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On Cographic Matroids and Signed-Graphic Matroids

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Abstract

We prove that a connected cographic matroid of a graph G is the bias matroid of a signed graph Σ iff G imbeds in the projective plane. In the case that G is nonplanar, we also show that Σ must be the projective-planar dual signed graph of an actual imbedding of G in the projective plane. As a corollary we get that, if $G_1, \dots, G_{29}$ denote the 29 nonseparable forbidden minors for projective-planar graphs, then the cographic matroids of $G_1, \dots, G_{29}$ are among the forbidden minors for the class of bias matroids of signed graphs. We will obtain other structural results about bias matroids of signed graphs along the way.

1 Introduction

Throughout this paper we assume that the reader is familiar with matroid theory as in [3]. So let G denote a graph, $M(G)$ the graphic matroid of G, and $M^*(G)$ the cographic matroid of G. Theorem 1 is a result of Hassler Whitney from [6].

Theorem 1 (Whitney). *If G denotes a graph, then $M^*(G) = M(H)$ for some graph H iff G is planar; furthermore, if G is imbedded in the plane with planar dual graph G^*, then $M^*(G) = M(G^*)$.*

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So planarity of graphs precisely determines the intersection of the class of cographic matroids with the class of graphic matroids. But what happens when G is nonplanar? In this paper we will gain some insight into the class of cographic matroids coming from nonplanar graphs by studying their relationship with bias matroids of signed graphs, which we informally define in the following paragraph. For brevity and style we will call the bias matroid of a signed graph a *signed-graphic matroid*. Signed-graphic matroids were introduced in [7, §5]. They are exactly the minors of Dowling geometries for the 2-element group.

A *signed graph* is a pair $\Sigma = (G, \sigma)$ in which G is a graph and σ is a labelling of the edges of G with elements of the set $\{-1, +1\}$. A circle (i.e., simple closed path) in Σ is called *positive* if the product of signs on its edges is positive, otherwise the circle is called *negative*. The signed-graphic matroid of Σ, denoted by $M(\Sigma)$, has as elements the edges of Σ and as circuits the edge sets of the following three types of subgraphs: positive circles, two edge-disjoint negative circles intersecting in only one vertex, and two vertex-disjoint negative circles along with a minimal connecting path.

Our main result in this paper is that projective-planarity of graphs precisely determines the intersection of the class of cographic matroids with the class of signed-graphic matroids. As a corollary we will show that, if $G_1, \ldots, G_{29}$ denote the 29 nonseparable forbidden minors for projective-planar graphs, then the cographic matroids of $G_1, \ldots, G_{29}$ are among the forbidden minors for the class of signed-graphic matroids. We will obtain other structural results about signed-graphic matroids along the way.

In the remainder of this introduction we will formally state and discuss the results of this paper.

Consider a polyhedral imbedding (i.e., an open 2-cell imbedding) of a graph G in the projective plane with projective-planar dual graph G^*. Choose some circle C in G that is a nonseparating closed curve in the projective plane. Let σ be a signing on the edges of G^* that is negative only on the edges corresponding to C. It is known that, given the imbedding of G, the signed graph $\Sigma = (G^*, \sigma)$ is uniquely defined up to switching. That is, if C' is another nonseparating circle in G, then the signed graph (G^*, σ') is switching equivalent to (G, σ). (*Switching* a signed graph is accomplished by choosing a subset X of the vertices and reversing the signs of the links with one endpoint in X and the other not in X.) We call $\Sigma = (G^*, \sigma)$ the *projective-planar* *dual signed graph* of the imbedded graph G. A circle in Σ is negative iff it is a nonseparating closed curve in the projective plane. Theorem 2 below is found in [5, §2].

