N} C_{jk}\hat{a}_j^\dagger \hat{a}_k$ satisfy the commutation relation $[\hat{A}, \hat{B}] = \hat{C}$. If matrices A, B and C are 2×2 Pauli matrices this statement provides possibility to construct all the spin-states with $S = 0, 1/2, 1, 3/2, \ldots$ in terms of two oscillator states $|n_1, n_2\rangle$, where $n_1 + n_2 = 2s + 1$. The original idea was to exploit the solutions of the non-stationary Schrödinger equation related to two-mode parametric oscillator to map them into solutions of the Schrödinger equation related to non-stationary Hamiltonians linear in the generators of the SU(2) group.

In this chapter, instead, we adopted exactly the opposite strategy. Through the Jordan-Schwinger mathematical trick, within each invariant subspace, it is possible to map the dynamical problem of the oscillator-amplifier system into that of a single spin-j (the value of j depends on the dimension of the subspace) characterized by a Hamiltonian linear in the su(2) generators. Thanks to the knowledge of the formal expression of the SU(2)-group elements (representing the time evolution operators solution of the dynamical problem of the general single spin-j) we constructed the time evolution operator of the quantum oscillator-amplifier system. Moreover, on the basis of the knowledge of exact solutions related to specific time-dependent scenarios, we studied the exact dynamics of the system under such specific time-dependent models. Following the same approach, we solved and analysed also the dynamics of two interacting quantum harmonic oscillators, making a comparison between the two systems which brought to light relevant physical analogies and differences. We emphasize moreover that other exact or approximated solutions of the single qubit dynamical problem [35, 36, 37, 38, 200] may be exploited to study the dynamics of the system under different physical conditions with possible useful applications.

Finally, we wish to point out that the same strategy can be used for arbitrarily Hamiltonians presenting a linear form in the generators of any Lie algebra. Also these generators, indeed, can be expressed in terms of bosonic or fermionic creation and annihilation operators as quadratic forms in operators with time-dependent coefficients. In that case, it results of basic importance the knowledge of exact solutions of dynamical problems characterized by different symmetries. In this respect, it is interesting to underline that a solution method for dynamical problems related to su(1,1) Hamiltonians is presented in the following chapter. Such kind of Hamiltonians are very useful and important to treat and study open quantum systems living in finite Hilbert spaces and described by pseudo-Hermitian Hamiltonians, such as PT-symmetry physical systems [11, 27, 28, 29, 30]. Moreover, it is interesting to stress that in case of infinite dimensional Hilbert spaces, like quantum oscillators and amplifier, the representation of the SU(1,1) group results unitary and then appropriate to describe coherent dynamics of closed physical systems.

The results discussed in this chapter have been reported in Ref. [201].

Chapter 5

Non-Hermitian *PT*-symmetry su(1,1) Dynamical Problems

The interest towards the study of non-Hermitian Hamiltonians (NHH) has grown exponentially in the last decades and it is still growing. This is due not only to the applications they have in many different fields of physics [110], but also to the relevant role played in understanding and developing fundamental aspects of Quantum Mechanics.

To appreciate this point it is enough to observe that particular closed systems may often be described by a non-Hermitian Hamiltonian invariant under the space-time inversion (PT-symmetry), implying a possible extension of Quantum Mechanics [25, 26]. Many decades ago Feshbach employed for the first time non-Hermitian Hamiltonians to represent effectively the coupling between a discrete level and a continuum of states of a given quantum system [24]. Such an approach is still largely adopted nowadays to bring to light several physical aspects of open quantum systems [202], as for example phase transitions and exceptional points [111]. An effective non-Hermitian Hamiltonian is characterized by a secular equation with real coefficients, giving thus rise to either real or pairs of complex-conjugated eigenvalues [203]. Such a property guarantees that a non-Hermitian Hamiltonian belongs to the class of pseudo-Hermitian operators, provided it is diagonalizable and possesses a discrete spectrum [25]. This fact paved a way to significant research on specific non-Hermitian Hamiltonians [204], whose physical implementations may be found in different contexts, like optical microspiral cavities [205], microcavities perturbed by particles [206], or modelling the propagation of light in perturbed medium [207, 208].

We are mainly interested towards dynamical problems of two-level systems described by a time-dependent pseudo-Hermitian su(1,1) Hamiltonians. Such nonautonomous systems were rarely studied in the context of pseudo-Hermitian dynamics. As we show, they may be of experimental interest, and one of our aims is finding special classes of new exactly solvable cases. The reason why we concentrate mainly on the dynamics of a two-level system stems from the fact that the dynamical problem of an N-level system characterized by an su(1,1) Hamiltonian may be always traced back to that of a two-level system [209]. This implies that we may construct the solution of the N-level system by knowing that of the related two-level system [209]. Furthermore, we know that in conventional quantum mechanics a variety of complicated quantum-mechanical problems can be reduced to the two-level model [12]. In many contexts, for example nuclear magnetic resonance [13], quantum information processing [9] and polarization optics [14], essential changes in the system may be described in terms of a two-state dynamics. The interest towards $\mathrm{su}(1,1)$-symmetric dynamical problem finds its reasons in the fact that many physical scenarios exhibit such a kind of symmetry in their Hamiltonian operators. For example, the dynamics of a $N = 2j+1$-level atom in a cascade coupling with a laser beam with time-dependent intensity and in the resonance condition (vanishing detuning) is characterized by a time-dependent Hamiltonian embedded in the $\mathrm{su}(1,1)$ algebra [209]. Another important $\mathrm{su}(1,1)$ physical scenario may be identified in the treatment of squeezed states of the electromagnetic field and scattering of projectiles from simple diatomic molecules [15]. All these systems have a group structure resulting in subdynamics with a su(1,1)-symmetry.

Moreover, a connection between su(1,1)-symmetric and PT-symmetric Hamiltonians can be easily found. The most general 2x2 null-trace matrix representing a Hamiltonian that meets all the conditions of PT quantum mechanics has the following form [210]

$$\begin{pmatrix} \alpha & i\beta \\ i\beta & -\alpha \end{pmatrix}, \tag{5.1}$$

with real α and β. Non-Hermitian matrices of this structure comprise a proper sub-class of the set of all $\mathrm{su}(1,1)$ matrices. An important application of PT-symmetric Hamiltonians is the study and description of the dynamics of the so-called gain and loss systems [111, 211], which may be encountered and realized in different physical contexts, exhibiting several interesting properties. In particular, such systems can undergo a phase transition related to the PT-symmetry breaking [11, 27, 28, 29, 30]. In the cited papers an emphasis is put on how the phase transition may be governed experimentally by manipulating the gain and loss parameters and how it can be related to the change of the energy spectrum from a real to a complex one. One may, therefore, ask what happens if the parameter(s) governing the reality/complexity of the spectrum (and, consequently, the symmetry properties of the Hamiltonian) are time-dependent. From a theoretical point view several efforts are yet necessary for a total comprehension and unifying description of dynamics related to time-dependent non-Hermitian Hamiltonians. Quite recently, proposals and investigations of fundamental issues have been done [212, 213, 214, 215] and important physical aspects concerning time-dependent non-Hermitian Hamiltonians have been brought to light [68, 112]. However, very few attempts concerning the identification of classes of exactly solvable scenarios for physical systems described by time-dependent non-Hermitian Hamiltonians were presented in the literature.

5.1 Exactly solvable time-dependent su(1,1) Hamiltonian models

The SU(1,1) group is not compact and as such it does not have finite-dimensional unitary representations. Its lowest-dimensional faithful matrix representation consists of the set of all 2×2 unit-determinant complex matrices U, satisfying the relation

$$\hat{\sigma}^z U^\dagger \hat{\sigma}^z = U^{-1}, \tag{5.2}$$

$\hat{\sigma}^x, \hat{\sigma}^y, \hat{\sigma}^z$ being the standard Pauli matrices. Generators of this non-unitary representation (i.e. a basis of the corresponding representation of the su(1,1) algebra) can be chosen as

$$\hat{K}^0 = \frac{\hat{\sigma}^z}{2}, \quad \hat{K}^1 = -i\frac{\hat{\sigma}^y}{2}, \quad \hat{K}^2 = i\frac{\hat{\sigma}^x}{2}. \tag{5.3}$$

They satisfy the relations [216]

$$[\hat{K}^1,\hat{K}^2] = -i\hat{K}^0, \quad [\hat{K}^1,\hat{K}^0] = -i\hat{K}^2, \quad [\hat{K}^2,\hat{K}^0] = i\hat{K}^1. \tag{5.4}$$

A t-dependent (in general, t is a generic parameter) null-trace 2x2 su(1,1) matrix is a linear combination, with real t-dependent coefficients $\omega_0(t), \omega_1(t)$ and $\omega_2(t)$, of the generators $\hat{K}^0$, $\hat{K}^1$ and $\hat{K}^2$, namely

$$H(t) = \omega_0(t)\hat{K}^0 + \omega_1(t)\hat{K}^1 + \omega_2(t)\hat{K}^2. \tag{5.5}$$

In terms of Pauli matrices it can be cast as

$$\begin{aligned} H(t) &= \Omega(t)\hat{\sigma}^z + i\omega_x(t)\hat{\sigma}^x - i\omega_y(t)\hat{\sigma}^y \\ &= \Omega(t)\hat{\sigma}^z - \omega(t)\hat{\sigma}^+ + \omega^*(t)\hat{\sigma}^-, \end{aligned} \tag{5.6}$$

where, conventionally, $\hat{\sigma}^{\pm} = (\hat{\sigma}^x \pm i\hat{\sigma}^y)/2$ and $\omega(t)$ is a complex parameter defined by $\omega(t) = \omega_y - i\omega_x \equiv |\omega(t)|e^{i\phi_\omega(t)}$, and $\Omega(t) = \omega_0(t)/2, \omega_x(t) = \omega_2(t)/2, \omega_y(t) = \omega_1(t)/2$. In this way, in the basis of $\hat{\sigma}^z$, the matrix representation of a general non-Hermitian operator $H(t)$ belonging to the su(1,1) algebra reads

$$H(t) = \begin{pmatrix} \Omega(t) & -\omega(t) \\ \omega^*(t) & -\Omega(t) \end{pmatrix}. \tag{5.7}$$

From Eq. (5.1) we see that the subclass of PT-symmetric su(1,1) Hamiltonians is identified by $\phi_\omega = \pm\pi/2$ or equivalently by $\omega_y = 0$.

It is important to underline that su(1,1)-symmetric Hamiltonians are pseudo-Hermitian, that is, by definition [25], there exists at least one non-singular Hermitian matrix η such that

$$H^\dagger(t) = \eta H(t)\eta^{-1}. \tag{5.8}$$

It is easy to see that the simplest matrix satisfying condition (5.8) is

$$\eta = \hat{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.9}$$

A diagonalizable operator is pseudo-Hermitian, if and only if its eigenvalues are either real or grouped in complex-conjugated pairs [25]. This fact is physically relevant, since it turns out to be the feature possessed by the non-Hermitian Hamiltonians resulting by the procedure provided by Feshbach [24] to describe effectively a quantum system with a discrete spectrum coupled to a continuum. Pseudo-Hermitian Hamiltonians, thus, play a very important role in the study of open quantum system [111, 212, 213, 214, 215], especially in description of their particular experimentally detectable physical aspects [111, 11, 27, 28, 29, 30].

The t-parameter-dependent spectrum of $H(t)$ reads $E_\pm(t) = \pm\sqrt{\Omega^2(t) - |\omega(t)|^2}$, which is real under the condition $|\omega(t)|^2 < \Omega^2(t)$. The reality of the spectrum is a sufficient and necessary condition for $H(t)$ to be quasi-Hermitian [25]; the condition of quasi-Hermiticity consists in the existence of a positive-definite matrix η_+ in the set of the matrices η accomplishing the equality in Eq. (5.8) [25]. It can be verified that such a matrix reads

$$\eta_+ = \begin{pmatrix} 1 & -\omega(t)/\Omega(t) \\ -\omega^*(t)/\Omega(t) & 1 \end{pmatrix}, \tag{5.10}$$

which is positive-definite for $|\omega(t)|^2 < \Omega^2(t)$. In this case we may identify a new Hilbert space in which $H(t)$ is Hermitian or, in other words, we may define a new scalar product $\langle\cdot|\cdot\rangle_{\eta_+}$ (defining the new Hilbert space), namely $\langle\cdot|\eta_+\cdot\rangle$ (where $\langle\cdot|\cdot\rangle$ is the standard Euclidean scalar product), with respect to which $H(t)$ is Hermitian. However, if the parameter t represents the time, this condition is not sufficient for our Hamiltonian to describe a closed quantum physical system. In fact, it can be shown, that a quasi-Hermitian time-dependent Hamiltonian describes a closed quantum system characterized by a (pseudo-) unitary dynamics only if the positive-definite matrix η_+ is time-independent [217]. This implies that a su(1,1) Hamiltonian (5.7) can describe a closed quantum system only if $\phi_\omega(t) = const.$ and $\Omega(t)$ and $|\omega(t)|$ have the same time-dependence, namely $\omega(t) = |\omega_0| f(t)$ and $\Omega(t) = \Omega_0 f(t)$, with $|\omega_0|^2 < \Omega_0^2$.

For all these reasons, in view of possible dynamical applications of finite dimensional su(1,1)-symmetry Hamiltonians $H(t)$ in either classical or quantum contexts, we search solutions of the Cauchy problem ($\hbar = 1$)

$$i\dot{U}(t) = H(t)U(t), \quad U(0) = \mathbb{1}, \tag{5.11}$$

which is the standard equation for the time evolution operator $U(t)$. For a Hermitian Hamiltonian, when the evolution of a state vector $|\psi(t)\rangle$ of the system in question is given by the the Schrödinger equation $i|\dot{\psi}(t)\rangle = H(t)|\psi(t)\rangle$, the operator $U(t)$ provides the solution $|\psi(t)\rangle = U(t)|\psi(0)\rangle$. Correspondingly, if a state of the system is described by a density matrix $\rho(t)$ evolving according to $i\dot{\rho}(t) = [H(t), \rho(t)]$, the evolution is given as $\rho(t) = U(t)\rho(0)U^\dagger(t)$.

Since $H(t)$ is an element of the Lie algebra su(1,1), the evolution operator $U(t)$ is an element of the corresponding Lie group SU(1,1). It can be thus written as an exponential of a (time-dependent) element $X(t) \in$ su(1,1), i.e. $U(t) = \exp(-iX(t))$. To find $X(t)$ for a time dependent $H(t)$, we have, in general, to evaluate a time-ordered integral. Wei and Norman [218] proposed an alternative way consisting in representing $U(t)$ in the form of a product of exponentials, each involving a single generator of the Lie algebra multiplied by a time-dependent coefficient. The resulting equations for these coefficients depend on the order of exponentials, but one can chose a canonical one that simplifies the equations maximally for all interesting algebras [219]. In particular, in our case we write the non-unitary operator $U(t)$ in the form,

$$\begin{aligned} U(t) &\equiv e^{u_1(t)\hat{\sigma}^+} e^{-u_2(t)\hat{\sigma}^z} e^{u_3(t)\hat{\sigma}^-} \\ &= \begin{pmatrix} e^{-u_2(t)} + u_1(t)e^{u_2(t)}u_3(t) & u_1(t)e^{u_2(t)} \\ e^{u_2(t)}u_3(t) & e^{u_2(t)} \end{pmatrix}, \end{aligned} \tag{5.12}$$

getting from Eq. (5.11) the following system of differential equations

$$\begin{cases} \dot{u}_1(t) = i\omega(t) - 2i\Omega(t)u_1(t) + i\omega^*(t)u_1^2(t), \\ \dot{u}_2(t) = i\Omega(t) - i\omega^*(t)u_1(t), \\ \dot{u}_3(t) = -i\omega^*(t)e^{-u_2(t)}, \end{cases} \tag{5.13}$$

to be associated with the initial conditions $u_j(0) = 0$ ($j = 1, 2, 3$). Once the first Riccati equation is solved, the remaining two can be simply integrated so that the whole su(1,1)-symmetry Hamiltonian problem may be exactly solved. A similar Riccati equation may be obtained when the analogous problem for the su(2) case is treated and an interesting interplay between Physics and Mathematics has recently been reported [200].

Since no general method is available to solve the system (5.13) for arbitrary $\Omega(t)$ and $\omega(t)$, we look for specific relations of physical interest between the Hamiltonian entries so that the Riccati equation under scrutiny can be solved analytically. To this end let us consider the following change of variable

$$u_1(t) = ie^{i\phi_\omega(t)}Y(t). \tag{5.14}$$

Plugging this expression into the Riccati equation in Eq. (5.13), we arrive at the following Riccati-Cauchy problem for the variable $Y(t)$:

$$\dot{Y}(t) = -|\omega(t)|Y^2(t) - i[2\Omega(t) + \dot{\phi}_\omega(t)]Y(t) + |\omega(t)|, \qquad Y(0) = 0. \tag{5.15}$$

It is quite clear, then, that under the analytical constraint

$$2\Omega(t) + \dot{\phi}_\omega(t) = 2\nu|\omega(t)|, \tag{5.16}$$

with ν a time-independent real non-negative dimensionless parameter, Eq. (5.15) becomes exactly solvable.

It is possible to adopt a different point of view to examine in depth the meaning of Eq. (5.16) if we consider the evolution of a state vector $|\psi(t)\rangle$ of the system. To this end, inspired by the seminal paper [32], we transform $|\psi(t)\rangle$ from the old basis to $|\tilde{\psi}(t)\rangle$ in a new basis,

$$|\psi(t)\rangle = \exp\{i\phi_\omega(t)\hat{\sigma}^z/2\}|\tilde{\psi}(t)\rangle, \tag{5.17}$$

getting the following new time-dependent Schrödinger equation

$$i|\dot{\tilde{\psi}}(t)\rangle = H_{eff}(t)|\tilde{\psi}(t)\rangle, \tag{5.18}$$

with

$$H_{eff}(t) = \left[\Omega(t) + \frac{\dot{\phi}_\omega(t)}{2}\right]\hat{\sigma}^z - i|\omega(t)|\hat{\sigma}^y. \tag{5.19}$$

From this expression it is clear why the relation (5.16) is a solvability condition for our problem. Indeed, the corresponding Schrödinger equation

$$i|\dot{\tilde{\psi}}(t)\rangle = |\omega(t)|\left[2\nu\hat{\sigma}^z - i\hat{\sigma}^y\right]|\tilde{\psi}(t)\rangle, \tag{5.20}$$

may be easily solved, even if the effective Hamiltonian is time-dependent.

