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Euler characteristics of moduli spaces of curves

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Abstract. Let $\mathcal{M}_g^n$ be the moduli space of n-pointed Riemann surfaces of genus g. Denote by $\overline{\mathcal{M}}_g^n$ the Deligne–Mumford compactification of $\mathcal{M}_g^n$. In the present paper, we calculate the orbifold and the ordinary Euler characteristics of $\overline{\mathcal{M}}_g^n$ for any g and n such that $n > 2 - 2g$.

1. Introduction

The moduli space of n-pointed Riemann surfaces of genus g, $\mathcal{M}_g^n$, is an object of much importance in several branches of mathematics and theoretical physics. It parameterizes algebraic curves of genus g with n points, or equivalently Riemann surfaces of genus g with n points. Deligne and Mumford in [5] defined a natural compactification of $\mathcal{M}_g^n$, $\overline{\mathcal{M}}_g^n$, by adjoining stable curves at infinity. These spaces serve as classifying spaces in algebraic geometry, so it is very important to understand their topological structure, especially their homology and cohomology. Madsen and Weiss [13] proved the Mumford conjecture by calculating the stable cohomology of $\mathcal{M}_g^n$. In the current paper we work in the unstable range, calculating both the orbifold and the ordinary Euler characteristics of $\overline{\mathcal{M}}_g^n$ for any g and n with $2g - 2 + n > 0$.

Related work. In [12], Harer and Zagier calculated the orbifold Euler characteristic of $\mathcal{M}_g^n$. Moreover, they computed the ordinary Euler characteristic of $\mathcal{M}_g^n$ for any g and $n = 0, 1$. Since then, there have been some results on the ordinary Euler characteristic of $\overline{\mathcal{M}}_g^n$ only for low values of g. The reader is referred to [9] and [14] for $g = 0$, to [7] for $g = 1$, to [2], [6], [8] for $g = 2$ and to [10] for $g = 3$.

Results. The main results of this paper are formulae for the Euler characteristics of $\mathcal{M}_g^n$ and $\overline{\mathcal{M}}_g^n$. These are given in Theorems 3.2, 4.3 and 4.5. These theorems are proven by combining the techniques of generating functions, integral representations and Wick's lemma that are well established in the field with other counting methods, including a

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formula of Serre and Brown [3] (also used in [12]) for computing actual Euler characteristics in terms of orbifold ones. We compute tables of values in all cases (these were generated using Maple). Our results agree with previous computations in the papers mentioned above.

Outline. Section 2 describes the stratification of $\overline{\mathcal{M}}_g^n$ in terms of stable curves. In Section 3 we review the results of [12], and compute the orbifold Euler characteristic of $\overline{\mathcal{M}}_g^n$. Section 4.1 concludes the paper by calculating the actual Euler characteristic of $\mathcal{M}_g^n$ and of $\overline{\mathcal{M}}_g^n$. Tables of values are given in each case.

2. The graph-type stratification of $\overline{\mathcal{M}}_g^n$

In this section, we review some basic facts and definitions we shall use in the rest of the paper. The moduli space of stable curves $\overline{\mathcal{M}}_g^n$ admits a stratification, which is determined by the configuration of nodes and by irreducible components of n-pointed genus g stable curves. This stratification may be described via stable graphs. For the sake of completeness, we briefly recall their definition.

Definition 2.1. Let g, n be nonnegative integers such that $n > 2 - 2g$. A *stable graph of type* (g, n) is given by the following data:

(SG1) two finite sets V_G and L_G;
(SG2) a partition $\mathcal{P}$ of L_G into subsets with one or two elements;
(SG3) a map γ from V_G to the set of integers $\{0, \dots, g\}$ such that

$$g = \sum_{v \in V_G} \gamma(v) + h^1(G);$$

(SG4) a subset $L(v) \subset L_G$, $v \in V_G$, such that $2\gamma(v) - 2 + |L(v)| > 0$;
(SG5) a map ν from subsets of L_G with one element to $\{1, \dots, n\}$.

The elements of V_G are the *vertices* of G, whereas the elements of L_G are the *half-edges* of G. Moreover, we define *legs* to be the subsets of $\mathcal{P}$ with one element, and *edges* to be the subsets of $\mathcal{P}$ with two elements. **(SG3)** relates g to the structure of G. In fact, $h^1(G) = 1 - v(G) + e(G)$, where $v(G)$ and $e(G)$ denote the number of vertices and the number of edges of the stable graph, respectively. The automorphism group $Aut(G)$ of G is the set of bijections which map V_G to V_G, L_G to L_G, and preserve all data of Definition 2.1. In what follows, for the sake of simplicity, we shall denote V_G and L_G by V and L, respectively. Moreover, by abuse of language, we will call $\gamma(v)$ the *genus of* v, and g the *genus of* G.

Given a stable graph G of type (g, n), choose an ordering of $L(v)$ for each vertex v. Next, consider the morphism

$$\xi_G : \prod_{v \in V} \overline{\mathcal{M}}_{\gamma(v)}^{l(v)} \to \overline{\mathcal{M}}_g^n, \tag{1}$$

where $l(v) = |L(v)|$. A point in the domain is the datum of an $l(v)$-pointed curve C_v for each v. The image point is the n-pointed genus g curve which is obtained as follows: identify the marked points of C_v corresponding to the half-edges of G which are connected by an edge. By definition, the map ξ_G is independent of the ordering of the sets $L(v)$. Analogously to (1), define the morphism

$$\xi_G^o : \mathcal{M}_G := \prod_{v \in V} \mathcal{M}_{\gamma(v)}^{l(v)} \to \overline{\mathcal{M}}_g^n. \tag{2}$$

Set $\Delta_G^o = \xi_G^o(\mathcal{M}_G)$ and denote by Δ_G its closure in $\overline{\mathcal{M}}_g^n$. By definition of (1), two elements in a fiber of ξ_G^o differ by an automorphism of G. This means that

$$\Delta_G^o \simeq \mathcal{M}_G / Aut(G). \tag{3}$$

We recall that a stable graph G of type (g, n) *degenerates* to a stable graph G' of the same type if G' can be obtained from G by a chain of the following moves:

1. collapse an edge that joins two different vertices v_1 and v_2 and label the new vertex v with $\gamma(v) = \gamma(v_1) + \gamma(v_2)$,
2. collapse a loop to a vertex v and increase the genus of v by one.

In that case, we write $G' < G$. Thus the following holds:

$$\overline{\mathcal{M}}_g^n = \bigcup_G \Delta_G^o, \tag{4}$$

where the union is over stable graphs of type (g, n).

The locally closed strata Δ_G^o are equipped with an orbifold structure. Fix a topological oriented surface $S_{\gamma(v),l(v)}$ for each vertex v of G. Such a surface has genus $\gamma(v)$ and $l(v)$ marked points, where $l(v)$ is the number of half-edges outgoing from v. Denote by $a(v)$ the number of half-edges outgoing from v that are legs, and set $b(v) := l(v) - a(v)$. For each vertex, consider the Teichmüller space $\mathcal{T}_{\gamma(v)}^{l(v)}$ and the mapping class group $\Gamma_{\gamma(v)}^{l(v)}$. If

$$T(G) := \prod_{v \in V_G} \mathcal{T}_{\gamma(v)}^{l(v)}, \tag{5}$$

the orbifold structure of Δ_G^o can be described as follows. We recall that the elements of $Aut(G)$ are obtained as compositions of permutations of vertices v_1 and v_2 (when $\gamma(v_1) = \gamma(v_2)$, $a(v_1) = a(v_2) = 0$, $b(v_1) = b(v_2)$) or permutations of half-edges of G. Any permutation of the $l(v)$ half-edges outgoing from v induces a permutation of the $l(v)$ marked points of $S_{\gamma(v),l(v)}$. Accordingly, $Aut(G)$ acts on $T(G)$ as follows. Fix a vertex v and, for simplicity, denote by $[C; x_1, \dots, x_l, [f]]$, $l = l(v)$, an element of the Teichmüller space associated with v. Choose τ in $Aut(G)$ which permutes the half-edges outgoing from v. Clearly, it corresponds to a permutation of the marked points of $S_{\gamma(v),l(v)}$. Thus τ maps $[C; x_1, \dots, x_l, [f]]$ to $[C; x_{\tau(1)}, \dots, x_{\tau(l)}, [\tau \circ f \circ \tau^{-1}]]$. On the other hand, take τ in $Aut(G)$ such that $\tau(v_1) = v_2$, where v_1 and v_2 are vertices of G

with the same genus and $a(v_1) = a(v_2) = 0$, $b(v_1) = b(v_2)$. Set $l_1 = l(v_1) = l(v_2)$, and consider the elements

$$[C_1; x_1, \ldots, x_{l_1}, [f_1]], \qquad [C_2; y_1, \ldots, y_{l_1}, [f_2]]. \tag{6}$$

These two are exchanged by τ in the product $T(G)$.

Definition 2.2.

$$\Gamma(G) := \prod_v \Gamma^{l(v)}_{\gamma(v)} \rtimes Aut(G). \tag{7}$$

We now define an action of $Aut(G)$ on $\prod_v \Gamma_{\gamma(v),l(v)}$ so that $\Gamma(G)$ acts on $T(G)$ and $T(G)/\Gamma(G) \cong \Delta^o_G$. If $\tau \in Aut(G)$ permutes the half-edges of G and $(\prod_v [\mathfrak{h}_v]) \in \prod_v \Gamma^{l(v)}_{g(v)}$, set

$$\tau \cdot \Big(\prod_v [\mathfrak{h}_v]\Big) = \Big(\prod_v [\tau \circ \mathfrak{h}_v \circ \tau^{-1}]\Big). \tag{8}$$

On the other hand, if $\tau \in Aut(G)$ permutes two vertices v_1 and v_2 with the same genus and $a(v_1) = a(v_2) = 0$, $b(v_1) = b(v_2)$, consider two elements as in (6). If $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are elements in the mapping class groups associated with v_1 and v_2, then τ acts on the group $\Gamma(G)$ since it swaps $\mathfrak{h}_1$ and $\mathfrak{h}_2$. As a result, the semidirect product $\Gamma(G)$ in 2.2 is well defined and acts on $T(G)$ in the following way. First, let us consider the case of an automorphism τ which permutes the half-edges of G that stem from a vertex v. If $[C; x_1, \ldots, x_l, [f]]$ and $\mathfrak{h}$ belong to the Teichmüller space and to the mapping class group associated with v, $[C; x_1, \ldots, x_l, [f]]$ is mapped to $[C; x_{\tau(1)}, \ldots, x_{\tau(l)}, [\tau \circ \mathfrak{h} \circ f \circ \tau^{-1}]]$. Second, let τ permute two vertices of G, v_1 and v_2, with no legs, the same genus, and the same number of half-edges. Consider two elements as the ones in (6), which belong to the Teichmüller spaces corresponding to v_1 and v_2, and two classes $\mathfrak{h}_1$ and $\mathfrak{h}_2$ in the mapping class groups associated with v_1 and v_2. Then the action of $\Gamma(G)$ exchanges $[C_1; x_1, \ldots, x_{l_1}, [\mathfrak{h}_1 \circ f_1]]$ with $[C_2; y_1, \ldots, y_{l_1}, [\mathfrak{h}_2 \circ f_2]]$.

