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*Default risk: Poisson mixture and the business cycle*

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# Default risk: Poisson mixture and the business cycle

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## Abstract

As emphasized by the introduction of Basel II, the macroeconomic factors strongly affect credit risk variables. In order to account for the business cycle in a forward-looking way, a macroeconomic forecast can be introduced in the estimation of credit risk variables. This work proposes to model the distribution of the default rate as a mixture distribution which accounts for a binary representation of the business cycle: the distribution changes according to the estimated probability of recession over the credit horizon considered.

## 1 Introduction

The measurement and management of credit risk cannot avoid to deal with the state of the business cycle: it is in fact well recognized that credit risk components correlate with macroeconomic conditions (e.g. [12], [2], [10] among many others). If the dependence on the economy is accounted for in credit risk measurement and therefore it is included in the definition of the economic capital, it is likely that the latter fluctuates over business cycle.

Relatively to the introduction of Basel II, the phenomenon of procyclicality has been widely discussed: since a risk-sensitive capital requirement may be high in recessions and low in expansions, it may exacerbate the business cycle through the lending channel. As a consequence, the capital requirement based on measures of credit risk may be high in recessions and low in expansions, possibly exacerbating the business cycle through the lending channel. In order to avoid this dangerous side-effect, Basel II implicitly requires ratings being assigned in a "through the cycle logic" (see [3], par 415) and the probabilities of default (PDs) being estimated as long-run averages. This solution is somewhat in contrast with the purpose of making the capital requirements

risk-sensitive in that the time dimension of risk is neglected. This solution is also criticized (e.g. [8]) because it causes a loss of transparency.

It is instead important that credit risk measures reflect the economic conditions prevailing over the credit horizon considered: procyclicality is particularly exacerbated if credit risk measures reflect the current economic conditions, but the phenomenon can be dampen by introducing some economic forecast. Specifically, a risk measure which increase in anticipation of a recession over the credit horizon and viceversa decreases in anticipation of an expansion, should determine a reduction of the risky exposures before the recession and an increase in lending at the bottom of the cycle, helping the economy out of the recession.

This purpose could be achieved with a model including a business cycle forecast, such as the model proposed in [13]. In [13] business cycle dependent PDs are estimated and included in the computation of the capital requirement as defined in Basel II.

In this work the same modelisation of the business cycle dependence is introduced in a framework such as the one of the proprietary model CreditRisk+ (CreditSuisse). Section 2 recalls the modelisation of the PDs and the business cycle proposed in [13]. Section 3 introduces this modelisation in the CreditRisk+ framework and the modified model is applied to US data in Section 4 in order to provide an example. The last Section concludes.

## 2 Business cycle dependent PDs

In this section the modelisation of the PDs in relation to the business cycle presented in [13] is recalled since it will be applied to the CreditRisk+ framework in the next Section. The modelisation rests on the empirical evidence, presented in [2] for US data, that the transition matrix and especially the PDs reflect the expansion and recession regimes of the business cycle.

The business cycle state is modelled as a dicotomic variable  $X$ :

$$X = \begin{cases} R & P(R) \\ E & P(E) \end{cases} \quad (1)$$

where  $R$  and  $E$  represent the recession and expansion state respectively and  $P(R)$  and  $P(E) = 1 - P(R)$  the relative probabilities.

Therefore the default rate  $DR$  is assumed to follow a state-dependent distribution with probability density function:

$$f(DR|X) = \begin{cases} f_R(DR) & \text{if } X = R \\ f_E(DR) & \text{if } X = E \end{cases} \quad (2)$$

The business cycle states probabilities are quantified by means of a probit model using financial variables as predictors. At time  $t$  the probability of a recession over the period  $[t, t + k]$  is defined as:

$$P_t(R) = P(X_{t+k} = R) = \Phi(\beta Y_t) \quad (3)$$

where  $Y_t$  are the chosen explanatory variables and  $\Phi$  is the standard normal distribution function.