The solution $Y_\nu(t)$ of the particular Riccati equation, related to a specific value of ν, reads

$$Y_\nu(t) = \frac{\sqrt{\nu^2-1}\,\tan[\sqrt{\nu^2-1}\,\chi(t)] - i\nu\tan^2[\sqrt{\nu^2-1}\,\chi(t)]}{\nu^2\sec^2[\sqrt{\nu^2-1}\,\chi(t)] - 1}, \tag{5.21}$$

where the time-dependent positive function $\chi(t)$ is defined as

$$\chi(t) = \int_0^t |\omega(\tau)|d\tau. \tag{5.22}$$

We may identify different classes and related different solutions depending on the value of the parameter ν. The case $\nu > 1$ defines the trigonometric regime with solution $Y_\nu^t(t)$ in the form (5.21). For $0 < \nu < 1$ the solution $Y_\nu(t)$ is in the hyperbolic regime having the form

$$Y_\nu^h(t) = \frac{\sqrt{1-\nu^2}\tanh[\sqrt{1-\nu^2}\,\chi(t)] - i\nu\tanh^2[\sqrt{1-\nu^2}\,\chi(t)]}{1-\nu^2\,\mathrm{sech}^2[\sqrt{1-\nu^2}\,\chi(t)]}. \tag{5.23}$$

The case $\nu = 1$ defines the rational regime with

$$Y_\nu^r(t) = \frac{\chi(t) - i\chi^2(t)}{\chi^2(t)+1}. \tag{5.24}$$

Finally, for $\nu = 0$ we have a real solution,

$$Y_0(t) = \tanh\left[\chi(t)\right]. \tag{5.25}$$

In this way, through Eq. (5.14), we may construct the time evolution operator of Eq. (5.12) for our exactly solvable scenario. To this end, it is important to point out that the SU(1,1) group elements and then the time evolution operators generated by the Hamiltonians in Eq. (5.7) depend on only two complex parameters. Indeed, the Caley-Klein parametrization for the SU(1,1) group elements reads

$$U(t) = \begin{pmatrix} a(t) & b(t) \\ b^*(t) & a^*(t) \end{pmatrix}, \tag{5.26}$$

with $|a(t)|^2 - |b(t)|^2 = 1$. Comparing this form with the one given in Eq. (5.12) it is easy to derive the following relations

$$u_1 = \frac{b}{a^*}, \quad u_2 = \log(a^*), \quad u_3 = \frac{b^*}{a^*}, \tag{5.27}$$

allowing us to simplify the matrix representation of the time evolution operator in terms of the $u_j s$ parameters as follows:

$$U(t) = \begin{pmatrix} \exp[u_2^*(t)] & u_1(t)\exp[u_2(t)] \\ u_1^*(t)\exp[u_2^*(t)] & \exp[u_2(t)] \end{pmatrix}, \tag{5.28}$$

with $e^{u_2^*(t)}e^{u_2(t)}(1-|u_1(t)|^2) = 1$. We see that in this case the expressions of the entries are easily readable and symmetric. Moreover, only two out of the three initial parameters appear. Then, the evolution operator for our general exactly solvable scenario may be written down as

$$U_\nu(t) = \begin{pmatrix} \exp[\mathfrak{r}_\nu(t)]\exp[-i\mathfrak{s}_\nu(t)] & |Y_\nu(t)|\exp[\mathfrak{r}_\nu(t)]\exp\left[i(\mathfrak{s}_\nu(t)+\mathfrak{y}_\nu(t))\right] \\ |Y_\nu(t)|\exp[\mathfrak{r}_\nu(t)]\exp\left[-i(\mathfrak{s}_\nu(t)+\mathfrak{y}_\nu(t))\right] & \exp[\mathfrak{r}_\nu(t)]\exp[i\mathfrak{s}_\nu(t)] \end{pmatrix}, \tag{5.29}$$

with

$$\mathfrak{r}_\nu(t) = \int_0^t |\omega(\tau)|\,\mathrm{Re}[Y_\nu(\tau)]d\tau, \tag{5.30a}$$

$$\mathfrak{s}_\nu(t) = \int_0^t \Omega(\tau)d\tau + \int_0^t |\omega(\tau)|\,\mathrm{Im}[Y_\nu(\tau)]d\tau, \tag{5.30b}$$

$$\mathfrak{y}_\nu(t) = \frac{\pi}{2} + 2\nu\int_0^t |\omega(\tau)|d\tau - 2\int_0^t \Omega(\tau)d\tau + \varphi_\nu(t), \tag{5.30c}$$

$$\varphi_\nu(t) = -\arctan\left[\frac{\nu\tan[\sqrt{\nu^2-1}\chi(t)]}{\sqrt{\nu^2-1}}\right]. \tag{5.30d}$$

Finally, it is easy to verify that the following identity

$$\det[U_\nu(t)] = \exp[2\mathfrak{r}_\nu(t)]\left(1-|Y_\nu(t)|^2\right) = 1, \tag{5.31}$$

is fulfilled at any time instant t for arbitrary ν.

5.2 Non-Hermitian quantum mechanics issues

In what follows we aim at exploring the applicability of our results in a quantum dynamical context. We underline that such an objective is not trivial since in the non-Hermitian Hamiltonian-based quantum dynamics conceptual difficulties in the physical interpretation of the mathematical results, may occur.

5.2.1 Trace and positivity preserving non-linear equation of motion

In the two-dimensional su(1,1) case, in contrast to the su(2) one, the complex entries $a(t)$ and $b(t)$ appearing in the operator $U(t)$, [the solution of the Cauchy problem (5.11)] lack they usual physical meaning. Indeed, in the su(2) case we interpret $|a(t)|^2$ and $|b(t)|^2$ as probabilities, while for the su(1,1) case, considered in this paper, $|a(t)|^2 \geq 1$ since $|a(t)|^2 - |b(t)|^2 = 1$ and, consequently, $U(t)$ cannot be identified as the unitary time evolution generator of a closed Hamiltonian system. This is intrinsically related to the dynamics generated by a su(1,1) finite-dimensional Hamiltonian. Indeed, as already mentioned earlier, only the infinite dimensional representations of SU(1,1) may be unitary.

A direct consequence of the non-unitarity of $U(t)$ obscuring the physical interpretation of the mathematical results we get from the study of the Cauchy problem (5.11), is that the Schrödinger-type time evolution of a density matrix $\rho'(t) = U(t)\rho'(0)U^\dagger(t)$ does not preserve the trace of ρ'. To recover the necessary normalization condition at any time instant, following the approach introduced in Ref. [212], we put

$$\rho(t) = \frac{\rho'(t)}{\mathrm{Tr}\{\rho'(t)\}}, \tag{5.32}$$

where $\rho'(t) = U(t)\rho'(0)U^\dagger(t)$ and $\dot{U}(t) = -iH(t)U(t)$.

This choice leads to a "new dynamics", that is, to a new Liouville-von Neumann equation governing the dynamics of our system, obtained by differentiating Eq. (5.32), namely

$$\dot{\rho}(t) = -i[H_0(t),\rho(t)] - \{\Gamma(t),\rho(t)\} + 2\rho(t)\mathrm{Tr}\{\rho(t)\Gamma(t)\}, \tag{5.33}$$

where we put

$$H(t) = H_0(t) - i\Gamma(t), \tag{5.34}$$

with $H_0^\dagger(t) = H_0(t)$ and $\Gamma^\dagger(t) = \Gamma(t)$. In fact, it is easy to check that the general su(1,1) Hamiltonian of the form (5.7), may be written in this way with

$$H_0(t) = \Omega(t)\hat{\sigma}^z, \qquad \Gamma(t) = -\omega_x(t)\hat{\sigma}^x + \omega_y(t)\hat{\sigma}^y, \tag{5.35}$$

and $\omega(t) \equiv \omega_y(t) - i\omega_x(t) \equiv |\omega(t)|e^{i\phi_\omega(t)}$.

From a physical point of view, Eq.(5.33) possesses interesting properties [215] which make it a valid candidate to describe the quantum dynamics of physical systems characterized by a non-Hermitian

Hamiltonian like PT-symmetric systems [29, 30]. The three most important properties to be pointed out are: 1) a pure state remain pure at any time, while the purity of a mixed state, in general, changes in time; 2) the trace and positivity are preserved at any time since the new equation was constructed *ad hoc* to satisfy this condition in order to recover the concept of probability and a statistical interpretation of the quantum dynamics related to Hermitian Hamiltonians; 3) the general solution of Eq. (5.33) reads, of course,

$$\rho(t) = \frac{U(t)\rho'(0)U^\dagger(t)}{\mathrm{Tr}\{U(t)\rho'(0)U^\dagger(t)\}}, \tag{5.36}$$

where $U(t)$ is the (non-unitary) operator satisfying Eq. (5.11). Thus the solution of the non-linear problem (5.33) is traced back to solve our original problem (5.11). This circumstance means that, through the procedure exposed in Sec. 5.1, we are able to solve the generalized Liouville-von Neumann non-linear equation (5.33) for the class of time-dependent scenarios identified by the relation (5.16), whose time evolution operator $U_\nu(t)$ is reported in Eq. (5.29).

5.2.2 Semigroups of nonlinear positivity and trace preserving maps

Equation (5.33) was constructed, to some extent, *ad hoc*, merely postulating a way of keeping the proper normalization of the density matrix during the whole evolution. However, one can argue that the form of an evolution equation is dictated by the fact that it should describe a one-parameter positivity preserving semigroup modeling time evolution of the density matrix of a quantum system. Let us consider a general, positivity preserving map,

$$\phi_t(\rho) = U(t)\rho U^\dagger(t). \tag{5.37}$$

When $U(t)$ is a one-parameter semigroup $U(s+t) = U(s)U(t)$, then a reasonable demand is that $\phi_t(\rho)$ has the semigroup property, i.e. $\phi_s \circ \phi_t = \phi_{s+t}$. This means that evolution for time 0 to $s+t$ is composed from the evolution from 0 to t followed by the evolution from t to $t+s$.

Let us now consider the following map,

$$\hat{\phi}_t(\rho) = \frac{\phi_t(\rho)}{\mathrm{Tr}\left(\phi_t(\rho)\right)}. \tag{5.38}$$

Such a map $\hat{\phi}_t$ happens also to describe a reasonable quantum evolution. It is clearly positivity- and trace-preserving and, moreover, has the semigroup property $\hat{\phi}_s \circ \hat{\phi}_t = \hat{\phi}_{s+t}$. Indeed [220],

$$\begin{aligned}
\hat{\phi}_s \circ \hat{\phi}_t(\rho) &= \hat{\phi}_s\left(\frac{\phi_t(\rho)}{\mathrm{Tr}\left(\phi_t(\rho)\right)}\right) = \frac{\phi_s\left(\frac{\phi_t(\rho)}{\mathrm{Tr}(\phi_t(\rho))}\right)}{\mathrm{Tr}\left(\phi_s\left(\frac{\phi_t(\rho)}{\mathrm{Tr}(\phi_t(\rho))}\right)\right)} \\
&= \frac{\frac{1}{\mathrm{Tr}(\phi_t(\rho))}\phi_s\left(\phi_t(\rho)\right)}{\mathrm{Tr}\left(\frac{1}{\mathrm{Tr}(\phi_t(\rho))}\phi_s\left(\phi_t(\rho)\right)\right)} \frac{\phi_s\left(\phi_t(\rho)\right)}{\mathrm{Tr}\left(\phi_s\left(\phi_t(\rho)\right)\right)} \\
&= \frac{\phi_{s+t}(\rho)}{\mathrm{Tr}\left(\phi_{s+t}(\rho)\right)} = \hat{\phi}_{s+t}(\rho),
\end{aligned} \tag{5.39}$$

from the linearity of ϕ_t and the trace. Moreover,

$$\hat{\phi}_t(\alpha\rho) = \hat{\phi}_t(\rho). \tag{5.40}$$

The transition from the map (5.37) to the map (5.32) that preserves the semigroup property can be generalized as follows. Let f be an arbitrary scalar function, $f : \mathscr{P} \to \mathbb{R}^+$, from the cone of positive operators $\mathscr{P}$ into positive reals $\mathbb{R}^+$. Then the map,

$$\tilde{\phi}_t(\rho) = f\left(\hat{\phi}_t(\rho)\right)\hat{\phi}_t(\rho), \tag{5.41}$$

has also the semigroup property. Indeed, using (5.40), we get,

$$\begin{aligned}
\tilde{\phi}_s \circ \tilde{\phi}_t(\rho) &= \tilde{\phi}_s(\tilde{\phi}_t(\rho)) = \tilde{\phi}_s\left(f(\hat{\phi}_t(\rho))\hat{\phi}_t(\rho)\right) \\
&= f\left(\hat{\phi}_s\left(f(\hat{\phi}_t(\rho))\hat{\phi}_t(\rho)\right)\right)\hat{\phi}_s\left(f(\hat{\phi}_t(\rho))\hat{\phi}_t(\rho)\right) \\
&= f\left(\hat{\phi}_s(\hat{\phi}_t(\rho))\right)\hat{\phi}_s(\hat{\phi}_t(\rho)) = f(\hat{\phi}_{s+t}(\rho))\,\hat{\phi}_{s+t}(\rho) \\
&= \tilde{\phi}_{s+t}(\rho),
\end{aligned} \tag{5.42}$$

Obviously, the above reasoning does not eliminate other generalizations of (5.38) preserving the semigroup property, but if we restrict (5.41) to the subspace $\mathscr{D}(\mathscr{H})$ of all density matrices upon the Hilbert space $\mathscr{H}$ of the system, we must have $f = 1$, since at $t = 0$, $\tilde{\phi}_{t=0}(\rho) = \rho = f(\rho)\rho$ whatever ρ is. Therefore, Eq.(5.32) itself is less *ad hoc* than it seems and in addition it derives from the same dynamical map generating the von Neumann-Liouville equation when $H = H^\dagger$.

5.2.3 Quantum dynamics of a su(1,1) "Rabi" scenario

In order to appreciate better the physical aspects of the generalized Liouville-von Neumann non-linear equation (5.33), we want to study now the "Rabi" scenario for the case of su(1,1) Hamiltonians and to point out differences and analogies with the su(2) case by bringing to light intriguing dynamical aspects. We know that the 'standard' Rabi scenario describes a spin-1/2 subjected to a time-dependent magnetic field precessing around the $\hat{z}$-axis. The matrix representation of a general su(2) Hamiltonian may written as

$$\tilde{H}(t) = \tilde{\Omega}(t)\hat{\sigma}^z + \tilde{\omega}_x(t)\hat{\sigma}^x + \tilde{\omega}_y(t)\hat{\sigma}^y = \begin{pmatrix} \tilde{\Omega}(t) & \tilde{\omega}(t) \\ \tilde{\omega}^*(t) & -\tilde{\Omega}(t) \end{pmatrix}, \tag{5.43}$$

with $\tilde{\omega}(t) \equiv \tilde{\omega}_x(t) - i\tilde{\omega}_y(t) \equiv |\tilde{\omega}(t)|e^{i\phi_{\tilde{\omega}}(t)}$ and where $\hat{\sigma}^k$ ($k = x, y, z$) are the Pauli matrices represented in the eigenbasis $\{|\pm\rangle\}$ of $\hat{\sigma}^z$: $\hat{\sigma}^z|\pm\rangle = \pm|\pm\rangle$. It is easy to see that the consideration of a magnetic field precessing around the $\hat{z}$-axis amounts to consider the three parameters $\tilde{\Omega}$, $|\tilde{\omega}|$ and $\dot{\phi}_{\tilde{\omega}}$ time independent. Further, the well known Rabi's resonance condition, ensuring a complete periodic population transfer between the two states $|+\rangle$ and $|-\rangle$, acquires the form $\tilde{\Omega} + \dot{\phi}_{\tilde{\omega}}/2 = 0$. From Sec. 1.3.1 we know that, also when the three parameters are time-dependent, the so-called generalized Rabi's resonance condition $\tilde{\Omega}(t) + \dot{\phi}_{\tilde{\omega}}(t)/2 = 0$ is a necessary condition to obtain periodic oscillations with maximum amplitude.

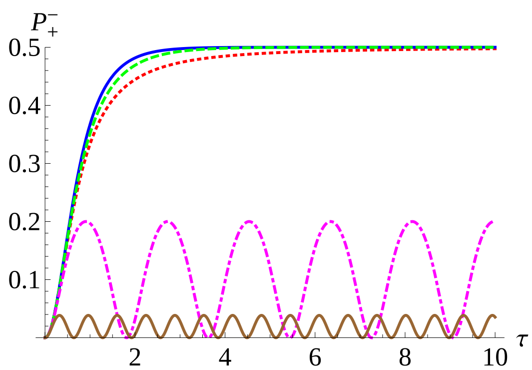
We see from (5.35) that we may interpret the su(1,1) Hamiltonian as a Rabi problem with a complex transverse magnetic field. Analogously to the su(2) case, we may define the Rabi-like scenario for a su(1,1) dynamical problem the case in which the three parameters Ω, $|\omega|$ and $\dot{\phi}_\omega$ are time independent.

Thus, the related solution for the quantum dynamics is given by Eqs. (5.29), (5.30a), (5.30b) and (5.30c) with $\Omega(\tau) = \Omega_0$, $|\omega(\tau)| = |\omega_0|$ and $\dot{\phi}_\omega(t) = \dot{\phi}_\omega^0$.

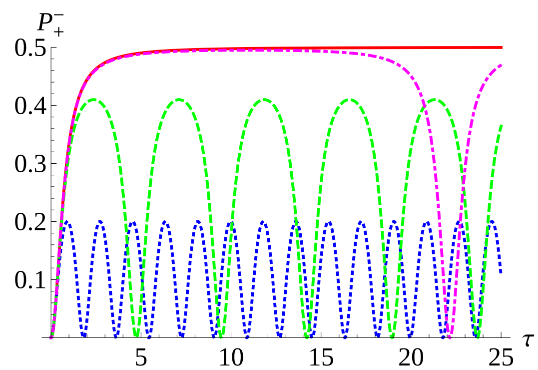
Let us now study the time behavior of the Rabi's transition probability $P_+^-(t)$, that is the probability to find the system in the state $|-\rangle$ at time t when it is initialized at time $t = 0$ in the state $|+\rangle$. Starting thus form the initial state $\rho(0) = \rho'(0) = |+\rangle\langle+|$ and employing the non-linear equation of motion (5.36) discussed before with the non-unitary operator $U(t)$ given by (5.29), we obtain for $P_+^-(t) = \rho_{22}(t)$,

$$P_+^-(t) = \frac{|b(t)|^2}{|a(t)|^2 + |b(t)|^2} = \frac{|Y_\nu(t)|^2}{1 + |Y_\nu(t)|^2}. \tag{5.44}$$

In Figs. 5.2b and 5.1b we depict the transition probability P_+^-, against the dimensionless time $\tau = |\omega_0|t$ for different values of the parameter ν. This is done in the case of a Rabi-like scenario, which amounts, as explained before, to considering the two parameters Ω and $|\omega|$, defining the operator $U(t)$ by Eqs. (5.29), (5.30a), (5.30b) and (5.30c), as independent of time.



(a)



(b)

Figure 5.1: (Color online) a) Time dependence of the transition probability P_+^- as a function of $\tau = |\omega_0|t$, for different values of ν: $\nu = 0; 0.7; 1; 2; 5$ correspond to the up-full blue, dashed green, dotted red, dot-dashed magenta and down-full brown curve, respectively; b) The plot illustrates ($\nu = 2; 1.2; 1.01; 1 \to$ dotted blue, dashed green, dot-dashed magenta, full red) the passage of $P_+^-(\tau)$ from the oscillatory regime to the plateau regime.