It is easy to check that the elements in $T(G)/\Gamma(G)$ are obtained by looking at the orbit of pointed stable curves under the action of $Aut(G)$. Thus,

$$T(G)/\Gamma(G) \cong \Delta^o_G. \tag{9}$$

Furthermore, Δ^o_G has an orbifold structure, since the action of $\Gamma(G)$ is properly discontinuous and with finite stabilizers, as can be readily checked.

3. The orbifold Euler characteristic of $\overline{\mathcal{M}}^n_g$

The orbifold structure of $\overline{\mathcal{M}}^n_g$ naturally induces an orbifold Euler characteristic, which will be hereafter denoted by $\chi(\overline{\mathcal{M}}^n_g)$. In this section, we use the stratification described in (4) to determine generating functions for the rational numbers $\chi(\overline{\mathcal{M}}^n_g)$.

Suppose an orbifold M admits a manifold $\widetilde{M}$ as a finite branched covering $\pi : \widetilde{M} \to M$ of degree d. Then $\chi(M)$ turns out to be $e(\widetilde{M})/d$, where $e(\widetilde{M})$ is the ordinary

Euler characteristic of $\widetilde{M}$. Recall that the Euler characteristic of a virtually torsion free group H is defined similarly, i.e., $\chi(H) = \chi(\widetilde{H})/d$, where $\widetilde{H}$ is a torsion free subgroup of index d in H. We shall use this group-theoretic analogy to compute $\chi(\Delta_G^o)$.

First, observe that $\Gamma(G)$ contains torsion free subgroups $\widehat{\Gamma}(G)$ which act freely on $T(G)$. This follows from well known facts about level structures of algebraic curves. As a consequence, $T(G)/\widehat{\Gamma}(G)$ is a finite branched covering of Δ_G^o of degree $[\Gamma(G) : \widehat{\Gamma}(G)]$. Therefore, $\chi(\Delta_G^o) = \chi(\Gamma(G))$. By the short exact sequence of groups

$$1 \to \prod_v \Gamma_{\gamma(v)}^{l(v)} \to \Gamma(G) \to Aut(G) \to 1,$$

we get

$$\chi(\Gamma(G)) = \Big(\prod_v \chi(\mathcal{M}_{\gamma(v)}^{l(v)})\Big)/|Aut(G)|. \tag{10}$$

Thus,

$$\chi(\overline{\mathcal{M}}_g^n) = \sum_G \frac{\prod_v \chi(\mathcal{M}_{\gamma(v)}^{l(v)})}{|Aut(G)|}.$$

The orbifold Euler characteristic of the moduli space $\mathcal{M}_g^n$ has been computed in [12]. More precisely, the following holds.

Theorem 3.1 ([12]). *For nonnegative integers g, n with $n > 2 - 2g$, the orbifold Euler characteristic of $\mathcal{M}_g^n$ is*

$$\chi(\mathcal{M}_g^n) = (-1)^n \frac{(2g-1)B_{2g}}{(2g)!}(2g+n-3)!,$$

where B_{2g} is the $(2g)$-th Bernoulli number.

In order to compute $\chi(\overline{\mathcal{M}}_g^n)$, we introduce the power series

$$F(x, \hbar) := \sum_{g \geq 0} F_g(x)\hbar^{g-1},$$

where

$$F_g(x) := \sum_{n>2-2g,\, n\geq 0} \chi(\overline{\mathcal{M}}_g^n)\frac{x^n}{n!}.$$

We will express the formal power series $F(x, \hbar)$ in terms of the known generating series

$$\Omega(x, \hbar) := \sum_{g\geq 0} \sum_{\substack{n>2-2g \\ n\geq 0}} \chi(\mathcal{M}_g^n)\frac{x^n}{n!}.$$

Standard techniques in asymptotic theory will yield closed formulae for $F_g(x)$.

Theorem 3.2.

$$\exp(F(x, \hbar)) = \int_{\mathbb{R}} \exp\left(-\frac{(x-y)^2}{2\hbar} + \Omega(y, \hbar)\right)\frac{dy}{\sqrt{2\pi\hbar}}. \tag{11}$$

Proof. If we make the substitution $y - x = z\sqrt{\hbar}$, the integral on the right-hand side of (11) reduces to a one-dimensional Gaussian integral which can be computed directly. Moreover, if the exponential to be integrated is expanded as a power series, we get

$$1 + \sum_{k \geq 1} \sum_{g_1, \ldots, g_k \geq 0} \sum_{\substack{r_1, \ldots, r_k \\ r_j > 2 - 2g_j}} \prod_{j=1}^{k} \chi(\mathcal{M}_{g_j, r_j})$$

$$\cdot \sum_{\substack{t_1, \ldots t_k = 0 \\ \sum t_j \text{ even}}}^{r_1, \ldots, r_k} \frac{(t_1 + \cdots + t_k - 1)!!}{k! t_1! \ldots t_k!} \frac{x^{\sum_{j=1}^{k}(r_j - t_j)}}{\prod_j (r_j - t_j)!} \hbar^{\sum_{j=1}^{k}(g_i - 1) + \frac{1}{2}\sum_j t_j}. \tag{12}$$

The claim will follow if the sum in (12) can be rewritten as a sum over stable graphs. For this purpose, consider k ($k \geq 1$) stable graphs $G_1, \ldots, G_k$. Each graph G_j has one vertex of genus g_j and r_j legs. If we choose t_j legs, $0 \leq t_j \leq r_j$, from each G_j, there are $(t_1 + \cdots + t_k - 1)!!$ possible ways of interconnecting them, provided $\sum t_j$ is even. Such a pairing yields a disconnected stable graph G of type (h_k, n_k), where

$$h_k = \sum_{j=1}^{k} g_j + 1 - k + \frac{1}{2}\sum_{j=1}^{k} t_j, \quad n_k = \sum_{j=1}^{k}(r_j - t_j).$$

Conversely, if we fix nonnegative integers g and n ($n > 2 - 2g$) and a disconnected stable graph of type (g, n), we can determine a collection of integers $k, t_1, \ldots, t_k$, $r_1, \ldots, r_k$ as in the sum which appears in (12). This sum can therefore be rearranged as

$$1 + \sum_{g \geq 0} \sum_{\substack{n \geq 0 \\ n > 2 - 2g}} \sum_{G \in \mathcal{G}_{g,n}} \frac{\chi(\mathcal{M}_G)}{|Aut(G)|} \hbar^{g-1}, \tag{13}$$

where $\mathcal{G}_{g,n}$ is the set of disconnected stable graphs of genus g with n legs. By standard combinatorial arguments, the theorem is completely proved. □

3.1. *Asymptotic formulae For $F_g(x)$*

In order to deduce formulae for $F_g(x)$ we perform a *semiclassical expansion* of the integral on the right-hand side of (12). In other words, we substitute

$$U(x, y, \hbar) := -\frac{(x - y)^2}{2\hbar} + \Omega(y, \hbar)$$

with its formal power series centered at the solution of

$$\overline{y} = x + \sum_{g \geq 0} \Omega'_g(\overline{y}) \hbar^g, \tag{14}$$

where the prime denotes the j-th derivative with respect to the variable y. We thus look for a solution of (14) of the form

$$\overline{y}(x, \hbar) := \sum_{g \geq 0} y_g(x) \hbar^g.$$

This yields the recursive relations

$$
\begin{aligned}
y_0(x) &= x + \sum_{n\geq 2} \chi(\mathcal{M}_{0,n+1}) \frac{y_0^n(x)}{n!}, \\
y_g(x) &= \sum_{s=0}^{g} \sum_{\substack{n>1-2s \\ n\geq 0}} \chi(\mathcal{M}_{s,n+1}) \sum_{\substack{m_1+2m_2+\cdots+gm_g=g-s \\ m_0+m_1+\cdots+m_g=n}} \frac{y_0^{m_0}(x)\dots y_g^{m_g}(x)}{m_0!\dots m_g!}.
\end{aligned}
\tag{15}
$$

The function $y_0(x)$ can be computed via the differential equation

$$
\frac{dy_0(x)}{dx}(1 - \log(1 + y_0(x))) = 1.
$$

This yields the power series

$$
y_0(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{7}{24}x^4 + \frac{17}{60}x^5 + \frac{71}{240}x^6 + \frac{163}{504}x^7 + o(x^8). \tag{16}
$$

Since $y_0'(0) = 0$, the $y_g(x)$'s are uniquely defined via the recursive relations

$$
\frac{y_g(x)}{y_0'(x)} = \sum_{s=1}^{g} \sum_{\substack{n>1-2s \\ n\geq 0}} \chi(\mathcal{M}_{s,n+1}) \cdot \sum_{\substack{m_1+2m_2+\cdots+gm_g=g-s \\ m_0+m_1+\cdots+m_g=g}} \frac{y_0^{m_0}(x)\dots y_g^{m_g}(x)}{m_0!\dots m_g!}.
$$

Let us now expand the function $U(x, y, \hbar)$ about the point $\overline{y}(x, \hbar)$, and set $w = y - \overline{y}$. Thus, we get

$$
-\frac{(x-\overline{y})^2}{2\hbar} + \Omega(\overline{y}, \hbar) - \frac{1}{2\hbar} w^2 \Big(1 - \sum_{g\geq 0} \Omega_g^{(2)}(\overline{y}) \hbar^g \Big) + \sum_{k\geq 3} \frac{1}{k!} w^k \Big(\sum_{g\geq 0} \Omega_g^{(k)}(\overline{y}) \hbar^{g-1} \Big),
$$

where the superscript (j), $j \geq 2$, denotes the derivative with respect to the variable y. For the sake of simplicity, we set

$$
G(x, \overline{y}(x,\hbar)) = \sum_{g\geq 0} \Omega_g^{(2)}(\overline{y}(x,\hbar))\hbar^g, \qquad S_k(x, \overline{y}(x,\hbar)) = \sum_{g\geq 0} \Omega_g^{(k)}(\overline{y}(x,\hbar))\hbar^{g-1},
$$

and

$$
\begin{aligned}
A(x, \overline{y}(x,\hbar)) = \sum_{r\geq 1} \sum_{\substack{k_1,\dots,k_r\geq 3 \\ \sum k_i \text{ even}}} & \frac{(k_1+\cdots+k_r-1)!!}{k_1!\dots k_r!} \\
& \cdot \frac{S_{k_1}(x, \overline{y}(x,\hbar)) \cdot \dots \cdot S_{k_r}(x, \overline{y}(x,\hbar))}{\sqrt{1 - G(x, \overline{y}(x,\hbar))^{k_1+\cdots+k_r+1}}} \hbar^{\frac{1}{2}\sum k_i}.
\end{aligned}
$$

Then the following holds.