Therefore the unconditional distribution of the default rate at time  $t$  is a mixture of the two conditional distribution, weighted by the business cycle states probabilities. The probability function is:

$$f(DR_t) = P_t(R) \times f_R(DR) + P_t(E) \times f_E(DR) \quad (4)$$

### 3 Poisson distribution and the business cycle

When introducing the modelisation defined in Section 2 into the CreditRisk+ (CR+ in the following) framework, the generic conditional distributions of the default rate (or more precisely of the number of defaults) are modelled as Poisson distributions and the unconditional distribution is a mixture of two Poissons. It is worthwhile at this point to recall the main features of the CR+ model in order to present the modification to the original model proposed in this work.

#### 3.1 The CreditRisk+ framework

CR+ provides the probability distribution of the default losses on a portfolio: by grouping the obligors in the portfolio by exposure size and modelling the number of default within each group as a Poisson, a recurrence relation defining the portfolio loss distribution is obtained.

Specifically, at a first stage CR+ models the number of default in a group as a stochastic variable distributed as a Poisson with the parameter, representing the expected number of defaults, defined as the sum of the PDs of the obligors belonging to the group. Since CR+ treats each group separately and aggregates them when computing the portfolio loss distribution, in the following we will generically refer to the obligors without distinction between groups and portfolio. The probability function of the number of defaults  $D$  is:

$$P(D = n) = \frac{e^{-\mu} \mu^n}{n!} \quad (5)$$

where  $\mu = \sum PD_i$   $i = 1, 2, \dots, m$  and  $m$  is the number of obligors considered.

In this setting it is implicitly assumed that the obligors are independent: this hypothesis is inconsistent with the empirical evidence. The Poisson distribution is characterised by the parameter  $\mu$  that represents both the mean and the variance: empirically the variance of the default is higher than the

mean. To account for this empirical observation and to deal with the correlation among obligors, CR+ modifies the Poisson distribution to be a Gamma mixture of Poissons. The parameter  $\mu$  is no more fixed but it is a stochastic variable itself and it follows a Gamma distribution. Let's call  $X$  the stochastic variable representing the expected number of defaults<sup>1</sup> and  $\sigma$  its standard deviation. The variable  $X$  has a Gamma distribution<sup>2</sup>. By making the expected number of default stochastic the correlation among obligors is implicitly introduced. The obligors are independent conditionally on a particular value of  $X$ . Therefore the conditional distribution of  $D$  is still a Poisson with parameter  $X = x$ , but the unconditional distribution is a mixture:

$$P(D = n) = \int_0^{+\infty} P(D = n|X = x)g_X(x)dx = \int_0^{+\infty} \frac{e^{-x}x^n}{n!}g_X(x)dx \quad (6)$$

It is a standard result in statistics that the probability function of a Gamma mixture of Poisson distributions can be expressed in a closed form as a Negative Binomial probability function<sup>3</sup>.

The distribution deriving from the introduction of the stochastic variable  $X$  is more dispersed compared to the Poisson distribution: in particular, the left tail is fatter, accounting for more extreme events of losses. Figure 3.1 shows the comparison between the two distributions.

## 3.2 Business cycle regimes in the default distribution

In this Section we will show the model proposed to define the default distribution. The idea is of exploiting the CR+ structure and introduce a business cycle forecast into the credit risk measurement framework. According to the modelisation presented in Section 2, the number of default  $D$  is characterised by two conditional distributions:

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<sup>1</sup>We are neglecting the distinction by sectors which characterises CR+ in order to simplify the exposition and the notation.

<sup>2</sup>The Gamma distribution is defined as follows:

$$g_X(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} e^{-\frac{x}{\beta}} x^{\alpha-1}$$

where:

- $\Gamma(\alpha) = \int_0^{+\infty} e^{-x}x^{\alpha-1}dx$
- $\alpha = \frac{\mu^2}{\sigma^2}, \beta = \frac{\sigma^2}{\mu}$