We note that we have oscillations when $\nu \geq 1$ of decreasing amplitude and period as long as ν increases; for $0 \leq \nu < 1$, instead, an asymptotic regime appears. This constitutes a deep difference between the Rabi scenario in the su(2) and in the su(1,1) case. In the former, the behaviour of the transition probability $P_+^-(t)$ is always oscillatory in time and different values of ν are related to different amplitudes of the oscillations. In the latter, instead, two kinds of time behaviour appear depending on the value of the parameter ν, with 1 as value of separation between the two regimes. It is important to highlight at this point that the existence of the two regimes, in general, is not related to the reality or complexity of the Hamiltonian spectrum. The latter, indeed, concerning the "Rabi" scenario we are analysing, is t-independent, the eigenvalues take the values $\pm\sqrt{\Omega_0^2 - |\omega_0|^2}$, and within the solvability condition (5.16), they are real if (without loosing generality that $\Omega_0 > 0$)

$$\nu > 1 + \frac{\dot{\phi}_\omega^0}{\Omega_0}. \tag{5.45}$$

We see, then, that only if $\dot{\phi}_\omega^0 = 0$, the ν-dependent transition between the two dynamical regimes coincides with the passage from a real to a complex spectrum. This happens to be the case for the generic su(1,1) 2x2 PT-symmetry matrix in Eq. (5.1) for which $\phi_\omega(t) = \pi/2$, or for a t-independent su(1,1) matrix. Conversely, if $\dot{\phi}_\omega^0 \neq 0$, two possible interesting cases arise. Namely, if $\dot{\phi}_\omega^0 < 0$ it means that the transition between the two dynamical regimes ($\nu > 1 \rightarrow \nu < 1$) occurs while the spectrum keeps its reality, since, in this case, $1 + \dot{\phi}_\omega^0/\Omega_0 < 1$. On the other hand, if $\dot{\phi}_\omega^0 > 0$, there is a range of values of ν, namely $1 < \nu < 1 + \dot{\phi}_\omega^0/\Omega_0$, for which the spectrum becomes complex without any appreciable evidence in the dynamical behavior of the system.

As a last remark we want to highlight a common feature of the su(2) and the su(1,1) cases. Note that the Rabi-like resonance condition $\Omega + \dot{\phi}_\omega/2 = 0$ amounts to putting $\nu = 0$ and the related up-full blue curve in Fig. 5.2b is the top limit one. We know that in the su(2) case this condition ensures the complete periodic population transfer between the two levels of the system, that is oscillations with maximum amplitude. Therefore, also in the su(1,1) case, the scenario related to the Rabi's resonance condition is the one with the maximum value for the transition probability at any time. However, it is important to note that in the su(1,1) case the transition probability, defined according to the framework delineated in Refs. [212] and [215], cannot overcome the value of 1/2, meaning that, in this instance, we cannot have the complete population transfer.

5.3 Analytically solvable 2×2 PT-symmetry dynamics

This paragraph deals with physical systems living in a two-dimensional Hilbert space and describable by 2×2 time-dependent quasi-Hermitian PT-symmetry matrices. The PT-symmetry two-level model describes a general sink-source or gain-loss system. It proves to be very useful for the comprehension of basic theoretical concepts [221] and of many experimental results, e.g. [28] and [29]. Other physical systems, e.g. coupled waveguides [11, 207, 224], are exactly described by a two-dimensional PT-symmetry Hamiltonian; in such cases, like in other photonic structures [10], the dynamics of the physical system is governed by a Schrödinger-like equation where the time variable is substituted with the spatial one (the propagation direction).

The matrix representation of a sink-source system may be cast as follows

$$\tilde{H} = \begin{pmatrix} -ir\sin\theta & \gamma \\ \gamma & ir\sin\theta \end{pmatrix}, \tag{5.46}$$

where r and θ are real parameters. The diagonal entry describes the energy time evolution in the sink-source [221]. The off-diagonal parameter γ may be interpreted as the coupling existing between the one-state sink and the one-state source [221] and in the present work is thought to be time-dependent.

It has been highlighted [221] that this two-box model is able to capture in its parameter space the passage from a condition where the exchange of energy between the two boxes takes place enabling the system to reach equilibrium to the opposite physical situation. The existence of such a radical change has been experimentally realized by parametrically changing the coupling constant γ [11, 28, 29, 221, 224] and may be interpreted as an evidence of the transition from unbroken to broken PT-symmetry phase.

In the following we concentrate in time-dependent PT-symmetry Hamiltonian problems. To this end we introduce a prescribed time-variation of the coupling constant γ. Investigating this kind of problems may find applications whenever one is interested in controlling a gain-loss physical system, representable by the Hamiltonian (5.46). To find the time-evolution operator generated by $\tilde{H}(t)$ is, generally speaking, a challenging problem strongly dependent on the off-diagonal time-dependent element. For this reason we search a general mathematical protocol providing examples of time-dependent coupling parameters leading to the prediction of the exact quantum dynamics of the corresponding physical system. To this end, in the following, we strategically construct a tool aimed at solving the general dynamical problem generated by the class of 2×2 quasi-Hermitian su(1,1) Hamiltonians. This procedure is immediately exploitable for treating dynamical problems characterized by PT-symmetry since they constitute a sub-class of the more general class of bi-dimensional su(1,1) dynamical problems.

The approach we use is analogous to the one reported in Ref. [38] aimed at individuating exactly solvable two-dimensional su(2) dynamical problems. On the basis of the result [38] and taking account of the 'affinity' existing between the SU(2) and the SU(1,1) symmetry-groups (both are sub-groups of the more general SL(2,$\mathbb{C}$) group), we extend the constructive method, successful for su(2) problems, to the su(1,1) case (see Appendix A.2). Thus, what is reported in the following possesses a twofold interest. We identify classes of exactly solvable dynamical problems governed by time-dependent bi-dimensional su(1,1) Hamiltonians, which is physically relevant in its own within the framework of pseudo- and quasi-Hermitian matrices. Moreover, through this general procedure, we get analytically solvable PT-symmetry dynamics with direct physical meaning thanks to the application to the general sink-source model and the gain-loss wave-guide scenarios.

5.3.1 Parametric solutions of the su(1,1) dynamical problem

It is important to observe that the dynamical problem described by Eq. (5.33) can be reduced to the solution of the Schrödinger equation $i\dot{U}=HU$, giving rise to a system of linear differential equations which may be put in the form (hereafter we omit the explicit time-dependences, if not necessary)

$$\Omega = i[\dot{a}a^* - \dot{b}b^*], \quad \omega = i[-\dot{a}b + \dot{b}a]. \tag{5.47}$$

Following the approach reported in Ref. [38], with appropriate changes to the class of 2×2 su(1,1) matrices (see Appendix A.2), the two entries a and b defining the time evolution matrix U in Eq. (5.26) may be represented as follows

$$a = \cosh\left[\Lambda_\theta\right]\exp\left[i\left(\frac{\phi_\omega(t)-\phi_\omega(0)}{2}-\frac{\Theta}{2}-\mathscr{R}\right)\right], \tag{5.48a}$$

$$b = -i\sinh\left[\Lambda_\theta\right]\exp\left[i\left(\frac{\phi_\omega(t)+\phi_\omega(0)}{2}-\frac{\Theta}{2}+\mathscr{R}\right)\right], \tag{5.48b}$$

where

$$\Lambda_\theta = \int_0^t |\omega| \cos[\Theta] \; dt', \tag{5.49a}$$

$$\mathscr{R} = \int_0^t \frac{|\omega| \sin[\Theta]}{\sinh[2\Lambda_\theta]} \; dt'. \tag{5.49b}$$

Θ is an arbitrary, real, time-dependent function such that $\Theta(0) = 0$ and Ω and ω appearing in H are related through the following relation

$$\dot{\Theta} + 2|\omega| \sin[\Theta] \coth[2\Lambda_\theta] = 2\Omega + \dot{\phi}_\omega. \tag{5.50}$$

Equation (5.50) practically serves as a recipe in the sense that, choosing at will the function Θ ($\Theta(0) = 0$), it determines Ω (ω) in terms of Θ and ω (Ω) making the dynamical problem (5.47) exactly solvable.

Such a recipe may be even and easily exploited in the treatment of the dynamical problem generated by the Hamiltonian given by Eq. (5.46). The already discussed link between the PT Hamiltonian in Eq. (5.1) and the su$(1,1)$ one in Eq. (5.46), in view of Eq. (5.50), provides indeed a possible time variation of the parameter γ, under the constraint $|r \sin\theta| = const$.

We emphasize also that Eqs. (5.48) and (5.50) are parametric solutions of the Schrödinger equation and this means that they are valid and may be reinterpreted also for problems whose dynamics is ruled by a Schrödinger-like equation. In guided wave optics, for example, a space-dependent Schrödinger equation appears, as shown in Refs. [11, 224]; in such a case the space variable represents the propagation direction of the waves in the guides and a space-dependent coupling may be reached by varying the distance between the guides [10]. Thus, e.g. for the physical systems studied in Refs. [11, 224], we may interpret Eq. (5.50) as a prescription how to vary over space the coupling between the guides in such a way to have an exactly solvable system of space-dependent differential equations for the amplitudes of the waves propagating in the guides. Given an initial condition of the amplitudes, the latter may be written at a certain space point in terms of the solutions in Eqs. (5.48), provided that the time variable is appropriately substituted by the spatial one.

5.3.2 Exact PT-symmetry examples

In the following examples we consider PT-symmetry cases, that is, we set $\phi_\omega = \pi/2$.

Example 1

If we choose such a parameter Θ that, given $|\omega|$, satisfies

$$\int_0^t |\omega| \cos[\Theta] \; dt' = \frac{1}{2} \operatorname{arcsinh}[\kappa], \quad \kappa = 2\int_0^t |\omega| dt', \tag{5.51}$$

from Eqs. (5.48) we get

$$a = \sqrt{\frac{\sqrt{1+\kappa^2}+1}{2}} \exp\left[-i\left(\frac{\Theta}{2} + \mathscr{R}\right)\right], \tag{5.52a}$$

$$b = \sqrt{\frac{\sqrt{1+\kappa^2}-1}{2}} \exp\left[-i\left(\frac{\Theta}{2} - \mathscr{R}\right)\right], \tag{5.52b}$$

where

$$\mathscr{R} = \frac{\text{arcsinh}[\kappa]}{2}. \tag{5.53}$$

In this instance, we actually have $\Theta = \arctan[\kappa]$ and the relation between the Hamiltonian parameters reads

$$\Omega = \frac{|\omega|}{2}\left[2 + \frac{1}{1+\kappa^2}\right]. \tag{5.54}$$

In the framework of gain-loss models we may interpret this relation as a prescription how to vary over time the coupling parameter γ in Eq. (5.46) between the sink and the source in such a way that the dynamical problem is solved by Eqs. (5.52). Supposing $|r\sin\theta| = const.$ (in this case $0<\theta<\pi/2$, in view of the choice $\phi_\omega = \pi/2$; for $\pi/2 < \theta < \pi$ we have to set $\phi_\omega = -\pi/2$), we may write Eq. (5.54) as

$$\gamma(\tau) = \frac{|r\sin\theta|}{2}\left[2 + \frac{1}{1+\tau^2}\right], \tag{5.55}$$

with $\tau = r\sin(\theta)\,t$. The parameter γ is plotted in Fig. 5.2a as a function of the dimensionless parameter τ.

If we suppose our system initially prepared in the state $\rho_0 = |-\rangle\langle-|$, the probability of finding it in the opposite state, according to Eq. (5.36), is

$$P_-^+ = \rho_{11} = \frac{|b|^2}{1+2|b|^2}, \tag{5.56}$$

where ρ_{11} is the (1,1)-element of the matrix $\rho = U\rho_0 U^\dagger/\text{Tr}\{U\rho_0U^\dagger\}$, solution of the equation (5.33). The plot of P_-^+ is reported in Fig. 5.2b against the dimensionless time τ. We note that the value

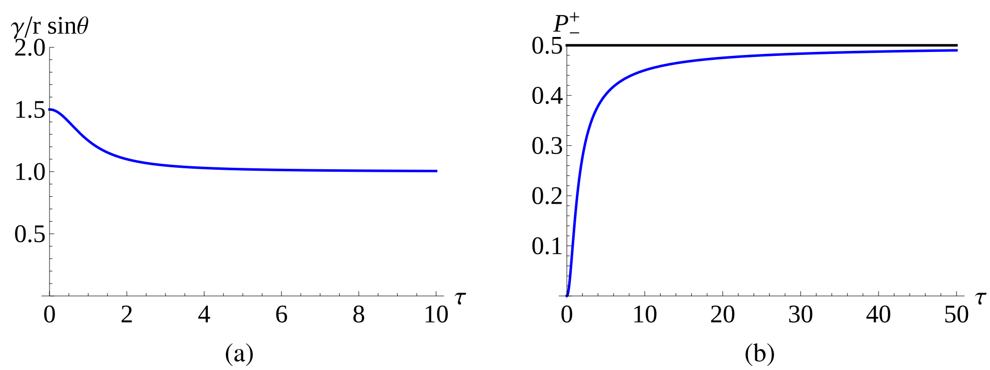


Figure 5.2: (Color online) a) Time evolution of the coupling parameter γ in Eq. (5.55) as a function of $\tau = r\sin(\theta)\,t$; b) Time dependence of the transition probability P_-^+ in Eq. (5.56) (for $|b|$ in Eq. (5.52b)) as a function of $\tau = r\sin(\theta)\,t$. The upper line corresponds to $P_-^+ = 1/2$.

reached asymptotically by the transition probability is 1/2. By Eq. (5.56) we see, indeed, that 1/2 is the maximum value reachable by the transition probability, precisely when $|b| \gg 1$.

A direct physical application of current interest my be found in the framework of coupled wave guides. In Ref. [11], for example, a PT-symmetry physical scenario of optical beam propagation is investigated. Under appropriate conditions, the optical-field dynamics of the two coupled wave-guides is described by the following space-dependent Schrödinger-like equation

$$i\frac{d}{dz}\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} i\varepsilon & -\gamma \\ -\gamma & -i\varepsilon \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \tag{5.57}$$

where $E_{1,2}$ are the field amplitudes in the first and the second guide, respectively, ε is the effective gain coefficient, γ is the coupling constant and z represents the one-dimensional location of the signals in the guides. In our case, we suppose a spatial configuration of the wave-guides such that the coupling parameter exhibits a spatial-dependence $\gamma \equiv \gamma(z)$ (reachable by varying the distance between he guides [10]). It is easy to see that, if we apply a unitary transformation accomplishing $\hat{\sigma}_z \to \hat{\sigma}_x$, $\hat{\sigma}_x \to -\hat{\sigma}_z$ and $\hat{\sigma}_y \to \hat{\sigma}_y$, the PT-symmetry 'Hamiltonian' governing the dynamics of the optical system becomes of the form (5.46). Then, in this instance, Eq. (5.54) reads

$$\gamma(\varepsilon, z) = \frac{\varepsilon}{2}\left[2 + \frac{1}{1+(\varepsilon\, z)^2}\right], \tag{5.58}$$

giving us a prescription how to spatially vary the coupling parameter between the two guides (in terms of the constant gain parameter), so that the system (5.57) may be analytically solved. In such a case the probability P_+^- represents the possibility to transfer the signal in the second wave-guide when it is initially injected in the first one. The space-dependence of the coupling parameter we are discussing has, then, the effect of asymptotically transferring half part of the initial optical signal from the first channel to the second one.

Example 2

By choosing, instead, Θ such that

$$\int_0^t |\omega| \cos[\Theta]\, dt' = \text{arcsinh}\,[\kappa/2], \tag{5.59}$$

it is easy to verify that we get

$$\mathscr{R} = \frac{1}{2}\arctan[\kappa/2]. \tag{5.60}$$

In this case we have

$$a = \sqrt{1+\frac{\kappa^2}{4}}\exp\left[-i\left(\frac{\Theta}{2}+\mathscr{R}\right)\right], \tag{5.61a}$$

$$b = \frac{\kappa}{2}\exp\left[-i\left(\frac{\Theta}{2}-\mathscr{R}\right)\right], \tag{5.61b}$$

and the necessary condition that the Hamiltonian parameters must satisfy results in

$$\Omega = |\omega|. \tag{5.62}$$

We see that this second choice turns out to be a trivial case. However, two interesting observations may be developed. First, it is worth noticing that, developing the same calculation within the framework of su(1,1) matrices, Eq. (5.62) becomes $2\Omega + \dot{\phi}_\omega = 2|\omega|$, which, in turn, is a particular case of the more general solvability condition reported in Eq. (5.16). It is possible to show that, within the class of su(1,1) matrices, this more general relation may be derived through our approach as reported in Appendix A.2.1. Second, the last example shows how a slight change in the choice of the function Θ might lead us to a significantly different scenario and then to a substantially different dynamics of the physical system. Then, this fact underlines the potentiality of the method [38], here proposed for su(1,1) matrices, in identifying exactly solvable scenarios of possible experimental interest.

Before closing this section, we emphasize that in the examples discussed before we were able to find the closed form of all the necessary quantities in order to solve the dynamical problem. However, if we were interested only in specific physical observables, it might happen that we are requested to find the explicit form of only few quantities. For example, if we were interested in the study of the transition probability P_-^+ or in the knowledge of

$$\langle \hat{\sigma}^z \rangle = \mathrm{Tr}\{\rho \hat{\sigma}^z\} = -1/(|a|^2 + |b|^2), \tag{5.63a}$$

$$\langle \hat{\sigma}^x \rangle = \sqrt{\frac{\kappa^2}{1+\kappa^2}} \cos[\phi_\omega(t) - \Theta(t) - \pi/2], \tag{5.63b}$$

for $\rho(0) = |-\rangle\langle -|$, it is sufficient to analytically write the expression of $|a|$ and $|b|$ (Θ is chosen at will). In this way, we are not obliged to solve analytically the expression of the integral $\mathscr{R}$ [Eq. (5.49b)] involved in the exponentials in Eqs. (5.52b) and (5.61b), which results in some cases very hard to solve. Thus, this fact means that, concentrating only on specific physical quantities, the choices of Θ we may perform and then the classes of exactly solvable models we may identify become wider and wider. For example, if we choose $|\omega| = |\omega_0| \cos^2(\tau)$ and $\Theta = \tau$ with $\tau = |\omega_0|\, t$, although we made relatively simple choices, we are not able to find the analytical expression of the integral $\mathscr{R}$ in Eq. (5.49b). However, we may derive the exact form of the transition probability P_-^+ in Eq. (5.56) plotted in Fig. 5.3a and, through Eq. (5.50), we may write the exact time-dependence of Ω plotted in Fig. 5.3b for $\dot{\phi}_\omega = 0$.

5.4 Summary and remarks

In this chapter we have first identified a non-trivial class of su(1,1) time-dependent Hamiltonian models for which exact solutions of the "dynamical" problem: $i\dot{U}(t) = H(t)U(t)$, may be provided. Secondly, we have constructed step by step a reasonable frame within which the knowledge of the non-unitary solution of the above mentioned equation may be legitimately exploited as a source for generating the time evolution of a generic initial state of the system represented by $H(t)$. Here "legitimately" means that the new dynamical equation for ρ introduced in [212], rests on the introduction of a good simple dynamical map generating the standard von Neumann-Liouville equation when the system is described by a Hermitian Hamiltonian.