Theorem 3.3. *Let*

$$F(x,\hbar) := \sum_{g\geq 0} \sum_{\substack{n\geq 0\\ n>2-2g}} \chi(\overline{\mathcal{M}}_g^n)\frac{x^n}{n!}\hbar^{g-1}$$

be the generating function of $\chi(\overline{\mathcal{M}}_g^n)$. *An asymptotic expansion of* $F(x,\hbar)$ *is given by*

$$\frac{(-x+\overline{y}(x,\hbar))^2}{2\hbar} + \sum_{g\geq 0} \Omega_g(\overline{y}(x,\hbar))\hbar^{g-1} \tag{17}$$

$$-\frac{1}{2}\log(1-G(x,\overline{y}(x,\hbar))) \tag{18}$$

$$+\log(1+A(x,\overline{y}(x,\hbar))). \tag{19}$$

Proof. The claim follows by rewriting the integral appearing on the right-hand side of (11) as

$$\exp\left(\frac{(-x+\overline{y}(x,\hbar))^2}{2\hbar} + \sum_{g\geq 0}\Omega_g(\overline{y}(x,\hbar))\hbar^{g-1}\right)$$
$$\cdot \int_{\mathbb{R}} \exp\Big(-\frac{1}{2\hbar}w^2\Big(1-\sum_{g\geq 0}\Omega_g^{(2)}(\overline{y})\hbar^g\Big)\Big)\exp\Big(\sum_{k\geq 3}\frac{1}{k!}w^k\Big(\sum_{g\geq 0}\Omega_g^{(k)}(\overline{y})\hbar^{g-1}\Big)\Big)\frac{dw}{\sqrt{2\pi\hbar}}. \tag{20}$$

By Wick's Lemma (see [1]), note that the term involving $A(x,\hbar)$ originates from the expansion of the exponential in the integral (20). □

The semiclassical expansion used in Theorem 3.3 can be interpreted as a *loopwise expansion*, i.e., as an expansion with respect to the first Betti number of a stable graph of type (g,n). Thus, we shall describe the $y_g(x)$'s, $g\geq 0$, from a combinatorial point of view.

Definition 3.4. A *stable tree T of genus h* ($h\geq 0$) is a tree such that

T1. there exists a map $\gamma : V_T \to \{0,\dots,h\}$ with $\sum_v \gamma(v) = h$,
T2. there are n ($n\geq 0$) numbered leaves and j unnumbered leaves going into the root such that $j\geq 1$, $n+j>2-2h$,
T3. for every vertex v of the tree, the number of outgoing edges (including the leaves) is greater than $2-2\gamma(v)$.

In what follows, we shall denote by $\mathcal{G}'_{h,n+j}$ the collection of stable trees of genus h with n numbered leaves and j unnumbered leaves going into the root. A graph in $\mathcal{G}'_{h,n+j}$ is by definition a stable graph of type $(h,n+j)$. Take now the two generating functions

$$\xi_0(x) := x + \sum_{n\geq 2}\sum_{T\in\mathcal{G}'_{0,n+1}}\chi(\Delta_T^o)\frac{x^n}{n!}, \qquad \xi_g(x) := \sum_{h=0}^{g}\sum_{n>1-2h}\sum_{T\in\mathcal{G}'_{h,n+1}}\chi(\Delta_T^o)\frac{x^n}{n!},$$

where $\chi(\Delta_T^o)$ is the orbifold Euler characteristic of the open stratum defined as the image of the morphism ξ_T^o in (2).

Proposition 3.5. *For each $g \geq 0$, $\xi_g(x) = y_g(x)$.*

Proof. Choose a graph $T \in \mathcal{G}'_{h,n+1}$. The root of T corresponds to a vertex v with an unnumbered leg, t outgoing edges, and $\gamma(v) = h$, $0 \leq h \leq g$. If we cut T along these edges, we get m graphs $T_1, \dots, T_m$ each of which belongs to $\mathcal{G}'_{a,b+1}$ with some $a \in \{0, \dots, g-h\}$ and $b > 1 - 2a$. The claim follows by (10). □

Let $\mathbb{G}_{(g,l,n)}$ be the set of stable graphs with genus g, $h^1(G) = l$, and n legs. Theorem 3.3 can be interpreted in the following way.

Proposition 3.6.

(i) $$\frac{(-x+\overline{y}(x,\hbar))^2}{2\hbar} + \sum_{g\geq 0} \Omega_g(\overline{y}(x,\hbar))\hbar^{g-1} = \sum_{g\geq 0} \sum_{\substack{n>2-2g\\ n\geq 0}} \sum_{G\in\mathbb{G}_{(g,0,n)}} \chi(\Delta^o_G)\frac{x^n}{n!}\hbar^{g-1};$$

(ii) $$-\frac{1}{2}\log(1 - G(x,\overline{y}(x,\hbar))) = \sum_{g\geq 1} \sum_{\substack{n>2-2g\\ n\geq 0}} \sum_{G\in\mathbb{G}_{(g,1,n)}} \chi(\Delta^o_G)\frac{x^n}{n!}\hbar^{g-1};$$

(iii) $$\log(1 + A(x,\overline{y}(x,\hbar))) = \sum_{l\geq 2} \sum_{g\geq l} \sum_{\substack{n>2-2g\\ n\geq 0}} \sum_{G\in\mathbb{G}_{(g,l,n)}} \chi(\Delta^o_G)\frac{x^n}{n!}\hbar^{g-1}.$$

Proof. (i) Since
$$-\frac{(x-\overline{y})^2}{2\hbar} = -\frac{1}{2\hbar}\Big(\sum_{g\geq 0} \Omega'_g(\overline{y})\hbar^{g-1}\Big)^2,$$
each contribution in (17) is of the form $\chi(\mathcal{M}_{h,r})P(x)$, where $P(x)$ is a polynomial in $y_g(x)$, $g \geq 0$. By Proposition 3.5, this is just a sum of the Euler characteristics $\chi(\Delta^o_G)$, where G is a stable graph and $h^1(G) = 0$.

(ii) Observe that
$$\Omega_g^{(j)}(\overline{y}) = \sum_{\substack{n\geq 0\\ n>2-j-2g}} \chi(\mathcal{M}_{g,n+j})\frac{\overline{y}^n}{n!}. \tag{21}$$

By using (14), we can describe (21) as a sum over oriented rooted trees of arbitrary genus having a root with j unnumbered leaves. Therefore, the product
$$\Omega_h^{(j)}(\overline{y})\Omega_t^{(k)}(\overline{y})$$
is a sum over stable graphs, which is obtained in the following way. We match the j unnumbered leaves outgoing from the root of $T_1 \in \mathcal{G}'_{a,j+n_1}$, $a \leq h$, with the k unnumbered leaves from the root of $T_2 \in \mathcal{G}'_{b,k+n_2}$, $b \leq t$, $n_1 + n_2 = n$. In other words, multiplying derivatives of $\Omega_g(\overline{y})$ gives rise to a sum over stable graphs with $h^1(G) \geq 1$. In particular, since in (18) there are only second derivatives, the contribution $-\frac{1}{2}\log(1-G(x,\overline{y}(x,\hbar)))$ can be rewritten as
$$\sum_{g\geq 1} \sum_{\substack{n>2-2g\\ n\geq 0}} \sum_{G\in\mathbb{G}_{(g,1,n)}} \chi(\Delta^o_G)\frac{x^n}{n!}\hbar^{g-1}.$$

(iii) Analogously to the case $h^1(G) = 1$, the contribution in (19) can be interpreted as

$$\sum_{l\geq 2}\sum_{g\geq l}\sum_{\substack{n>2-2g\\ n\geq 0}}\sum_{G\in\mathbb{G}_{(g,l,n)}} \chi(\Delta^o_G)\frac{x^n}{n!}\hbar^{g-1}. \qquad \square$$

By Proposition 3.6 we also have

Theorem 3.7. *Let $\mathcal{M}^c_{g,n}$ be the moduli space of stable genus g curves with n marked points whose associated stable graph G is a tree. Then*

$$\sum_{g\geq 0}\sum_{\substack{n\geq 0\\ n>2-2g}} \chi(\mathcal{M}^c_{g,n})\frac{x^n}{n!}\hbar^{g-1} = \frac{(-x+\overline{y}(x,\hbar))^2}{2\hbar} + \sum_{g\geq 0}\Omega_g(\overline{y}(x,\hbar))\hbar^{g-1}.$$

Example 3.8. The combinatorial interpretation carried out in Proposition 3.6 yields explicit formulae for the functions $F_g(x)$. In the genus zero case,

$$F_0(x) = \Omega_0(y_0) - \frac{1}{2}\Omega_0'(y_0),$$

where $y_0(x)$ is defined in (16). Since the function $y_0(x)$ satisfies the identity

$$(1+y_0)\log(1+y_0) = 2y_0 - x,$$

the power series $x + F_0'(x)$ coincides with the one given in [14] in an implicit form.