Theorem 2. *If G denotes a polyhedral imbedding of a connected graph in the projective plane with projective-planar dual signed graph (G^*, σ), then $M^*(G) = M(G^*, \sigma)$.*

Theorem 5.3 in [11] is similar to but weaker than Theorem 2. It says that if a signed graph Σ is the projective-planar dual signed graph of a graph G imbedded in the projective plane, then $M(\Sigma) = M^*(H)$ for some graph H. We cannot conclude from the theorem that H is equal or isomorphic to G.

A graph G is called *nonseparable* if G is connected and is not the one-vertex join of two graphs with nonempty edge sets. The main result of this paper is Theorem 3. Theorem 3 along with Theorem 2 provide an analogue to Whitney's Theorem for the projective plane.

Theorem 3. *If G denotes a nonseparable graph satisfying $M^*(G) = M(\Sigma)$ for some signed graph Σ, then the following are true.*

(1) G is projective-planar.

(2) If G is nonplanar, then Σ is the projective-planar dual signed graph of some imbedding of G (after removing isolated vertices from each).

(3) If G is nonplanar and imbedded in the projective plane with projective-planar dual signed graph Σ, then the edges incident to any vertex of G correspond in the imbedding to the edges of a positive circle of Σ.

The assumption of nonseparability in Theorem 3 is necessary because the class of signed-graphic matroids is closed under direct sums while the class of projective-planar graphs is not closed under disjoint unions and one-vertex joins. That is, if G is the disjoint union or one-vertex join of two nonplanar yet projective-planar graphs, say G_1 and G_2, then G is not projective-planar but the matroid $M^*(G) = (M(G_1) \oplus M(G_2))^* = M^*(G_1) \oplus M^*(G_2)$ will be signed-graphic, by Theorem 2.

A result stronger than Theorem 3(1) was proven independently by H. Qin and T. Dowling in [4]. Their proof uses the list of 35 forbidden minors for projective-planar graphs (see [1]). Our proof will use different techniques. The main tools we use are Theorem 4 and

Corollary 5 which are results of J. Edmonds from [2]. They are quoted here nearly verbatim.

Theorem 4 (Edmonds). *A one-to-one correspondence between the edges of two connected graphs is a duality with respect to some polyhedral surface imbedding iff for each vertex v of each graph, the edges which meet v correspond in the other graph to the edges of a subgraph G_v which is connected and which has an even number of edge ends to each of its vertices (where the image in G_v of a loop at v is counted twice).*

Corollary 5 (Edmonds). *A necessary and sufficient condition for a graph G to have a polyhedral surface imbedding in a surface of euler characteristic χ is that it have an edge correspondence with another graph G^* for which*

(1) the conditions of Theorem 4 are satisfied and

(2) $|V(G)| - |E(G)| + |V(G^*)| = \chi$.

Of the 35 forbidden minors for projective-planar graphs, 29 are nonseparable. Theorem 6 gives us 29 forbidden minors for the class of signed-graphic matroids. The complete list of forbidden minors for the class of signed-graphic matroids is not known and it is possible that it is quite long. We comment further on the complete list in the concluding section.

Theorem 6. *If $G_1, \ldots, G_{29}$ are the nonseparable forbidden minors for projective-planar graphs, then $M^*(G_1), \ldots, M^*(G_{29})$ are forbidden minors for the class of signed-graphic matroids.*

Proof. If G is a nonseparable forbidden minor for the class of projective-planar graphs, then for any $e \in E(G)$, $G\backslash e$ and G/e are both connected, projective-planar graphs. Thus Theorem 3(1) implies that $M^*(G)$ is not signed-graphic, while Theorem 2 implies that $M^*(G)\backslash e = M^*(G/e)$ and $M^*(G)/e = M^*(G\backslash e)$ are signed-graphic. $\square$

Theorem 7 contains several structural results about cographic signed-graphic matroids. These results are used in the proof of Theorem 3 yet they are also worth noting themselves. A signed graph is called *separable* if its underlying graph is separable.

Theorem 7. *If G is a nonplanar graph such that $M^*(G) = M(\Sigma)$ for some signed graph Σ, then*

(1) Σ has no balancing vertex.