Exploiting this new point of view, we have treated the dynamics of an su(1,1) "Rabi" system generating results interpretable within the quantum context. We have evaluated analytically the transition

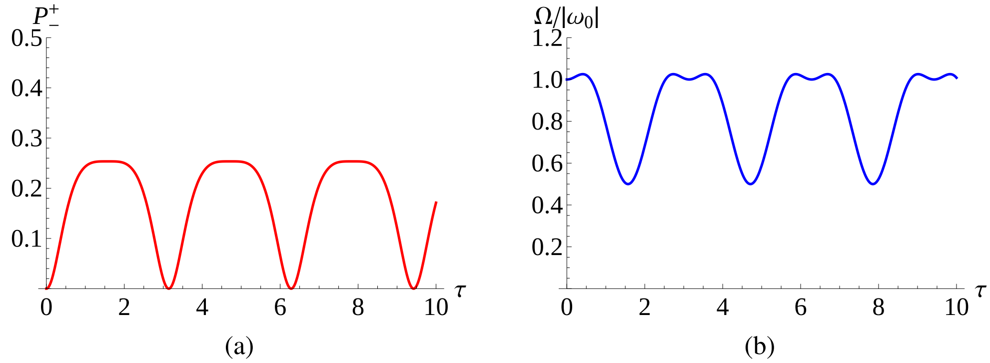


Figure 5.3: (Color online) Time dependence of a) the transition probability P_-^+ and b) the parameter Ω against the dimensionless time $\tau = |\omega_0|t$ for $|\omega| = |\omega_0|\cos^2(\tau)$, $\Theta = \tau$ and $\dot{\phi}_\omega = 0$.

probability $P_+^-(t)$ under three different regimes, evidencing remarkable differences of the time behavior exhibited by the same probabilities in the Rabi su(2) problem. In addition, we have clarified that the passage from a ν-regime to another one is governed by a condition on this parameter that does not coincide with the one ruling the transition from a real (time-independent) energy spectrum to a complex one. This result makes evident that such a coincidence might at most be only a particular case ($\dot{\phi}_\omega = 0$) of a wider scenario, where a direct link between the regime transition and the change in the spectrum of the Hamiltonian does not generally occur.

Moreover, get inspired by previous results in other contexts [38], we have developed a protocol through which we found a parametrization for the solutions of the dynamical problem related to two-dimensional time-dependent quasi-Hermitian su(1,1) Hamiltonians. Such a result turns out to be of physical interest at the light of the fact that 2×2 PT-symmetry Hamiltonians, describing sink-source or gain-loss systems [221], are a special sub-class of the su(1,1) matrices.

This fact has allowed us to interpret transparently the solvability condition [Eq. (5.50)] of the dynamical problem from a physical point of view. Such a relation may be read as the prescription how to vary over time the coupling between the sink and the source in order to controllably drive the dynamics of the whole gain-loss system. Moreover, we have brought to light also the relevance of the result in guided wave optics scenarios [10, 11, 207, 224]. In such cases, the dynamical problem is converted in a space-dependent one obeying to a space-dependent Schrödinger-like equation. Thus, provided that the time variable is substituted with the spatial one, our solutions keep their validity. We emphasize that, as a byproduct, we then get prescriptions on how to vary the coupling (space distance) between the wave-guides as to have an analytically solvable model.

Our analytical results might furnish new experimentally stimulating and physically realizable scenarios. In this respect, we are confident in such a possibility, by taking account, for example, of time-dependent exactly solvable models reproduced through coupled waveguides systems. The Landau-Majorana-Stuckelberg-Zener dynamics and the STIRAP process as showed in [10] are two of such examples.

Finally, we emphasize that the parametrized solutions of a 2×2 su(1,1) dynamical problem here reported possess a more general value. Indeed, as it happens in the su(2) dynamical case, a higher dimensional su(1,1) dynamical problem may be reduced to the 2×2 one and its solution may be written in terms of the two parameters a and b defining the Cayley-Klein parametrization of the evolution operator in Eq. (5.26) [209].

The results discussed in this chapter have been published in Refs. [222, 223].

Conclusions

Since its appearing Quantum Mechanics (QM) has provoked animated interpretative debates which have highlighted its profound philosophical implications. QM, in fact, with its basic principles has revolutionized Physics and the whole Science in both its ontological and epistemological aspects. The greatest innovation lies in the lost of the most important characteristic of the fundamental hard science: the determinism. QM is a non-deterministic but causal theory. The non-deterministic character stems from the fact that QM does not predict 'what is' or 'what happens', but only what can be observed with a certain probability once the experimental conditions are fixed. The Schrödinger equation, that is the fundamental equation of motion governing the dynamics of quantum physical systems, instead, introduces into the theory the aspect of mechanical causation. This causality, however, characterizes the evolution of the state of the quantum system in its Hilbert space and not in the real one, that is in the space of probability amplitudes and not in the actual physical one. In this respect, Heisenberg emphasized that QM has revived the concept of *potentia* [2], born with the Aristotelian philosophy: QM foresees the *tendencies*, that is the probabilities of the different scenarios that can arise from the evolution and the observation (measurement process) of a quantum physical system.

It is important to emphasize that the concept of probability in QM, however, is not related (only) to the epistemological uncertainties stemming from the operational limits in carrying out an experiment, as it is the case for Classical Physics (CP). It has a deeper character because it directly enters into the definition of the state of a quantum system. This shocking difference on the ontological level between QM and CP is what has always left perplexed one of the most important scientist of all times, Albert Einstein. In this respect, he clearly claimed the famous sentence: "*God does not play dice*" to externalize all his disagreement with such a drastic principle change. Einstein did not agree and did not accept that the object of scientific knowledge could be characterized by an intrinsic uncertainty. This uncertainty, that gives rise to the probabilistic character of QM, is intimately connected to the fact that the ontological definition of the object of study is affected by the circumstance that in QM an objective division between subject and object is not possible. As Heisenberg says, QM has lost that utopian ideal (typical of CP) consisting in the fundamental principle of the Cartesian philosophy that characterized the whole Science until the early 1900s: the separation between (God,) I and World. Heisenberg says precisely [2] "*The mechanics of Newton and all the other parts of classical physics constructed after its model started from the assumption that one can describe the world without speaking about God or ourselves. This possibility soon seemed almost a necessary condition for natural science in general.*"; and he continues: "*[...] in the Copenhagen interpretation of quantum theory we can indeed proceed without mentioning ourselves as individuals, but we cannot disregard the fact that natural science is formed by men. Natural science does not simply describe and explain nature; it is a part of the interplay between nature and ourselves; it describes nature as exposed to our method of questioning. This was* *a possibility of which Descartes could not have thought, but it makes the sharp separation between the world and the I impossible.*". Concerning such a question, to sustain the Heisenberg's thesis, C. F. F. von Weizsäcker said indeed: "***Nature is earlier than man, but man is earlier than natural science***" [2]. What we do is Science, being our understanding of Nature in relation to our methods and approaches. In order to succinctly and elegantly summarize this concept, N. Bohr referred to the old adage according to which "***we are both actors and audience in the great drama of life***"[1, 2].

The indetermination in the definition of the state of a quantum system to which we refer is that, for example, we have in the so-called coherent superpositions of quantum states. This type of states, in bipartite or multipartite systems, gives rise to the phenomenon of entanglement consisting in a correlation of quantum nature getting established between the subsystems. The peculiarity of the entanglement lies in the fact that two related subsystems can continue to "feel" each other even at very large distances. In fact, the measurement of the state of one of the two systems, causing its collapse, will immediately cause as well the collapse of the other subsystem in entanglement with the first one. Even this further aspect left Einstein totally unsatisfied, who, in this regard, spoke of "spooky action at distance". Einstein was the first to bring to light the physical relevance of these states; he, however, tried to exploit them to demonstrate the incompleteness of quantum theory to sustain the famous idea of the 'hidden variables theory'. Later, however, it was demonstrated that it is not possible to exchange information through the entanglement making signals travel above the speed of light, in accordance with the principle of relativity. Furthermore, thanks to the Bell's inequality, theoretically predicted in 1964 [225] and experimentally verified in 2015 through a loophole-free experiment [226], the descriptive completeness of quantum theory was confirmed, against the possibility of a 'wider' theory based on the existence of hidden local variables.

It is worth noticing that the development of sophisticated nano-technologies has been fundamental to verify and to deeply understand the basic principles of quantum theory, previously briefly discussed. To date, in fact, it is possible to measure with great accuracy the quantum states of physical systems, highlighting non-classical effects such as quantum interference as predicted by the theory. Furthermore, thanks to the development of increasingly accurate techniques for the coherent manipulation of quantum systems, it has been possible to generate overlaps of states such as 'Schroedinger cat' states [227] and thus, for multipartite systems, to generate entangled states [228].

In this sense, the physical scenarios consisting of trapped ions and atoms, nuclear and electronic spins in condensed matter and superconducting circuits based on the well-known Josephson junctions, have been and still are of fundamental importance. The great importance of these physical systems stems also from their versatility and controllability which makes them the most promising candidates as building blocks for the realization of possible future quantum computers. Indeed, these systems show their full potentiality as quantum simulators, that is, as systems capable of simulating the quantum dynamics of other quantum physical systems: the brilliant idea introduced for the first time by R. P. Feynman [181]. Using these technologies, for example, in N.I.S.T. laboratories were realized for the first time entangled states between quasi-classical states of harmonic oscillators and atomic states [228]; at the University of Innsbruck the "Cirac-Zoller C-Not" quantum gate was instead implemented [229]; finally, in both centres it was possible to achieve the teleportation of quantum states [230, 231].

The relevance of the work reported in this thesis with the scenarios described above lies in the fact that such physical systems are modelled, that is, formally described and mathematically treated, through the formalism of the spin variables. Trapped ions and atoms and superconducting circuits, in fact, under appropriate experimental conditions, actually behave as N-level (qudit) interacting systems.

In this thesis work, in addition to studying the effects of the interaction between effective qudits, it has been reported the analysis of the dynamics of these systems when they are subjected to the external action of time-dependent classical electromagnetic fields too. The theoretical study of such interacting spin systems, embracing a great variety of physical scenarios and possibilities of realization, allows, therefore, both to investigate basic aspects of quantum theory and to study and predict physical effects turning out to be relevant from an applicative point of view within information sciences and quantum computing.

This thesis work is basically focused on showing how it is possible to exploit the knowledge of exact solutions of the dynamical problem of a two-level system to obtain exactly solvable scenarios for more complex systems of interacting spin-qudits subjected to classical time-dependent fields. If we want to summarize in very few words the method we have used, we could borrow the famous Latin motto '***divide et impera***'. The study of the models we consider starts from the analysis of the symmetries possessed by the Hamiltonian operator that allows the identification of invariant Hilbert subspaces. Within each subspace we describe the dynamics of the original system under scrutiny in terms of fictitious spin variables. For example, in case of subspaces within which the fictitious Hamiltonian is characterized by a SU(2) symmetry, we can describe the problem in terms of a single spin J (whose value depends on the size of the subspace). Thanks to the Group Theory, we can formally write the structure of the time evolution operator in terms of the solutions of the analogous problem relative to a single spin-1/2 [39, 173]. In this way, we can take advantage of the knowledge of exactly solvable single-qubit scenarios to solve the original problem (concerning many interacting spins) within the subspace under scrutiny. Our approach, therefore, consists in decomposing the original dynamical problem into independent dynamical sub-problems rewritten in terms of fictitious variables, in order to make it analytically treatable and solvable.

We showed how this approach allows to obtain analytical results in the case of two qubits (Chapter 2), two qutrit (Section 3.1), two qudits (Section 3.2) and N qubits (section 3.3), interacting with each other and subjected to local or uniform time-dependent fields.

In the first two cases we have highlighted the possibility of obtaining coupling-based LMSZ transitions by applying only a longitudinal field linearly varying over time (the coupling plays the role of a fictitious transverse field that generates the avoided crossing). An application of interest that exploits such a physical effect consists in generating entangled states of the spins by suitably setting the slope variation of the field. It is curious that it is required a non-adiabatic dynamic, that is a sufficiently fast variation of the field, for the generation of these entangled states.

In the third case, instead, it is interesting to see how the symmetry of both the coupling (isotropic exchange) and the field (homogeneous) gives rise to subspaces all characterized by SU(2) symmetry. This circumstance, in addition to making the dynamical problem solvable for certain temporal scenarios, made it possible to identify classes of IFE states [174, 175, 176]. Such states are types of initial conditions for which the two qudits evolve over time in a completely independent manner, i.e., as if the other spin with which they are coupled were not present.

In the latter case, the peculiar characteristic of the considered model is the presence of N-order interaction terms, i.e., interactions that simultaneously involve all the spins present in the system. These interactions, although far from the standard physical contexts (nuclear, atomic and molecular), can be easily reproduced through quantum simulation technologies such as trapped ions [41] and superconducting circuits composed of interacting transom qubits [8]. We have shown how these systems can be of relevant interest for applications. In fact, thanks to this type of interactions, it is possible, for example, to propagate the dynamics of a single spin, manipulated through the application of a time-dependent local field, to all the other spins of the chain.

We have also shown that the dynamical decomposition technique can be very useful for exactly solving the dynamics of systems living infinite-dimensional Hilbert spaces. Indeed, we have solved and compared (Chapter 4) the exact dynamics of a system composed of a quantum harmonic oscillator and a Glauber amplifier (inverted quantum harmonic oscillator) with that of the analogous system composed of two standard quantum harmonic oscillators.

Finally, the resolutive approach [38] used in the first chapter to identify new temporal scenarios of interest for a single qubit, has been exploited for the identification of time-dependent two-dimensional non-Hermitian su(1,1)-symmetric Hamiltonians for which it is possible to solve the related dynamical problem. In this case, the dynamics of the system is based on a non-linear equation of motion which generalizes the Liouville-von Neumann equation to the case of non-Hermitian Hamiltonians. We have highlighted how the study of su(1,1)-symmetry matrices finds an interesting application in the context of PT-symmetric Hamiltonians, being used to describe gain-loss open quantum systems such as, for example, coupled waveguide systems [10, 11].

In conclusion, it is important to underline that, in the analysed interacting spin models, we have taken into account the presence of an environment by considering noisy components of the field [177]. We have shown that, even in presence of these components varying randomly over time, it is possible, in some cases, to carry out the dynamical decomposition and solve the original problem in terms of relatively simpler problems. A possible perspective of the work reported in this thesis could consist in considering the spin systems coupled to one or more baths of harmonic oscillators. Also in this case it might be possible to identify fictitious dynamical sub-problems consisting of one or more interacting spins which in turn interact with (fictitious) baths. In this way, however, an exact resolution may be difficult to obtain. A possibility could consist in the numerical approach based on the quantum-classical Liouville-von Neuman equation, derivable through the partial Wigner transform [23]. In this case, the dynamical decomposition, in addition to identifying the physical properties of the system, would always be a useful tool also to make the numerical resolution procedure more efficient. Through this analysis it would be possible, therefore, to foresee undesired effects stemming from the coupling of the system with the environment so as to be able to minimize them in the experimental phase or perhaps to take advantage by exploiting them depending on the application tasks.

Appendix A

Parametric solutions of 2x2 dynamical problems

A.1 Solution approach of Ref. [38] for su(2) Hamitonians

The matrix representation of the general 2×2 su(2) Hamiltonian describing a spin-1/2 subjected to a time-dependent magnetic field may be cast in the following form

$$H = \begin{pmatrix} \Omega(t) & \omega(t) \\ -\omega^*(t) & -\Omega(t) \end{pmatrix} \tag{A.1}$$

up to a constant term having no physical relevance. The unitary operator solution of the Schrödinger equation $i\hbar\dot{U}(t) = H(t)U(t)$ can be formally put as

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \tag{A.2}$$

where a and b are two complex valued time-dependent functions satisfying $|a|^2 + |b|^2 = 1$. By the Schrödinger equation we get the following system of linear differential equations:

$$\begin{cases} i\hbar\dot{a} = \Omega a - \omega b^*, \\ i\hbar\dot{b} = \omega a^* + \Omega b, \\ a(0) = 1, \quad b(0) = 0. \end{cases} \tag{A.3}$$

In accordance with the traditional procedure of resolution of a linear system of differential equations of the first order, one seeks the second-order linear non-autonomous differential equation in $a(t)$ or $b(t)$. Unfortunately, the resulting equation even if linear, cannot be solved unless some special links among its variable coefficients are given. The approach reported in [38], successfully reaches the objective of providing a useful strategy for "constructing" solvable SU(2) problems, togehther their exact solutions. The method consists in introducing an auxiliary function $X(t)$ enabling the explicit analytical representation of both $a(t)$ and $b(t)$ at the cost of precisely defining the specific Hamiltonian of the exactly solvable SU(2) problem within the same resolution protocol. In other words one may claim that $X(t)$ generates too the peculiar link between the time dependent components of the applied magnetic field from the consistent construction of part of the solution. It is worthy to emphasize that, generally speaking, if such a link were given at will, there would be no certainty of being able to exactly solve the corresponding dynamical problem, for example starting from equations (A.3). Thus, the merit of the approach under scrutiny is just that of furnishing a self-consistent recipe to single out solvable SU(2) problems and to solve them.

Let us analyse in detail the above described method. Rewriting the Schrödinger equation as $H(t) = i\hbar\dot{U}(t)U^{\dagger}(t)$ we get the following system of linear differential equations:

$$\begin{cases} \Omega(t) = i\hbar[\dot{a}(t)a^{*}(t) + \dot{b}(t)b^{*}(t)] \\ \omega(t) = i\hbar[a(t)\dot{b}(t) - \dot{a}(t)b(t)] \end{cases}. \tag{A.4}$$

Under the following conditions

$$U(0) = \mathbb{1} \quad \text{o} \quad a(0) = 1, \quad b(0) = 0 \tag{A.5}$$

the second equation can be expressed in terms of b

$$b = -i\frac{a}{\hbar}\int_0^t \frac{\omega}{a^2} \tag{A.6}$$

and the first one can be written as

$$\dot{a} = -\left(\frac{i}{\hbar}\Omega + \frac{\dot{X}X^{*}}{\hbar^2 + |X|^2}\right)a, \tag{A.7}$$

where we used the expressions

$$b(t) = \frac{1}{\hbar}aX \quad \text{and} \quad \omega(t) = a^2\dot{X}, \tag{A.8}$$

with

$$X \equiv \int_0^t \frac{\omega}{a^2} \tag{A.9}$$

being an auxiliary arbitrary function which can be chosen at will. In this way, the solution of the differential equation for a reads

$$a(t) = \frac{\hbar}{(\hbar^2 + |X|^2)^{1/2}}\exp\left\{-\frac{i}{\hbar}\int_0^t \Omega(t')dt' - i\int_0^t \frac{\mathscr{I}[\dot{X}X^{*}]}{\hbar^2 + |X|^2}\right\}. \tag{A.10}$$

Thus, prescribing the longitudinal component of the magnetic field and choosing at will the auxiliary function X we may construct both the other Hamiltonian parameter ω and the parameters of the related time evolution operator. In this way, as explained before, the protocol enables in singling out time-dependent Hamiltonians for which we are able to solve the related dynamical problem and to explicitly construct the time-evolutio. operator.