When $g = 1$,

$$F_1(x) = \Omega_1(y_0) - \frac{1}{2}\log(1-\Omega_0^{(2)}(y_0)).$$

When $g = 2$,

$$F_2(x) = \Omega_2(y_0) - y_1^2 y_0' - \frac{y_1^2}{2} + \frac{\Omega_1^{(2)}(y_0)}{2(1-\Omega_0^{(2)}(y_0))} - \frac{1}{8(1-\Omega_0^{(2)}(y_0))^2(1+y_0)^2}$$
$$+ \frac{1}{12(1+y_0)^2(1-\Omega_0^{(2)}(y_0))} + \frac{1}{8(1+y_0)^2(1-\Omega_0^{(2)}(y_0))^3}.$$

4. The ordinary Euler characteristic of $\overline{\mathcal{M}}^n_g$

In this section, we will express the ordinary Euler characteristic of $\overline{\mathcal{M}}^n_g$ in terms of $\chi(\mathcal{M}^n_g)$. This amounts to the computation of $e(\Delta^o_G)$ for any stable graph G. For this purpose, we shall pursue previous work in [12] and apply some results of group cohomology theory. In fact, Δ^o_G is a rational $K(\Gamma(G), 1)$; hence we have

$$e(\Delta^o_G) = e(\Gamma(G)).$$

Thus, the computation of $e(\Gamma(G))$ will follow from a result in [3]. Define a group K to be *geometrically WFL* if there is a contractible, finite-dimensional, proper K-complex Y such that there are only finitely many cells of Y under the action of K. Suppose, further, that K has finitely many conjugacy classes of elements of finite order and for every element σ in K the centralizer $Z_K(\sigma)$ is geometrically WFL. Then the following holds.

Theorem 4.1 ([3]). *For each σ of finite order in K,*

$$e(K) = \sum_{C_\sigma} \chi(Z_K(\sigma)),$$

where the sum is over all conjugacy classes C_σ of elements of finite order in K, and $\chi(Z_K(\sigma))$ is the Euler characteristic of the group $Z_K(\sigma)$ in the sense of Wall (cf. [4]).

We shall apply Theorem 4.1 to the group $\Gamma(G)$ for any stable graph G. Let Y_g^n, $n \geq 1$, be the CW-complex introduced in [11]. Y_g^n is a contractible, finite-dimensional complex such that the mapping class group Γ_g^n acts cellularly, with finite stabilizers and finitely many orbits. For a graph G, consider the CW-complex given by the product

$$Y(G) := \prod_{v \in V_G} Y_{g(v)}^{l(v)}. \tag{22}$$

By the properties of $Y(G)$, the group $\Gamma(G)$ is geometrically WFL. In Corollary 4.9 we shall prove that centralizers of elements of finite order in $\Gamma(G)$ are geometrically WFL. Since, as we shall see, $\chi(Z_G(\sigma))$ can be computed in terms of the characteristic of a group which is a finite extension of products of mapping class groups, the ordinary Euler characteristic of the stratum Δ_G^o is determined by $\chi(\mathcal{M}_g^n)$. Various algebraic manipulations will yield the final result.

4.1. $e(\mathcal{M}_g^{n+1})$

To exemplify the strategy above, we consider first the stable graph G with one vertex and $n+1$ legs. This will yield a formula for the open locus of smooth pointed curves. In this section we restrict our attention to curves with at least two marked points. The remaining cases are dealt with in [12].

Fix a genus g topological oriented surface $S_{g,n+1}$ with $n+1$ marked points. Let $\sigma \in \Gamma_g^{n+1}$ be an element of finite order. As proved in [15], σ may be represented by a periodic homeomorphism f of order k which fixes $p_i \in S_{g,n+1}$, $i = 1, \dots, n+1$. Such a homeomorphism defines a branched covering

$$\psi_{\mathfrak{f}} : S_{g,n+1} \to H_{g,n+1} := S_{g,n+1}/\langle \mathfrak{f} \rangle,$$

where $H_{g,n}$ has a natural structure of orbifold of genus h. If $p_1, \dots, p_n, p_{n+1}, \dots, p_{n+d+1}$ denote the ramification points, then by the Riemann–Hurwitz formula we have

$$2g - 1 + n = k(2h - 1 + n + d) - \sum_i M_i, \tag{23}$$

where $k \geq 1$, $h \geq 0$ and $1 \leq M_r < k$, $M_r \mid k$. The unramified covering corresponding to $\psi_{\mathfrak{f}}$ is clearly determined by a group homomorphism

$$\omega_\sigma : H_1(H_{g,n+1} - B) \to \mathbb{Z}/k\mathbb{Z}, \tag{24}$$

where B is the branch locus of $\psi_{\mathfrak{f}}$. As a result, an element of finite order in Γ_g^{n+1} determines a homomorphism ω_σ and integers h, k, d, M_i satisfying (23). On the other hand, it is easy to check that data $\{h, k, d, M_i\}$ are sufficient to have an element of order k in Γ_g^{n+1}. Define now $\Gamma(H_{g,n+1})$ to be the group of all isotopy classes of homeomorphisms of $H_{g,n+1}$ which fix the set $\{p_1, \ldots, p_{n+1}\}$ and may permute p_i and p_j for $i, j \geq n+2$—when they have the same monodromy. Set, further,

$$\Gamma_g^{n+1}(\mathfrak{f}) := \{\mathfrak{h} \in \Gamma(H_{g,n+1}) : \omega_\sigma \circ \mathfrak{h} = \omega_\sigma\}.$$

Let Z_σ and N_σ be the centralizer and the normalizer of σ in Γ_g^{n+1}, respectively. Analogously to Lemma 3 in [12], the following holds.

Lemma 4.2. *The groups N_σ and $\Gamma_g^{n+1}(\mathfrak{f})$ are related via the short exact sequence*

$$1 \to \mathbb{Z}/k\mathbb{Z} \to N_\sigma \to \Gamma_g^{n+1}(f) \to 1.$$

Moreover, the groups N_σ and Z_σ are geometrically WFL.

By Lemma 4.2, $\chi(Z_\sigma)$ is well defined and can be computed in terms of the Euler characteristic of N_σ. Similar arguments to those in [12] yield a closed formula for $e(\mathcal{M}_g^{n+1})$. Let us now recall some conventional notation. As is customary, we denote by ϕ and μ the Euler and the Möbius arithmetic functions, respectively. Additionally, for any triple of nonnegative integers k, l, δ such that $l \mid k$ and $\delta \mid k$, we set

$$c(k, l, \delta) = \frac{\phi(k/l)}{\phi(\delta/(\delta, l))} \mu(\delta/(\delta, l)), \tag{25}$$

where (δ, l) is the g.c.d. of δ and l. Then the following holds.

Theorem 4.3. *For nonnegative integers g, n such that $2g-1+n > 0$, the ordinary Euler characteristic of $\mathcal{M}_g^{n+1}$ is*

$$e(\mathcal{M}_g^{n+1}) = \sum_{h,k,M_1,\ldots,M_d} \frac{\phi(k)}{k} \frac{\chi(\mathcal{M}_h^{d+1+n})}{d!} k^{2h-1} \sum_{\delta \mid k} \mu(\delta)\,(c(k,1,\delta))^n \cdot \prod_{r=1}^{d} c(k, M_r, \delta), \tag{26}$$

where $h, k, d, M_1, \ldots, M_d$ satisfy the following conditions:

$$k \geq 1, \quad h \geq 0, \quad 1 \leq M_r < k, \quad M_r \mid k,$$
$$2g - 1 + n = k(2h - 1 + n + d) - M_1 - \cdots - M_d.$$

Remark 4.4. Note that for $n = 0$ formula (26) coincides with the one given in [12].

In Table 1 we give some values of $e(\mathcal{M}_g^{n+1})$ for $3 \leq g \leq 10$ and $1 \leq n \leq 8$. In fact, all the values we get for $g = 0, 1, 2$ coincide with the known ones. Finally, we further remark that formula (26) generates the same numbers as $\chi(\mathcal{M}_g^{n+1})$ for $n \geq 2g + 2$. This is consistent with the general fact that a smooth curve with at least $2g + 3$ marked points is automorphism free; hence the two Euler characteristics coincide.

Table 1. Some values of $e(\mathcal{M}_g^{n+1})$

g \ n	1	2	3	4	5	6
3	8	6	4	−10	30	−660
4	−2	−10	−24	−24	−360	2352
5	12	26	92	182	1674	−16716
6	0	−46	−206	188	−7512	124296
7	38	120	676	−1862	71866	−1058676
8	−166	−630	−5362	16108	−680616	12234600
9	748	2132	29632	−323546	7462326	−164522628
10	−1994	6078	−213066	4673496	−106944744	2559934440

4.2. The general case

Analogously to Section 4.1, we shall give a formula for $e(\overline{\mathcal{M}}_g^n)$. First of all, we arrange all such numbers in the generating function

$$f(\lambda, y) := \sum_{\substack{g\geq 0\\ n\geq 1\\ 2g+n\geq 3}} e(\overline{\mathcal{M}}_g^n)\lambda^{2g-2+n}\frac{y^n}{n!} \tag{27}$$

and express f in terms of matrix integrals. To state the main formula, we need some more notation. For any nonnegative integers k and $\delta \mid k$, define $V(k, \delta)$ to be the polynomial

$$\begin{aligned} V(k,\delta) = c(k,1,\delta)\lambda^k y &+ \sum_{1\leq m<k} c(k,m,\delta)\lambda^{k-m} + T(k,\delta)\lambda^k \\ &+ \sum_{\substack{1\leq r\leq k\\ r\mid k}} c(k,r,\delta)x_r\lambda^k \end{aligned} \tag{28}$$

in the variables $\lambda, y, x_1, \dots, x_k$. In (28), note that

$$T(k,\delta) = \begin{cases} k/2, & k\equiv 0 \bmod 2,\ \delta = 1, 2,\\ 0, & \text{otherwise,}\end{cases} \tag{29}$$

and the other coefficients are defined in (25). Set, further,

$$Q(\lambda, y, \underline{x}) = \sum_{\delta\mid k}\sum_{\substack{2h+s\geq 3\\ h,s}} \phi(\delta)\chi(\mathcal{M}_h^s)(k\lambda^k)^{2h-2}\frac{V^s(k,\delta)}{s!}. \tag{30}$$

Then the following holds.