Furthermore, if G is also nonseparable, then the following are true as well.

(2) Σ contains no loops, no loose edges, no half edges, and no two vertex-disjoint negative circles.

(3) Σ is nonseparable.

(4) The set of edges incident to any vertex v of Σ is a cocircuit of $M(\Sigma)$.

2 Definitions and Preliminaries

We review some notions concerning graphs, signed graphs, and their matroids in an effort to make the presentation more self-contained.

Graphs We denote the vertex set of a graph G by $V(G)$ and its edge set by $E(G)$. A graph has four types of edges: *links*, *loops*, *half edges*, and *loose edges*. Links have their ends attached to distinct vertices, loops have both ends attached to the same vertex, half edges have one end attached to a vertex and the other unattached, and loose edges have both ends unattached. A graph containing neither half edges nor loose edges is called an *ordinary* graph.

If $X \subseteq E(G)$, then we denote the subgraph of G consisting of the edges in X and all vertices incident to an edge in X by $G{:}X$. A graph G is called *separable* if there is a bipartition (X, Y) of $E(G)$ with nonempty parts such that $|V(G{:}X) \cap V(G{:}Y)| \leq 1$. Note that all edges of a nonseparable graph are links. A nonseparable graph is connected save for isolated vertices, if any. A *block* is a maximal subgraph that is either an isolated vertex or nonseparable. A *circle* is a simple closed path.

Graphic Matroids Given an ordinary graph G, the *graphic matroid* $M(G)$ is the matroid whose element set is $E(G)$ and whose circuits are the edge sets of circles in G. If $X \subseteq E(G)$, then $r(X) = |V(G{:}X)| - c(G{:}X)$ where $c(G{:}X)$ denotes the number of components of $G{:}X$. The graphic matroid $M(G)$ is connected iff G is nonseparable. A critical notion in the study of graphic matroids is that matroid contraction and deletion correspond to the usual notions of contraction and deletion of edges in ordinary graphs. This means that $M(G\backslash e) = M(G)\backslash e$ and $M(G/e) = M(G)/e$ for any edge e.

Signed Graphs For those who may be unfamiliar with signed graphs and signed-graphic matroids, a good reference for them is [7]. Given a graph G, let $E'(G)$ denote the collection of links and loops of G. A *signed graph* is a pair $\Sigma = (G, \sigma)$ in which $\sigma : E'(G) \to \{+1, -1\}$. A circle in a signed graph Σ is called *positive* if the product of signs on its edges is positive, otherwise the circle is called *negative*. If H is a subgraph of Σ, then H is called *balanced* if it has no half edges and all circles in H are positive. A *balancing vertex* is a vertex of an unbalanced signed graph whose removal leaves a balanced subgraph. Not all unbalanced signed graphs have balancing vertices. When drawing signed graphs, positive edges are represented by solid curves and negative edges by dashed curves. We write $\|\Sigma\|$ to denote the underlying graph of Σ.

A *switching function* on a signed graph $\Sigma = (G, \sigma)$ is a function $\eta : V(\Sigma) \to \{+1, -1\}$. The signed graph $\Sigma^\eta = (G, \sigma^\eta)$ has sign function σ^η on $E'(G)$ defined by $\sigma^\eta(e) = \eta(v)\sigma(e)\eta(w)$ where v and w are the end vertices (or end vertex) of the link or loop e. The signed graphs Σ and Σ^η have the same list of positive circles. When two signed graphs Σ and Σ_2 satisfy $\Sigma^\eta = \Sigma_2$ for some switching function η, the two signed graphs are said to be *switching equivalent*. An important notion in the study of signed graphs is that two signed graphs with the same underlying graph are switching equivalent iff they have the same list of positive circles (see [7, Proposition 3.2]).