It is possible to see that the following link between the Hamiltonian parameters

$$\frac{\Omega}{\hbar} + \frac{\dot{\phi}_{\omega}}{2} = \frac{|\omega|}{c} \tag{A.11}$$

with the related solutions

$$a(t) = \left(\cos[\Phi(t)] - i\frac{1}{\sqrt{1+c^2}}\sin[\Phi(t)]\right)\exp\left\{i\frac{\phi_\omega(t)}{2}\right\}, \tag{A.12a}$$

$$b(t) = \frac{c}{\sqrt{1+c^2}}\sin[\Phi(t)]\exp\left\{i\left(\frac{\phi_\omega(t)}{2} - \frac{\pi}{2}\right)\right\}, \tag{A.12b}$$

where

$$\Phi(t) = \frac{\sqrt{1+c^2}}{c}\int_0^t \frac{|\omega(t')|}{\hbar}dt', \tag{A.13}$$

can be derived by choosing

$$X = c\sin(\phi)e^{i\phi}. \tag{A.14}$$

By considering instead a general complex-valued X function we get the most general relation

$$\frac{1}{2}\dot{\Theta}(t) + \frac{|\omega(t)|}{\hbar}\sin\Theta(t)\cot\left[\frac{2}{\hbar}\int_0^t |\omega(t')|\cos\Theta(t')dt'\right] = \frac{\Omega(t)}{\hbar} + \frac{\dot{\phi}_\omega(t)}{2}, \tag{A.15}$$

and the solutions of the dynamical problem

$$a(t) = \cos\left[\frac{1}{\hbar}\int_0^t |\omega(t')|\cos\left[\Theta(t')\right]dt'\right] \times \exp\left\{i\left(\frac{\phi_\omega(t)-\phi_\omega(0)}{2} - \frac{\Theta(t)}{2} - \mathscr{R}(t)\right)\right\}, \tag{A.16a}$$

$$b(t) = \sin\left[\frac{1}{\hbar}\int_0^t |\omega(t')|\cos\left[\Theta(t')\right]dt'\right] \times \exp\left\{i\left(\frac{\phi_\omega(t)+\phi_\omega(0)}{2} - \frac{\Theta(t)}{2} + \mathscr{R}(t) - \frac{\pi}{2}\right)\right\}, \tag{A.16b}$$

with

$$\mathscr{R}(t) = \int_0^t \frac{|\omega(t')|\sin[\Theta(t')]}{\sin\left[2\int_0^{t'}|\omega(t'')|\cos[\Theta(t'')]dt''\right]}dt'. \tag{A.17}$$

A.2 Solution approach for su(1,1) dynamical problems

In the following we apply the method reported in ref. [38] to the class of 2×2 su(1,1) matrices. The Schrödinger equation $i\dot{U} = HU$, with H and U defined in Eq. (5.7) and (5.26), respectively, gives rise to a system of linear differential equations which may be put in the form

$$\Omega = i[\dot{a}a^* - \dot{b}b^*], \quad \omega = i[-\dot{a}b + \dot{b}a]. \tag{A.18}$$

Let us introduce the following function

$$X = \int_0^t \frac{\omega}{a^2}dt'. \tag{A.19}$$

By such a position and by the second equation in (A.18), we may write respectively

$$\omega = a^2\dot{X}, \quad b = -iaX. \tag{A.20}$$

Thus, the relation which the two functions a and b have to satisfy reads $|a|^2[1-|X|^2]=1$, implying $|X|^2 \leq 1$. In this way, the first equation in (5.47) becomes a closed integral-differential equation for a, namely

$$\dot{a} = \left(-i\Omega + \frac{\dot{X}X^*}{1-|X|^2}\right) a, \tag{A.21}$$

which is solved to yield

$$a = \frac{1}{[1-|X|^2]^{1/2}} \exp\left[-i\int_0^t \Omega \, dt' + i\int_0^t \frac{\mathrm{Im}[\dot{X}X^*]}{1-|X|^2} \, dt'\right]. \tag{A.22}$$

Let us consider an arbitrary complex function X in the form

$$X = A\exp[i\phi], \qquad A(0)=0, \tag{A.23}$$

with ϕ and A real functions of time. The latter must satisfy the condition $A^2 \leq 1$ due to the condition $|X|^2 \leq 1$. The function $\dot{\phi}(t)$ has to satisfy

$$\dot{\phi} = \frac{\dot{A}}{A}\tan[\Theta], \tag{A.24}$$

where Θ is a real function of time t, defined by

$$\Theta = \phi_\omega + \phi + 2\int_0^t \Omega \, dt' - 2\int_0^t \frac{\dot{\phi}}{1-A^2} \, dt' - 2\phi(0). \tag{A.25}$$

By Eq. (A.24) we derive

$$\dot{A}^2 + \dot{\phi}^2 A^2 = \dot{A}^2(1+\tan^2[\Theta]) = |\omega|^2(1-|X|^2), \tag{A.26}$$

implying

$$A = \tanh[\Lambda_\theta], \quad \dot{\phi} = \frac{2|\omega|\sin[\Theta]}{\sinh[2\Lambda_\theta]}, \tag{A.27}$$

with

$$\Lambda_\theta = \int_0^t |\omega|\cos[\Theta] \, dt', \tag{A.28}$$

so that we have to put $\sin[\Theta(0)]=0$, which is assured by assuming $\phi_\omega(0)=\phi(0)$, if we want to keep parameters well behaved. We see that the function A in Eq. (A.27) satisfies the condition $A^2 \leq 1$. From the equations in (A.27) we find that Θ must satisfy the following integral-differential equation

$$\dot{\Theta} + 2|\omega|\sin[\Theta]\coth[2\Lambda_\theta] = 2\Omega + \dot{\phi}_\omega. \tag{A.29}$$

The solutions a and b may be written as

$$a = \cosh[\Lambda_\theta]\exp\left[i\left(\frac{\phi_\omega(t)-\phi_\omega(0)}{2} - \frac{\Theta}{2} - \mathscr{R}\right)\right], \tag{A.30a}$$

$$b = -i\sinh[\Lambda_\theta]\exp\left[i\left(\frac{\phi_\omega(t)+\phi_\omega(0)}{2} - \frac{\Theta}{2} + \mathscr{R}\right)\right], \tag{A.30b}$$

with

$$\mathscr{R} = \int_0^t \frac{|\omega|\sin[\Theta]}{\sinh[2\Lambda_\theta]} \, dt', \tag{A.31}$$

and putting $\Theta(0)=0$.

A.2.1 Special Case

To recover the solvability condition $\Omega(t)+\dot{\phi}_\omega(t)/2=\nu|\omega(t)|$, let us assume the function X as

$$X_\nu = \nu^{-1}\sin\phi_\nu \exp\left[i(\phi_\nu+\phi_\omega^0)\right], \quad \phi_\nu(0)=0, \tag{A.32}$$

with $\phi_\omega^0=$ const., and then it is necessary that $\sin^2\phi_\nu(t)<\nu^2$. Under this assumption and according to the general theory, the transverse field has to be put

$$\omega = \frac{\nu\dot{\phi}_\nu}{\nu^2-\sin^2\phi_\nu}\times\exp\left\{-2i\int_0^t \Omega\, dt' + 2i\int_0^t \frac{\nu^2\dot{\phi}_\nu}{\nu^2-\sin^2\phi_\nu}\, dt' + i\phi_\omega^0\right\}. \tag{A.33}$$

Assuming ν and $\dot{\phi}$ positive we may write

$$|\omega| = \frac{\nu\dot{\phi}_\nu}{\nu^2-\sin^2\phi_\nu}, \tag{A.34}$$

and from Eq. (A.33) it is possible to derive

$$2\Omega+\dot{\phi}_\omega = 2\nu\,|\omega|\,, \tag{A.35}$$

being nothing but the relation we were looking for, got through an other approach in Sec. 5.1 where its physical reason has been brought to light. This relation is valid for the dynamical regimes $\nu<1$ and $\nu\geq 1$ with $\nu\geqslant 0$ and the consistency of the procedure leads to the following expression for ϕ_ν

$$\phi_\nu = \arctan\left[\frac{\nu}{\sqrt{\nu^2-1}}\tan\left(\sqrt{\nu^2-1}\int_0^t |\omega|\, dt'\right)\right]. \tag{A.36}$$

For this case the solutions a_ν and b_ν can be constructed and acquire different expressions depending on the value of ν. In the regime $\nu>1$, we get

$$a_\nu = \left[\cos(\Lambda_\nu) - i\frac{\nu}{\sqrt{\nu^2-1}}\sin(\Lambda_\nu)\right]\exp\left\{i\frac{\phi_\omega(t)-\phi_\omega^0}{2}\right\}, \tag{A.37a}$$

$$b_\nu = \frac{-i\sin(\Lambda_\nu)}{\sqrt{\nu^2-1}}\exp\left\{i\frac{\phi_\omega(t)+\phi_\omega^0}{2}\right\}, \tag{A.37b}$$

while, in the regime $0\leq\nu<1$, we have

$$a_\nu = \left[\cosh(\Lambda'_\nu) - i\frac{\nu}{\sqrt{1-\nu^2}}\sinh(\Lambda'_\nu)\right]\exp\left\{i\frac{\phi_\omega(t)-\phi_\omega^0}{2}\right\}, \tag{A.38a}$$

$$b_\nu = -i\frac{\sinh\left[\Lambda'_\nu(t)\right]}{\sqrt{1-\nu^2}}\exp\left\{i\frac{\phi_\omega(t)+\phi_\omega^0}{2}\right\}, \tag{A.38b}$$

with

$$\Lambda_\nu = \sqrt{\nu^2-1}\int_0^t |\omega|\, dt', \quad \Lambda'_\nu = \sqrt{1-\nu^2}\int_0^t |\omega|\, dt'. \tag{A.39}$$

Appendix B

Two-qubits exactly solvable time-dependent scenarios

If we choose the two magnetic fields acting upon the two spin-1/2's as follows

$$\hbar\omega_{1/2}(t) = \frac{|\Gamma_+|}{\cosh(2\tau_+)} \pm \frac{|\Gamma_-|}{\cosh(2\tau_-)} \tag{B.1}$$

the solutions for the entries of the time evolution operator (2.21) are

$$\begin{aligned}
&|a_+(t)| = \sqrt{\frac{\cosh(2\tau_+)+1}{2\cosh(2\tau_+)}}, \qquad |b_+(t)| = \sqrt{\frac{\cosh(2\tau_+)-1}{2\cosh(2\tau_+)}}, \\
&\phi_a^+(t) = -\arctan[\tanh(\tau_+)] - \tau_+ \qquad \phi_b^+(t) = \phi_{\Gamma_+} - \arctan[\tanh(\tau_+)] + \tau_+ - \frac{\pi}{2} \\
&|a_-(t)| = \sqrt{\frac{\cosh(2\tau_-)+1}{2\cosh(2\tau_-)}}, \qquad |b_-(t)| = \sqrt{\frac{\cosh(2\tau_-)-1}{2\cosh(2\tau_-)}}, \\
&\phi_a^-(t) = -\arctan[\tanh(\tau_-)] - \tau_- \qquad \phi_b^-(t) = \phi_{\Gamma_-} - \arctan[\tanh(\tau_-)] + \tau_- - \frac{\pi}{2}.
\end{aligned} \tag{B.2}$$

If, instead, the two local magnetic fields change in time as

$$\hbar\omega_{1/2}(t) = \frac{|\Gamma_+|}{\cosh(2\tau_+)} \pm \frac{|\Gamma_-|}{4}\left[\frac{3}{\cosh(\tau_-)} - \cosh(\tau_-)\right], \tag{B.3}$$

the solutions, in this case, read

$$\begin{aligned}
&|a_+(t)| = \sqrt{\frac{\cosh(2\tau_+)+1}{2\cosh(2\tau_+)}}, \qquad |b_+(t)| = \sqrt{\frac{\cosh(2\tau_+)-1}{2\cosh(2\tau_+)}}, \\
&\phi_a^+(t) = -\arctan[\tanh(\tau_+)] - \tau_+ \qquad \phi_b^+(t) = \phi_{\Gamma_+} - \arctan[\tanh(\tau_+)] + \tau_+ - \frac{\pi}{2} \\
&|a_-(t)| = \frac{1}{\cosh(\tau_-)}, \qquad |b_-(t)| = \tanh(\tau_-) \\
&\phi_a^-(t) = -\arctan\left[\tanh\left(\frac{\tau_-}{2}\right)\right] - \frac{1}{2}\sinh(\tau_-), \qquad \phi_b^-(t) = \phi_{\Gamma_-} - \arctan\left[\tanh\left(\frac{\tau_-}{2}\right)\right] + \frac{1}{2}\sinh(\tau_-) - \frac{\pi}{2}.
\end{aligned} \tag{B.4}$$

In the previous expressions we put

$$\tau_\pm = \frac{|\Gamma_\pm|}{\hbar}t, \qquad |\Gamma_\pm| = \sqrt{(\gamma_{xx} \mp \gamma_{yy})^2 + (\pm\gamma_{xy} + \gamma_{yx})^2}, \qquad \phi_{\Gamma_\pm} = -\arctan\left[\frac{\pm\gamma_{xy} + \gamma_{yx}}{\gamma_{xx} \mp \gamma_{yy}}\right]. \tag{B.5}$$

In the previous formulas, τ_+ and τ_- are scaled dimensionless times acting as independent variables; ϕ_{Γ_+} and ϕ_{Γ_-} are true parameters strictly related to the microscopic model. In our calculations we consider $\gamma_{xx} = \gamma_{yy} = \beta\gamma_{xy} = \beta\gamma_{yx} = c$ with $\beta = 2$; we get $|\Gamma_+| = c = |\Gamma_-|/2$ and $\phi_{\Gamma_+} = -\pi/2$, $\phi_{\Gamma_-} = 0$ and then $\tau_- = 2\tau_+$.

It is worth emphasizing that other two possible exactly solvable scenarios may be constructed, namely when the magnetic fields are

$$\begin{aligned}
\hbar\omega_{1/2}(t) &= \frac{|\Gamma_+|}{4}\left[\frac{3}{\cosh(\tau_+)} - \cosh(\tau_+)\right] \pm \frac{|\Gamma_-|}{\cosh(2\tau_-)}, \\
\hbar\omega_{1/2}(t) &= \frac{|\Gamma_+|}{4}\left[\frac{3}{\cosh(\tau_+)} - \cosh(\tau_+)\right] \pm \frac{|\Gamma_-|}{4}\left[\frac{3}{\cosh(\tau_-)} - \cosh(\tau_-)\right].
\end{aligned} \tag{B.6}$$

Appendix C

Quantum Discord for two-qubit X-states

C.1 Quantum Discord

The main feature of the quantum world, discriminating it from the classical one, is the possibility of representing a pure state as superposition of pure states. Indeed, while a quantum state of a bipartite system is not necessarily writeable as a tensor product of two independent pure states of the two subsystems, a pure state of a classical bipartite system turns out to be always factorizable since the superposition principle does not hold in this context [114]. This crucial difference leads to the following remarkable physical consequences. 1) Contrary to what happens in classical Physics, the knowledge that a non-factorizable state of a quantum bipartite system is pure does not lead to pure states of the two subsystems. 2) Non-factorizable states of a quantum bipartite system lead to non-locality effects christened by Schrödinger as entanglement.

Quantum entanglement represents one of the most important resources in quantum information [9, 115]. At the same time, there exist quantum correlations, different from entanglement, with potential applications in quantum information tasks, for instance quantum nonlocality without entanglement [115]. Likewise, it was shown that there exist separable states which can produce a speeding of some protocols, in comparison to the classical states [119].

Such a kind of nonlocal correlation, introduced by Ollivier and Zurek [116, 117], is quantum discord, that received a lot of attention in the recent years [117, 123]. Quantum discord is defined as the difference between two different quantum analogues of classically equivalent expressions of the quantum mutual information, which is a measure of all correlations in a quantum state. In a bipartite state discord measures the total quantum correlations, without restricting to entanglement. Discord coincides with the entropy of entanglement for pure entangled states. Some mixed separable states can have non-zero discord, so that it is considered to represent a characteristic of the quantumness of such separable states.

The measure of the total correlations in a bipartite system AB is given by the quantum mutual information [232]

$$\mathscr{I}(\rho^{AB}) = S(\rho^{A}) + S(\rho^{B}) - S(\rho^{AB}). \tag{C.1}$$

where $S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho)$ is the von Neuman entropy. ρ^{AB} represents the density operator of the compound system $A+B$, whilst $\rho^{A(B)} = \mathrm{Tr}_{B(A)}(\rho^{AB})$ is the reduced density matrix of the subsystem

A (B). Quantum discord was determined [117] by using a measurement-based conditional density operator in order to generalize the classical mutual information. The considered von Neumann-type measurement consists of one-dimensional local projectors summing to identity. The quantum mutual information corresponding to the quantum conditional entropy associated to a measurement

$$S(\rho|B_k) = \sum_k p_k S(\rho_k), \tag{C.2}$$

is given by [117]

$$\mathscr{I}(\rho|B_k) = S(\rho^A) - S(\rho|B_k). \tag{C.3}$$

Here B_k is the set of the projectors which perform the measurement on the subsystem B and $p_k = \mathrm{Tr}(I \otimes B_k)\rho(I \otimes B_k)$ is the measurement probability for the kth projector. We may write, indeed, the conditional density operator ρ_k, denoting the reduced density operator of subsystem A after the local measurements and which is associated with the measurement outcome k, in the following form (I denotes the identity operator on the subsystem A):

$$\rho_k = \frac{1}{p_k}(I \otimes B_k)\rho(I \otimes B_k). \tag{C.4}$$

Quantum discord is interpreted as a measure of quantum correlations since it is defined by the difference between the mutual information $\mathscr{I}(\rho)$ and the classical correlations $\mathscr{C}(\rho)$

$$D(\rho) = \mathscr{I}(\rho) - \mathscr{C}(\rho). \tag{C.5}$$

The measure of bipartite classical correlations $\mathscr{C}(\rho) = \sup_{B_k} \mathscr{I}(\rho|B_k)$ (sup is taken over all possible von Neumann local measurements B_k) represents the quantum mutual information induced by measurement.

When a bipartite system is in a pure state, entanglement and quantum discord give the same information on quantum correlations, while in a mixed state there might be quantum correlations - discord, even if the two subsystems are not entangled. Quantum discord has the peculiarity to be strictly related to the subsystem under measurement to investigate the existence of quantum correlations. This means that for a bipartite system composed by two subsystems A and B, we speak of quantum discord with respect to the subsystem A, D_A, and B, D_B. It is useful to remind that quantum discord is zero for a general state of a bipartite system, $D_B(\rho_{AB}) = 0$, if and only if the state can be written as

$$\rho_{AB} = \sum_i p_i \rho_A^i \otimes |i\rangle\langle i|_B, \qquad \sum_i p_i = 1, \quad p_i \geq 0, \tag{C.6}$$

that is if there exist an orthonormal basis for the subsystem with respect to which we calculate the quantum discord (B in this case) such that its state results diagonal.

