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Inequality, mobility and the financial accumulation process: A computational economic analysis*

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Abstract

Our computational economic analysis investigates the relationship between inequality, mobility and the financial accumulation process. Extending the baseline model by Levy et al., we characterise the economic process trough stylised return structures generating alternative evolutions of income and wealth through historical time. First we explore the limited heuristic contribution of one and two factors models comprising one single stock (capital wealth) and one single flow factor (labour) as pure drivers of income and wealth generation and allocation over time. Then we introduce heuristic modes of taxation in line with the baseline approach. Our computational economic analysis corroborates that the financial accumulation process featuring compound returns plays a significant role as source of inequality, while institutional configurations including taxation play a significant role in framing and shaping the aggregate economic process that evolves over socioeconomic space and time.

Keywords: inequality, economic process, compound interest, simple interest, taxation, minimal institution, computational economics, econophysics

JEL Codes: C46, C63, D31, E02, E21, E27, D63, H22

1 Introduction and Literature Review

Capital wealth accumulation is an evergreen matter of economic analysis and policy.[1] Economic analysts and policy advisors generally deal with notions of production, income and capital wealth that relate to some macroeconomic models, which foreshadow the aggregate income and wealth evolutions over time (Bertola et al. 2006; Blanchard 2011; Snowdon and Vane 2005). Since the fifties, the standard representation of growth reproduces a multiplicative process that is commonplace in representing individual financial investments, as if a single capital stock could be measured and reinvested for the aggregate economy over time (Perroux 1949; Stone 1986).

Income and wealth distributions have been addressed by public finance and financial macroeconomics (Bertola et al. 2006; Snowdon and Vane 2005). Recent advances in dynamic macroeconomic modelling based upon the representative agent hypothesis appear to have disregarded the aggregate dimension featured by collective and dynamic phenomena (Gallegati and Kirman 1999). Therefore, income and wealth distributions have been somewhat neglected, although they have gained socioeconomic moment in the aftermath of the

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[1] Hereafter, the term "capital wealth" combines concepts of capital and wealth to stress the productive nature of the wealth share that is considered by our economic analysis (which especially points to financial investments). Indeed durable assets held for consumption are excluded from our analysis.

Global Financial Crisis, including through the 99% movement in US (Haldane et al. 2014; Alvaredo et al. 2013). This movement has claimed that increased financialisation of economy and society has involved an increased appropriation of income and wealth by the so-called richest 1% of the population at the detriment of the remaining 99%, leading to more unequal and allegedly unfair distributions of income and wealth. This distributional issue has raised political attention under the Obama administration, as well as renewed theoretical interest through the influential positions taken by leading economists (Krugman 2013, 2014a,b; Stiglitz 2012; Solow 2014) as well as by policy-makers (Haldane 2014), and through the publication of the historical economic studies conducted by Thomas Piketty and Emmanuel Saez among others, reconstructing long-run statistical time series of income and wealth distributions in US and abroad (Atkinson et al. 2011; Piketty and Saez 2014).

According to Haldane et al. (2014), "as ever, dispute rages about the precise statistics. But the long-term patterns are clear enough - and remarkable. Almost half of the growth in US national income between 1975 and 2007 accrued to the top 1% (OECD 2014). In the UK and US, the top 1%'s share of the income pie has more than doubled since 1980 to around 15% and their share of the wealth pie has been estimated at up to a third - more than the whole bottom half of the population put together (ONS 2013; Wolff 2012). The five richest households in the UK have greater wealth than the bottom fifth of the population (Oxfam 2014)".[2]

The theoretical issue of income and wealth distributions is well-known since classic economic theorists in the XIX century at least, when the leading economist J.S. Mill (1861) considered "fair and reasonable that the general policy of the State should favour the diffusion rather than the concentration of wealth." At the beginning of the XX century, the leading economist and sociologist V. Pareto argued for the so-called Pareto (power-law) wealth distribution as an empirical regularity (Dagum 1990; Drăgulescu and Yakovenko 2001; Persky 1992; Kirman 1987), while the economic statistician C. Gini developed ingenious statistical measurement techniques to capture this inequality through the so-called Gini index (Gini 1912).

Recent advances in econophysics especially point to the functional forms of statistical distributions of income and wealth (Lux 2005). In particular, some students try to capture empirical regularities through modelling simple, elegant additive economic processes (Angle 2006; Richmond and Solomon 2001; Solomon and Richmond 2002); while other students purport to explain the fat tail of these distributions (that concerned with the high-range of aggregate income and wealth) through multiplicative economic processes leading to the emergence of power laws (Levy 2005; Levy and Levy 2003; Milakovic 2003). Some recent contributions suggest the form of a *deformed* exponential function that seems to capture well the empirical regularities of the income distribution at the low-middle range as well as its power law tail (Kaniadakis 2001, 2002). These modelling attempts have raised a lively debate with some economists worrying about allegedly poor socioeconomic understanding and lack of theoretical economic underpinnings (Gallegati et al. 2006; Lux 2005).

In this context, generalising Champernowne (1953), Levy (2005) and Levy and Levy 2003 (Levy et al. thereafter) have developed an elegant modelling strategy purporting to explain the power law tail of income and wealth distributions through financial market efficiency, and the stochastic distribution of financial returns across individuals active in this market.

In sum, theoretical and societal attention paid to the economic inequality issue raises the question of the source of this inequality. Whichever tentative response to this question has profound socioeconomic implications. It raises further theoretical and applied concerns that go beyond the functional form of statistical distributions of income and wealth across individuals. From this broader perspective, our contribution purports to address two featuring dimensions:

(i) the distributional dimension related to the inequality of income and wealth across individuals, and its evolution over time;

(ii) the significance of collective institutional mechanisms including taxation that actively frame and shape this economic process that generates and allocates income and wealth across individuals over time.

In particular, our modelling strategy consists in extending and improving on existing literature by elaborating the Levy et al. model to consider these three dimensions. Levy et al. provide a convenient baseline since their model subsumes the basic assumptions that characterise widespread economic modelling on these matters.

[2]See also CBO (2011).

Fernholz and Fernholz (2014) and Bertola et al. review and develop more sophisticated models that maintain similar background assumptions. In this context, Levy et al. model has the advantage to reduce the model structure to its minimal, synthetic and simple formulation. By elaborating on the Levy et al. model, our computational economic analysis will show the relevance of the financial accumulation process that features compound return investment over time. This peculiar accumulation process explains qualitatively both the increasing inequality across individuals, and decreasing social mobility empirically observed in modern societies.

The rest of the article is organised as follows. The second section introduces the financial accumulation process model in the Levy et al. model as baseline case. The third section shows the implications of this model for the evolution of inequality and social mobility in time and assess their sensitivity to changes in variance and to non-normal distributions of returns. The fourth section extends the baseline model by introducing decreasing returns and simple return structure. Especially this latter structure corroborates that, without financial accumulation, inequality would not be generated over time in the baseline scenario. The fifth section considers a second flow factor (labour income) along with the stock factor (capital wealth) introduced by Levy et al. The introduction of flow factor may involve an income-saving process that complements and integrates the financial accumulation process driven by inherited wealth. Overall, the analysis developed in the first five sections makes clear that distributional effects have been neglected by received literature, and that they depend on aggregate configurations. This preliminary conclusion paves the way to introducing minimal institutions (à la Shubik) that denote collective mechanisms related to income and wealth distributions. In particular, the sixth section introduces simple centralised modes of taxation, featuring a proportional taxation model (proportional taxation of periodic net income, uniformly redistributed through provision of universal public service), and a progressive taxation model (progressive taxation of periodic net income, redistributed in a regressive way through direct transfers). A summary of main results concludes.

2 A capital wealth model of aggregate economic process

Levy et al. develop a simple model of aggregate economic process based upon one stock factor (wealth) generating a pure compound rate of return $r_{i,t}$ stochastically distributed across individuals and time periods. This model appears to capture the Pareto law shape in the high-wealth (and high-income) range of the aggregate distribution, where "changes in wealth are mainly due to financial investment, and are, therefore, typically multiplicative" (Levy 2005, p. 105). This modelling strategy is based on a stochastic multiplicative process of wealth accumulation with lower bound on wealth and homogeneous accumulation talent. According to the authors, this framework implies that "the only reason for inequality is the stochastic process - chance. This implies that there is no differential ability in asset selection or in timing the [financial] market, which is in line with the efficient-market hypothesis. [...] Homogeneous accumulation talent means that all investors draw their returns randomly from the same distribution (the realized return, however, generally differs from one investor to another)" (Levy and Levy 2003, p. 709 and 711).[3] Let formalise the Levy et al. model of the financial economic process through the familiar structure of compound return where the wealth W_{t+1} of agent i at time $t+1$ is computed as:

$$W_{i,t+1} = (1 + r_{i,t})W_{i,t} \quad \text{for } r > -1 \tag{1}$$

or

$$W_{i,T} = W_{i,1} \prod_{t=1}^{T} (1 + r_{i,t}) \tag{2}$$

where each individual i draws his actual return $r_{i,t}$ at time t, T from the same statistical distribution defined as follows: $r_{i,t} \sim N(\mu_r, \sigma_r)$ with $\mu_r, \sigma_r > 0$. We take σ_r sufficiently large to enable the possibility of financial investment losses.

[3] In fact, Levy 2005 (chapter, p. 111, footnote 13) concedes that even joint accumulation processes with heterogeneous accumulation talents are asymptotically Paretian, with the faster-increasing multiplicative process dominating the high-range in the long run. Fernholz and Fernholz (2014) maintain that, in their model, "luck alone - in the form of high realised investment returns - [...] creates divergent levels of wealth."

In the degenerated case with r constant, the Eq. 2 becomes the classic formula of compound return over time:

$$W_T = W_1(1+r)^T \tag{3}$$

where for $-1 < r < 0 : W_t \to_t 0$ and for $r > 0 : W_t \to_t +\infty$.

This stylised model does not pretend to reproduce economic reality in its totality. In particular, it does not introduce consumption, overlapping generations, or windfall losses due to wars or accidents. However, it captures one featuring element of the aggregate economic process: financial accumulation opportunities. Compound return features financial investment dynamics and related institutions. Financial institutions such as investment funds and widespread measures of financial performance base upon compound return as reference logic. It seems then particularly significant to analyse its impact distinctively. Aggregate economic process is increasingly managed through corporate forms that live indefinitely and can then go on performing financial accumulation. On the one hand, financial investment is conducted by institutional investors which are driven by and assessed against compound return. On the other hand, eventual redistribution of their financial proceeds are often received by corporate recipients that go on reinvesting those proceeds over time, in a self-referential financial accumulation dynamics.

Throughout all our computational analysis, we assume an initial uniform distribution of wealth $W_{i,t=1} = 10 \; \forall \; i$ across all individuals at initial time $t = 1$. This implies that inequality effects depend entirely on the specifications of the economic process. For sake of simulation, when not mentioned otherwise, we define a population of $N = 5000$ individuals having an initial uniform wealth endowment $W_{t=1}$ of 10 wealth units each and each specific simulation is run for $t_{max} = 5000$ steps.

Our computational economic analysis disentangles two featuring dimensions to be analysed: wealth inequality across individuals; and social mobility relative to wealth dimension.

Wealth inequality is captured through the Gini index, defined as follows:

$$G_t = \left[(N+1) - 2\left(\frac{\sum_{k=1}^{N}(N+1-k)w_{k,t}}{\sum_{k=1}^{N} w_{k,t}}\right)\right]\frac{1}{N-1} \;\; \text{with} \;\; 0 \leq G_t \leq 1 \tag{4}$$

where N is the number of individuals and $w_{k,t} \leq w_{k+1,t}$ denotes the ranked vector of $W_{i,t}$ at time t. Accordingly, $G_t \to 0$ when individual wealth become more equal, while $G_t \to 1$ when richer individuals tend to acquire a larger share of aggregate wealth. In order to further corroborate the results obtained observing the Gini index we also study other measures of inequality such as the Theil index, the absolute and relative share of income of the top 1% of the population as well as the evolution of the proportion of wealth appropriated by different deciles of wealth. All these measures essentially confirms the results and are thus relegated to supplementary material.

Concerning wealth mobility, our Weighted Mobility Index M_t denotes the relative change in wealth position by agent i between two adjacent time periods $t-1$ and t. We consider the average of this index across individuals at each period t. Weighted Mobility Index M_t is computed as follows:

$$M_t = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{|Dec\,[W_j]_{i,t-1} - Dec\,[W_j]_{i,t}\,|}{Dec\,[W_{j=1}]_{i,t} - Dec\,[W_{j=10}]_{i,t}}\right] \tag{5}$$

where $Dec\,[W_j]_{j,t}$ represents the median wealth at time t for the decile j in which agent i was at time t, while $[W_{j=1}]_{i,t}$ and $[W_{j=10}]_{i,t}$ denote the median wealth for the first and the last decile respectively at time t. This index captures the relative movement accomplished by the individual i whenever he moves across deciles, relative to the maximum relative wealth distance between the first and the last decile. By taking its mean for each period across population N, we denote then the average wealth-weighted individual capacity to move across deciles period after period. Again, in order to corroborate the results obtained with this index, we introduce other measures of mobility. These additional measures confirm the deduction that can be drawned from M_t and are thus relegated to supplementary material.