<sup>3</sup>The Negative Binomial is defined as follows:

$$P(D = n) = \binom{n + \alpha - 1}{n} q^n (1 - q)^\alpha$$

with  $q = \frac{\beta}{1+\beta}$

$$\begin{cases} P(D = n|X = R) = \frac{e^{-\mu_R} \mu_R^n}{n!} \\ P(D = n|X = E) = \frac{e^{-\mu_E} \mu_E^n}{n!} \end{cases} \quad (7)$$

where  $\mu_R$  and  $\mu_E$  are the parameters of the two conditional distributions, representing the expected number of defaults in state of recession and expansion respectively. The unconditional distribution is a mixture of two Poisson distributions, and the mixing variable  $X$  represents the business cycle state:

$$P(D = n) = P(R) \frac{e^{-\mu_R} \mu_R^n}{n!} + P(E) \frac{e^{-\mu_E} \mu_E^n}{n!} \quad (8)$$

Let's call the distribution in 8 E-R Poisson mixture. Based on the distribution chosen for D, CR+ derives the loss distribution as a recurrence relation. We refer to [5] for the derivation of the loss distribution in both cases of fixed or stochastic parameter. A similar recurrence relation can be derived for the E-R Poisson mixture default distribution (see Appendix).

### 3.3 The E-R Poisson mixture compared to alternatives

The E-R mixture distribution is more dispersed compared to the simple Poisson. In the following a graphical and numerical comparison is presented, where the conditional PDs have been taken from a Standard&Poor's default data set used in [2]. We used the expansion and recession PD for rating B estimated as averages over the period 1981-1998 according to the NBER business cycle chronology. Figure 2 represents the default distribution when the recession probability is assumed to be equal to the proportion of recessions over the sample considered (i.e. about 0,1): in this case the two distributions are very close. If the recession probability were higher the difference in the left tail would be more pronounced, as it is evident in Figure 3. Beyond the static comparison, we can think about a situation where the default distribution is estimated based on the information available at a certain point in time. Suppose that default series are available from 1981 to 1998 and we want to estimate the default distribution for 1999. The data set allows to estimate conditional and unconditional PDs: the Poisson distribution is defined just by the unconditional PD, which depends on the proportion of recessions in the sample considered. On the other hand, the E-R Poisson mixture depends on the recession probability estimated for the credit horizon considered: higher losses are more likely when the recession probability is higher. The recession probability affects both the mean and the variance of the default distribution and the Value at Risk (VaR)<sup>4</sup>. Table 1 shows some numerical examples.

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<sup>4</sup>The concept of VaR refers to the loss distribution: however, if we simplify the portfolio to a unique type of obligors with unit exposure, the VaR of the

For any value of the recession probability:

$E(D) = P(E) \times \mu_E + P(R) \times \mu_R$ ;  $V(D) = E(D^2) - [E(D)]^2 = P(E) \times \mu_E + P(R) \times \mu_R + (\mu_R - \mu_E)^2 \times P(E) \times P(R)$ ; VaR is approximated by the entire number corresponding to the 95<sup>th</sup> percentile; relative VaR is the excess VaR over the expected number of defaults

While  $E(D)$  obviously increases with the recession probability, the variance  $V(D)$  is higher when is higher the uncertainty about the business cycle state prevailing over the credit horizon considered. The absolute VaR increases with the recession probability, due both to the shift of the distribution (i.e. increase in the mean) and to an increase in the dispersion. Compared to the Poisson distribution, the E-R Poisson mixture accounts for the higher capital needed to face losses during a recession.

We can now compare the E-R Poisson mixture and the Gamma mixture, as both account for a higher dispersion (variance) compared to the Poisson distribution with fixed parameters. The relative behaviour of these two distributions depends on the values used for the relevant parameters. Specifically, the shape of the E-R mixture depends on the estimated recession probability over the considered horizon as discussed above. The shape of the Gamma mixture depends on the volatility of the default rates: the higher the volatility the fatter is the left tail and eventually the higher is VaR. Figure 4 and 5 show the comparison between the two distributions for two couples of parameters. Figure 4 compares the E-R mixture with  $P(R)=0,1$  to the Gamma mixture with two different volatility values: while for low volatility the two distributions approximatively overlap, for high volatility the Gamma mixture left tail definitely dominates the E-R mixture. Figure 5 compares the Gamma mixture to the E-R mixture with different values of the recession probability: the higher is the estimated recession probability the more the left tail of the E-R mixture tends to dominate the Gamma mixture. In terms of VaR, the two distributions give different results depending on the recession probability and the volatility of the default rates: therefore, in order to draw sensible conclusions, the comparison of the two approaches needs to be developed on real portfolio and default data over several periods characterized by different economic conditions.