The difficulty in calculating quantum discord consists in the complexity of the maximization procedure for computing the classical correlations, due to the fact that maximization has to be performed over all possible von Neumann measurements on party B. Analytical expressions for classical correlations and quantum discord are known for two-qubit Bell diagonal state and for some kinds of two-qubit X states [123].

C.2 X-states: Fano parametrization and analytical discord

The Bloch generalization of the density operator of a qubit to the case of two-qubit systems is given by the parametrization introduced by Fano [233]. The general expression of a two-qubit density operator acting in the Hilbert space $\mathscr{H}_A \otimes \mathscr{H}_B$ is [233, 234]:

$$\rho = \frac{1}{4}\left(I \otimes I + \mathbf{r} \cdot \boldsymbol{\sigma} \otimes I + I \otimes \mathbf{s} \cdot \boldsymbol{\sigma} + \sum_{m,n=1}^{3} t_{mn} \sigma_m \otimes \sigma_n \right), \tag{C.7}$$

where σ_j, with j = 1, 2, 3 are the Pauli operators. Eq. (C.7) represents the Fano parametrization of ρ. The vectors $\mathbf{r}$ and $\mathbf{s}$ are real, their expressions being $r_j = \mathrm{Tr}(\rho\sigma_j \otimes I)$ and $s_j = \mathrm{Tr}(\rho I \otimes \sigma_j)$. In addition, the matrix T defined by t_{mn} is also a real matrix, with $t_{mn} = \mathrm{Tr}(\rho\sigma_m \otimes \sigma_n)$, where m and n = 1, 2, 3.

Let us briefly discuss the transformation of a two-qubit density operator under a local unitary transformation. One knows that for any unitary transformation U there is a unique rotation O such that:

$$U\,\mathbf{n} \cdot \boldsymbol{\sigma} U^\dagger = (O\mathbf{n}) \cdot \boldsymbol{\sigma}. \tag{C.8}$$

Let us denote by $\tilde{\rho}$ the transformed density operator obtained by applying a local unitary transformation $U_A \otimes U_B$:

$$\tilde{\rho} = U_A \otimes U_B \rho U_A^\dagger \otimes U_B^\dagger. \tag{C.9}$$

Hence, the parameters $\mathbf{r}$, $\mathbf{s}$, and T transform as [234]:

$$\tilde{\mathbf{r}} = O_A\,\mathbf{r}; \quad \tilde{\mathbf{s}} = O_B\,\mathbf{s}, \tag{C.10}$$

$$\tilde{T} = O_A\,T\,O_B^T, \tag{C.11}$$

where O_A and O_B are related to U_A and U_B, respectively, through Eq. (C.8).

A generic X-state of a system of two spin-1/2's, A and B, may be cast in the following form

$$\rho_X = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}. \tag{C.12}$$

The unit trace and positivity conditions read $\sum_{i=1}^{4} \rho_{ii} = 1$, $\rho_{11}\rho_{44} \geqslant |\rho_{14}|^2$ and $\rho_{22}\rho_{33} \geqslant |\rho_{23}|^2$, assuming in general $\rho_{14} = |\rho_{14}|e^{i\phi_{14}}$ and $\rho_{23} = |\rho_{23}|e^{i\phi_{23}}$. The Fano parametrization of an X state is given by:

$$\begin{aligned} \mathbf{r}_\mathrm{X} :& \quad 0, 0, r; \\ \mathbf{s}_\mathrm{X} :& \quad 0, 0, s; \\ T_\mathrm{X} =& \begin{pmatrix} t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & 0 \\ 0 & 0 & t_{33} \end{pmatrix}. \end{aligned} \tag{C.13}$$

The link between the general form of the X state (C.12) and its Fano parametrization (C.13) is given by

[**?**]:

$$
\begin{aligned}
r &= \rho_{11}+\rho_{22}-\rho_{33}-\rho_{44}, \\
s &= \rho_{11}-\rho_{22}+\rho_{33}-\rho_{44}, \\
T_{11} &= 2\,\mathrm{R}e[\rho_{23}+\rho_{14}], \\
T_{22} &= 2\,\mathrm{R}e[\rho_{23}-\rho_{14}], \\
T_{33} &= \rho_{11}-\rho_{22}-\rho_{33}+\rho_{44}, \\
T_{12} &= 2\,\mathrm{I}m[\rho_{23}-\rho_{14}] \\
T_{21} &= -2\,\mathrm{I}m[\rho_{23}+\rho_{14}].
\end{aligned}
$$

One can diagonalize T by applying two rotations O_A and O_B along the Ox_3-axis, associated to the following local unitary operation, according to Eqs. (C.8) and (C.11) [235]:

$$U_A \otimes U_B = e^{-i(\varphi_{14}+\varphi_{23})\sigma_3/4} \otimes e^{-i(\varphi_{14}-\varphi_{23})\sigma_3/4}. \tag{C.14}$$

This transformation leads to the *canonical form* of a general X state, i.e. $\rho_{\mathrm{X}}^{can} = U_A \otimes U_B \rho_{\mathrm{X}} U_A^\dagger \otimes U_B^\dagger$ [235]:

$$\rho_{\mathrm{X}}^{can} = \begin{pmatrix} \rho_{11} & 0 & 0 & |\rho_{14}| \\ 0 & \rho_{22} & |\rho_{23}| & 0 \\ 0 & |\rho_{32}| & \rho_{33} & 0 \\ |\rho_{41}| & 0 & 0 & \rho_{44} \end{pmatrix}. \tag{C.15}$$

Accordingly, the Fano parametrization of the canonical form of the X state (C.15) is given by $T = \mathrm{diag}(c_1, c_2, c_3)$:

$$
\begin{aligned}
r^{can} &= r = \rho_{11}+\rho_{22}-\rho_{33}-\rho_{44}, \\
s^{can} &= s = \rho_{11}-\rho_{22}+\rho_{33}-\rho_{44}, \\
c_1 &= T_{11}^{can} = 2\,(|\rho_{23}|+|\rho_{14}|), \\
c_2 &= T_{22}^{can} = 2\,(|\rho_{23}|-|\rho_{14}|), \\
c_3 &= T_{33}^{can} = T_{33} = \rho_{11}-\rho_{22}-\rho_{33}+\rho_{44}.
\end{aligned} \tag{C.16}
$$

Therefore, the canonical form of the Fano parametrization of the density operator of an X state is given by:

$$\rho_{\mathrm{X}}^{can} = \frac{1}{4}\left(I \otimes I + r\sigma_3 \otimes I + sI \otimes \sigma_3 + \sum_{j=1}^{3} c_j \sigma_j \otimes \sigma_j \right). \tag{C.17}$$

Thus, it is possible to parametrize the generic X-state just with five parameters in the following way

$$\rho_X = \frac{1}{4}\begin{pmatrix} 1+r+s+c_3 & 0 & 0 & c_1-c_2 \\ 0 & 1+r-s-c_3 & c_1+c_2 & 0 \\ 0 & c_1+c_2 & 1-r+s-c_3 & 0 \\ c_1-c_2 & 0 & 0 & 1-r-s+c_3 \end{pmatrix}. \tag{C.18}$$

We underline that if **r**=**s**=0, ρ becomes the Bell diagonal state.

As a measure of entanglement we shall use Wootter's concurrence (entanglement of formation [101] is a monotonically increasing function of the concurrence), which can be calculated by using the eigenvalues of $\rho\widetilde{\rho}$, where $\widetilde{\rho} = \sigma^y \otimes \sigma^y \rho^* \sigma^y \otimes \sigma^y$. The eigenvalues of $\rho\widetilde{\rho}$ for the state (C.18) are

$$\lambda_1 = \frac{1}{16}(c_1 - c_2 - \sqrt{(1+c_3)^2 - (r+s)^2})^2, \qquad \lambda_2 = \frac{1}{16}(c_1 - c_2 + \sqrt{(1+c_3)^2 - (r+s)^2})^2,$$

$$\lambda_3 = \frac{1}{16}(c_1 + c_2 - \sqrt{(1-c_3)^2 - (r-s)^2})^2, \qquad \lambda_4 = \frac{1}{16}(c_1 + c_2 + \sqrt{(1-c_3)^2 - (r-s)^2})^2$$

and the concurrence is given by

$$C(\rho) = \max\{2\max\{\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}, \sqrt{\lambda_4}\} - \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}, 0\}. \tag{C.19}$$

For fixed r and s, the previous states and their corresponding concurrence depend on three parameters.

For two-qubit X-states with density matrices of the form (C.18), the quantum discord can be computed analytically according to the procedure elaborated in Refs. [123, 124], and shortly described in the following. The quantum mutual information can be expressed in the form

$$\mathscr{I}(\rho) = S(\rho^A) + S(\rho^B) + u_+ \log_2 u_+ + u_- \log_2 u_- + v_+ \log_2 v_+ + v_- \log_2 v_-, \tag{C.20}$$

with

$$S(\rho^A) = 1 + f(r), \quad S(\rho^B) = 1 + f(s), \qquad f(t) = -\frac{1-t}{2}\log_2(1-t) - \frac{1+t}{2}\log_2(1+t), \quad 0 \le t \le 1 \tag{C.21}$$

and

$$u_\pm = \frac{1}{4}[1 - c_3 \pm \sqrt{(r-s)^2 + (c_1+c_2)^2}], \quad v_\pm = \frac{1}{4}[1 + c_3 \pm \sqrt{(r+s)^2 + (c_1-c_2)^2}]. \tag{C.22}$$

being the two eigenvalues of ρ_X in Eq. (C.18). After performing the von Neumann measurement B_i, $i = 0, 1$ for the subsystem B, one obtains the ensemble $\{\rho_i, p_i\}$ and then the classical correlations $\mathscr{C}(\rho)$ can be evaluated by

$$\mathscr{C}(\rho) = \sup_{B_i} \mathscr{I}(\rho|B_i) = S(\rho^A) - \min_{B_i} S(\rho|B_i), \tag{C.23}$$

where

$$S(\rho|B_i) = p_0 S(\rho_0) + p_1 S(\rho_1). \tag{C.24}$$

According to Refs. [123, 124] the minimum of the quantum conditional entropy (C.24) has to be taken over the following expressions:

$$\begin{aligned} S_1 = &-\frac{1+r+s+c_3}{4}\log_2\frac{1+r+s+c_3}{2(1+s)} - \frac{1-r+s-c_3}{4}\log_2\frac{1-r+s-c_3}{2(1+s)} \\ &-\frac{1+r-s-c_3}{4}\log_2\frac{1+r-s-c_3}{2(1-s)} - \frac{1-r-s+c_3}{4}\log_2\frac{1-r-s+c_3}{2(1-s)}, \end{aligned} \tag{C.25}$$

$$S_2 = 1 + f(\sqrt{r^2 + c_1^2}), \qquad S_3 = 1 + f(\sqrt{r^2 + c_2^2}). \tag{C.26}$$

Finally, Li [124] formulated the following

Theorem: For any state ρ of the form (C.18), the classical correlations of ρ are given by

$$\mathscr{C}(\rho) = S(\rho^A) - \min\{S_1, S_2, S_3\}, \tag{C.27}$$

where S_1, S_2, S_3 are defined in Eqs. (C.25) and (C.26), respectively. The quantum discord is then given by

$$D(\rho) = \mathscr{I}(\rho) - \mathscr{C}(\rho), \tag{C.28}$$

with $\mathscr{I}(\rho)$ given by Eq. (C.20). The same results are obtained by generalizing von Neumann measurements to POVM [123].

Appendix D

Exact treatment of the N-qubit model

D.1 Transformation procedure

Let us consider the following N-spin model

$$H = \sum_{k=1}^{N} \hbar\omega_k \hat{\sigma}_k^z + \gamma_x \bigotimes_{k=1}^{N} \hat{\sigma}_k^x + \gamma_y \bigotimes_{k=1}^{N} \hat{\sigma}_k^y + \gamma_z \bigotimes_{k=1}^{N} \hat{\sigma}_k^z, \tag{D.1}$$

describing N distinguishable spins subjected, in general, to different magnetic fields and interacting between them only through N-wise interaction terms, that is each interaction term involves all the N-spins at the same time. $\hat{\sigma}^x$, $\hat{\sigma}^y$ and $\hat{\sigma}^z$ are the standard Pauli matrices.

This model may be exactly diagonalized by a process consisting in a chain of unitary transformations. To this end it is useful to start by considering the easiest case of two interacting spin 1/2's. In this instance the Hamiltonian reads

$$H_2 = \hbar\omega_1 \hat{\sigma}_1^z + \hbar\omega_2 \hat{\sigma}_2^z + \gamma_x \hat{\sigma}_1^x \hat{\sigma}_2^x + \gamma_y \hat{\sigma}_1^y \hat{\sigma}_2^y + \gamma_z \hat{\sigma}_1^z \hat{\sigma}_2^z \tag{D.2}$$

and it is possible to verify that $[H_2, \hat{\sigma}_1^z \hat{\sigma}_2^z] = 0$. Transforming H_2 through the following unitary and hermitian operator ($\mathbb{1}$ is the identity operator in the four dimensional Hilbert subspace)

$$T_{12} = \frac{1}{2}\left[\mathbb{1} + \hat{\sigma}_1^z + \hat{\sigma}_2^x - \hat{\sigma}_1^z \hat{\sigma}_2^x\right], \tag{D.3}$$

we get

$$T_{12}^{\dagger} H_2 T_{12} = \tilde{H}_2 = \hbar\left(\omega_1 + \omega_2 \hat{\sigma}_2^z\right) \hat{\sigma}_1^z + \gamma_x \hat{\sigma}_1^x - \gamma_y \hat{\sigma}_2^z \hat{\sigma}_1^x + \gamma_z \sigma_2^z. \tag{D.4}$$

It is easy to see that $\hat{\sigma}_2^z$ is constant of motion for $\tilde{H}$ and thus it may be treated as a parameter ($= \pm 1$), rewriting

$$\tilde{H}_{\sigma_2^z} = \hbar\left(\omega_1 + \omega_2 \sigma_2^z\right) \hat{\sigma}_1^z + \left(\gamma_x - \gamma_y \sigma_2^z\right) \hat{\sigma}_1^x + \gamma_z \sigma_2^z. \tag{D.5}$$

This means that we have got two Hamiltonians of single spin-1/2, each one related to one of the two eigenvalues of $\hat{\sigma}_2^z$, ± 1. So, in this manner, we have reduced the two-interacting-spin problem into two independent single-spin-1/2 problems, easier to be solved. Furthermore, it is worth to underline that each single-spin-1/2 Hamiltonian governs the dynamics of our two-spin system in one of the two dynamically invariant Hilbert subspace related to the two eigenvalue of $\hat{\sigma}_2^z$.

If we now consider the case of three spins, the Hamiltonian (D.1) reads

$$H_3 = \hbar\omega_1 \hat{\sigma}_1^z + \hbar\omega_2 \hat{\sigma}_2^z + \hbar\omega_3 \hat{\sigma}_3^z + \gamma_x \hat{\sigma}_1^x \hat{\sigma}_2^x \hat{\sigma}_3^x + \gamma_y \hat{\sigma}_1^y \hat{\sigma}_2^y \hat{\sigma}_3^y + \gamma_z \hat{\sigma}_1^z \hat{\sigma}_2^z \hat{\sigma}_3^z. \tag{D.6}$$

Now, it is possible to convince oneself that $\hat{\sigma}_2^z \hat{\sigma}_3^z$ is constant of motion and then if we apply the procedure previously used to the two spins 2 and 3 in H_3, we get the following new Hamiltonian

$$T_{23}^\dagger H_3 T_{23} = \tilde{H}_3 = \hbar\omega_1 \hat{\sigma}_1^z + \hbar\left(\omega_2 + \omega_3 \sigma_3^z\right) \hat{\sigma}_2^z + \gamma_x \hat{\sigma}_1^x \hat{\sigma}_2^x - \gamma_y \sigma_3^z \hat{\sigma}_1^y \hat{\sigma}_2^x + \gamma_z \sigma_3^z \hat{\sigma}_1^z, \tag{D.7}$$

where σ_3^z (integral of motion) appears as parameter and so we have two different Hamiltonians of two interacting spin 1/2's. This time the unitary and hermitian operator accomplishing the transformation is

$$T_{23} = \frac{1}{2}\left[\mathbb{1} + \hat{\sigma}_2^z + \hat{\sigma}_3^x - \hat{\sigma}_2^z \hat{\sigma}_3^x\right], \tag{D.8}$$

in accordance with the form of T_{12}. It is immediate, at this point, to understand that we may apply one more time the same procedure for $\tilde{H}_3$, using the operator written in Eq. (D.3) since $\hat{\sigma}_1^z \hat{\sigma}_2^z$ is constant of motion for $\tilde{H}_3$. Thus, we get

$$T_{12}^\dagger \tilde{H}_3 T_{12} = T_{123}^\dagger H_3 T_{123} = \tilde{\tilde{H}}_3 = \hbar\left(\omega_1 + \omega_2 \sigma_2^z + \omega_3 \sigma_2^z \sigma_3^z\right) \hat{\sigma}_1^z + \gamma_x \hat{\sigma}_1^x - \gamma_y \sigma_3^z \hat{\sigma}_1^y + \gamma_z \sigma_3^z \hat{\sigma}_1^z, \tag{D.9}$$

where we put $T_{123} = T_{23} T_{12}$. In this case we have two parameters, σ_2^z and σ_3^z, and so we have four Hamiltonians of single spin-1/2 governing the dynamics of the three spin system in each of the four dynamically invariant subspaces related to the four pairs of the eigenvalues of the two constant of motion $\hat{\sigma}_1^z \hat{\sigma}_2^z$ and $\hat{\sigma}_2^z \hat{\sigma}_3^z$. Therefore, also in this case, we have reduced the initial dynamical problem of three interacting spins into independent problems of a single spin-1/2.

Basing on this last result we understand that, for the case of N spins, if we apply the procedure previously exposed for three spins, to the last three spins, we obtain a new Hamiltonian characterized by the same structure of the original one with the parameters redefined and depending only on the first $N-2$ spins (the last two spins appear as parameter). One can imagine to iterate this procedure for each spin-triplet until the Hamiltonian is completely reduced to that of a single spin 1/2. More precisely, it means that if we had, e.g., ten spins we could consider firstly the spin-triplet (8 9 10) and diagonalize the Hamiltonian with respect to these three spins, obtaining a new Hamiltonian depending on the dynamical variables of the spin 8 and those of the other spins not involved in the transformation; the spins 9 and 10 would appear only through σ_9^z and σ_{10}^z having the role of parameters. At this point we should proceed by considering the spin-triplets (6 7 8), (4 5 6) and so on, diagonalizing every time with respect to the spin-triplet under consideration until we get a final Hamiltonian depending only on one spin-1/2, actually the first spin for the example taken into account. It is important to underline that in the case of odd number of spins, through this technique, we get directly a final Hamiltonian of a single spin-1/2, while for an even number of spin we get firstly a Hamiltonian of two spins which can be treated analogously to get the final one depending on just one spin.