3 The baseline case

The dynamics of wealth concentration across individuals over time is impressive under the baseline case developed by Levy et al. Wealth distributions become highly skewed under various compound return structures where actual individual returns by period are extracted from normal and gamma distributions. All these distributions show that the upper tail of wealth distribution goes on appropriating an increasing share of aggregate wealth over time. This dynamic effect has implications for wealth inequality (Figure 1). In particular, the Gini index shows that wealth inequality is magnified under the baseline case, asymptotically tending to its maximal value of one. In particular, drawing upon Fernholz and Fernholz (2014)' proof, we introduce the following Proposition 1 (see proof in our Appendix 7.1)

Proposition 1. *The Gini index based upon time-average wealth asymptotically tends almost surely to its maximum value of one for* $t \to \infty$

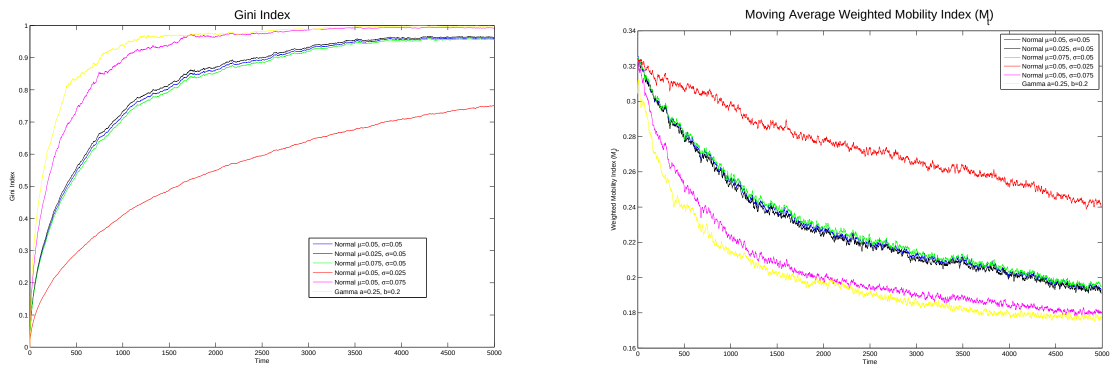

Figure 1: Left Panel: Gini Index (Equation 4) over time periods for wealth distribution of 5000 individuals under different return structures: $r_{i,t} \sim N(\mu_r, \sigma_r)$ and $gamma(a, b)$. The Gini Index asympotically tends to one (see Appendix 7.1). Right Panel: Moving average over ten periods of Weighted Mobility Index defined in Equation 5. The index is computed under various return structures : $r_{i,t} \sim N(\mu_r, \sigma_r)$ and $gamma(a, b)$.

Furthermore, the Gini index is increasing along with the variance of the underlying return structure (Figure 2). This concentration effect could be counterbalanced by social mobility relative to wealth, involving the actual capacity of individuals to move across the wealth distribution through historical time. Our wealth mobility index M_t assesses the capacity of individuals to change their wealth level relative to social wealth range in terms of relative position across deciles (M_t) over time. In the baseline scenario, this index shows that wealth mobility is rapidly decreasing over time (Figure 1) and in the variance of returns (Figure 2).

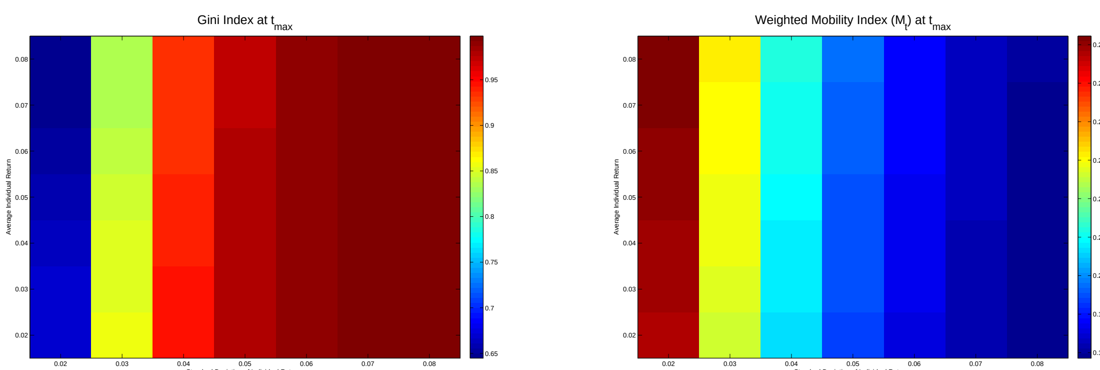

Figure 2: Left Panel: Gini Index G_t value at time $t_{max} = 5000$ under various return structures $r_{i,t} \sim N(\mu_r, \sigma_r)$. Right Panel: Moving average over ten periods of Weighted Mobility Index as defined in Equation 5. The index is computed under various return structures : $r_{i,t} \sim N(\mu_r, \sigma_r)$ and $gamma(a, b)$.

Our result on wealth mobility is further reinforced by visualising the mobility of the 1% richest individuals at different points of time (Figure 3). As time passes, richest individuals at given time tend to remain the among the richest. Indeed, while individuals that are among the richest at $t = 10, 100$ may still revert their lower position over time (being replaced by previously poorer individuals), individuals that are rich at $t = 1000, 2000$ tend to remain in the top decile of the wealth distribution. Finally, individuals that are rich at $t = 3000, 4000$ tend to remain in the centile of wealth distribution, perpetuating their social position relative to wealth over time.

Both results for wealth inequality and wealth mobility prove to depend especially on return variances (Figure 2). Coeteris paribus, the power law slope for the upper 10% tail, Gini Index, and the Wealth Mobility Index are increasing with return variance σ_r under normally distributed return structures(Figure 4, Left Panel). Interestingly, under our baseline case, increased wealth inequality does not depend on misalignement between mean individual financial returns and aggregate growth. As expected, average aggregate growth remains in line with average individual return in our baseline case, as Figure 4 (Right Panel) shows.

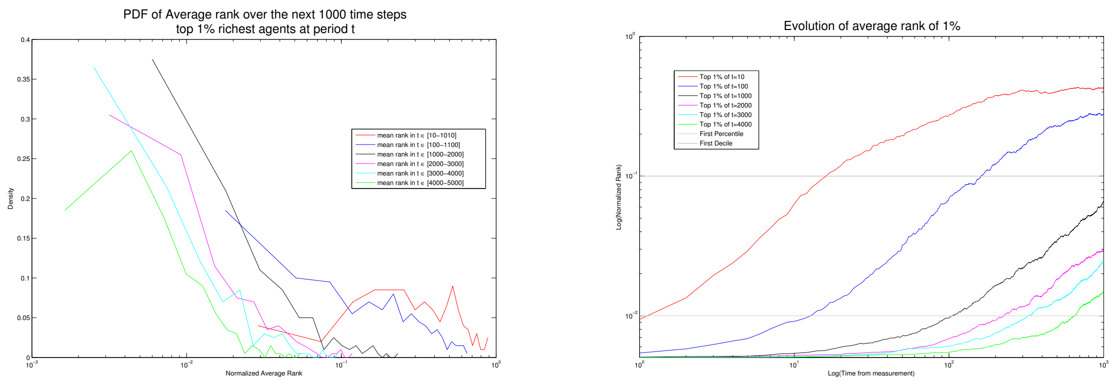

Figure 3: Left Panel: Distribution of average normalized ranking of agents belonging to the top 1% at certain periods of time. Right Panel: Evolution of average normalized ranking of agents belonging to the top 1% at certain periods of time as function of time from that period. For both panels $N = 20000$.

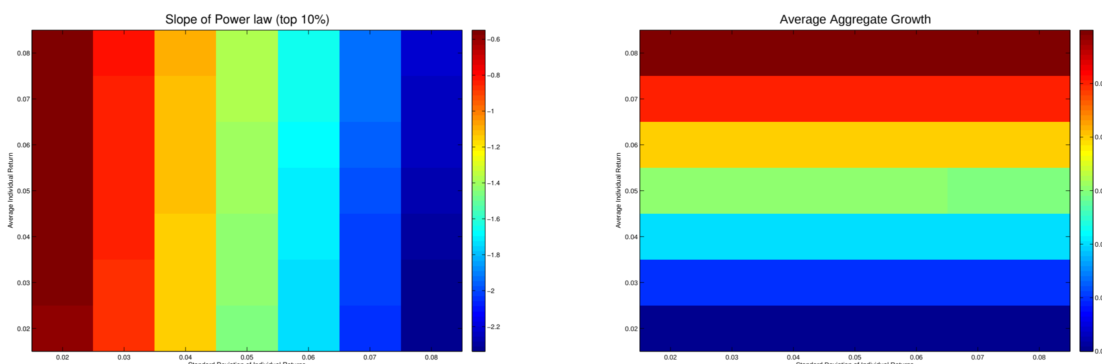

Figure 4: Left Panel: Power law slope α in $Log(W_{i,t}) = c + \alpha \cdot Log(Rank_{i,t})$ at $t = t_{max}$ with 5000 individuals under various return structures $r_{i,t} \sim N(\mu_r, \sigma_r)$. Estimation is performed over the upper 10% tail of wealth distribution under the baseline case. Right Panel: Relationship between average aggregate growth and mean individual return. Aggregate Growth is defined as in Equation 11 while mean individual return is μ_r and the standard deviation of individual returns is σ_r. By assumption, returns are then dispersed according to the normal distribution $r_{i,t} \sim N(\mu_r, \sigma_r)$, in the baseline case.

This sensibility to return variance in the baseline case may be magnified by financial market dynamics. Empirical evidence from the actual behaviour of financial markets shows that actual market price and return series are not normally distributed, featuring fat tails and extreme events (Biondi and Righi 2015). For sake of simulation, the normal distribution of returns is then replaced by a gamma distribution having the same mean as the baseline case but featuring extreme events, i.e., $\gamma(a, b)$ with $a = 0.25$ and $b = 0.2$ where $a \cdot b = \mu_r = 0.05$ and $a \cdot b^2 = \sigma_r = 0.01$. Computational results show that a gamma distribution of returns reinforces wealth concentration and inequality, while undermining wealth mobility over time. In particular, Gini index is always superior at each period of time (Figure 2, Left Panel), while the Wealth Mobility Index is always inferior (Figure 2, Right Panel). This result foreshadows that the non-normally distributed dynamics of financial market may have an inequality-enhancing impact, favouring skewed accumulation of wealth across individuals and time.

4 Decreasing compound returns and simple return structures: History matters

Levy et al. insist on the stochastic and multiplicative nature of their financial investment process. Our computational economic analysis further points to its cumulative nature over time, depending on the peculiar deployment of compound returns. Along with stochastic extraction of the actual return r for investor i at each time t, the financial investment process is further featured by the cumulative impact of the series of compound returns over accumulated wealth trough historical time (Figure 1). A quick glance at the deterministic reduction of the process model in Equation 2 shows that being richer at time t almost assures becoming richer at a further time $t + n$ with $n >> 0$. Coeteris paribus, this evolutionary structure tends to favour investors that become richer earlier in time, that is, investors that accumulate net gains before (and net losses after) the others, since every gain compounds positively, while every loss compounds negatively through historical time. This cumulative process is exacerbated by constant compound return to wealth - that is, by mean r being constant in time. Therefore, rather than 'being lucky', this cumulative process shows that 'history matters'. This financial accumulation process has important implications for the evolution of wealth through socioeconomic space and time. In a similar vein, Keynes (1933) would "trace the beginnings of British foreign investment to the treasure which Drake stole from Spain in 1580", reinvested at annual compound return of 3.25% over the next centuries to 1930, while remembering its connection to "avarice and usury and precaution that must be our gods for a little longer still [... to] lead us out of the tunnel of economic necessity into daylight". Hysteresis and path dependency play an important role in explaining inequality generated by the financial accumulation process. Wealth concentration is ever-increasing over time (Figure 1), while relative social mobility is undermined by increasing differences in total wealth[4], as showed by our wealth mobility index (Figures 2). In sum, idiosyncratic compound returns over investment over time prove to have cumulative effects, making the aggregate distribution of wealth not stationary. In particular, this distribution becomes increasingly right-skewed over time and tends to a limit in which wealth is concentrated entirely at the top (see also Fernholz and Fernholz 2014). This power-law tail of wealth distribution proves to depend especially on the second order of return distributions at a certain time period t (the variance σ_r in the case of returns that are normally distributed).

This section assesses the relationship of wealth inequality and mobility with the temporal evolution of returns through time. A first extension consists in exploring further economic processes characterised by decreasing returns to wealth. For simulation purpose, decreasing returns to aggregate wealth can be introduced by imposing an external constraint on all the returns $r_{i,t}$ as follows:

$$(1 + r_{i,t}) = \frac{1 + r_{i,t}}{\log(1 + TW_t)} \quad \forall t > 1 \tag{6}$$

where

$$TW_t = \sum_{i=1}^{N} W_{i,t} \tag{7}$$

Accordingly, all actual returns $r_{i,t}$ $\forall i, t$ decrease in proportion to aggregate wealth, which is increasing on average over time by assumption. Possible and actual net gains and losses are then progressively reduced over historical time. Under our baseline assumptions, they tend to zero in the long run, that is, $1 + r_{i,t} \to 1$ for $t \to +\infty$. Decreasing returns are relevant here to test the sensitivity of wealth inequality and mobility

[4] Levy and Levy (2003) p. 7 prove that "the actual wealth distribution converges to a Pareto distribution [...] with minimum wealth, average wealth, and variance that grow over time".

to time. In addition, positive constant returns do not appear to be a reasonable hypothesis asymptotically, since some constraints on multiplying wealth may appear, as well as limits in natural and human resources. Decreasing returns may further capture the case of absent or decaying technological development.