In a signed graph $\Sigma = (G, \sigma)$, the deletion of e from Σ is defined as $\Sigma\backslash e = (G\backslash e, \sigma)$ where σ is understood to be restricted to the domain $E'(G) \setminus e$. The contraction of an edge e is defined for three distinct cases. If e is a link, then $\Sigma/e = (G/e, \sigma^\eta)$ where η is a switching function satisfying $\sigma^\eta(e) = +$. (Again, the domain of σ^η is understood to be restricted to $E'(G) \setminus e$.) If e is a positive loop or loose edge, then $\Sigma/e = \Sigma\backslash e$. If e is a negative loop or half edge with endpoint v, then Σ/e is the signed graph on $V(\Sigma) \setminus v$ obtained from Σ as follows: delete e, links incident to v become half edges incident to their other endpoint, loops and half edges incident to v become loose edges, and edges not incident to v remain unchanged. Contraction is defined so that it corresponds to contraction in signed-graphic matroids.

Signed-graphic Matroids The bias matroid of a signed graph was introduced in [7]. Within this paper we call the bias matroid of a signed graph a *signed-graphic matroid.* Although other matroids associated with a signed graph may also rightly be called "signed-graphic", our definition of a signed-graphic matroid should not cause confusion because we are only discussing the bias matroid in this paper.

We denote the signed-graphic matroid of Σ by $M(\Sigma)$. The element set of $M(\Sigma)$ is $E(\Sigma)$ and a circuit is either a loose edge, the edge set of a positive circle, or the edge set of a subgraph in which all circles are negative and which is a subdivision of one of the two graphs shown in Figure 8 where a negative loop may be replaced by a half edge.

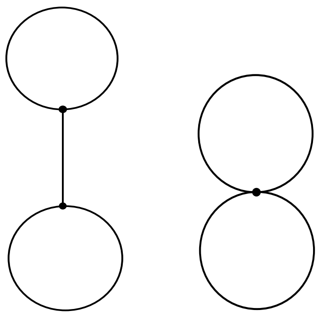


Figure 8.

Since switching a signed graph does not change the list of positive circles, $M(\Sigma) = M(\Sigma^{\eta})$ for any switching function η. Conversely, if $\|\Sigma\| = \|\Sigma'\|$ and $M(\Sigma) = M(\Sigma')$, then Σ and Σ' must have the same list of positive circles. Thus Σ and Σ' are switching equivalent.

If $X \subseteq E(\Sigma)$, then then the rank of X is $r(X) = |V(\Sigma{:}X)| - b(\Sigma{:}X)$ where $b(\Sigma{:}X)$ denotes the number of balanced components of $\Sigma{:}X$ (see [7, Theorem 5.1(j)]). Loose edges do not contribute to the number of balanced components. Three situations in which a signed-graphic matroid $M(\Sigma)$ is not connected are when Σ has a loose edge, Σ is disconnected after removing isolated vertices, and Σ is the one-vertex join of Σ_1 and Σ_2 with Σ_1 balanced. We use these facts about connectivity freely in this paper.

With our definition of deletions and contractions in signed graphs, $M(\Sigma\backslash e) = M(\Sigma)\backslash e$ and $M(\Sigma/e) = M(\Sigma)/e$ for any $e \in E(\Sigma)$ (see [7, Theorem 5.2]).

Imbeddings An imbedding of a graph G in a surface is called *polyhedral* if the interior of each face of G in the surface is homeomorphic to an open disk. (Some call a polyhedral imbedding an *open 2-cell* imbedding.) The topological dual graph of G imbedded in S is denoted by G^*.

Lemmas 9 and 10 are special properties of nonplanar graphs imbedded in the projective plane. Lemma 10 is used in the proof of Theorem 3(3).