It is appropriate to define and make clear what we intend for "diagonalize with respect to a spin-triplet". Considering the generic spin-triplet (i, j, k) (with $i < j < k$), diagonalizing with respect the three spins i, j and k means to transform the Hamiltonian through the following operator

$$T_{ijk} = \frac{1}{4}\left[\mathbb{1} + \hat{\sigma}_j^z + \hat{\sigma}_k^x - \hat{\sigma}_j^z \hat{\sigma}_k^x\right]\left[\mathbb{1} + \hat{\sigma}_i^z + \hat{\sigma}_j^x - \hat{\sigma}_i^z \hat{\sigma}_j^x\right] \tag{D.10}$$

acting only upon the dynamical variables of the three spins under consideration. As shown and explained before, this transformation leaves the Hamiltonian dependent on the dynamical variables of the first spin of the triplet (i-th spin) and on those of all the other spins not affected by the transformation. The spins j and k appears only with $\hat{\sigma}_j^z$ and $\hat{\sigma}_k^z$ which, being constant of motion, may be treated as parameters and substituted with their eigenvalues in the expression of the transformed Hamiltonian.

It is useful now to observe what are the effects on the Hamiltonian after a diagonalization with respect to a spin-triplet:

- a -1 factor appears in the interaction term in γ_y;
- the σ^z operator (parameter) of the last spin in the triplet appears in the interaction terms in γ_y and γ_z;
- the Pauli spin operators ($\hat{\sigma}^x$, $\hat{\sigma}^y$ and $\hat{\sigma}^z$) of the first spin of the triplet under consideration remain unchanged in each relative interaction term (γ_x, γ_y and γ_z).

We observe also that, from Eqs. (D.4) and (D.9), it is easy to conjecture the general form of the factor multiplying $\hat{\sigma}_1^z$ and depending on the ω_k parameters, namely

$$\omega_1 + \sum_{k=2}^{N} \omega_k \prod_{k'=2}^{k} \sigma_{k'}^z. \tag{D.11}$$

For, we are able, via an induction procedure, to write the argued form of the final single-spin-1/2 Hamiltonian. In the case of an odd number of spins it reads

$$\tilde{H} = \hbar \left[\omega_1 + \sum_{k=2}^{N} \omega_k \prod_{k'=2}^{k} \sigma_{k'}^z \right] \hat{\sigma}_1^z + \gamma_x \hat{\sigma}_1^x + \left[(-1)^{\frac{N-1}{2}} \gamma_y \prod_{k=1}^{(N-1)/2} \sigma_{2k+1}^z \right] \hat{\sigma}_1^y + \left[\gamma_z \prod_{k=1}^{(N-1)/2} \sigma_{2k+1}^z \right] \hat{\sigma}_1^z, \tag{D.12}$$

whereas for an even number of spins we have

$$\tilde{H} = \hbar \left[\omega_1 + \sum_{k=2}^{N} \omega_k \prod_{k'=2}^{k} \sigma_{k'}^z \right] \hat{\sigma}_1^z + \gamma_x \hat{\sigma}_1^x + \left[(-1)^{\frac{N}{2}} \gamma_y \prod_{k=1}^{N/2} \sigma_{2k}^z \right] \hat{\sigma}_1^x + \gamma_z \prod_{k=1}^{N/2} \sigma_{2k}^z. \tag{D.13}$$

It is of relevance to underline that $(N-1)/2$ and $N/2$, appearing respectively in Eq. (D.12) and (D.13), are the numbers of transformations to be applied to the original Hamiltonian in Eq. (D.1) to get the final ones. The total unitary operator accomplishing this chained transformations may be written as

$$T = \frac{1}{2^{N-1}} \prod_{k=0}^{N-2} \left[\mathbb{1} + \hat{\sigma}_{N-(k-1)}^z + \hat{\sigma}_{N-k}^x - \hat{\sigma}_{N-(k+1)}^z \hat{\sigma}_{N-k}^x \right]. \tag{D.14}$$

D.2 Eigenvectors and breaking down of the Schrödinger equation

To understand the eigenvectors structure, let us consider, for the sake of simplicity, the simplest case of two spin-1/2's. By Eqs. (D.4) and (D.5), it is easy to understand that we may write the eigenvectors of $\tilde{H}$ as follows

$$|\tilde{\psi}_{ij}\rangle = |\phi_{ij}\rangle \otimes |\sigma_2^z = i\rangle \tag{D.15}$$

with $i=\pm1,\ j=1,2,\ |\sigma_2^z=1\rangle=(1,0)^T$ and $|\sigma_2^z=-1\rangle=(0,1)^T$. In the previous expressions, $|\phi_{1i}\rangle$ ($|\phi_{-1i}\rangle$) are the two eigenvectors of $\tilde{H}_{+1}$ ($\tilde{H}_{-1}$). Finally, the eigenvectors of H are easily derived by the relation

$$T|\tilde{\psi_{ij}}\rangle=|\psi_i\rangle. \tag{D.16}$$

If the Hamiltonian H is time-dependent, we have to study the time-dependent Schrödinger equation, namely

$$i\hbar|\dot{\psi}(t)\rangle=H(t)|\psi(t)\rangle. \tag{D.17}$$

Since $\frac{\partial}{\partial t}T=0$, it is easy to verify that we may write

$$i\hbar|\dot{\tilde{\psi}}(t)\rangle=\tilde{H}(t)|\tilde{\psi}(t)\rangle. \tag{D.18}$$

By writing a general initial condition as follows

$$|\tilde{\psi}(0)\rangle=\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}a\\c\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}b\\d\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}, \tag{D.19}$$

since $[\tilde{H}(t),\sigma_2^z]=0$, we may write the evolved state at time t as

$$\begin{aligned}|\tilde{\psi}(t)\rangle=\begin{pmatrix}a(t)\\b(t)\\c(t)\\d(t)\end{pmatrix}&=\begin{pmatrix}a(t)\\c(t)\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}b(t)\\d(t)\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}=\\&=|\tilde{\phi}_1\rangle_1\otimes|\sigma_2^z=1\rangle_2+|\tilde{\phi}_{-1}\rangle_1\otimes|\sigma_2^z=-1\rangle_2\end{aligned} \tag{D.20}$$

where $|\tilde{\phi}_{\pm1}\rangle_1$ satisfy the following dynamical problems

$$i\hbar|\dot{\tilde{\phi}}_{\pm1}(t)\rangle=\tilde{H}_{\pm1}(t)|\tilde{\phi}_{\pm1}(t)\rangle \tag{D.21}$$

being nothing but two independent single spin-1/2 time-dependent Schrödinger equations.

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Bibliography

[1] N. Bohr, *I quanti e la vita*, 2012 Bollati Boringhieri editore.

[2] W. Heisenberg, *Fisica e Filosofia*, il Saggiatore/Net, Milano 2003.

[3] John A. Weil, James R. Bolton, Electron Paramagnetic Resonance - Elementary Theory and Practical Applications (Second Edition), John Wiley & Sons, Hoboken, New Jersey (2007).

[4] R. Calvo, J. E. Abud, R. P. Sartoris and R. C. Santana, Phys. Rev. B **84** 104433 (2011).

[5] S. Haroche, J. M. Raimond, *Exploring the Quantum: Atoms Cavities And Photons*, Oxford University Press, Oxford, UK 2006.

[6] J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **74**, 4091 (1995); D. Porras and J. I. Cirac, Phys. Rev. Lett. **92**, 207901 (2004).

[7] A. Blais, Ren-Shou Huang, A. Wallraff, S. M. Girvin, and R. J. Schoelkopf, Phys. Rev. A **69**, 062320 (2004).

[8] A. Mezzacapo, L. Lamata, S. Filipp, and E. Solano, Phys. Rev. Lett. **113**, 050501 (2014).

[9] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 1990).

[10] S. Longhi, Laser & Photon. Rev. **3**, 243 (2009).

[11] C. E. Ruter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, and D. Kip, Nat. Phys. **6**, 192 (2010).

[12] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, New York, 1964), Vol. III; L. Allen and J. H. Eberly, *Optical Resonance and Two Level Atoms* (Dover, Mineola, NY, 1987).

[13] A. Abragham, *The Principles of Nuclear Magnetism* (Clarendon, Oxford, 1961).

[14] M. Born and E. Wolf, *Principles of Optics*, 7th ed. (Cambridge University Press, Cambridge, 1999).

[15] R. Gilmore, *Lie groups, physics, and geometry: an introduction for physicists, engineers and chemists*, Cambridge University Press, I Ed. (2008); R. Gilmore and J. M. Yuan, J. Chem. Phys. **86**, 130 (1987); R. Gilmore and J. M. Yuan, J. Chem. Phys. **91**, 917 (1989).

[16] Y. Makhlin, G. Schn, and A. Shnirman, Rev. Mod. Phys. **73**, 357 (2001).

[17] J. Clarke, and F. K. Wilhelm, Nature **453**, 1031-1042 (2008).

[18] A. Blais, J. Gambetta, A. Wallraff, D. I. Schuster, S. M. Girvin, M. H. Devoret, and R. J. Schoelkopf, Phys. Rev. A **75**, 032329 (2007)

[19] J. Gambetta, A. Blais, D. I. Schuster, A. Wallraff, L. Frunzio, J. Majer, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf, Phys. Rev. A **74**, 042318 (2006).

[20] A. J. Leggett, S. Chakravarty, A. T. Dorsey, Matthew P. A. Fisher, Anupam Garg, and W. Zwerger, Rev. Mod. Phys. **59**, 1 (1987); Erratum Rev. Mod. Phys. **67**, 725 (1995).

[21] M. Anderlini, P. J. Lee, B. L. Brown, J. Sebby-Strabley, W. D. Phillips, and J. V. Porto, Nature **448** (7152) 452 (2007).

[22] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, J. Math. Phys. **17**, 821 (1976); G. Lindblad, Commun. Math. Phys. **48**, 119 (1976).

[23] R. Kapral, G. Ciccotti, J. Chem. Phys. **110**, 8919-8929 (1999); R. Kapral, J. Phys. Chem. A **105**, 2885-2889 (2001); S. Nielsen; R. Kapral, G. Ciccotti, J. Chem. Phys. **115**, 5805-5815 (2001).

[24] H. Feshbach, Ann. Phys. **5**, 357 (1958); H. Feshbach, Ann. Phys. **19**, 287 (1962).

[25] A. Mostafazadeh, J. Math. Phys. **43**, 205 (2002); A. Mostafazadeh, J. Math. Phys. **44**, 974 (2003); A. Mostafazadeh, J. Phys. A: Math. Gen. **36**, 7081 (2003); A. Mostafazadeh and A. Batal, J. Phys. A: Math. Gen. **37**, 11645 (2004); A. Mostafazadeh, Phys. Scr. **82**, 038110 (2010); A. Mostafazadeh, Int. J. Geom. Methods Mod. Phys. **07**, 1191 (2010).

[26] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998); C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett. **89**, 270401 (2002); C. M. Bender, Rep. Prog. Phys. **70**, 947 (2007).

[27] M. Liertzer, L. Ge, A. Cerjan, A. D. Stone, H. E. Tureci, and S. Rotter, Phys. Rev. Lett. **108**, 173901 (2012).

[28] S. Bittner, B. Dietz, U. Gunther, H. L. Harney, M. MiskiOglu, A. Richter, and F. Schafer, Phys. Rev. Lett. **108**, 024101 (2012).

[29] J. Schindler, A. Li, M. C. Zheng, F. M. Ellis, and T. Kottos, Phys. Rev. A **84**, 040101(R) (2011).

[30] V. Tripathi, A. Galda, H. Barman, and V. M. Vinokur, Phys. Rev. B **94**, 041104(R) (2016).

[31] I. I. Rabi, Phys. Rev. **51** 652 (1937).

[32] I. I. Rabi, N. F. Ramsey and J. Schwinger, Rev. Mod. Phys. **26** 167 (1954).

[33] L. D. Landau, Phys. Z. Sowjetunion **2**, 46 (1932); E. Majorana, Nuovo Cimento **9**, 43 (1932); E. C. G. Stückelberg, Helv. Phys. Acta **5**, 369 (1932); C. Zener, Proc. R. Soc. London, Ser. A **137**, 696 (1932).

[34] K. Bergmann, H. Theuer, and B. W. Shore, Rev. Mod. Phys. **70**, 1003 (1998).

[35] V. G. Bagrov, D. M. Gitman, M. C. Baldiotti, and A. D. Levin, Annalen der Physik **14** (11), 764 (2005).

[36] M. Kuna and J. Naudts, Reports on Mathematical Physics **65** (1) 77 (2010).

[37] E. Barnes and S. Das Sarma, Phys. Rev. Lett. **109** 060401 (2012).

[38] A. Messina and H. Nakazato, J. Phys. A: Math. Theor. **47** 445302 (2014).

[39] Weissbluth, Mitchel. Atoms and molecules. Elsevier, 2012.

[40] F. T. Hioe, J. Opt. Soc. Am. B **4**, 8 (1987).

[41] M. Müller, K. Hammerer, Y. L. Zhou, C. F. Roos and P. Zoller, New J. Phys. **13**, 085007 (2011).

[42] J. Casanova, A. Mezzacapo, L. Lamata, and E. Solano, Phys. Rev. Lett. **108**, 190502 (2012).

[43] J. J. Sakurai and J. Napolitano, *Modern quantum mechanics*, Pearson Harlow (2014).

[44] J. Schwinger, Phys. Rev. **51** 648 (1937).

[45] L. M. K. Vandersypen and I. L. Chuang, Rev. of Mod. Phys. **76** 1037 (2004).

[46] D. Braak, Q. H. Chen, M. T. Batchelor, E. Solano, J. Phys. A: Math. Theor. **49**, 300301 (2016).

[47] Y. B. Borozdina, E. Mostovich V. Enkelmann, B. Wolf, P. T. Cong, U. Tutsch, M. Lang and M. Baumgarten, J. Mater. Chem. C **2** 6618 (2014).

[48] J. R. Petta, A. C. Johnson, J. M. Taylor, E. A. Laird,1 A. Yacoby, M. D. Lukin, C. M. Marcus, M. P. Hanson, A. C. Gossard, Science **309** (5744) 2180 (2005).

[49] H. Bluhm, S. Foletti, I. Neder, M. Rudner, D. Mahalu, V. Umansky and A. Yacoby, Nat. Phys. **7** 109 (2011).

[50] Xin Wang, L. S. Bishop, J. P. Kestner, E. Barnes, Kai Sun, and S. Das Sarma, Nat. Comm. **3** 997 (2012).

[51] R. Grimaudo, A. S. M. de Castro, H. Nakazato, and A. Messina, Ann. Phys. (Berlin) **530** (12), 1800198 (2018).

[52] L. Thomas, F. Lionti, R. Ballou, D. Gatteshi, R. Sessoni and B. Barbara, Nature (London) **383**, 145 (1996).

[53] V. Calbucci, Ph.D. dissertation: Metodi matematici nelle scienze fisiche. Praxis: software per la simulazione del comportamento delle molecole magnetiche (2012).

[54] L. M. B. Napolitano, O. R. Nascimento, S. Cabaleiro, J. Castro and R. Calvo, Phys. Rev. B **77**, 214423 (2008).

[55] M. C. Baldiotti, V. G. Bagrov and D. M. Gitman, Physics of Particles and Nuclei Letters, **6**, 7 (2009).

[56] Van Hieu Nguyen, J. Phys.: Condens. Matter **21**, 273201 (2009).

[57] A. Imamoglu, D. D. Awschalom, G. Burkard, D. P. DiVincenzo, D. Loss, M. Sherwin, and A. Small, Phys. Rev. Lett. **83**, 4204 (1999).

[58] Shi-Biao Zheng and Guang-Can Guo, Phys. Rev. Lett. **85**, 2392 (2000).

[59] Xiaoguang Wang, Phys. Rev. A **64**, 012313 (2001).

[60] M. C. Arnesen, S. Bose, and V. Vedral, Phys. Rev. Lett. **87**, 017901 (2001).

[61] X. X. Yi, H. T. Cui, and L. C. Wang, Phys. Rev. A **74**, 054102 (2006).

[62] D. Porras and J. I. Cirac, Phys. Rev. Lett. **92**, 207901 (2004).

[63] I. Dzyaloshinskii, J. Phys Chem. Solids **4**, 241 (1958).

[64] T. Moriya, Phys. Rev. **120**, 91 (1960).

[65] E. Albayrak, Eur. Phys. J. B **72**, 491496 (2009).

[66] R. J. Guerrero and M. F. Rojas, Quantum Inf Process **14**, 1973-1996 (2015).

[67] G. S. Vasilev and S. S. Ivanov and N. V. Vitanov, Phys. Rev. A **75**, 013417 (2007).

[68] B. T. Torosov and N. V. Vitanov, Phys. Rev. A **84**, 063411 (2011).

[69] N. V. Vitanov and B. M. Garraway, Phys. Rev. A **53**, 6 (1996).

[70] L. Allen and J. H. Eberly, Optical Resonance and Two-Level Atoms (Dover, New York, 1987); F. T. Hioe, Phys. Rev. A **30**, 2100 (1984).

[71] Y. N. Demkov and M. Kunike, Vestn. Leningr. Univ. Fiz. Khim. **16**, 39 (1969).

[72] G. S. Vasilev and N. V. Vitanov, Phys. Rev. A **73**, 023416 (2006).

[73] J. M. S. Lehto and K.-A. Suominen, Phys. Rev. A **94**, 013404 (2016).

[74] S.N. Shevchenko, S. Ashhab, Franco Nori, Phys. Rep. **492**, 1 (2010).

[75] S. S. Ivanov and N. V. Vitanov, Phys. Rev. A **77**, 023406 (2008).

[76] N. A. Sinitsyn, Phys. Rev. A **87**, 032701 (2013).

[77] B. D. Militello and N. V. Vitanov, Phys. Rev. A **91**, 053402 (2015).

[78] S. Thiele, F. Balestro, R. Ballou, S. Klyatskaya, M. Ruben, W. Wernsdorfer1, Science **344** 6188 (2014).

[79] D. J. Reilly, J. M. Taylor, J. R. Petta, C. M. Marcus, M. P. Hanson, A. C. Gossard, Science **321**, 817 (2008).

[80] P. Huang, J. Zhou, F. Fang, X. Kong, X. Xu, C. Ju, and J. Du, Phys. Rev. X **1**, 011003 (2011).

[81] J. Randall, A. M. Lawrence, S. C. Webster, S. Weidt, N. V. Vitanov, and W. K. Hensinger1, Phys. Rev. A **98**, 043414 (2018).

[82] R. Wieser, V. Caciuc, C. Lazo, H. Hölscher, E. Y. Vedmedenko and R. Wiesendanger, New J. Phys. **15** 013011 (2013).

[83] I. N. Sivkov, D. I. Bazhanov and V. S. Stepanyuk, Sc. Rep. **7**, 2759 (2017).

[84] V. M. Akulin and W. P. Schleich, Phys. Rev. A **46**, 7 (1992).

[85] N. V. Vitanov and S. Stenholm, Phys. Rev. A **55**, 4 (1997).

[86] V. L. Pokrovsky and N. A. Sinitsyn, Phys Rev. B **67**, 144303 (2003).

[87] P. A. Ivanov and N. V. Vitanov, Phys. Rev. A **71**, 063407 (2005).