Computational results show that wealth inequality is materially reduced, while wealth mobility is improved. In particular, Gini Index is consistently lower across time, while Weighted Mobility Index, although decreasing, remains asymptotically superior over time, relative to the baseline case (Figure 5). Therefore, individuals are always less unequal and more able to move across wealth relative levels in this framework, relative to the baseline case. Decreasing returns over time progressively reduce the opportunity by individuals to gain or lose from their wealth investment and accumulation. Therefore, individuals do not have enough time to accumulate both in absolute and relative terms over time. Decreasing returns reshape both the first and the second order of return distribution, reducing both the total wealth and its dispersion across individuals over time. This result shows that wealth inequality and mobility depend on temporal evolution of returns.

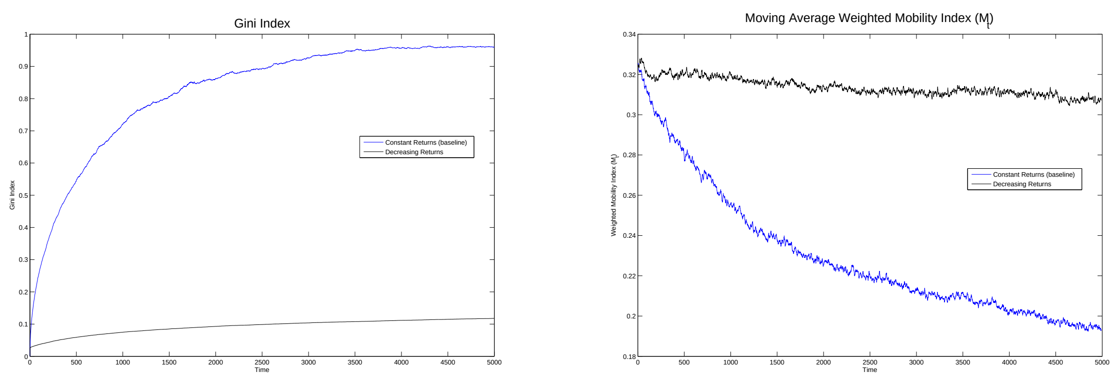

Figure 5: Left Panel: Comparison of Gini Index over time under constant and decreasing returns to aggregate wealth. Right Panel: Comparison of Weighted Mobility Index M_t over time under constant and decreasing returns to aggregate wealth.

This preliminary conclusion is corroborated by analysing simple return structure that enables exploring the distinctive impact of the cumulative dimension of the baseline economic process. Simple return constitutes a reference logic which is diametrically opposite to compound return. It heuristically corresponds to the 'financial capital maintenance' rule that is applied by corporate accounting systems. This rule computes the maintenance of invested shareholders' equity from one past period to the next one. The net difference is then reported as net earnings that can be distributed to shareholders (if positive), the undistributed part being reinvested as retained earnings. At the microeconomic level, simple return means that only net wealth is reinvested over time, while eventual net gains are consumed period after period. This heuristically corresponds to investing the same absolute amount in bonds and gilts repeatedly, but not reinvesting proceeds through time (Biondi 2011). At the macroeconomic level, simple return may correspond to a capital formation process that is stationary and constrained over reinvestment. Capital stock is then reproduced more than accumulated, its net contribution to total income being consummated over time. Under simple return structure, financial capital is remunerated as productive factor but it is not financially accumulated over time. It is then treated as a flow factor (and thus made analogous to labor in this respect) involving full consumption of all the net positive proceeds generated by capital wealth at each period of time. Formally:

$$W_{i,T} = W_{i,1} \sum_{t=1}^{T} (1 + r_{i,t}) \text{ with } r_{i,t} > -1 \;\forall\; i, t \tag{8}$$

For sake of simulation, we compute actual simple returns under two return structures: $N(\mu_r, \sigma_r)$ and $gamma(a, b)$. Computational results (Figure 6) show that wealth inequality does not accumulate under simple return structures, while wealth mobility remains stronger over time. Individuals go on adding their profit and loss to the initial invested capital, without reinvesting the proceeds. Asymptotically, those profits and losses compensate each other because they are generated by a stochastic process applied over the same initial amount. In particular, the Gini Index is decreasing and asymptotically near to zero, while the wealth mobility index remains asymptotically higher than the baseline case. In Appendix 7.2, we analytically prove the following proposition (as visualised by simulations in Figure 6):

Proposition 2. *The Gini Index tends to 0 for $t \to \infty$, under simple return structure, involving perfect equality among individuals.*

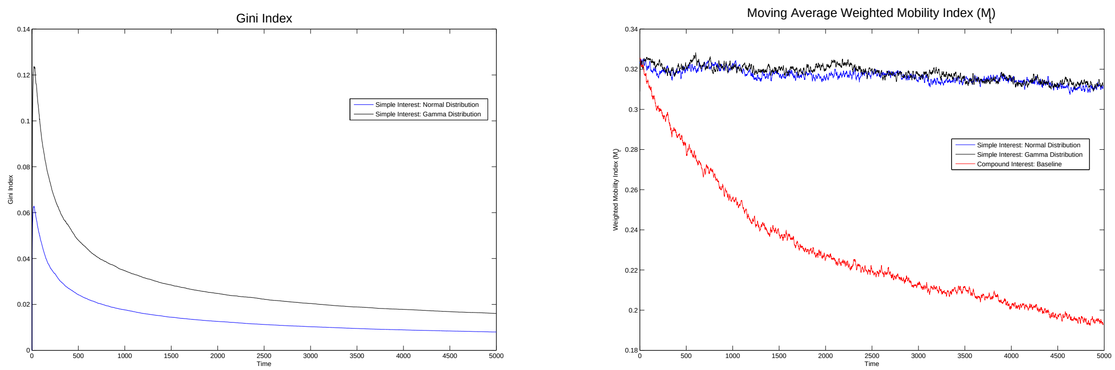

Figure 6: Left Panel: Gini Index over time under simple return structure under normal distribution $N(\mu_r, \sigma_r)$, and simple return structure under Gamma distribution $gamma(a, b)$. The Gini Index asympotically tends to zero (see Appendix 7.2). Right Panel: Moving mean of Weighted Mobility Index under compound return structure (baseline case), simple return structure under normal distribution $N(\mu_r, \sigma_r)$, and simple return structure under Gamma distribution $gamma(a, b)$.

Therefore, wealth inequality proves to be crucially dependent on the accumulation of returns through time, while this accumulation further undermines social mobility relative to wealth relative levels.

In sum, the economic process modelled by Levy et al. denotes a significant connection between inequality and the financial accumulation process. This accumulation through time proves to be conducive to increased wealth inequality and decreased wealth mobility over historical time. However, its assumption of constant average returns to aggregate wealth under compound return structure seems unsustainable, because indefinite compounding cannot realistically hold in the long-run (Voinov and Farley 2007; Biondi 2011). Moreover, IMF studies (Berg and Osrty 2013; Ostry et al. 2014) show that inequality affects growth, higher inequality being associated with lower growth. Its seems then unrealistic to assume a structurally stable economic process while the upper wealth tail goes on appropriating an ever-increasing part of total wealth, with the Gini index asymptotically reaching the maximum value of one in the long-run. However, our model clearly shows the logical and institutional tensions between the multiplicative logic embedded in compound return and this potential trade-off between inequality and growth. Let alone, such a multiplicative process may involve a self-fulfilling decrease in returns, reducing then social welfare.

In order to extend this baseline one-factor model, the next section shall introduce a two-factors model of the economic process, adding a flow factor featuring an additive evolution along with the stock factor characterised by a cumulative evolution through time.

5 A model of aggregate economic process combining capital wealth (stock) and labour (flow) factors

According to Oulton (1976), policy attitudes towards the inequality of wealth depend on views of its two paradigmatic causes. Wealth inequality can then be explained either "because income was unequally distributed and hence some people saved more, in consequence accumulating more wealth", or because wealth inheritance (accumulation) through time. Textbook macroeconomics introduces a stylised economic process that combines both economic factors: one stock factor (capital wealth) and another flow factor (labour). In this context Cobb-Douglas is a classic function of production for the aggregate economy, featuring factor returns to scale in its parameter space. Extending the baseline model by Levy et al., a second flow factor can be introduced as follows. Total income comprises the sum of wealth income and labour income as follows:

$$Y_{i,T}^{T} = Y_{i,T}^{W} + Y_{i,T}^{L} \tag{9}$$

Labour income can be consumed (for a share $1 - s_{i,t}$) or saved (for a share $s_{i,t}$) at every period t. Total wealth comprises then accumulated wealth at compound returns and cumulated saved income as follows:

$$W_{i,T}^{SUM} = W_{i,t=1} \prod_{t=1}^{T} (1 + r_{i,t}) + \sum_{i,T} (s_{i,t} \cdot Y_{i,t}^{L}) \tag{10}$$

with $-1 \leq r_{i,t}$ denoting pure wealth yield and $0 \leq s_{i,t} < 1$ denoting saving share.

In this context, total wealth growth can be defined as follows:

$$\text{Growth}_t = \frac{TW_t - TW_{t-1}}{TW_{t-1}} \tag{11}$$

where TW_t applies Equation 7 to $W_{i,T}^{SUM}$ defined in Equation 10. For sake of generality and comparability, we maintain a fair condition between capital wealth yield and labour income saving by granting such a uniform initial capital wealth that would yield a weighted mean permanent rent equal to the weighted mean saved income at each period of time t. The perpetual rent value is computed by the ratio between the period rent income and its period return rate. In our case, the wealth return $r_{i,t}$ defines the wealth yield $Y_{i,t}^{W} = r_{i,t} W_{i,t-1}$. The fair condition imposes that this wealth income is equal on average to the saved income of the period. The latter is defined as $Y_{i,t}^{L} = s_{i,t} Y_{t}^{L}$.

Accordingly, the fair condition imposes that:

$$W_{t=1} \equiv \frac{\sum_{i=1}^{N} (s_{i,t} \cdot Y_{i,t}^{L})}{\sum_{i=1}^{N} (r_{i,t})} \tag{12}$$

Under this condition, our parameter space does not introduce any bias between the two factor incomes of the period on average (Figure 7). Both factors contribute then equally to total income on average at the onset.

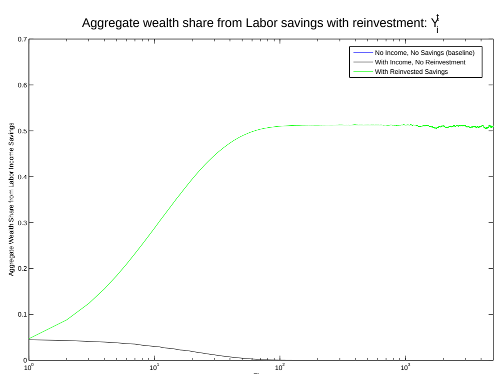

Figure 7: Time evolution of the share of wealth from income and savings.

On this basis, we introduce labour income and saving though the same logic that Levy et al. apply to capital wealth. We introduce a stochastic saving rate $s_{i,t} \in [0, 1]$ for which:

(i) all saving rates are equally likely, and

(ii) saving rate mean value is equal for all the individuals through time.

Under this assumption, taking $Y_{i,t}^L = 1$ and $s_{i,t} \sim U[0,1]\ \forall i,t$ and recalling that: $\frac{1}{N}\sum_{i=1}^{N}(s_{i,t}Y_{i,t}^L) = s_{mean} \equiv 0.5$ while $\frac{1}{N}\sum_{i=1}^{N}(r_{i,t}) = \mu_r \equiv 0.05$, the fair condition imposes $W_{t=1} \equiv \frac{0.5}{0.05} = 10$ consistently with our assumption for initial wealth $W_{i,t=1} = 10$ for all individuals.

Our computational analysis shows that the presence of a labor factor does not reshape the baseline economic process concerning wealth inequality and mobility. In particular, either if labour income cannot be saved (that is, $s_{i,t} = 0\ \forall i,t$), or if savings cannot be reinvested (as in the previous Equation 10), the labour factor additive process cannot match the multiplicative process of wealth accumulation. The latter continues then to dominate the aggregate economic process in the long run. Wealth inequality and mobility are not reshaped by the presence of that additive factor (Figure 8).