Lemma 9. *Let G be a connected graph imbedded in the projective plane. If G has no isthmus and is nonplanar, then G^* is loopless.*

Proof. Let P denote the projective plane and suppose that e is a loop in G^*. Then $G^*{:}e$ is either a separating or nonseparating curve in P. If $G^*{:}e$ is separating, then e must be a separating edge of G since G only intersects G^* in P at the transverse crossings of curves corresponding to the same edge. This contradicts the assumption that G has no isthmus, so it must be that $G^*{:}e$ is a nonseparating curve in P. Cutting P along $G^*{:}e$ yields a disk in which $G \setminus e$ is imbedded; furthermore, the endpoints of e in G must be on the boundary of the outer region of $G \setminus e$ in the disk. Thus we can redraw e on a disk without crossing any edges of $G \setminus e$, showing that G is planar. This contradicts our assumption that G is not planar. Thus G^* is loopless. □

Lemma 10. *Let G denote a graph imbedded in the projective plane. If G is nonseparable and nonplanar, then the edges bounding a face of G in the projective plane are the edges of a circle in G (that is, the representativity of the imbedding is at least 2).*

Proof. Since G is nonseparable, G does not contain an isthmus. So Lemma 9 implies that G^* is loopless. Thus no face boundary walk of G repeats an edge and so what would prevent the edges of a face boundary walk from being the edges of a circle in G would be that a vertex was repeated in the walk. By way of contradiction, assume that v is a vertex repeated in the boundary walk of face F. Since no face boundary walk of G repeats an edge, the vertex v has degree at least four. Obtain graph G_0 imbedded in the projective plane by splitting the vertex $v \in V(G)$ into two adjacent vertices v_1 and $v_2 \in V(G_0)$ while keeping the face structure unchanged except for the fact that the face of G_0 corresponding to F now has a repeated edge in its face boundary walk, specifically the new edge linking v_1 and v_2, call it e. Note that each v_i has degree at least three. Now G_0^* has a loop and, since $G = G_0/e$ and G is nonplanar, G_0 must be nonplanar. This contradicts Lemma 9 as long as we can show that G_0 has no isthmus.

Assume that G_0 has an isthmus f. If $f \neq e$, then f is an isthmus of $G = G_0/e$, a contradiction. If $f = e$, then $G = G_0/e$ is separable, a contradiction, or e is a link with an endpoint of degree one, a contradiction. Thus G_0 has no isthmus. □

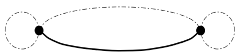
3 Proofs of our main results.

Addition and deletion of isolated vertices in graphs and signed graphs has no affect on their matroids. So we will always delete isolated vertices unless otherwise noted.

Proof of Theorem 7. In the proof of each part, G denotes a nonplanar graph and Σ a signed graph such that $M^*(G) = M(\Sigma)$. We will also use the term *joint* to mean an edge that is either a negative loop or a half edge. In the proof of Parts (2)–(4) we also assume that G is nonseparable.

Part (1) Assume that Σ does have a balancing vertex, call it v. Thus Σ is switching equivalent to a signed graph Σ' in which all negative edges and half edges are incident to v. Since Σ and Σ' are switching equivalent, $M(\Sigma) = M(\Sigma')$. Let G' be the graph obtained from Σ' by splitting v into two vertices v_1 and v_2 where the positive links attached to v in Σ become links attached to v_1 in G', the negative links attached to v in Σ' become links attached v_2 in G', positive loops attached to v in Σ' become loops attached to v_1 in G', and joints attached to v in Σ' become links connecting v_1 and v_2 in G'. Note that G' is an ordinary graph aside from perhaps some loose edges. Since loose edges are matroid loops, $M(G')$ is a graphic matroid. By checking circuits of $M(\Sigma')$ and $M(G')$ we see that $M(\Sigma') = M(G')$. Thus $M^*(G) = M(G')$. So Whitney's Theorem (Theorem 1) guarantees that G is planar, a contradiction.