Theorem 4.5.

$$f(\lambda, y) + \sum_{g\geq 2}\Big(\sum_{\substack{h=1\\ h|(g-1)}}^{g-1} e(\overline{\mathcal{M}}_{h+1}) - e(\overline{\mathcal{M}}_g)\Big)\lambda^{2g-2}$$
$$= \log\left\{\lim_{M\to\infty}\frac{1}{(2\pi)^{M/2}}\int_{\mathbb{R}^M}\exp(Q)\,d\mu_M\right\}, \qquad (31)$$

where

$$d\mu_M = \exp\left(-\frac{1}{2}(x_1^2+\cdots+x_M^2)\right)dx_1\ldots dx_M. \qquad (32)$$

Remark 4.6. Note that the results in Section 4.1 can be expressed in terms of generating functions, too. In fact, set

$$f_0(\lambda, y) = \sum_{\substack{g,n\geq 0\\ 2g+n\geq 3}} \frac{e(\mathcal{M}_g^n)}{n!}\lambda^{2g-2+n}y^n,$$
$$V_0(k,\delta) = c(k,1,\delta)\lambda^k y + \sum_{\substack{1\leq m<k\\ m|k}} c(k,m,\delta)\lambda^{k-m}.$$

Then

$$f_0(\lambda, y) - \sum_{g\geq 2}\Big(\sum_{\substack{h=1\\ h|g-1}}^{g-1} e(\mathcal{M}_{h+1}) - e(\mathcal{M}_g)\Big)\lambda^{2g-2}$$
$$= \sum_{\substack{2h+s\geq 3\\ h,s\geq 0}}\sum_{1\leq\delta|k}\phi(\delta)\chi(\mathcal{M}_h^s)(k\lambda^k)^{2h-2}\frac{V_0^s(k,\delta)}{s!}.$$

As sketched before, the proof of Theorem 4.5 is organized as follows. To begin, we will prove that centralizers of elements of finite order in $\Gamma(G)$ are geometrically WFL—see Section 4.2.2. Henceforth, we assume G is a stable graph with at least two vertices. Next, we will obtain a formula for $e(\overline{\mathcal{M}}_g^n)$ in terms of $\chi(\mathcal{M}_h^r)$ for suitable values of h and r—see Section 4.2.2. Finally, we will deduce formula (31) by standard techniques in matrix integral theory.

4.2.1. Centralizers of elements of finite order in $\Gamma(G)$. The elements of finite order in $\Gamma(G)$, and their centralizers, are better understood if we describe the group $\Gamma(G)$ in an alternative way. For any stable graph G, let $S(G)$ be the surface

$$S(G) = \bigsqcup_{v\in V_G} S_{g(v),a(v)+b(v)}, \qquad (33)$$

where $S_{g(v),a(v)+b(v)}$ is defined in Section 2. For nonnegative integers $\mathfrak{g},\mathfrak{b}$ denote by $V_G^{(\mathfrak{g},\mathfrak{b})}$ the subset of vertices of G such that $g(v)=\mathfrak{g}$, $a(v)=0$ and $b(v)=\mathfrak{b}$. Any nonempty $V_G^{(\mathfrak{g},\mathfrak{b})}$ is equipped with a permutation action of the symmetric group $\mathfrak{S}_{(\mathfrak{g},\mathfrak{b})}$. Define $\Gamma(S(G))$ to be the semidirect product

$$\widetilde{\Gamma} \rtimes \prod_{V_G^{(\mathfrak{g},\mathfrak{b})}\neq\emptyset} \mathfrak{S}_{(\mathfrak{g},\mathfrak{b})},$$

where $\widetilde{\Gamma}$ consists of the isotopy classes of the orientation-preserving diffeomorphisms of $S(G)$ which, for each vertex v, fix the $a(v)$ points, but may permute the $b(v)$ points.

We now show that $\Gamma(G)$ is isomorphic to a subgroup L of $\Gamma(S(G))$. Let L be the set of elements of $\Gamma(S(G))$ that are compatible with the identifications of the marked points, which are induced by G. In other words, if q_i and q_j are identified in $S(G)$, then any $\mathfrak{h}$ in L identifies $\mathfrak{h}(q_i)$ and $\mathfrak{h}(q_j)$. It is easy to check that $\Gamma(G)$ is isomorphic to L. Clearly, any element of $\Gamma(G)$ induces an element in L. Conversely, note that $\mathfrak{h}$ in L satisfies $\mathfrak{h}(q_i) = q_{\theta(i)}$ and $\mathfrak{h}(q_j) = q_{\theta(j)}$, where θ is a permutation of the set of the $\sum_v b(v)$ marked points of $S(G)$. Accordingly, $\mathfrak{h}$ induces the element $(\widetilde{\mathfrak{h}}, \widetilde{\theta})$ in $\Gamma(G)$, where $\widetilde{\theta}$ is the automorphism of G induced by θ and $\widetilde{\mathfrak{h}} = \widetilde{\theta} \circ \mathfrak{h}$.

Since $\Gamma(G)$ can be viewed as a subgroup of $\Gamma(S(G))$, an element σ of finite order in $\Gamma(G)$ can be realized (cf. [15]) as a periodic diffeomorphism of $S(G)$. The quotient of $S(G)$ by the group generated by σ is a disconnected orbifold, X. Let us describe X in detail.

X has a finite number (say p) of connected components X_i each of which is an orbifold of genus h_i, $1 \le i \le p$. Each X_i has some natural marked points. The n marked points of $S(G)$ corresponding to legs of G are fixed by σ and thus descend to marked points of X. Hence, each X_i has a_i ($a_i \ge 0$) points of this type, where $\sum a_i = n$. Another set of marked points is given by the orbits of $\langle\sigma\rangle$ that contain points of $S(G)$ corresponding to half-edges of G. Additionally, we mark the points of X_i which come from orbits of length smaller than $|\langle\sigma\rangle|$. Each X_i may have d_i additional marked points of this type.

This leads quite naturally to the construction of an *orbi-graph* H in the following way. The set of vertices $\{v_1, \dots, v_p\}$ of H has order p. Each v_i corresponds to one of the X_i's. The edges of H are recovered from the edges of G if we respect some compatibility conditions. More explicitly, let β_i and β_j be two half-edges of H. Suppose they correspond to points x_i and x_j of X. If y_i and y_j are points of $S(G)$ which correspond to x_i and x_j, then β_i and β_j are paired if and only if the half-edges of G corresponding to y_i and y_j are paired too. Note that some edges of G may correspond to half-edges of H. Thus, we denote by b_i the number of marked points in X_i that correspond to the half-edges issuing from v_i and giving rise to edges of H. On the other hand, we denote by c_i the number of half-edges issuing from v_i and different from the $a_i + b_i + d_i$ half-edges listed so far. Finally, we denote by k_i the degree of the covering, say σ_i, of X_i. An element of finite order in $\Gamma(G)$ thus determines the following set of data:

$$\begin{aligned} &p,\ \{k_1, \dots, k_p\},\ \{h_1, \dots, h_p\},\ \{a_1, \dots, a_p\},\\ &\{b_1, \dots, b_p\},\ \{c_1, \dots, c_p\},\ \{d_1, \dots, d_p\},\ \pi, \end{aligned} \tag{34}$$

where π is the pairing among the $\sum_{i=1}^{p} b_i$ half-edges of H induced by the compatibility conditions described above.

Note that a stable graph G and the elements of finite order σ_i determine group homomorphisms from $H_1(X_i^0)$ to the cyclic group of order k_i, where X_i^0 is X_i with all the marked points removed. Moreover, these homomorphisms satisfy the following conditions. Below, when we refer to small loops around a marked point q of X_i^0 we mean an oriented loop (say counterclockwise) around q which is small enough to encircle only the point q.

(I) $(\omega_i(\alpha), k_i) = 1$ for a small loop α around each of the a_i marked points;
(II) $\omega_i(\delta) \neq 0$ for a small loop around each of the d_i marked points;
(III) $(\omega_i(\gamma), k_i) \equiv 0 \bmod 2$ for a small loop γ around each of the c_i marked points.

Conversely, given a graph H and data as in (34), we would like to recover a stable graph G' of type (g, n) for some g and n. Roughly speaking, we would like to reconstruct a topological surface from X. For these purposes, we need some extra information. First, we define group homomorphisms

$$\omega_i : H_1(X_i^0) \to \mathbb{Z}/k_i\mathbb{Z} \tag{35}$$

which satisfy conditions (I)–(III). A homomorphism ω_i in (35) determines a branched covering of X_i of degree k_i: the image of the loops around the marked points of X determine the local monodromy of the covering. Let

$$N_j^i, \quad 1 \leq j \leq b_i, \quad N_j^i \,|\, k_i, \quad 1 \leq N_j^i \leq k_i, \tag{36}$$

$$M_r^i, \quad 1 \leq r \leq d_i, \quad M_r^i \,|\, k_i, \quad 1 \leq M_r^i < k_i, \tag{37}$$

$$R_s^i, \quad 1 \leq s \leq c_i, \tag{38}$$

be the numbers of points lying over each of the b_i, d_i, c_i marked points of X_i. Notice that each vertex of the graph G' has

$$\eta_i = a_i + \sum_{j=1}^{b_i} N_j^i + \sum_{s=1}^{c_i} R_s^i \tag{39}$$

half-edges. Note that the sums in (39) should be considered empty when b_i (or c_i) are zero. Next, we introduce a pairing $\widetilde{\pi}$ among the points in the orbit of each of the b_i points. Clearly, this pairing has to be compatible with the pairing π of the graph H. Indeed, suppose that β_j^i and $\beta_{j'}^{i'}$ are paired via π. By abuse of notation, we still denote the marked points corresponding to them by β_j^i and $\beta_{j'}^{i'}$. If there are N_j^i (resp. $N_{j'}^{i'}$) points lying above β_j^i (resp. $\beta_{j'}^{i'}$), then $N_j^i = N_{j'}^{i'}$.

We can now associate a graph G' to the covering of X determined by the homomorphisms ω_i. The number, $v(G')$, of vertices of G' is $\sum_{i=1}^{p} P_i$, where P_i is the number of components lying over X_i. The total number, $e(G')$, of edges is determined as follows. There is only one way to lift the c_i half-edges of H, so G' has $\sum_{i,s} R_s^i/2$ edges of this type. Let e be an edge obtained by pairing two half-edges e_- and e_+. Denote by $N(e_-)$ and $N(e_+)$ the numbers of points over (the marked points corresponding to) e_- and e_+. By the discussion above, there exist $N(e_-) = N(e_+)$ possible pairings $\widetilde{\pi}$ for G'.