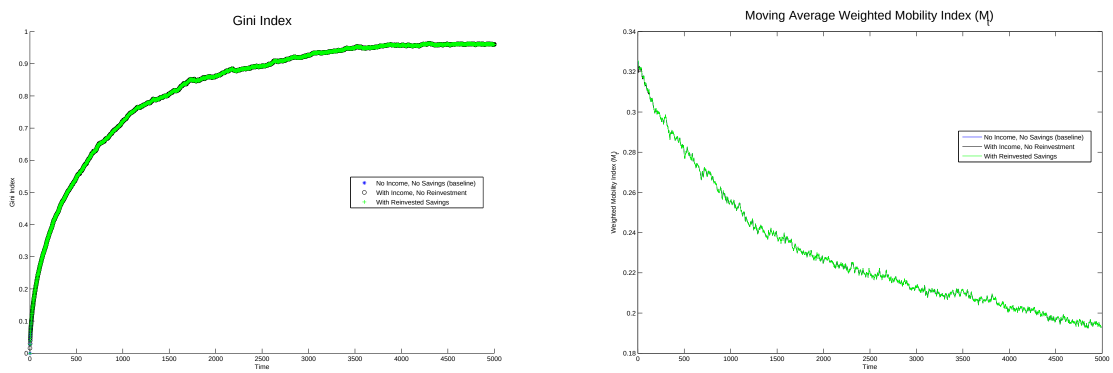

Figure 8: Left Panel: Gini Index value over time under baseline case (one-factor model), two-factors model without labor income savings, and two-factors model with reinvested savings over time Right Panel: Moving average of Weighted Mobility Index under baseline case (one-factor model), two-factors model without labor income savings, and two-factors model with reinvested savings over time.

According to our computational results, labour income alone cannot reshape wealth concentration, inequality and mobility dynamics through time. Nevertheless, some may wonder if reinvested savings from labour income could do it. Let introduce a progressive wealth accumulation driven by labour income savings. Formally:

$$W_{i,t+1} = W_{i,t}(1 + r_{i,t}) + (s_{i,t+1}Y_{i,t+1}^L) \tag{13}$$

In this case, the saved share of labour income Y_i^L is progressively accumulated at compound return along with inherited wealth across individuals and time. Total individual wealth through time includes both saved wealth and inherited wealth. Formally:

$$W_{i,t+1}^{SUM} = W_1 \prod_{h=1}^{t}(1 + r_{i,h}) + \sum_{j=1}^{t+1} s_{i,j}Y_{i,j}^L \prod_{k=j}^{t}(1 + r_{i,k}) \tag{14}$$

If labor income remains constant over time, i.e. $Y_{i,t}^L = Y_0\ \forall i,t$, this formula reduces to:

$$W_{i,t+1}^{SUM} = W_1 \prod_{h=1}^{t}(1 + r_{i,h}) + Y_0 \sum_{j=1}^{t+1} s_{i,j} \prod_{k=j}^{t}(1 + r_{i,k}) \tag{15}$$

In formula 15, we maintain that $r_{i,h}$ (returns over inherited wealth) have the same structure as $r_{i,k}$ (returns over accumulated savings). For sake of simulation, we continue to assume $Y_0 = 1\ \forall i,t$.

According to our computational results (Figure 8), even this progressive reinvestment of savings cannot reshape wealth inequality and mobility under the fair condition stated above. Wealth distributions and all the indexes maintain the same behaviour than under the baseline case. [5]

[5] The same results hold when actual returns are derived from simple return structure. Computational results are available under request.

Although each individual draws savings from labour income according to a uniform distribution, these savings are reinvested according to the same financial accumulation process as the inherited wealth that has an initial uniform distribution. Therefore, savings from labour income do not introduce an alternative reference logic or a complementary economic process. They cannot then reshape wealth distribution, inequality and mobility over time.

In sum, under both the Levy et al. model and the widespread two-factors model of the aggregate economic process, wealth concentration, inequality and mobility depend crucially on the compound return structure that characterises the accumulation of financial investment over time. The decomposition of wealth dynamics in his factors of productions does not drive its evolution.

In fact, our computational results further show emerging aggregate behaviour that prove distant from economic reality, echoing the (Knight, 1938, p. 81)'s claim that:

> The entire notion of 'factor of production' is an incubus on economic analysis, and should be eliminated from economic discussion as summarily as possible.

(Solow, 1976, p. 138) acknowledged the aggregation problem as follows:

> I have to insist again that anyone who reads my 1955 article [Solow (1955)] will see that I invoke the formal conditions for rigorous aggregation not in the hope that they would be applicable [. . .] but rather to suggest the hopelessness of any formal justification of an aggregate production function in capital and labor.

Nevertheless, our computational analysis shed some light on theoretical and applied implications that are implicitly assumed in widespread representations and models of the aggregate economic process, as reviewed and further developed by Fernholz and Fernholz (2014) and Bertola et al. (2006) among others. From this perspective, the following section shall expand upon our preliminary conclusions about inequality and the financial accumulation process. We shall introduce a stylised institutional configuration that typically frames and shapes income and wealth dynamics in economy and society: taxation.

6 The impact of taxation and redistribution

Taxation is a classic matter related to income and wealth distributions, including because available statistics extensively rely upon fiscal data for gathering evidence (Saez and Zucman 2014; Topritzhofer et al. 1970). It is then interesting to explore its effect on wealth concentration, inequality and mobility by expanding the baseline model by Levy et al. through featured modes of taxation. For simulation purpose, we introduce four stylised modes of taxation, all governed by a central authority that knows and intervenes over the individual positions of wealth and income (change in wealth) at the end of each period. These modes of taxation combine two methods of tax levy and two methods of tax distribution:

- Concerning tax levy, we assume either a uniform proportional tax rate for all individuals (uniform taxation), or a progressive tax rate increasing with the tax basis (progressive taxation).
- Concerning tax distribution, we assume either a uniform redistribution to all individuals (featuring the provision of universal public service), or a regressive redistribution that decreases with the tax basis (featuring provision of direct transfers for welfare through the polity).

Under proportional taxation and public service model (**Proportional & PS**), the tax authority levies a fixed universal share $\tau_{i,t} = \tau \; \forall \; i, t$ on positive net changes in wealth $\mathcal{T}_{i,t} = \max\{W_{i,t} - W_{i,t-1}; 0\}$, and redistributes the total levied amount equally among all the individuals. This model features the provision of public service to the polity through proportional taxation as follows:

$$Tax_{i,t} = \mathcal{T}_{i,t}\tau; \tag{16}$$

$$Subsidy_{i,t} = \frac{\sum_k Tax_{k,t}}{N} \tag{17}$$

Under proportional taxation and welfare model (**Proportional & Welfare**), the tax authority employs the proportional tax levy to provide direct transfers through the polity. These transfers are redistributed in regressing proportion to individual wealth. Individual tax is computed according to Equation 16, while the redistribution is managed according to the following formula:

$$Subsidy_{i,t} = \left[1 - \frac{\mathcal{T}_{i,t}}{\sum_{k=1}^{N} \mathcal{T}_{k,t}}\right] \frac{1}{N-1} \sum_{k} Tax_{k,t} \tag{18}$$

Under progressive taxation and public service model (**Progressive & PS**), the tax authority levies a progressive share $\mathcal{T}_{i,t}$ of net changes in wealth. Accordingly, the richest individual applies the maximum rate $\tau_{max} = \tau_{i,t}(\max \mathcal{T})\ \forall t$, while the poorest individual does not pay anything. On this basis, the tax authority provides a universal public service to all the individuals. Individual tax payment is defined as follows:

$$Tax_{i,t} = \mathcal{T}_{i,t} \cdot \tau_{max} \frac{\mathcal{T}_{i,t} - \min \mathcal{T}_{i,t}}{\max \mathcal{T}_{i,t} - \min \mathcal{T}_{i,t}} \tag{19}$$

On this basis, the individual i having the maximal tax basis $\mathcal{T}_{i,t}$ applies the maximal tax rate τ_{max}, while the individual i having the minimal tax basis $\mathcal{T}_{i,t}$ does not pay taxes (i.e., its tax rate $\tau_{i,t} = 0$ at time period t). The redistribution mechanism under this model follows Equation 17.

Under progressive taxation and welfare model (**Progressive & Welfare**), the tax authority employs the tax levy in Equation 19 to redistribute the total levied amount in a regressive proportion to wealth, according to Equation 18. This model features the provision of direct transfers thorough the polity, funded by progressive tax levy.

In sum, our computation analysis features four stylised modes of taxation, summarized in Table 1.

Tax redistribution **Tax Levy**	**Proportional** **Redistribution Rate**	**Regressive** **Redistribution Rate**
Proportional Tax rate	Proportional taxation & public service	Proportional taxation & welfare
Progressive Tax Rate	Progressive taxation & public service	Progressive taxation & welfare

Table 1: Stylised modes of taxation and redistribution

We assess the impact of taxation over wealth inequality and mobility under these stylised modes of taxation. For sake of simulation, we retain a tax rate $\tau = 0.05$ for the proportional tax system and a maximum tax rate of $\tau_{max} = 0.10$ for the progressive taxation. Under progressive tax regimes, this framework implies a tax rates structure that endogenously depends on the distributions of income (net wealth change) and wealth and their evolution over time. Computational results (Figure 9, 10) show that all modes of taxation and redistribution are effective in reducing and stabilising wealth inequality (Figure 9, Left Panel), while improving and stabilising wealth mobility (Figure 9, Right Panel). Contrary to savings from labour income, taxation effectively introduces an alternative economic logic and a complementary economic process in our miniature economy. This alternative and complementary collective action proves to be effective in compensating the impact of financial accumulation over wealth inequality and mobility.

In particular, Gini index is consistently and materially inferior to the baseline case, while remaining asymptotically far from one. Wealth mobility indexes is consistently and materially superior to the baseline case featured by the Levy et al. model.

The actual relative impact on wealth inequality and mobility across modes of taxation depends on the parameter space assumptions. Computational results (Figure 10) for mean tax and redistribution rates explain why progressive taxation is less effective than proportional taxation in reshaping wealth inequality and mobility, under the retained framework of analysis.As mentioned above, individual tax rates under progressive tax regimes depend on the distribution of wealth. Since the latter is increasingly right-skewed over time, this dependency involves a mean tax rate that progressively becomes and remains very low over time (materially inferior to the mean tax rate of 0.05 applied under proportional tax regimes). The impact of progressive tax regimes over wealth inequality is then materially reduced both in absolute terms, and relative to proportional tax regimes that apply an exogenous fixed tax rate in our framework. Moreover, since wealth distribution is increasingly and materially right-skewed over time, a relatively low tax rate is also sufficient to asymptotically stabilise the Gini Index (Figure 9, Left Panel). Wealth is so concentrated on

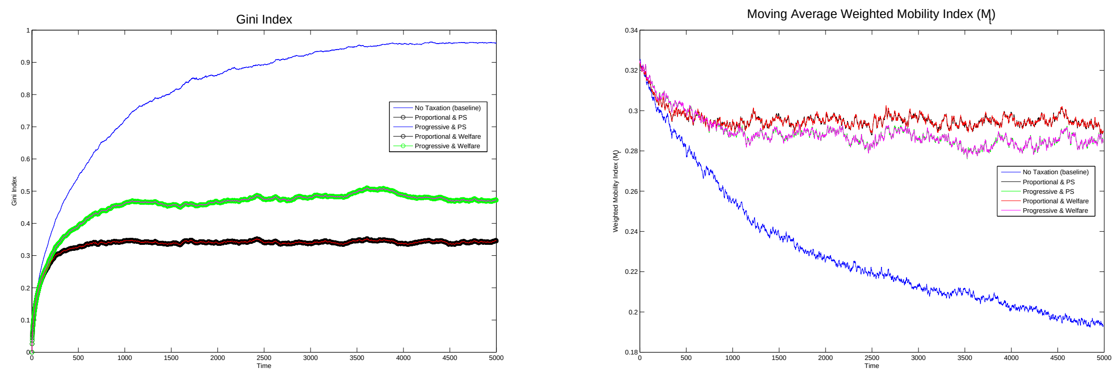

Figure 9: Left Panel: Gini Index over time under baseline case (no taxation), proportional taxation and public service model (**Proportional & PS**), proportional taxation and welfare model (**Proportional & Welfare**), progressive taxation and public service model (**Progressive & PS**) and progressive taxation and welfare model (**Progressive & Welfare**). Right Panel: Weighted Mobility Index over time under the same cases.