In particular, the E-R mixture approach can be appealing as long as the estimation of the recession probability is satisfactory. The advantage of this approach compared to the Gamma mixture approach lies in the feature that VaR increases in anticipation of recessions only; therefore it is extremely important the forecasting ability of model chosen to estimate the recession probability. Standard probit model with financial variables as predictors have quite satisfactory performances when applied to US business cycle data: [6] obtain good results by using the interest rates term spread and an equity index portfolio coincides with the VaR computed on the default distribution.

as explanatory variables. [7] review the literature on business cycle forecasting and argue that binary choice model are more accurate and stable in predicting the business cycle. [13] adopt a probit model to predict the US business cycle using the NBER chronology to define the dependent variable and selecting the interest rates term spread as the main predictor. While the results on US data are generally satisfactory, the same does not hold for other countries such as Italy: empirical works on Italian business cycle forecast are definitely more limited, but the results are generally characterized by scarce goodness of fit (see e.g. [11]). The problems lie in the choice of both the business cycle chronology and the predictors. As for the business cycle chronology, there is not a well accepted one as the NBER for US, and different chronologies lead to very different results both in terms of prediction and of representation of the default series. As for the predictors, for small economy international variables are generally useful in prediction.

## 4 Conclusions

Given the relevance of macroeconomic factors in credit risk and the importance of taking into account these factors in a forward-looking approach, this work presents a possible distribution to model the default event, which accounts for a prediction of the business cycle over the considered horizon. The distribution proposed is a mixture of two Poisson distribution conditioned to two possible states of the business cycle: expansion and recession. This is compared to the Gamma mixture adopted in the CreditRisk+ framework: the comparison depends on the parameters used and therefore the comparison is performed for different values of the default rates volatility and the recession probability. Since the appeal of the E-R mixture lies in the increases of VaR in anticipation of recessions and viceversa, the advantage of adopting this approach is strictly related to the performances of the model chosen to predict recessions. If a binary choice model is adopted, as in [7], it is extremely important to use a reliable chronology of the business cycle and explanatory variables with high predictive power.

This work compares two possible distribution for the default rates from a theoretical point of view and the numerical comparison performed has just exemplifying purpose. The next step would consist in applying the proposed E-R mixture distribution to real portfolio and business cycle data and compare it with the CreditRisk+ Gamma mixture distribution.



## Appendix: The loss distribution

Assuming that the obligors are independent conditionally on  $x$ , the probability generating function (pgf) for the E-R mixture distribution, based on the properties of the pgf (see [9]), is defined as in 9.

$$\begin{aligned}
 F(z) &= \int_x F(z|x)f(x)dx = F(z|E)P(E) + F(z|R)P(R) = \\
 &= P(E) \sum_{n=0}^{\infty} \frac{e^{-\mu(E)} \mu(E)^n}{n!} z^n + P(R) \sum_{n=0}^{\infty} \frac{e^{-\mu(R)} \mu(R)^n}{n!} z^n \\
 &= \sum_{n=0}^{\infty} \left[ P(E) \frac{e^{-\mu(E)} \mu(E)^n}{n!} + P(R) \frac{e^{-\mu(R)} \mu(R)^n}{n!} \right] z^n
 \end{aligned} \tag{9}$$

In order to account for the size of the portfolio losses, CreditRisk+ divides the portfolio adjusted exposures in bands of exposures of similar size: more precisely, after specifying a unit amount of exposure  $L$ , each exposure is divided by  $L$  and these obligor exposures expressed in terms of unit are rounded to the nearest integer. The portfolio divided in  $m$  bands can be represented by:

- band exposure  $v_j$ ,  $j = 1, \dots, m$ ;
- band expected number of defaults  $\mu_j$ ,  $j = 1, \dots, m$  (sum of the PDs of the obligors belonging to each band, or sum of the rating PDs multiplied by the number of obligors associated to each rating and belonging to each band)

The total expected number of default is the sum of the expected number of default for each band:  $\mu = \sum_{j=1, \dots, m} \mu_j$ .