[88] M. Scala, B. Militello, A. Messina and N. V. Vitanov, Phys. Rev. A **84**, 023416 (2011).

[89] A. V. Dodonov, B. Militello, A. Napoli, and A. Messina, Phys. Rev. A **93**, 052505 (2016).

[90] R. G. Unanyan, N.V. Vitanov, and K. Bergmann, Phys. Rev. Lett. **87**, 137902 (2001).

[91] H. Ribeiro and G. Burkard, Phys. Rev. Lett. **102**, 216802 (2009).

[92] H. Ribeiro, J. R. Petta, and G. Burkard, Phys. Rev. B **87**, 235318 (2013).

[93] M. J. Rancić, D. Stepanenko, Phys. Rev. B **94**, 241301(R) (2016).

[94] J. Larson, Eur. Phys. Lett. **107**, 30007 (2014).

[95] S. N. Shevchenko, A. I. Ryzhov, and Franco Nori, Phys. Rev. B **98**, 195434 (2018).

[96] S. Mhel, Phys. Rev. B **91**, 035430 (2015).

[97] A. A. Khajetoorians, J. Wiebe, B. Chilian, R. Wiesendanger, Science **332** (6033) 1062-1064 (2011).

[98] S. Yan, D.-J. Choi, J. A. J. Burgess, S. Rolf-Pissarczyk and S. Loth, Nature Nanotechnology **10**, 4045 (2015).

[99] B. Bryant, A. Spinelli, J. J. T. Wagenaar, M. Gerrits, and A. F. Otte, Phys. Rev. Lett. **111**, 127203 (2013).

[100] Kun Tao, V. S. Stepanyuk, W. Hergert, I. Rungger, S. Sanvito, and P. Bruno, Phys. Rev. Lett. **103**, 057202 (2009).

[101] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).

[102] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1964).

[103] S. Weidt, J. Randall, S.C. Webster, K. Lake, A.E. Webb, I. Cohen, T. Navickas, B. Lekitsch, A. Retzker, and W.K. Hensinger, Phys. Rev. Lett. **117** 220501 (2016).

[104] L. M. K. Vandersypen and I. L. Chuang, Rev. Mod. Phys. **76**, 1037 (2005).

[105] Michael Johanning, Andrés F Varń and Christof Wunderlich, J. Phys. B: At. Mol. Opt. Phys. **42** (2009) 154009.

[106] A. Gaëtan, Y. Miroshnychenko, T. Wilk, A. Chotia, M. Viteau, D. Comparat, P. Pillet, A. Browaeys & P. Grangier, Nature Phys. **5**, 115 (2009).

[107] E. Urban, T. A. Johnson, T. Henage, L. Isenhower, D. D. Yavuz, T. G. Walker & M. Saffman, Nature Phys. **5**, 110 (2009).

[108] M. Müller, L. Liang, I. Lesanovsky and P. Zoller, New J. Phys. **10**, 093009 (2008).

[109] C. Benedetti and M . Paris, Phys. Lett. A **378** 24952500 (2014).

[110] N. Moiseyev, Non-Hermitian Quantum Mechanics (Cambridge Univ. Press, Cambridge, 2011).

[111] I. Rotter and J. P. Bird, Rep. Prog. Phys. **78**, 114001, (2015); I. Rotter, J. Phys. A: Math. Theor. **42**, 153001, (2009).

[112] L. S. Simeonov and N. V. Vitanov, Phys. Rev. A **93**, 012123 (2016).

[113] B. T. Torosov and N. V. Vitanov, Phys Rev. A **96**, 013845 (2017).

[114] G. Adesso, T. R. Bromley, M. Cianciaruso, J. Phys. A: Math. Theor. **49**, 473001 (2016).

[115] R. Horodecki, P. Horodecki, M. Horodecki, K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).

[116] W. H. Zurek, Ann. Phys. (Leipzig) **9**, 853 (2000).

[117] H. Ollivier, W. H. Zurek, Phys. Rev. Lett. **88**, 017901 (2001).

[118] V. Madhok, A. Datta, Phys. Rev. A **83**, 032323 (2011).

[119] B. P. Lanyon, M. Barbieri, M. P. Almeida, A. G. White, Phys. Rev. Lett. **101**, 200501 (2008).

[120] Y. Huang, New J. Phys. **16** 033027 (2014).

[121] P. Giorda, M. G. A. Paris, Phys. Rev. Lett. **105**, 020503 (2010).

[122] J. Maziero, T. Werlang, F. F. Fanchini, L. C. Celeri, R. M. Serra, Phys. Rev. A **81**, 022116 (2010).

[123] M. Ali, A. R. P. Rau, and G. Alber, Phys. Rev. A **81**, 042105 (2010).

[124] B. Li, Z. X. Wang, S. M. Fei, Phys. Rev. A **83**, 022321 (2011).

[125] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).

[126] T. Yu, J. H. Eberly, Quantum Information and Computation **7**, 459 (2007)

[127] C. Weitenberg, M. Endres, J. F. Sherson, M. Cheneau, P. Schauss, T. Fukuhara, I. Bloch, and S. Kuhr, Nature **471**, 319 (2011).

[128] T. Monz, D. Nigg, E. A. Martinez, M. F. Brandl, P. Schindler, R. Rines, S. X. Wang, I. L. Chuang, R. Blatt, Science **351**, 1068 (2016).

[129] A. Bermudez, X. Xu, R. Nigmatullin, J. OGorman, V. Negnevitsky, P. Schindler, T. Monz, U.G. Poschinger, C. Hempel, J. Home, F. Schmidt-Kaler, M. Biercuk, R. Blatt, S. Benjamin, and M. Müller, Phys. Rev. X **7**, 041061 (2017).

[130] C. Piltz, T. Sriarunothai, A. F. Varon, and C. Wunderlich, Nature Commun. **5**, 4679, (2014).

[131] B. Lekitsch, S. Weidt, A. G. Fowler, K. Mølmer, S. J. Devitt, C. Wunderlich, and W. K. Hensinger, Science Advances **3**, e1601540 (2017).

[132] R. Grimaudo, A. Messina, and H. Nakazato, Phys. Rev. A **94** (2), 022108 (2016).

[133] R. Grimaudo, A. Isar, T. Mihaescu, I. Ghiu, and A Messina, Res. Phys. **13**, 102147 (2019).

[134] R. Grimaudo, N. V. Vitanov, and A. Messina, Phys. Rev. B **99** (17), 174416 (2019).

[135] I. Ghiu, R. Grimaudo, T. Mihaescu, A. Isar, and A. Messina, *"Sudden death and revival of entanglement for two interacting spin-qubit systems"*, *submitted to Proceedings of the London Royal Society A*.

[136] P. Jurcevic, B. P. Lanyon, P. Hauke, C. Hempel, P. Zoller, R. Blatt and C. F. Roos, Nature **511**, 202-205 (2014).

[137] P. Richerme, Zhe-Xuan Gong, A. Lee, C. Senko, J. Smith, M. Foss-Feig, S. Michalakis, A. V. Gorshkov and C. Monroe, Nature **511**, 198-201 (2014).

[138] T. Boness, S. Bose, T. S. Monteiro, Phys. Rev. Lett. **96**, 187201 (2006).

[139] S. Sahling, G. Remenyi, C. Paulsen, P. Monceau, V. Saligrama, C. Marin, A. Revcolevschi, L. P. Regnault, S. Raymond and J. E. Lorenzo, Nat. Phys. **11**, 255-260 (2015).

[140] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. **80**, 517 (2008).

[141] C. F. Hirjibehedin, C. P. Lutz, A. J. Heinrich, Science **312** (2006).

[142] F. D. M. Haldane, Phys. Rev. Lett. **50**, 1153 (1983).

[143] L.-M. Duan, E. Demler, and M. D. Lukin, Phys. Rev. Lett. **91**, 090402 (2003).

[144] A.-C. Ji, X. C. Xie, and W. M. Liu, Phys. Rev. Lett. **99**, 183602 (2007).

[145] P. A. Ivanov, N. I. Karchev, N. V. Vitanov, and D. G. Angelakis, Phys. Rev. A **90**, 012325 (2014).

[146] C. Brukner, M. Zukowski, and A. Zeilinger, Phys. Rev. Lett. **89**, 197901 (2002).

[147] G. Molina-Terriza, A. Vaziri, R. Ursin, and A. Zeilinger, Phys. Rev. Lett. **94**, 040501 (2005).

[148] A. Vaziri, G. Weihs, and A. Zeilinger, Phys. Rev. Lett. **89**, 240401 (2002).

[149] D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, Phys. Rev. Lett. **88**, 040404 (2002).

[150] D. Kaszlikowski, P. Gnacinski, M. Zukowski, W. Miklaszewski, and A. Zeilinger, Phys. Rev. Lett. **85**, 4418 (2000).

[151] H. Bechmann-Pasquinucci and A. Peres, Phys. Rev. Lett. **85**, 3313 (2000).

[152] N. J. Cerf, M. Bourennane, A. Karlsson, and N. Gisin, Phys. Rev. Lett. **88**, 127902 (2002).

[153] A. D. Greentree, S. G. Schirmer, F. Green, L. C. L. Hollenberg, A. R. Hamilton, and R. G. Clark, Phys. Rev. Lett. **92**, 097901 (2004).

[154] A. B. Klimov, R. Guzman, J. C. Retamal, and C. Saavedra, Phys. Rev. A **67**, 062313 (2003).

[155] N. V. Vitanov, Phys. Rev. A **85**, 032331 (2012).

[156] P. A. Ivanov, E. S. Kyoseva, and N. V. Vitanov, Phys. Rev. A **74**, 022323 (2006).

[157] S. K. Yip, Phys. Rev. Lett. **90**, 250402 (2003).

[158] M. M. He, C. T. Xu, and J. Q. Liang Phys. Lett. A **358**, 381 (2006).

[159] A. Bermudez, F. Jelezko, M. B. Plenio, and A. Retzker, Phys. Rev. Lett. **107**, 150503 (2011).

[160] I. Cohen and A. Retzker, Phys. Rev. Lett. **112**, 040503 (2014).

[161] C. Senko, P. Richerme, J. Smith, A. Lee, I. Cohen, A. Retzker, and C. Monroe, Phys. Rev. X **5**, 021026 (2015).

[162] W H Joyia, International Journal of Quantum Information Vol. **13**, No. 5, 1550035 (2015).

[163] Mojtaba Jafarpour Mehrzad Ashrafpour, Quantum Inf Process **12**: 761772 (2013).

[164] Xi-Wen Houa, Xiu-Fang Lei, and Bo Chen, Eur. Phys. J. D **67**: 106 (2013).

[165] G. Vidal, and R. F. Werner, Phys. Rev. A **65**,032314, (2002).

[166] S. Rai, and J. R. Luthra, arXiv:quant-ph/0507263.

[167] J. L. Cereceda, arXiv:quant-ph/0305043.

[168] T. Hayashi, T. Fujisawa, H. D. Cheong, Y. H. Jeong, and Y. Hirayama, (2003) Phys. Rev. lett., **91** (22), 226804.

[169] X. Hu,, and S. D. Sarma, (2000) Phys. Rev. A, **61** (6), 062301.

[170] J. Gorman, D. G. Hasko, and D. A. Williams, (2005) Phys. Rev. lett., **95** (9), 090502.

[171] N. Mason, M. J. Biercuk, and C. M. Marcus, (2004) Science, **303** (5658), 655-658.

[172] Xin Wang, F. A. Calderon-Vargas, M. S. Rana, J. P. Kestner, E. Barnes, and S. Das Sarma, Phys. Rev. B **90**, 155306, (2014).

[173] M. Hamermesh, Group Theory and Its Application to Physical Problems (Addison-Wesley, Reading, Mass., 1962), p. 354.

[174] A. Napoli, M. Guccione, A. Messina, and D. Chruściński, Phys. Rev. A **89**, 062104 (2014).

[175] D. Chruściński, A. Messina, B. Militello, and A. Napoli, Phys. Rev. A **91**, 042123 (2015).

[176] B. Militello, D. Chruściński, A. Messina, P. Należyty, and A. Napoli, Phys. Rev. A **93**, 022113 (2016).

[177] V. L. Pokrovsky and N. A. Sinitsyn, Phys Rev. B **69**, 104414 (2004).

[178] M. B. Kenmoe, H. N. Phien, M. N. Kiselev, and L. C. Fai, Phys. Rev. B, **87**, 224301 (2013).

[179] M. Anderlini, J. Sebby-Strabley, J. Kruse, J. V. Porto, and W. D. Phillips, (2006) J. Phys. B: Atomic, Molecular and Optical Physics, **39**(10), S199.

[180] M. H. Devoret and R. J. Schoelkopf, Science **339**, 1169 (2013).

[181] R. P. Feynman, Int. J. Theor. Phys. **21**, 467 (1982).

[182] S. Lloyd, Science **273**, 1073 (1996).

[183] J. Barreiro, M. Müller, P. Schindler, D. Nigg, T. Monz, M. Chwalla, M. Hennrich, C. F. Roos, P. Zoller and R. Blatt, Nature **470**, 48 (2011).

[184] K. Mølmer and A. Sørensen, Phys. Rev. Lett. **82**, 1835 (1999); A. Sørensen and K. Mølmer, Phys. Rev. A **62**, 022311 (2000).

[185] T. Monz, P. Schindler, J. T. Barreiro, M. Chwalla, D. Nigg, W. A. Coish, M. Harlander, W. Hänsel, M. Hennrich and R. Blatt, Phys. Rev. Lett. **106**, 130506 (2011).

[186] S. E. Nigg and S. M. Girvin, Phys. Rev. Lett. **110**, 243604 (2013).

[187] A. Kitaev, Ann. Phys. **303**, 2 (2003).

[188] J. Zylberberg and E. Shea-Brown, Phys. Rev. E **92**, 062707 (2015).

[189] E. Solano, Phys. Rev. A **71**, 013813 (2005).

[190] R. Grimaudo, A. Messina, P.A. Ivanov, and N.V. Vitanov, J. Phys. A: Math. and Theor. **50** (17), 175301 (2017).

[191] R. Grimaudo, N. V. Vitanov, and A. Messina, Phys. Rev. B **99** (21), 214406 (2019).

[192] R. Grimaudo, Y. Belousov, H. Nakazato, and A. Messina, Ann. Phys. (NY) **392**, 242 (2018).

[193] R. Grimaudo, L. Lamata, E. Solano, and A. Messina, Phys. Rev. A **98** (4), 042330 (2018).

[194] R. J. Glauber, Proceedings of the Second Zvenigorod Seminar on Group Theoretical Method in Physica, Vol. **I** p. 137 (1982).

[195] R. J. Glauber, Ann. New York Acad. Sci. **480**, 336 (1986).

[196] S. Tarzi, J. Phys. A: Math. Gen. **21** 3105-3111 (1988).

[197] S. Gentilini, M. C. Braidotti, G. Marcucci, E. Del Re, and C. Conti, Sci. Rep. **5**, 15816 (2015).

[198] R. Jordan, Z. Phys. 94, 531 (1939).

[199] J. Schwinger, *Quantum Theory of Angular Momentum*, (L. Biedenharn and H. Van Dam Eds.) Academic, New York 1965, p. 229.

[200] L. A. Markovich , R. Grimaudo, A. Messina and H. Nakazato, Ann. Phys. **385**, 522 (2017).

[201] R. Grimaudo, V. I. Man'ko, M.A. Man'ko, and A. Messina, Physica Scripta **95** (2), 024004 (2019).

[202] B. Peng, S. K. Oezdemir, F. Lei, F. Monifi, M. Gianfreda, G. L. long, S. Fan, F. Nori, C. M. Bender, and L. Yang, Nat. Phys. **10**, 394 (2014).

[203] M. M. Sternheim and J. F. Walker, Phys. Rev. C **6**, 114 (1972).

[204] G. Scolarici and L. Solombrino, Phys. Lett. A **303**, 239 (2002).

[205] J. Wiersig, S. W. Kim, and M. Hentschel, Phys. Rev. A **78**, 053809 (2008).

[206] J. Wiersig, Phys. Rev. A **84**, 063828 (2011).

[207] A. Yariv, IEEE J. Quantum Electron. **9**, 9 (1973).

[208] R. Boyd, *Nonlinear Optics*, 3rd ed. (Academic Press, SanDiego, 2008).

[209] D. Ellinas, Phys. Rev. A **45**, 1822 (1992).

[210] K. Jones-Smith and H. Mathur, Phys. Rev. A **82**, 042101 (2010).

[211] S. Croke, Phys. Rev. A **91**, 052113 (2015).

[212] A. Sergi and K. G. Zloshchastiev, Int. J. Mod. Phys. B, **27**, 1350163 (2013).

[213] A. Sergi, and K. G. Zloshchastiev, Phys. Rev. A **91**, 062108 (2015).

[214] A. Sergi, and K. G. Zloshchastiev, Journal of Statistical Mechanics: Theory and Experiment 2016.3: 033102 (2016).

[215] D. C. Brody and E. M. Graefe, Phys. Rev. Lett. **109**, 230405 (2012).

[216] A. B. Klimov, S. M. Chumakov, *A Group Theoretical Approach to Quantum Optics*, 2009 WILEY-VCH Verlag GmbH & Co.

[217] A. Mostafazadeh, Phys. Lett. B **650**, 208 (2007); J. Math. Phys. **45**, 932 (2004).

[218] J. Wei, E. Norman, J. Math. Phys. **4**, 575 (1963).

[219] S. Charzyński, M. Kuś, J. Diff. Equ. **259**, 1542 (2015).

[220] J. Grabowski, M. Kuś, and G. Marmo, Open Syst. Inf. Dyn. **13**, 343 (2006).

[221] C. M. Bender, B. K. Berntson, D. Parker and E. Samuel, American Journal of Physics **81**, 173 (2013).

[222] R. Grimaudo, A. S. M. de Castro, M. Kuś, and A. Messina, Phys. Rev. A **98** (3), 033835 (2018).

[223] R. Grimaudo, A. S. M. de Castro, H. Nakazato, and A. Messina, Phys. Rev. A **99** (5), 052103 (2019).

[224] A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides, Phys. Rev. Lett. **103**, 093902 (2009).

[225] J. S. Bell, Physics Vol. **1**, No. 3, pp. 195-290 (1964).

[226] B. Hensen *et al.*, Nature **526**, 682-686 (2015).

[227] C. Monroe, D. M. Meekhof, B. E. King, D. J. Wineland, Science **272**, 1131-1136 (1996).

[228] J. M. Raimond, M. Burne, and S. Haroche, Rev. Mod. Phys. **73**, 565 (2001).

[229] F. Schmidt-Kaler *et al.*, Nature **422**, 408 (2003).

[230] M. Riebe *et al.*, Nature **429**, 734 (2004).

[231] M. D. Barrett *et al.*, Nature **429**, 737 (2004).

[232] B. Groisman, S. Popescu, A. Winter, Phys. Rev. A **72**, 032317 (2005).

[233] U. Fano, Rev. Mod. Phys. **29**, 74 (1957).

[234] R. Horodecki, M. Horodecki, Phys. Rev. A **54**, 1838 (1996).

[235] K. K. Sabapathy, R. Simon, arXiv:1311.0210 [quant-ph] (2013).