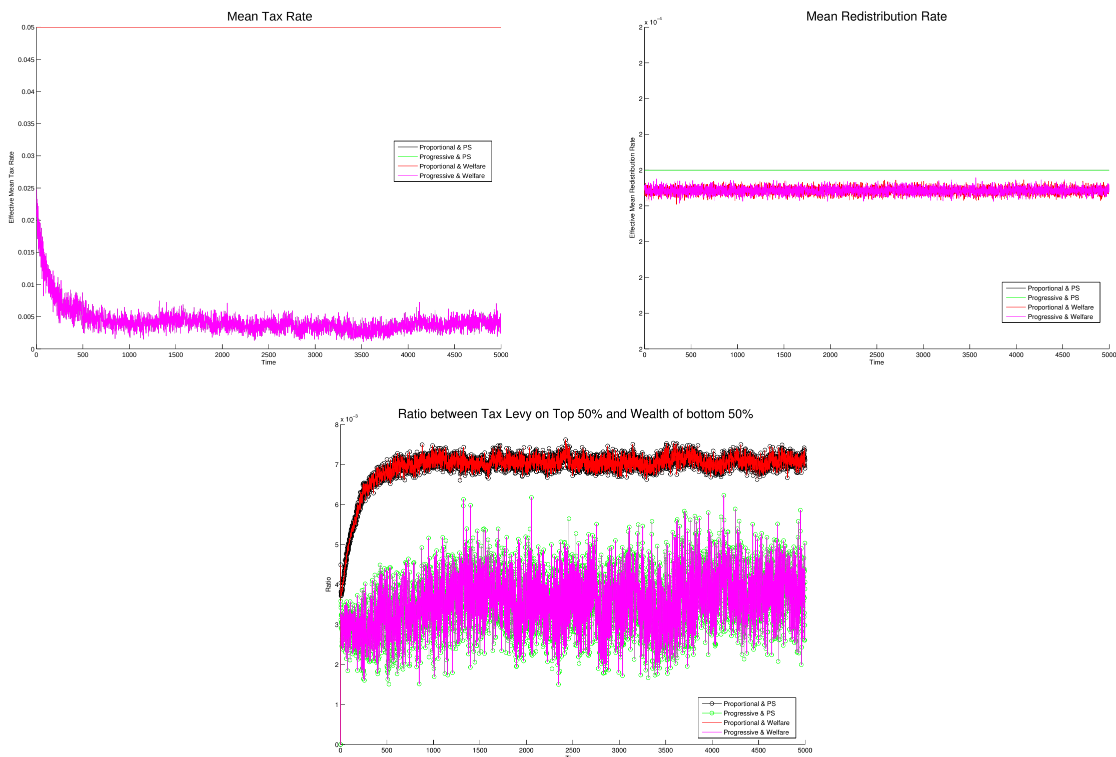

Figure 10: Left Panel: Mean Tax Rate over time under baseline case (no taxation), proportional taxation and public service model (**Proportional & PS**), proportional taxation and welfare model (**Proportional & Welfare**), progressive taxation and public service model (**Progressive & PS**) and progressive taxation and welfare model (**Progressive & Welfare**). Righi Panel: Mean Redistribution rate over time under the same cases. Bottom Panel: Ratio between the total tax levied on the richest 50% of the population, relative to the total wealth of the bottom 50% of the population.

the top (according to our Proposition 1) that a relatively low tax extraction from the richer is sufficient to materially increase wealth of the poorer, involving a stabilising effect of wealth inequality over time (Figure 10, Bottom Panel). A policy implication if this result is that effectiveness of fiscal systems depend on the underlying economic structure and process.

In conclusion, taxation materially reduces wealth concentration and inequality, compensating the financial accumulation process. Taxation proves then to be effective in counterbalancing the inequality effects of the financial accumulation process. This result is consistent with Fernholz and Fernholz (2014) arguing that "the presence of redistributive mechanisms then ensures the stability of the distribution of wealth over time".

7 Concluding remarks

The poet Trilussa mocked national statistics to be that national accounting method for which, one individual having eaten two chickens and another one just none, both would result to have eaten one chicken each.[6] Students of income and wealth distributions may keep this adage in mind while developing related macroeconomic modelling, especially when the representative agent applies.

Our computational economic analysis shows the significant connection between inequality and the financial accumulation process in the study of income and wealth distributions. This connection has been investigated through progressive extensions of the baseline model introduced by Levy et al. Our analysis shows the limited heuristic contribution of a two factors model comprising one single stock (capital wealth) and one single flow factor (labour) as pure drivers of aggregate income and wealth generation and allocation over time. We further show the theoretical contribution of minimal institutions (à la Shubik), to partly overcome this limitation. In particular, we investigate heuristic models of taxation in line with the baseline approach. Drawing upon our computational economic analysis, we can infer that the financial accumulation process plays a significant role as socioeconomic source of inequality, while institutional configurations including taxation play another significant role in framing and shaping the aggregate economic process that evolves over socioeconomic space and time. Our computational economic analysis bases upon a simple modelling strategy along with a calibration that is suitable for comparing alternative model configurations, not necessarily to fit empirical regularities. Therefore, we cannot infer empirical or forecasting predictions, but theory-driven implications that deserve further consideration from theoretical and applied viewpoints. Wealth inequality and wealth mobility are important socio-economic dimensions of our economy and society. Increased wealth inequality may raise fairness issues, undermining economic sustainability and development through historical time. Decreased wealth mobility may raise further fairness issues, undermining socio-economic incentives to entrepreneurship and workmanship. Concerning wealth inequality and mobility issues, our computational economic analysis points to featuring drivers that deserve further attention by analysts and policy-makers. First of all, the financial accumulation process appears to be the key driver of both issues, generated by the peculiar compound return structure that characterises financial investment in current institutional configurations. Its contribution to wealth inequality and mobility further appears to fundamentally depend on the financial market dynamics featuring volatility clustering and extreme events. Labour income and savings do not appear to be able to rebalance the impact of this financial accumulation through historical time. However, taxation appears to be effective in compensating its effect. Finally, according to our computational economic analysis, the causes of recent increases in wealth inequality may be sought in socioeconomic transformations of financial market dynamics and taxation (including fiscal niches exploitation and tax elusion) over the recent decades. From our theoretical perspective, return structure, volatility and exuberance in financial markets, as well as the working of fiscal systems are candidates to drive wealth inequality and wealth mobility in our economy and society.

Appendix

7.1 Gini evolution for Compound Interest Structure

Lemma 3. *For compound interest structure,* $G_t \to 1$ *as* $t \to +\infty$.

[6] "Me spiego: da li conti che se fanno \ seconno le statistiche d'adesso \ risurta che te tocca un pollo all'anno: \ e, se nun entra nelle spese tue, \ t'entra ne la statistica lo stesso \ perchèèè c'è un antro che ne magna due." (Trilussa, La statistica)

Proof. Our proof draws upon Fernholz and Fernholz (2014). Our model for compound return structure replicates the background structure of the Fernholz and Fernholz (2014) model, as represented by their Equation 10. Accordingly, with our notation:

$$W_{i,t} = W_{i,t=1} \cdot e^{r(t)} \tag{20}$$

This equation denotes continuously compound return structure over time, with return function $r(t)$ depending on a standard Brownian motion. In this context, Fernholz and Fernholz (2014)'s Theorem 2 proves that, if $\sigma_r > 0$, the time-averaged share of total wealth held by the wealthiest single household converges to one, almost surely (their Equation 13), although it is not the same household to maintain the leading position over time. Analytically:

$$\lim_{t\to+\infty} \int_0^t \theta_{i,t}(t)dt = 1 \text{ a.s.}$$

with $\theta_{i,t} = \frac{W_{i,t}}{\sum_t W_{i,t}}$ and where $\theta_{max,t} = \max_i[\theta_{i,t}]$.

Our model applies a discretely compound return structure as follows:

$$W_{i,t} = W_{i,t=1} \prod (1 + r_{i,t})$$

Without loss of generality, this structure can be made continuous by making the time change infinitesimal as follows:

$$W_{i,t} \to W_{i,t-1} \cdot e^{R(t)} \text{ with } dt \to 0 \tag{21}$$

$$R(t) = \ln(1 + r_{i,t}) \tag{22}$$

Where the $R(t)$ function transforms our return $r_{i,t}$ from discrete to continuous time. This formulation is analogous to Fernholz and Fernholz (2014)' formula, since $r_{i,t} \sim N(\mu_r; \sigma_r)$ by construction. Their proof applies then to it. ■

7.2 Gini evolution for Simple Interest Structure

Lemma 4. *For simple interest structure, $G_t \to 0$ as $t \to +\infty$.*

Proof. The Gini Index tend to 0 if the wealth of all individuals tends to be equal for $t \to +\infty$. By construction, $\forall t$

$$W_{i,t} = \sum_t [W_{i,1}(1 + r_{i,t})]$$

The equality condition imposes that $W_{i,t} = W_{j,t}$, thus:

$$\sum_t [W_{i,1}(1 + r_{i,t})] = \sum_t [W_{j,1}(1 + r_{j,t})]$$

or

$$\sum_t [W_{i,1}(1 + r_{i,t})] - \sum_t [W_{j,1}(1 + r_{j,t})] = 0 \tag{23}$$

Given that $W_{i,1} = W_j, 1 = W_1$, for each i, j by assumption (i.e., uniform initial wealth distribution), Equation 23 becomes:

$$W_1 t + W_1 \sum_t (W_1 r_{i,t}) - W_1 t - W_1 \sum_t (W_1 r_{j,t}) = 0 \tag{24}$$

or simplyfing:

$$W_1[\sum_t (r_{i,t}) - \sum_t (r_{j,t})] = 0$$

since $r \sim N(\mu_r, \sigma_r)$ by construction and since $\sum r_{h,t} \to t \cdot \mu_r$ if $t \to +\infty$ $\forall h = i, j$, then:

$$W_1 \left[\sum_t (r_{i,t}) - \sum_t (r_{j,t}) \right] \to 0 \text{ for } t \to +\infty$$

Q.D.E.

Remark 5. *The same result can be relaxed and hold if r has a distribution stably converging to its mean μ_r for $t \to +\infty$*

Remark 6. *If we introduce heterogeneous initial distribution of wealth (that is, $W_{i,1} \neq W_{j,1}$ for some i, j), it can be proved that the simple return dynamics tends to be neutral on the initial ranking for $t \to +\infty$, that is, the initial ranking is maintained in the long-run under the simple return structure.*

■

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Inequality, mobility and the financial accumulation process: A computational economic analysis Supplementary Material

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In order to further corroborate the results shown in the main text, we present here additional results for different indicators of wealth inequality and distribution as well as of social mobility.

Wealth Inequality. Concerning the wealth inequality, an alternative to the Gini index is the Theil Index (Theil 1967), which is defined as follows:

$$Th_t = \frac{\sum_{i=1}^{N} \frac{W_{i,t}}{\hat{W}_t} \log\left(\frac{W_i}{\hat{W}_t}\right)}{N\log(N)} \quad \text{with} \quad 0 \leq T_t \leq 1 \tag{1}$$

where $\hat{W}_t$ is the average wealth of the population of agents at that time step t. Results concerning the Theil Index and relative to the cases studied in the main text are shown in Figures 1, 2 and 3.

Our result on wealth inequality is further reinforced by visualising the share of wealth appropriated by the upper 1% of population, as well as the ratio of it over the share appropriated by the lower 50% (Figures 9, 10, 11, 12 and 13).

Finally we also reproduce in a Log-Log graph the wealth of individuals as function of their position in the population' wealth ranking (Figures 4, 5, 6, 7 and 8, Bottom Panels.) with the aim of reproducing the results of Levy and Levy (2003).

Social Mobility. Concerning the social mobility in the main text we introduced the Weighted Mobility index. To corroborate the results of this index, we here propose two alternative indexes (displayed in Figures 14, 15,16, 17, 18 and 19):

- The Mean Wealth Change Index V_t denotes the average of the relative change in wealth of each agent i between two adjacent time periods t and $t-1$, weighted by the aggregate range of wealth at time t (i.e., the difference between maximum and minimum wealth), as follows:

$$V_t = \frac{1}{N}\sum_{i=1}^{N} \frac{W_{i,t} - W_{i,t-1}}{\max_i W_{i,t} - \min_i W_{i,t}} \tag{2}$$

  This index captures the relative capacity by agent i to move across wealth positions relative to the maximum inequality that is present at time t. A positive index implies a relative improvement, and viceversa.

- The Absolute Wealth Change Index, which corresponds to the magnitude of V_t, defined as:

$$|V_t| = \frac{1}{N} \sum_{i=1}^{N} \left| \frac{W_{i,t} - W_{i,t-1}}{\max W_t - \min W_t} \right| \tag{3}$$

References

Levy, M. and Levy, H. (2003). Investment talent and the pareto wealth distribution: Theoretical and experimental analysis. *Review of Economics and Statistics*, 85(3):709–725.

Theil, H. (1967). *Economics and information theory*, volume 7. North-Holland Amsterdam.

0.1 Theil Index

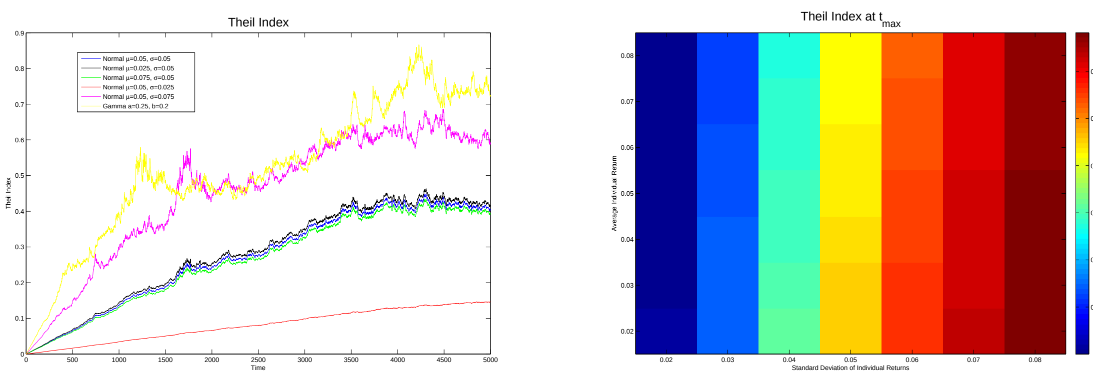

Figure 1: Left Panel: Theil index over time periods for wealth distribution of 5000 individuals under different return structures: $r_{i,t} \sim N(\mu_r, \sigma_r)$ and $gamma(a, b)$. Right Panel: Theil Index value Th_t at time $t_{max} = 5000$ under various return structures $r_{i,t} \sim N(\mu_r, \sigma_r)$.