Moving to the loss distribution, the bands have to be considered: each band is now characterised by a double parameter  $\mu_j(R)$  and  $\mu_j(E)$ .

The conditional pgf of losses is defined as in 10:

$$G(z|x) = \sum_{n=0}^{\infty} p(\text{loss} = nL|x)z^n = \exp\left(-\sum_j \mu_j(x) + \sum_j \mu_j(x)z^{v_j}\right) \tag{10}$$

Therefore the unconditional pgf is:

$$G(z) = P(E) \exp\left(-\sum_j \mu_j(E) + \sum_j \mu_j(E)z^{v_j}\right) + P(R) \exp\left(-\sum_j \mu_j(R) + \sum_j \mu_j(R)z^{v_j}\right) \tag{11}$$

Moving to the last step, the loss distribution is defined as in :

$$\begin{aligned}
A_n &= p(\text{loss} = nL) = \frac{1}{n!} \frac{d^n G(z)}{dz^n} \Big|_{z=0} = P(E) \frac{1}{n!} \frac{d^n G(z|E)}{dz^n} + P(R) \frac{1}{n!} \frac{d^n G(z|R)}{dz^n} \\
&= P(E) A_n(E) + P(R) A_n(R) = P(E) \sum_{j:v_j \leq n} \frac{\mu(E)_j v_j}{n} A_{n-v_j}(E) + P(R) \sum_{j:v_j \leq n} \frac{\mu(R)_j v_j}{n} A_{n-v_j}(R)
\end{aligned}$$

$$\text{with initialization } A_0 = P(E) \exp\left(-\sum \mu_j(E)\right) + P(R) \exp\left(-\sum \mu_j(R)\right). \tag{12}$$

## Tables and Figures

$P(R)$	$E(D)$	$V(D)$	$VaR\ 95\%$	$VaR\ 99\%$
0,1	4,33	6,24	10	12
0,2	4,79	8,19	11	14
0,5	6,18	11,49	13	15
0,8	7,56	10,96	14	16

Table 1: Default distributions for different recession probabilities

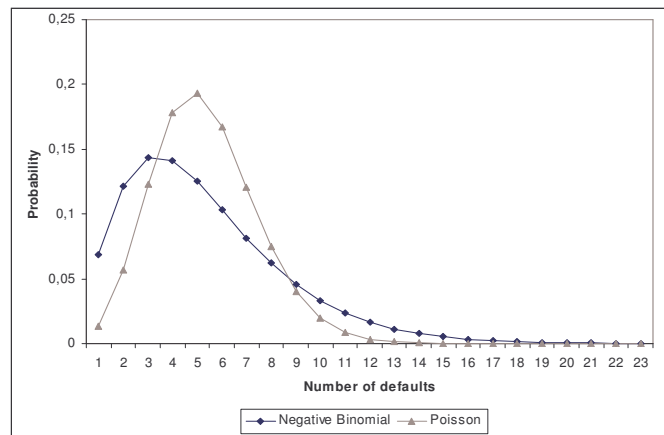


Figure 1: Comparison between Poisson and Negative Binomial with default probability equal to 4.3% and standard deviation equal to 2.5%.