As a consequence, there are $\frac{1}{2}\sum_{i,j} N_j^i$ edges of this type. We observe that each vertex of G' corresponds to a genus g_i covering of X_i, where g_i is determined by the Riemann–Hurwitz formula, namely

$$P_i(2-2g_i) = k_i(2-2h_i-a_i-b_i-c_i-d_i)+a_i+\sum_{j=1}^{b_i} N_j^i+\sum_{r=1}^{d_i} M_r^i+\sum_{s=1}^{c_i} R_s^i. \qquad (40)$$

If we set

$$n := \sum_{i=1}^{p} a_i, \quad g := \sum_{i=1}^{p} P_i g_i + e(G') - v(G') + 1, \qquad (41)$$

we get

$$2g-2+n = \Big(\sum_{i=1}^{p} k_i(2h_i-2+a_i+b_i+c_i+d_i)\Big) - \sum_{i,r} M_r^i. \qquad (42)$$

The graph G' does not have a total ordering of all the $a_1+\cdots+a_p$ points. Thus, we give a total ordering $\mathbf{O}$ to all these points and there are clearly $n!$ of these. We call the result G, and the following now holds.

Proposition 4.7. *The graph G is stable if and only if*

$$k_i(2h_i-2+a_i+b_i+c_i+d_i) - \sum M_r^i > 0 \qquad (43)$$

for all $1 \le i \le p$. Furthermore, this condition is equivalent to the stability of H except for the case where $h_i = a_i = c_i = 0$, $b_i = 1$, $d_i = 2$, k_i is even and $M_i^1 = M_i^2 = k_i/2$. If these values hold for some i, then G is not stable, even though H is.

Proof. Let $a_i+b_i+c_i+d_i$ be the number of half-edges at the i-th vertex of H and let η_i be the number of half-edges of G' as in (39). Suppose first that (43) holds. This means that $2h_i-2+a_i+b_i+c_i+d_i > 0$ for $1 \le i \le p$. By (40) we have

$$P_i(2g_i-2+\eta_i) \ge P_i(2g_i-2)+\eta_i = k_i(2h_i-2+a_i+b_i+c_i+d_i) - \sum M_r^i > 0,$$

so G is stable.

Conversely, suppose that H is not stable at a vertex v_i, $1 \le i \le p$. Then either

$$h_i = 1, \quad a_i+b_i+c_i+d_i = 0,$$

or

$$h_i = 0, \quad a_i+b_i+c_i+d_i \le 2.$$

Let $\widetilde{X}_i$ be one of the P_i components lying above X_i. In the former case, there are no branch points, so $\widetilde{X}_i$ must be a torus with no marked points, implying that G is also not stable. In the latter case, $\widetilde{X}_i$ must be a cover of X_i branched at most over two points and again G cannot be stable. □

In other words, a pair (G,σ) of a stable graph and an element of finite order in $\Gamma(G)$ is determined by (i) a stable graph H that satisfies (43), (ii) a collection of data as in

(34), (iii) group homomorphisms ω_i satisfying conditions (I)–(III), and finally (iv) a total ordering **O** and a pairing π.

This said, it is now easier to study properties of the centralizer $Z_G(\sigma)$ of an element σ of finite order in $\Gamma(G)$.

Let $N_G(\sigma)$ be the normalizer of σ in $\Gamma(G)$. Denote by H the graph associated with (G, σ). Since H is stable, we define the group $\Gamma(H)$ as in 2.2, i.e.

$$\prod_{i=1}^{p} \Gamma_{h_i, a_i+b_i+c_i+d_i} \rtimes Aut(H), \tag{44}$$

where $Aut(H)$ is the group of automorphisms of H generated by

- automorphisms of H which may permute two vertices v_i and v_j only if $h_i = h_j$, $a_i = a_j = 0$, $b_i = b_j$, $c_i = c_j$ and $d_i = d_j$,
- permutations of the d_i marked points for each i, $1 \leq i \leq p$.

Any element $\mathfrak{h}$ in $\Gamma(H)$ induces an automorphism of the fundamental group of X which we denote by $\mathfrak{h}_*$. Thus, we set

$$\Gamma(H, \omega_\sigma) := \{\mathfrak{h} \in \Gamma(H) : \omega_\sigma \circ \mathfrak{h}_* = \omega_\sigma\}, \tag{45}$$

where ω_σ is any of the group homomorphisms determined by σ.

Proposition 4.8. *The groups $N_G(\sigma)$ and $\Gamma(H, \omega)$ are related via the short exact sequence of groups*

$$1 \to C_k \xrightarrow{t_1} N_G(\sigma) \xrightarrow{t_2} \Gamma(H, \omega_\sigma) \to 1. \tag{46}$$

Proof. The group C_k is the cyclic group of order k generated by σ, and t_1 is the inclusion homomorphism. The map t_2 is defined as follows. Pick an element $\mathfrak{h}$ in $N_G(\sigma)$. By the same arguments in [12, Lemma 3], there exists an orientation-preserving diffeomorphism $\mathfrak{f}_\mathfrak{h}$ of $S(G)$ such that

$$\mathfrak{f}_\mathfrak{h} \sigma \mathfrak{f}_\mathfrak{h}^{-1} = \sigma^j \quad \text{for some } j. \tag{47}$$

Then $t_2(\mathfrak{h})$ is defined as the isotopy class of $\mathfrak{f}_\mathfrak{h}$. By condition (47) and the definition of $\Gamma(H, \omega_\sigma)$, $\mathfrak{f}_\mathfrak{h}$ yields an element in $\Gamma(H, \omega_\sigma)$. □

Corollary 4.9. *The centralizers of elements of finite order in $\Gamma(G)$ are geometrically WFL.*

Proof. The group $\Gamma(H, \omega_\sigma)$ is a finite index subgroup of $\Gamma(H)$ so it acts on the CW-complex $Y(G)$ introduced in (22). The exact sequence in (46) gives an action of $N_G(\sigma)$ and $Z_G(\sigma)$ on $Y(G)$; so they are both WFL. □

4.2.2. A formula for $e(\overline{\mathcal{M}}_g^n)$. In this section we give a formula for $e(\overline{\mathcal{M}}_g^n)$ when $n \geq 1$. This will be used in the next section for the generating function f. In what follows, we adopt the same notation as in Section 4.2.1.

By Theorem 4.1, we have

$$e(\overline{\mathcal{M}}_g^n) = \sum_G e(\Delta_G^o) = \sum_G \sum_{C_\sigma} \chi(Z_G(\sigma)). \tag{48}$$

Since each (G, σ) determines an orbi-graph H, we rewrite $e(\overline{\mathcal{M}}_g^n)$ as a sum over pairs (H, ω_i), where H is an orbi-graph and ω_i are the group homomorphisms introduced in (35). For each k_i we consider the action of $\Gamma(H)$ on the set $A_H(k_i)$ of all homomorphisms ω_i in (35) which satisfy conditions (I)–(III). In particular, if $\omega_i \in A_H(k_i)$, we denote by $\Gamma(H, \omega_i)$ the stabilizer of ω_i under the action of $\Gamma(H)$. Notice that when ω_i is induced by $\sigma \in \Gamma(G)$, $\Gamma(H, \omega_i)$ is the group in (45).

Fix an orbi-graph H and data $\mathbf{O}$ and π described in Section 4.2. Let Λ_i be a set of representatives of the conjugacy classes C_{σ_i}, and suppose that the quotient of $S(G)$ by the σ_i's is isomorphic to X. Thus

$$e(\overline{\mathcal{M}}_g^n) = \sum_p \sum_{k_1,\ldots,k_p} \sum_{C_{\sigma_1},\ldots,C_{\sigma_p}} \sum_{C_{\rho_1},\ldots C_{\rho_p}} \prod_{i=1}^p \chi(Z_G(\sigma_i)),$$

where S_{σ_i} is a set of representatives of the orbits of

$$G_{\sigma_i} = \{\sigma_i^n : (n, k) = 1\}$$

under the action of $N_G(\sigma_i)$ by conjugation. If $\mathcal{O}_j$ is such an orbit, then we get

$$\sum_{\rho \in S_{\sigma_i}} \chi(Z_G(\rho)) = \sum_{\rho \in S_\sigma} |\mathcal{O}_j| \chi(N_G(\rho)) = \phi(k)\chi(N_G(\sigma)). \tag{49}$$

Hence we have

$$\sum_p \sum_{k_1,\ldots,k_p} \sum_{C_{\sigma_1},\ldots,C_{\sigma_p}} \prod_{i=1}^p \frac{\phi(k_i)}{k_i} \chi(\Gamma(H, \omega_{\sigma_i})).$$

By the short exact sequence

$$1 \to \prod_{i=1}^p \Gamma_{h_i, a_i+b_i+c_i+d_i} \to \Gamma(H) \to Aut(H) \to 1,$$

we get

$$\begin{aligned} \chi(\Gamma(H, \omega_{\sigma_i})) &= [\Gamma(H) : \Gamma(H, \omega_{\sigma_i})] \frac{\prod_{i=1}^p \Gamma_{h_i, a_i+b_i+c_i+d_i}}{|Aut(H)|} \\ &= |\mathcal{O}(\omega_{\sigma_i})| \frac{\prod_{i=1}^p \Gamma_{h_i, a_i+b_i+c_i+d_i}}{|Aut(H)|}, \end{aligned} \tag{50}$$

where $\mathcal{O}(\omega_{\sigma_i})$ is the orbit of ω_{σ_i} under the action of $\Gamma(H)$ on $A_H(k_i)$.