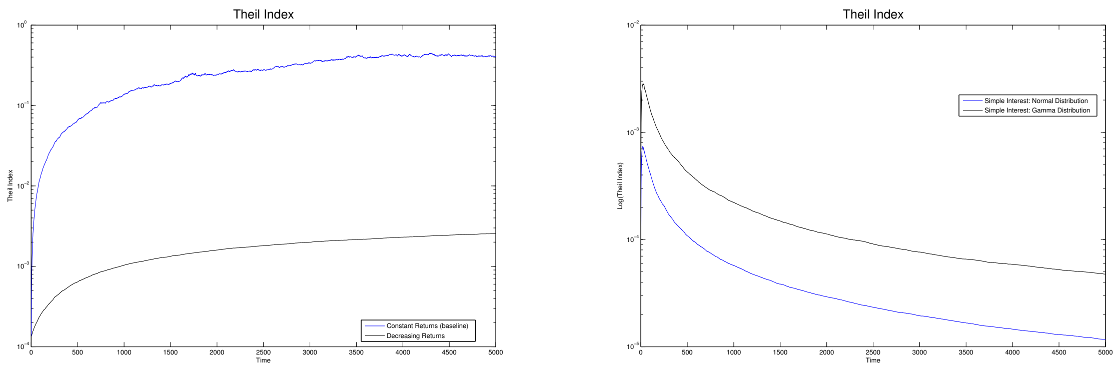

Figure 2: Left Panel: Comparison of Theil Index over time under constant and decreasing returns to aggregate wealth. Right Panel: Theil Index over time under simple return structure under normal distribution $N(\mu_r, \sigma_r)$, and simple return structure under Gamma distribution $gamma(a, b)$.

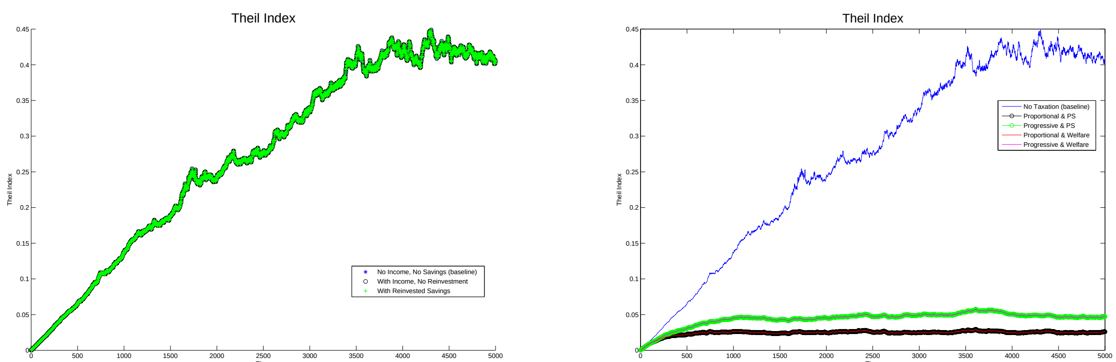

Figure 3: Left Panel: Theil Index value over time under baseline case (one-factor model), two-factors model without labor income savings, and two-factors model with reinvested savings over time. Right Panel: Theil Index over time under baseline case (no taxation), proportional taxation and public service model (**Proportional & PS**), proportional taxation and welfare model (**Proportional & Welfare**), progressive taxation and public service model (**Progressive & PS**) and progressive taxation and welfare model (**Progressive & Welfare**).

0.2 Evolution of wealth and its distribution

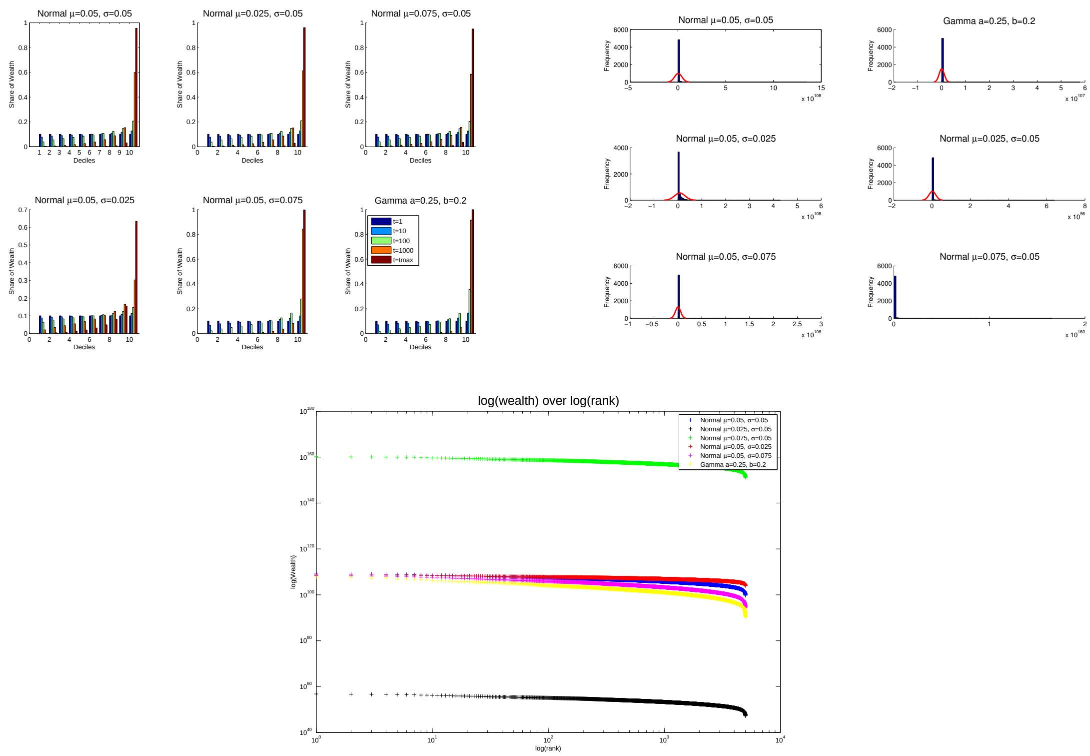

Figure 4: Upper Left Panel: Decile-based distribution of wealth for 5000 individuals under different return structures: $r_{i,t} \sim N(\mu_r, \sigma_r)$ and $gamma(a, b)$. Upper Right Panel: Frequency based distribution of wealths at $t_{max} = 5000$, for the same return structures. Lower Panel: Log-log plot of wealth-rank relationship at $t_{max} = 5000$ for the same return structures.

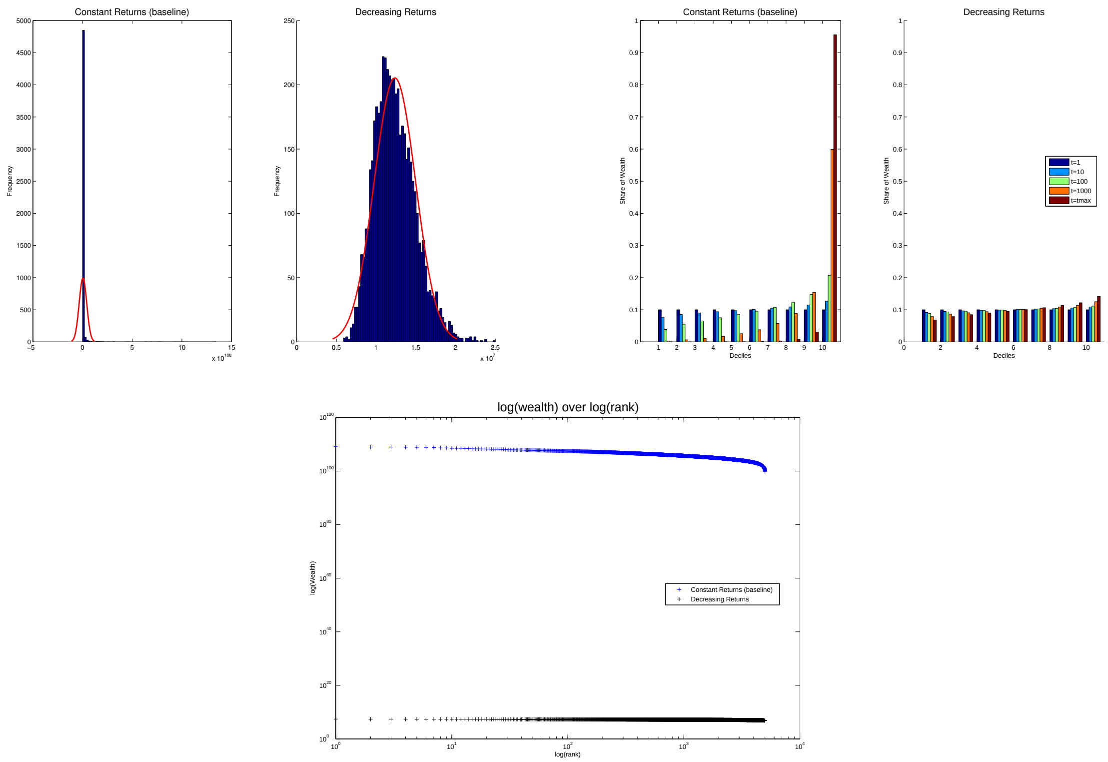

Figure 5: Top Left Panel: Comparison of wealth distribution at $t_{max} = 5000$ under constant returns (baseline) and decreasing returns to aggregate wealth. Top Right Panel: Comparison of decile-based representations of wealth distribution at $t_{max} = 5000$ under constant and decreasing returns to aggregate wealth. Bottom Panel: Log-log plot of wealth-rank relationship at $t_{max} = 5000$ under constant and decreasing returns to aggregate wealth.

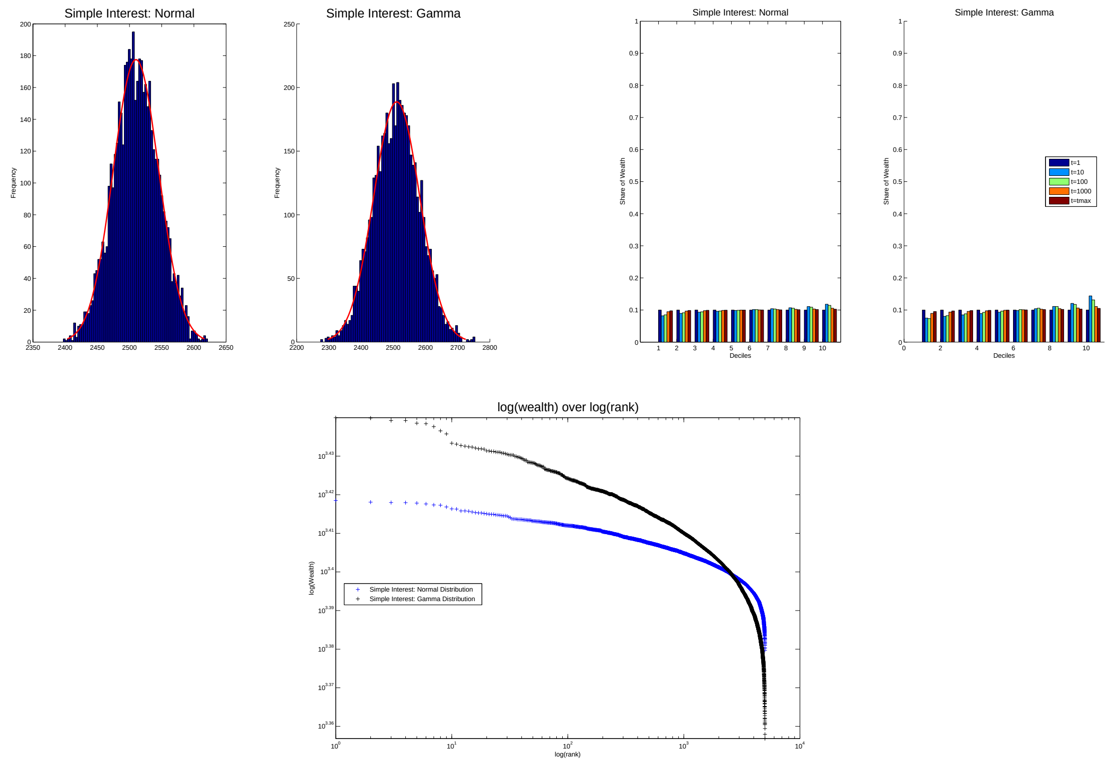

Figure 6: Top Left Panel: Comparison of wealth distributions under compound return structure (baseline case), simple return structure under normal distribution $N(\mu_r, \sigma_r)$, and simple return structure under Gamma distribution $gamma(a, b)$. Top Right Panel: Decile-based distributions under compound return structure (baseline case), simple return structure under normal distribution $N(\mu_r, \sigma_r)$, and simple return structure under Gamma distribution $gamma(a, b)$. Bottom Panel: Log-log plots of wealth-rank relationship under, simple return structure under normal distribution $N(\mu_r, \sigma_r)$, and simple return structure under Gamma distribution $gamma(a, b)$.