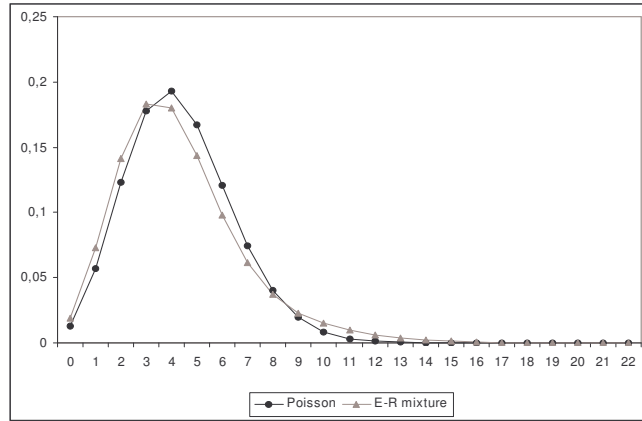


Figure 2: Comparison between Poisson distribution and E-R mixture distribution with  $P(R)=0,1$

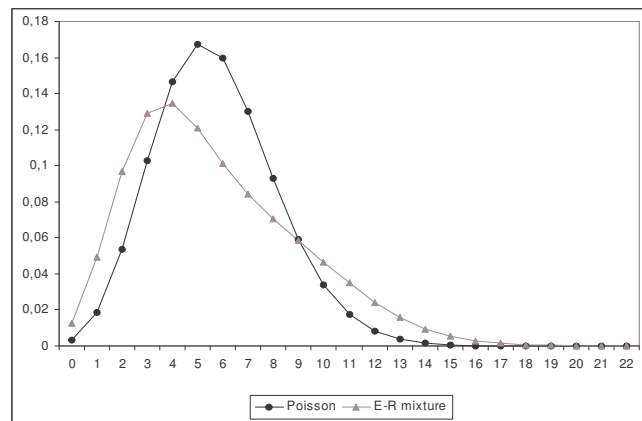
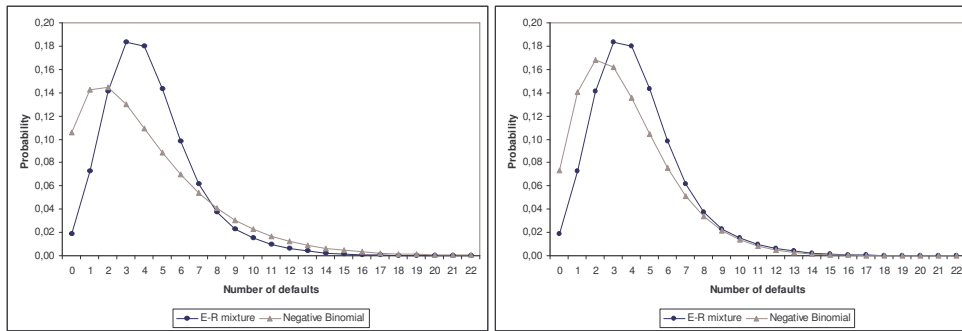


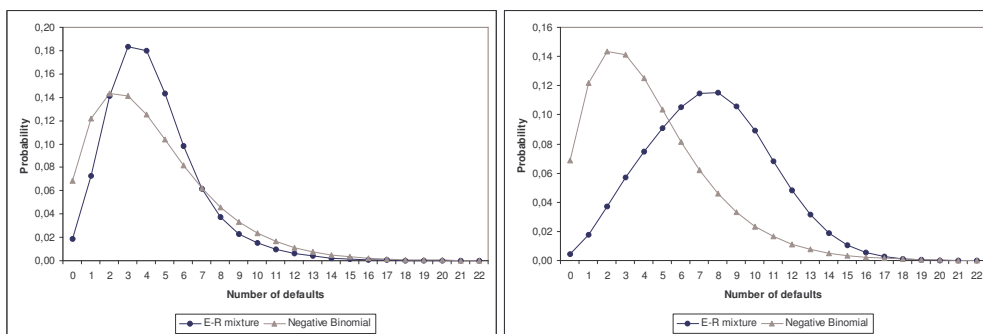
Figure 3: Comparison between the Poisson distribution and the E-R mixture distribution with  $P(R)=0,4$



$P(R)=0,1; \sigma=3\%$

$P(R)=0,1; \sigma=2\%$

Figure 4: Comparison between the E-R mixture and the Gamma mixture distribution for different volatility values



$P(R)=0,1; \sigma=2,5\%$

$P(R)=0,8; \sigma=2,5\%$

Figure 5: Comparison of the E-R mixture and the Gamma mixture distribution for different values of the recession probability

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