The expression in (50) can be further simplified. We say that H is *ordered* if it has an ordering on its vertices and an ordering on each collection of the half-edges going out of a vertex v. Let $\mathbf{H}'$ be the set of all ordered (disconnected) stable orbi-graphs. The group

$$\mathfrak{S}_p \ltimes \prod_{i=1}^{p} (\mathfrak{S}_{a_i} \times \mathfrak{S}_{b_i} \times \mathfrak{S}_{c_i} \times \mathfrak{S}_{d_i})$$

acts on $\mathbf{H}'$ with orbits equal to orbi-graphs. To determine the ordered orbi-graph H, we only need to enumerate

$$\begin{array}{ll} (h_1, \dots, h_p),\ h_i \geq 0, & (k_1, \dots, k_p),\ k_i \geq 1, \\ (a_1, \dots, a_p),\ a_i \geq 0, & (b_1, \dots, b_p),\ b_i \geq 0,\ \sum_{i=1}^{p} b_i \equiv 0 \bmod 2, \\ (c_1, \dots, c_p),\ c_i \geq 0, & (d_1, \dots, d_p),\ d_i \geq 0, \end{array}$$

together with a pairing π of the numbers $1, \dots, b_1 + \dots + b_p$. Denote by $\underline{N}$ the vector

$$(N_1^1, \dots, N_{b_1}^1, \dots, N_1^p, \dots, N_{b_p}^p),$$

where N_j^i is defined in (36). By (42), (41), (37), and (38), we get

$$e(\overline{\mathcal{M}}_g^n) = \sum_p \sum_{k_1,\dots,k_p} \frac{1}{p!} \sum_{\omega_i \in A_H(k_i)} \frac{\phi(k_i)}{k_i} \sum_{\substack{h_i, a_i, b_i, c_i, d_i \geq 0 \\ b_1 + \dots + b_p \ even \\ 2h_i + a_i + b_i + c_i + d_i \geq 3}} \prod_{i=1}^{p} \frac{\chi(\mathcal{M}_{h_i}^{a_i+b_i+c_i+d_i})}{a_i! b_i! c_i! d_i!}. \tag{51}$$

We can simplify the sum in (51) if we enumerate all the elements in $A_H(k_i)$. A homomorphism $\omega_i : H_1(X_i^0) \to \mathbb{Z}/k_i\mathbb{Z}$ is determined by assigning its values on a basis of $H_1(X_i)$ and on each small oriented loop around any of the marked points in X_i. Therefore, for any k_i, the number of such homomorphisms can be computed as follows. The number of values that ω_i can assume on a basis of $H_1(X_i)$ is $\prod_{i=1}^{p} k_i^{2h_i}$. For the values on the loops around the marked points we introduce the following notation. Denote by $<\alpha_l^i>, <\beta_j^i>, <\gamma_s^i>, <\delta_r^i>$ the loops around the a_i, b_i, c_i, d_i marked points. Then ω is determined by assigning the following elements of $\mathbb{Z}/k_i\mathbb{Z}$:

1. $\omega(<\alpha_l^i>) = A_l^i$, with $(A_l^i, k_i) = 1,\ 1 \leq l \leq a_i$,
2. $\omega(<\beta_j^i>) = B_j^i$, with $(B_j^i, k_i) = N_j^i,\ N_j^i \mid k_i,\ 1 \leq N_j^i \leq k,\ 1 \leq j \leq b_i$,
3. $\omega(<\gamma_s^i>) = C_s^i$, with $(C_s^i, k_i) \equiv 0 \bmod 2$, $1 \leq s \leq c_i$,
4. $\omega(<\delta_r^i>) = D_r^j$, with $(D_r^i, k) = M_r^i,\ M_r^i \mid k,\ 1 \leq M_r^i < k,\ 1 \leq r \leq d_i$.

These requirements depend on the conditions (I)–(III) satisfied by ω_i. Moreover, the image under ω_i of the relation among cycles in $H_1(X_i)$ yields the additional constraint

$$\sum_{i,l} A_l^i + \sum_{j,i} B_j^i + \sum_{s,i} C_s^i + \sum_{r,i} D_r^i \equiv 0 \bmod k_i.$$

Define $\mathbf{T}(\{a_i\}, \{b_i\}, \{c_i\}, \{d_i\}, \{N_j^i\}, \{M_r^i\})$ to be the cardinality of the set

$$\Big\{(A_1^1, A_2^1, \ldots, D_{d_p}^p) : A_1^1 = 1,\ (A_l^i, k_i) = 1,\ (B_j^i, k) = N_j^i, \\ (C_s^i, k_i) \equiv 0 \bmod 2,\ (D_r^i, k_i) = M_r^i, \\ \sum_{i,l} A_l^i + \sum_{j,i} B_j^i + \sum_{s,i} C_s^i + \sum_{r,i} D_r^i \equiv 0 \mod k_i \Big\}.$$

Accordingly, if $a_i \geq 1$, the number of homomorphisms ω_i is

$$\prod_{i=1}^{p} k_i^{2h_i} \phi(k_i) \mathbf{T}(\{a_i\}, \{b_i\}, \{c_i\}, \{d_i\}, \{N_j^i\}, \{M_r^i\}). \tag{52}$$

By standard facts in elementary number theory (see [12]) we have

$$\phi(k_i)\mathbf{T}(\{a_i\}, \{b_i\}, \{c_i\}, \{d_i\}, \{N_j^i\}, \{M_r^i\})$$
$$= \prod_{i=1}^{p} \frac{1}{k_i} \sum_{\substack{\zeta \\ \zeta^k=1}} \sum_{\substack{0\le A_l^i<k_i \\ (A_l^i,k_i)=1}} \sum_{\substack{0\le B_j^i<k_i \\ (B_j^i,k_i)=N_j^i}} \sum_{\substack{0\le C_s^i<k_i \\ (C_s^i,k)\equiv 0 \bmod 2}} \sum_{\substack{0\le D_r^i<k_i \\ (D_r^i,k)=M_r^i}} \zeta^{\sum_l A_l^i + \sum_j B_j^i + \sum_s C_s^i + \sum_r D_r^i}$$
$$= \prod_{i=1}^{p} \frac{1}{k_i} \sum_{\substack{\zeta \\ \zeta_i^k=1}} \prod_{l=1}^{a_i} \Big(\sum_{\substack{0\le s<k_i \\ (s,k_i)=1}} \zeta^s \Big) \prod_{j=1}^{b_i} \Big(\sum_{\substack{0\le s<k_i \\ (s,k_i)=N_j^i}} \zeta^s \Big) \prod_{l=1}^{c_i} \Big(\sum_{\substack{0\le s<k_i \\ (s,k_i)\equiv 0 \bmod 2}} \zeta^s \Big) \prod_{j=1}^{d_i} \Big(\sum_{\substack{0\le s<k_i \\ (s,k_i)=M_r^i}} \zeta^s \Big).$$

Lemma 4.10. (i) *If k is even,*

$$\sum_{\substack{(r,k)\equiv 0 \bmod 2 \\ 0\le r<k}} \zeta^r = \begin{cases} k/2 & \text{for } \zeta = 1, -1, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *For any pair l, δ of divisors of k and ζ a primitive δ-th root of unity, we have*

$$\sum_{\substack{(r,k)=l \\ 0\le r<k}} \zeta^r = c(k, l, \delta),$$

where $c(k, l, \delta)$ is defined in (25).

Proof. (i) Since k is even, we have

$$\sum_{\substack{(r,k)\equiv 0 \bmod 2 \\ 0\le r<k}} \zeta^r = 1 + \zeta^2 + \zeta^4 + \cdots + \zeta^{k-2},$$

which is zero unless $\zeta = 1$ or -1 in which case it equals $k/2$.

(ii) This follows from the definition of the Möbius function. □

By Lemma 4.10, we have

$$\begin{aligned}
&\phi(k_i)\mathbf{T}(\{a_i\},\{b_i\},\{c_i\},\{d_i\},\{N_j^i\},\{M_r^i\})\\
&\quad=\prod_{i=1}^{p}\frac{1}{k_i}\sum_{\substack{\delta|k_i\\ 1\le\delta<k_i}} c(k_i,1,\delta)^{a_i}\prod_{j=1}^{b_i}c(k_i,N_j^i,\delta)\gamma(k_i,\delta,c_i)\prod_{r=1}^{d_i}c(k_i,M_r^i,\delta), \qquad (53)
\end{aligned}$$

where

$$\gamma(k,\delta,c)=\begin{cases}\phi(\delta), & c=0,\\ 0, & k\equiv 1 \bmod 2,\ c>0,\\ 0, & k\equiv 0 \bmod 2,\ c>0,\ \delta>2,\\ (k/2)^c, & k\equiv 0 \bmod 2,\ c>0, \delta=1,2.\end{cases} \qquad (54)$$

As a result, the following holds.

Theorem 4.11. *For any integers $g\ge 0$ and $n\ge 1$ such that $n>2-2g$, the ordinary Euler characteristic of $\overline{\mathcal{M}}_g^n$ is given by*

$$\begin{aligned}
e(\overline{\mathcal{M}}_g^n)=n!&\sum_{p=1}^{2g-2+n}\frac{1}{p!}\sum_{\substack{b_1,\dots,b_p\\ b_1+\dots+b_p\ even}}\sum_{\underline{N}}\sum_{\pi}\prod_{e}N(e)\\
&\cdot\prod_i\sum_{\substack{h_i,a_i,d_i\\ 2h_i+a_i+b_i+c_i+d_i\ge 3}}\sum_{k_i,M_i^r}\frac{\chi(\mathcal{M}_{h_i}^{a_i+b_i+d_i+c_i})}{d_i!a_i!b_i!c_i!}\\
&\cdot k_i^{2h_i-2}\sum_{\delta_i|k_i}\Big(c(k_i,1,\delta_i)^{a_i}\prod_{j=1}^{b_i}c(k_i,N_i^j,\delta_i)\gamma(k_i,\delta_i,c_i)\prod_{r=1}^{d_i}c(k_i,M_i^r,\delta_i)\Big),
\end{aligned}$$

where

$$\begin{gathered}
2g-2+n=\sum_{i=1}^{p}k_i(2h_i-2+a_i+b_i+c_i+d_i)-\sum_{i=1}^{p}(M_i^1+\dots+M_i^{d_i});\\
M_i^r\,|\,k_i,\quad M_i^r<k_i,\quad k_i\le 1;\\
0\le h_i\le g,\quad \sum_{i=1}^{p}a_i=n;\\
1\le N_i^j\le k_i,\quad N_i^j\,|\,k_i,\quad N_i^j=N_{i'}^{j'};\\
\pi \text{ a connected pairing of the numbers } 1,\dots,b_1+\dots+b_p,\ b_i\ge 1.
\end{gathered}$$

4.2.3. The proof of Theorem 4.5. This section is devoted to proving Theorem 4.5. Basically, we shall apply Wick's Lemma to deduce (31).