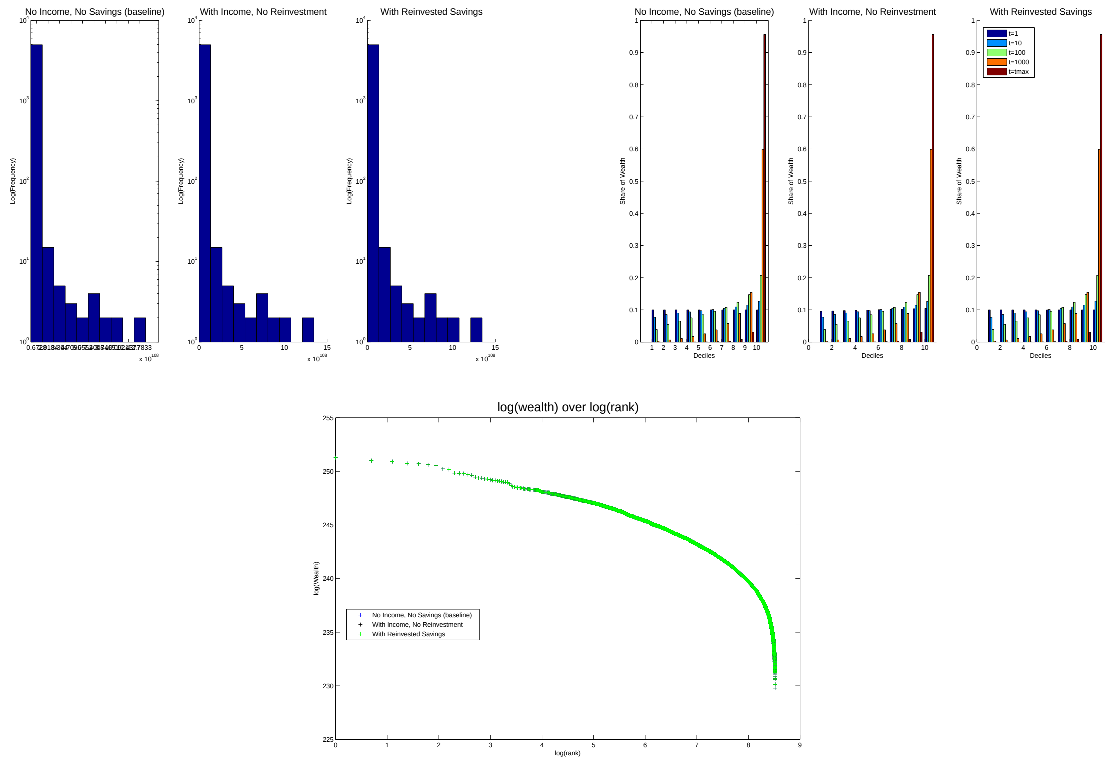

Figure 7: Top Left Panel: Comparison of wealth distributions under baseline case (one-factor model), two-factors model without labor income savings, and two-factors model with reinvested savings over time. Top Right Panel: Decile-based wealth distributions under baseline case (one-factor model), two-factors model without labor income savings, and two-factors model with reinvested savings over time Bottom Panel: Log-log plots of wealth-rank relationship under baseline case (one-factor model), two-factors model without labor income savings, and two-factors model with reinvested savings over time.

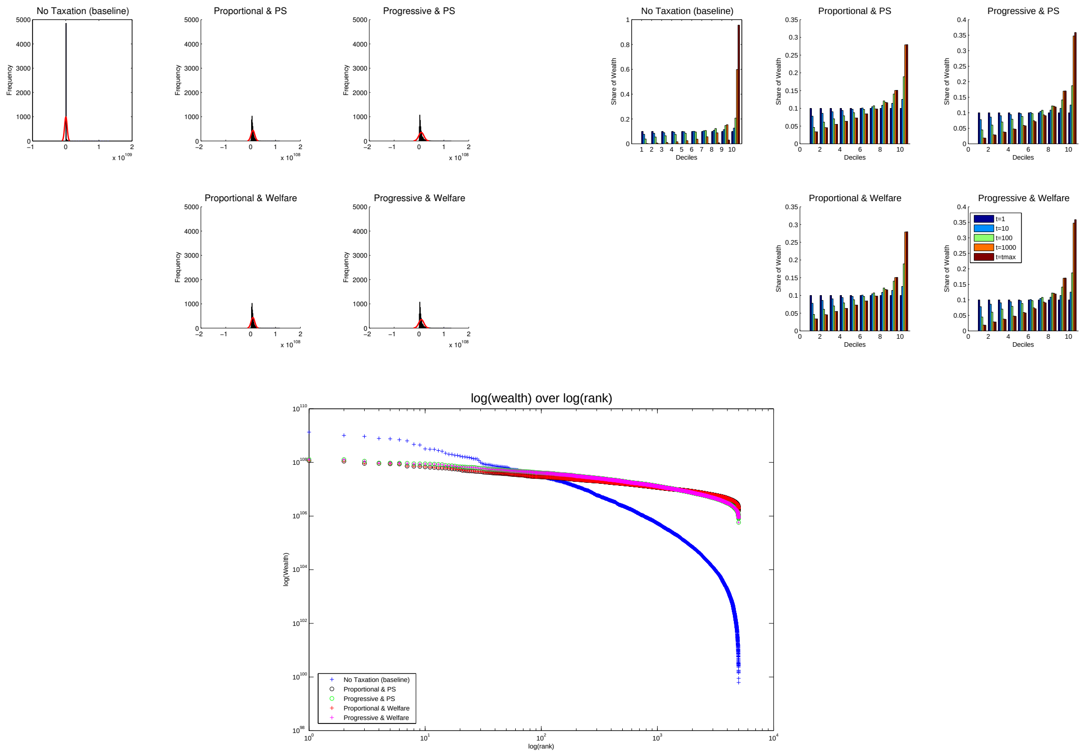

Figure 8: Top Left Panel: Wealth distributions under baseline case (no taxation), proportional taxation and public service model (**Proportional & PS**), proportional taxation and welfare model (**Proportional & Welfare**), progressive taxation and public service model (**Progressive & PS**) and progressive taxation and welfare model (**Progressive & Welfare**). Top Right Panel: Decile-based distributions under the same cases. Bottom Panel: Log-log plots of wealth-rank relationship under the same cases.

0.3 Absolute and Relative wealth of Top 1%

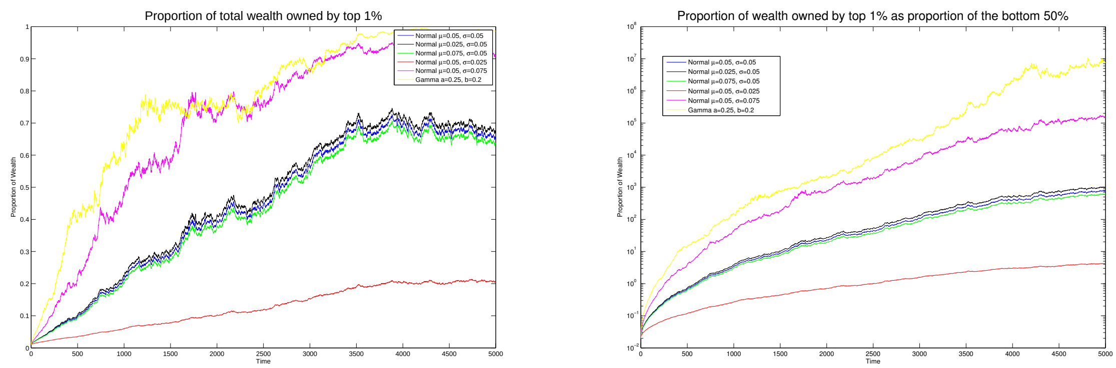

Figure 9: Left Panel: Proportion of wealth owned at each time by the richest 1% of the population as proportion of the total wealth (minimum value of 0.01). Right Panel: Proportion of wealth owned at each time by the richest 1% of the population as proportion of the wealth owned by the bottom 50% (minimum value of 0.02).

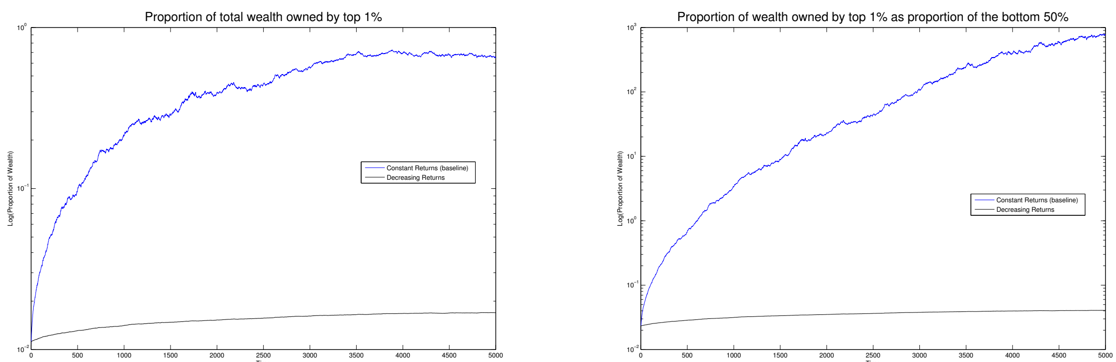

Figure 10: Left Panel: Proportion of wealth owned at each time by the richest 1% of the population as proportion of the total wealth (minimum value of 0.01). Right Panel: Proportion of wealth owned at each time by the richest 1% of the population as proportion of the wealth owned by the bottom 50% (minimum value of 0.02).

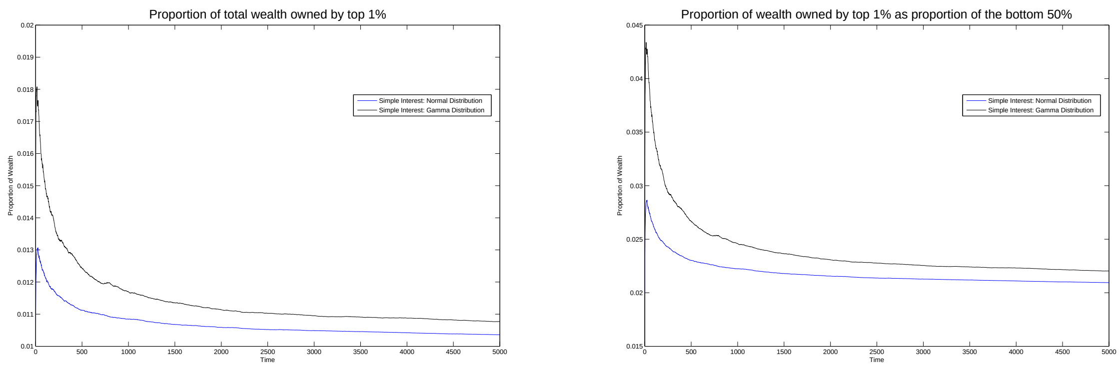

Figure 11: Left Panel: Proportion of wealth owned at each time by the richest 1% of the population as proportion of the total wealth (minimum value of 0.01). Right Panel: Proportion of wealth owned at each time by the richest 1% of the population as proportion of the wealth owned by the bottom 50% (minimum value of 0.02).

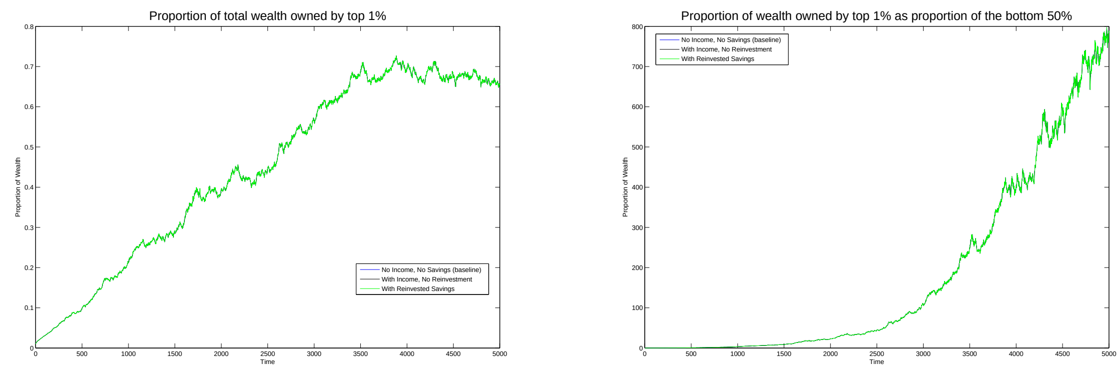

Figure 12: Left Panel: Proportion of wealth owned at each time by the richest 1% of the population as proportion of the total wealth (minimum value of 0.01). Right Panel: Proportion of wealth owned at each time by the richest 1% of the population as proportion of the wealth owned by the bottom 50% (minimum value of 0.02).