By (42) and Theorem 4.11, the generating series

$$\sum_{\substack{g\geq 1, n\geq 0\\ n\geq 2g+2}} e(\overline{\mathcal{M}}_g^n)\lambda^{2g-2}\frac{y^n}{n!}$$

is equal to

$$\sum_{p\geq 1}\frac{1}{p!}\sum_{k_1,\dots,k_p}\sum_{\substack{b_1,\dots,b_p\\ b_1+\cdots+b_p \text{ even}}}\sum_{\underline{N},\pi}\prod_e N(e) \tag{55}$$

$$\cdot\prod_i \sum_{\substack{h_i,a_i,d_i\\ 2h_i+a_i+b_i+c_i+d_i\geq 3}}\sum_{k_i,M_i^r}\frac{\chi(\mathcal{M}_{h_i,a_i+b_i+d_i+c_i})}{d_i!a_i!b_i!c_i!} \tag{56}$$

$$\cdot (k_i\lambda^{k_i})^{2h_i-2}\sum_{\delta_i|k_i}\Big((c(k_i,1,\delta_i)\lambda^{k_i}y)^{a_i} \tag{57}$$

$$\cdot\prod_{j=1}^{b_i}(c(k_i,N_i^j,\delta_i)\lambda^{k_i})(\gamma(k_i,\delta_i,c_i)\lambda^{k_i})\prod_{r=1}^{d_i}(c(k_i,M_i^r,\delta_i)\lambda^{k_i-M_r^i})\Big). \tag{58}$$

Note that

$$\gamma(k_i,\delta_i,c_i)\lambda^{k_i}=\phi(\delta_i)(T(k_i,\delta_i)\lambda^{k_i})^{c_i},$$

where $T(k_i,\delta_i)$ is defined in (29). Moreover, all the indices b_i in (55) are positive integers. Looking at the expansion above, we define the generating series

$$\begin{aligned}\widehat{f}(\lambda,y) = &\sum_{p\geq 1}\frac{1}{p!}\sum_{k_1,\dots,k_p}\sum_{\substack{b_1,\dots,b_p\geq 0\\ b_1+\cdots+b_p \text{ even}}}\sum_{\underline{N},\pi}\prod_e N(e)\\ &\cdot\prod_i \sum_{\substack{h_i,a_i,d_i\\ 2h_i+a_i+b_i+c_i+d_i\geq 3}}\sum_{k_i,M_i^r}\frac{\chi(\mathcal{M}_{h_i,a_i+b_i+d_i+c_i})}{d_i!a_i!b_i!c_i!}\\ &\cdot (k_i\lambda^{k_i})^{2h_i-2}\sum_{\delta_i|k_i}\Big((c(k_i,1,\delta_i)\lambda^{k_i}y)^{a_i}\\ &\cdot\prod_{j=1}^{b_i}(c(k_i,N_i^j,\delta_i)\lambda^{k_i})(\gamma(k_i,\delta_i,c_i)\lambda^{k_i})\prod_{r=1}^{d_i}(c(k_i,M_i^r,\delta_i)\lambda^{k_i-M_r^i})\Big).\end{aligned}$$

Clearly, we have

$$\widehat{f}(\lambda,y)=\sum_{g\geq 2}u_g\lambda^{2g-2}+\sum_{\substack{g\geq 1,\, n\geq 0\\ n\geq 2g+2}} e(\overline{\mathcal{M}}_g^n)\lambda^{2g-2}\frac{y^n}{n!},$$

whence

$$f(\lambda,y)=\widehat{f}(\lambda,y)-\sum_{g\geq 2}(u_g-e(\overline{\mathcal{M}}_g))\lambda^{2g-2}.$$

Theorem 4.5 will be completely proved if we show that (i) $\widehat{f}(\lambda, y)$ is equal to the right-hand side of (31) and (ii) the following holds:

$$u_g = \sum_{\substack{h=1 \\ h|(g-1)}}^{g-1} e(\overline{\mathcal{M}}_{h+1}). \tag{59}$$

To prove (i) we argue as follows. Set

$$\widehat{Q}(k, b, N_1, \dots, N_b) = \sum_{\delta|k} \phi(\delta) \prod_{j=1}^{b} \sqrt{N_j} c(k, N_j, \delta)\lambda^k$$
$$\cdot \sum_{\substack{2h+s+b\geq 3 \\ h,s}} \chi(\mathcal{M}_h^{b+s})(k\lambda^k)^{2h-2}\Big(c(k, 1, \delta)\lambda^k y + T(k, \delta)\lambda^k + \sum_{\substack{1\leq m<k \\ m|k}} c(k, m, \delta)\lambda^{k-m}\Big)^s.$$

Then we have

$$\widehat{f}(\lambda, y) = \sum_{p\geq 1} \frac{1}{p!} \sum_{k_1,\dots,k_p} \sum_{\substack{b_1,\dots,b_p\geq 0 \\ b_1+\cdots+b_p \text{ even}}} \sum_{\underline{N},\pi} \prod_{i=1}^{p} \widehat{Q}(k_i, b_i, N_i^1, \dots, N_i^p). \tag{60}$$

If we now expand $\exp(\widehat{f})$, we get

$$\sum_{q\geq 0} \frac{1}{q!} \sum_{k_1,\dots,k_q\geq 0} \sum_{\substack{N_i^j|k_i \\ b_1,\dots,b_q\geq 0}} \sum_{\pi} \prod_{i=1}^{q} \widehat{Q}(k_i, b_i, N_i^1, \dots, N_i^{b_i}), \tag{61}$$

where π is a not necessarily connected pairing. For any positive integer r denote by N_r the number of those N_i^j's that equal r. Set also

$$\mathcal{N}_r = \begin{cases} (N_r - 1)!!, & N_r \text{ even}, \\ 0, & \text{else}. \end{cases}$$

Formula (61) can be written as

$$\sum_{q\geq 0} \frac{1}{q!} \sum_{k_1,\dots,k_q\geq 0} \sum_{\substack{N_i^j|k_i \\ b_1,\dots,b_q\geq 0}} \prod_{r\geq 1} \mathcal{N}_r \prod_{i=1}^{q} \widehat{Q}(k_i, b_i, N_i^1, \dots, N_i^{b_i}).$$

For any integers $k, b, N_1, \dots, N_b, s$ define $R(\lambda, y, x_{N_1}, \dots, x_{N_b})$ to be the polynomial

$$R(\lambda, y, x_{N_1}, \dots, x_{N_b}) = \sum_{\delta|k} \phi(\delta) \prod_{j=1}^{b} \sqrt{N_j} c(k, N_j, \delta) x_{N_j} \lambda^k$$
$$\cdot \sum_{\substack{2h+s+b\geq 3 \\ h,s}} \chi(\mathcal{M}_h^{b+s})(k\lambda^k)^{2h-2}\Big(c(k, 1, \delta)\lambda^k y + T(k, \delta)\lambda^k + \sum_{\substack{1\leq m<k \\ m|k}} c(k, m, \delta)\lambda^{k-m}\Big)^s.$$

By Wick's Lemma, we get

$$\exp(\widehat{f}) = \lim_{M\to\infty} \frac{1}{\sqrt{(2\pi)^M}} \int_{\mathbb{R}^M} \exp\Big(\sum_{\substack{b\geq 0\\ k\geq 1}} \sum_{N_i|k} R(\lambda, y, x_{N_1}, \ldots, x_{N_b}) \Big)\, d\mu_M,$$

where $d\mu_M$ is the Gaussian measure as in (32). It is an easy exercise (left to the reader) to check that

$$\sum_{\substack{b\geq 0\\ k\geq 1}} \sum_{N_i|k} R(\lambda, y, x_{N_1}, \ldots, x_{N_b}) = Q(\lambda, y, \underline{x})$$

where $Q(\lambda, y, \underline{x})$ is defined in (30). Hence (i) is proved.

As for (59), it suffices to show that the following identity of generating series holds:

$$-\sum_{g\geq 2} e(\overline{\mathcal{M}}_g) \log(1-\lambda^{2g-2}) = \sum_{g\geq 2} u_g \lambda^{2g-2}. \tag{62}$$

$e(\overline{\mathcal{M}}_g)$ is the sum of the Euler characteristics $e(\Delta(G))$, where $\Delta(G)$ are the strata in $\overline{\mathcal{M}}_g$. Analogously to Section 4.2.2, $e(\Delta(G))$ can be computed by taking into account connected coverings of Riemann surfaces of genus g.

Let us now expand the left-hand side of (62). Clearly, we get

$$\sum_{g\geq 2} e(\overline{\mathcal{M}}_g) \sum_{m\geq 1} \frac{(\lambda^{2g-2})^m}{m}. \tag{63}$$

From what we recalled on $e(\overline{\mathcal{M}}_g)$, (63) can be interpreted as a generating series for coverings with more than one connected component. This is exactly what the numbers u_g enumerate.

Table 2. Some values of $e(\overline{\mathcal{M}}_g^n)$

$g \backslash n$	0	1	2	3	4	5	6
2	6	13	42	181	1004	6883	56392
3	32	102	454	2612	18515	156094	1526677
4	200	882	5214	37945	327584	3272624	37151502

For genus $g = 0, 1$ our numbers coincide with the known values. For $g = 2$ our method showed an incongruence with the values in [2]. In what follows, we adopt the same notation as in that paper. By Proposition 3.15, p. 507 in [2], the contribution of *graphs of type 5* should be

$$\frac{1}{24(1-E)(1+D)^2} + \frac{11+2D-3D^2}{24(1-E)} + \frac{1}{2} + \frac{3D}{2} + \frac{7D^2}{4} + \frac{7D^3}{6} + \frac{11D^4}{24} + \frac{D^5}{8} + \frac{D^6}{48}$$

and not

$$\frac{1}{24(1-E)(1+D)^2}+\frac{11+2D-3D^2}{24(1-E)}+\frac{1}{2}+\frac{3D}{2}+\frac{7D^2}{4}+\frac{7D^3}{6}+\frac{11D^4}{24}-\frac{D^5}{8}+\frac{D^6}{48}.$$

As a consequence, the final generating function $K_2(t)$ in [2] should be modified by adding $D^5/4$. This yields the same values we get in the present paper for $g=2$.

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