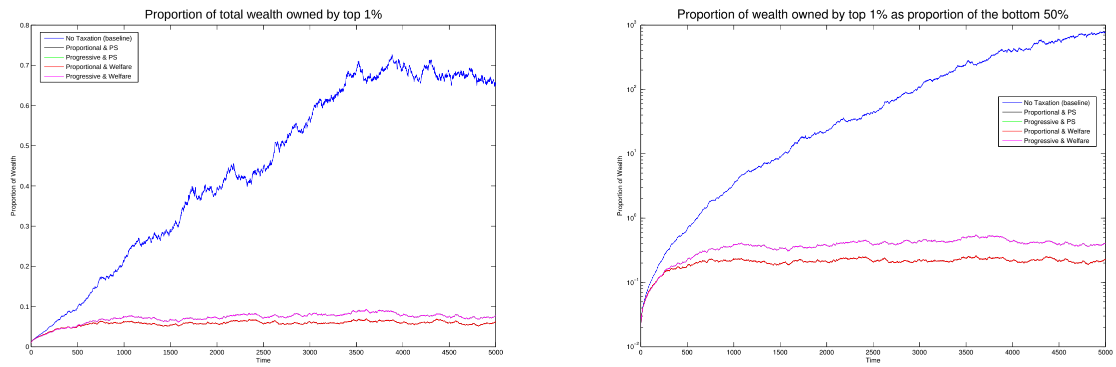

Figure 13: Left Panel: Proportion of wealth owned at each time by the richest 1% of the population as proportion of the total wealth (minimum value of 0.01). Right Panel: Proportion of wealth owned at each time by the richest 1% of the population as proportion of the wealth owned by the bottom 50% (minimum value of 0.02).

0.4 Wealth Change Indexes

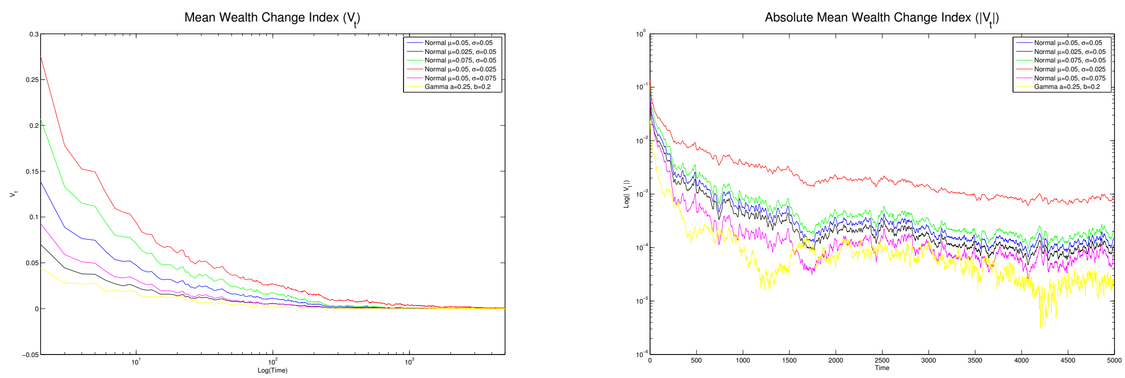

Figure 14: Left Panel: Log plot over time of Mean Wealth Change Index, defined in Equation 2. The index is computed under various return structures : $r_{i,t} \sim N(\mu_r, \sigma_r)$ and $gamma(a, b)$. Right Panel: Log plot over time of Absolute Mean Wealth Change Index defined in Equation 3. The index is computed under various return structures : $r_{i,t} \sim N(\mu_r, \sigma_r)$ and $gamma(a, b)$.

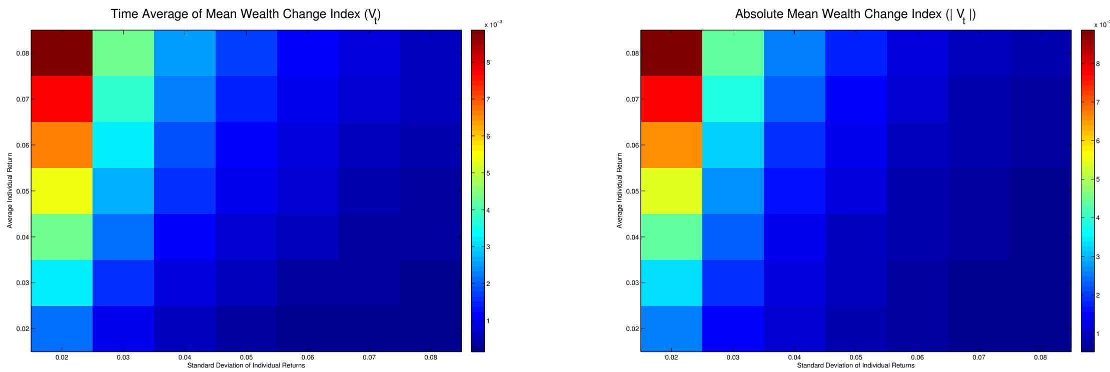

Figure 15: Left Panel: Mean Wealth Change Index V_t under various return structures $r_{i,t} \sim N(\mu_r, \sigma_r)$. Right Panel: Mean Absolute Wealth Change Index $|V_t|$ under various return structures $r_{i,t} \sim N(\mu_r, \sigma_r)$.

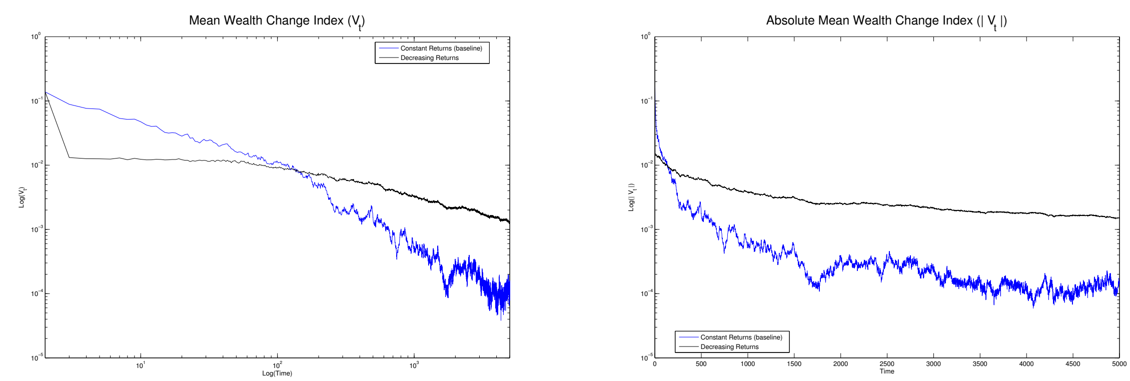

Figure 16: Top Right Panel: Comparison of Wealth Change Index V_t over time under constant and decreasing returns to aggregate wealth. Bottom Panel: Comparison of absolute Wealth Change Index $|V_t|$ over time under constant and decreasing returns to aggregate wealth

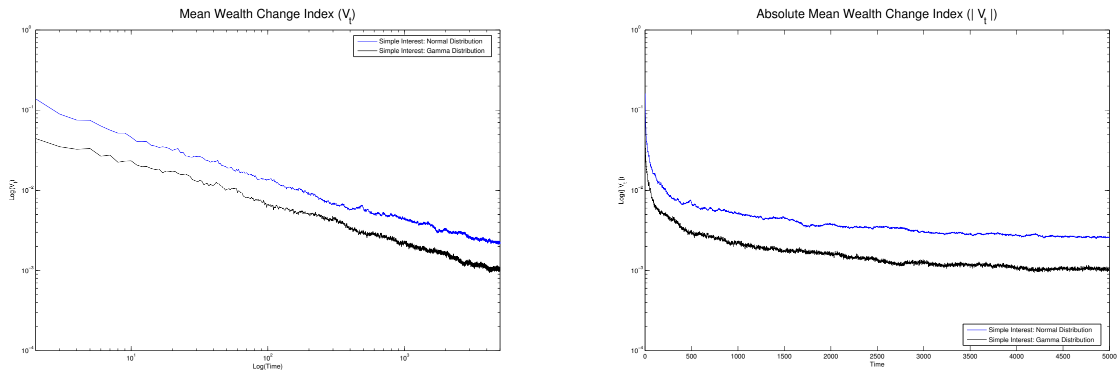

Figure 17: Left Panel:Mean Wealth Change Index under compound return structure (baseline case), simple return structure under normal distribution $N(\mu_r, \sigma_r)$, and simple return structure under Gamma distribution $gamma(a, b)$. Right Panel: Mean Absolute Wealth Change Index under under the same cases.

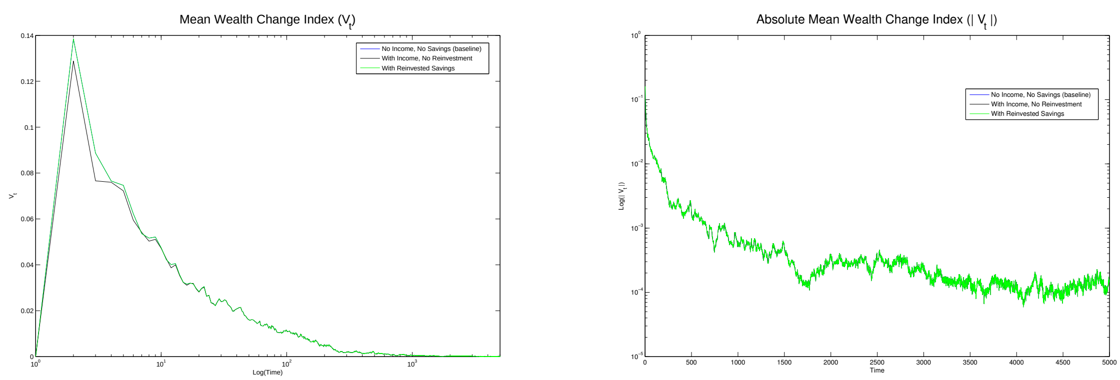

Figure 18: Left Panel: Mean Wealth Change Index under baseline case (one-factor model), two-factors model without labor income savings, and two-factors model with reinvested savings over time. Right Panel: Mean Absolute Wealth Change Index under baseline case (one-factor model), two-factors model without labor income savings, and two-factors model with reinvested savings over time.

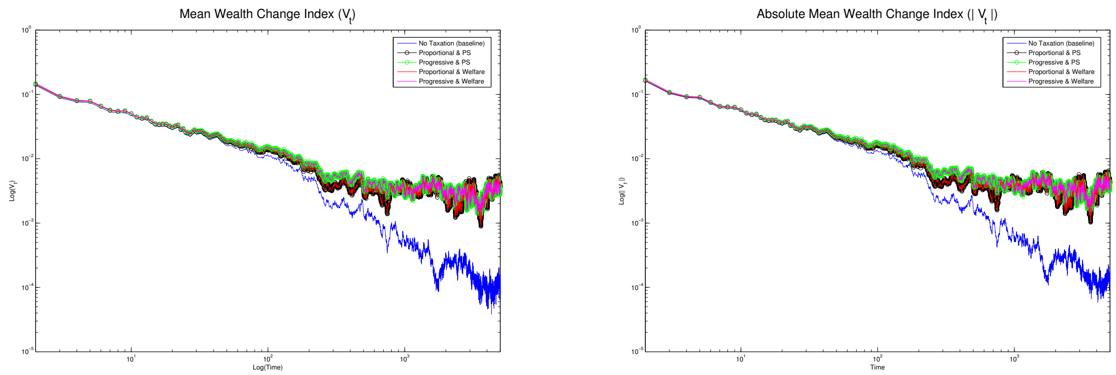

Figure 19: Top Left Panel: Mean Wealth Change Index over time under baseline case (no taxation), proportional taxation and public service model (**Proportional & PS**), proportional taxation and welfare model (**Proportional & Welfare**), progressive taxation and public service model (**Progressive & PS**) and progressive taxation and welfare model (**Progressive & Welfare**). Right Panel: Mean Absolute Wealth Change Index over time under the same cases.

0.5 Median Redistribution and Tax Rates

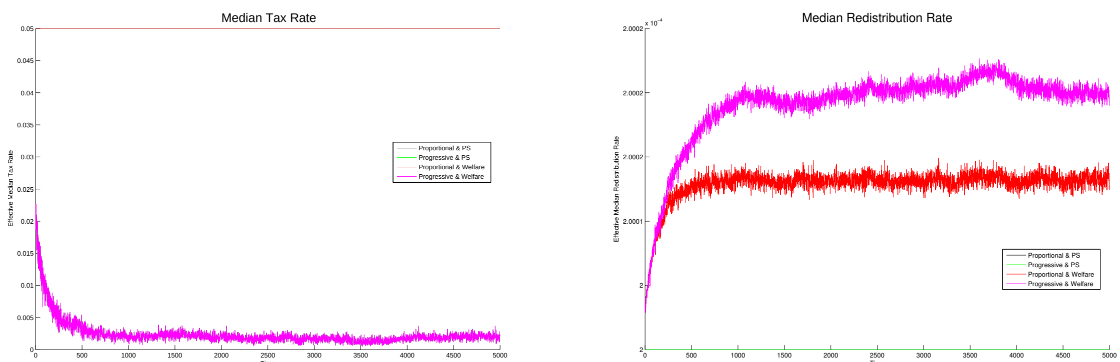

Figure 20: Left Panel: : Median Tax Rate over time under baseline case (no taxation), proportional taxation and public service model (**Proportional & PS**), proportional taxation and welfare model (**Proportional & Welfare**), progressive taxation and public service model (**Progressive & PS**) and progressive taxation and welfare model (**Progressive & Welfare**). Bottom Panel: Median Redistribution rate over time under the same cases.