Part (2) We first note some structural facts about the signed graph Σ and the matroids $M(G)$, $M^*(G)$, and $M(\Sigma)$. Because G is nonseparable, the matroids $M(G)$ and $M^*(G) = M(\Sigma)$ are connected. If a signed graph without isolated vertices is not connected then its signed-graphic matroid is not connected, so Σ is connected. Also, if a signed graph has a positive loop or loose edge, then its signed-graphic matroid is not connected because positive loops and loose edges in Σ are matroid loops in $M(\Sigma)$. Thus Σ has no positive loops and no loose edges. Since graphic matroids are regular and duals of regular matroids are regular, $M(\Sigma)$ is regular and so does not contain the uniform matroid $U_{2,4}$ as a minor. (Recall that a matroid is binary iff it does not contain $U_{2,4}$ as a minor.) Up to isomorphism and interchange of negative loops with half edges, the signed graph in Figure 11 is the only signed graph whose matroid is $U_{2,4}$. We will say that a signed graph has an Υ minor if it contains a signed graph whose matroid is $U_{2,4}$.

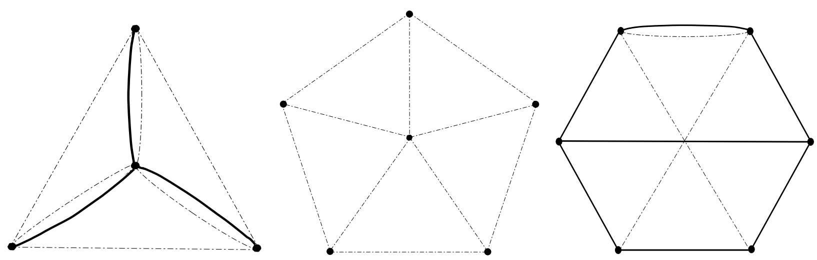
Figure 11. 

Thus $M(\Sigma)$ is binary iff Σ does not contain an Υ minor. We now proceed by contradiction, assuming $M(\Sigma)$ is binary and yet Σ either contains a joint or two vertex-disjoint negative circles. We will arrive at a contradiction by showing that Σ is forced to contain an Υ minor. We split this task into two cases: in the first case, Σ has two vertex-disjoint negative circles contained in a common block of Σ and in the second case, either Σ contains a joint or Σ contains two vertex-disjoint negative circles in distinct blocks.

Case 1: Consider a block of Σ containing two vertex-disjoint negative circles. Let C_1 and C_2 denote two such circles. Recall that a joint and its endpoint always form their own block in a graph. Thus C_1 and C_2 each have length at least two. Thus we can apply Menger's Theorem to find two vertex-disjoint paths γ_1 and γ_2 linking C_1 and C_2 together. Evidently, the subgraph $C_1 \cup C_2 \cup \gamma_1 \cup \gamma_2$ contracts to a copy of the signed graph in Figure 11. Thus Σ has an Υ minor.

Case 2: Since G is nonplanar, Kuratowski's Theorem guarantees that G contains a K_5 or $K_{3,3}$ minor. Thus $M(\Sigma)$ contains an $M^*(K_5)$ or $M^*(K_{3,3})$ minor. By [9, Proposition 4A], the only signed graphs, up to switching, that have matroids isomorphic to $M^*(K_5)$ or $M^*(K_{3,3})$ are the signed graphs Σ_1, Σ_2, and Σ_3 shown respectively from left to right in Figure 12.

Figure 12.



Thus Σ contains a minor that is switching equivalent to Σ_1, Σ_2, or Σ_3. Let Γ be a minimal subgraph of Σ that contracts to such a minor. Assume that Γ is nonseparable. (We will prove that Γ is nonseparable in the next paragraph.) Since Γ is nonseparable, it is contained entirely in one block of Σ. Since Σ either has joint or has two vertex-disjoint negative circles contained in two distinct blocks, Σ has a negative circle or joint in a block different from the block containing Γ. Pick one such joint or negative circle, call it C. Since Σ is connected, there is a path γ linking C to Γ. Evidently $\Gamma \cup \gamma \cup C$ contracts to a signed graph that is switching equivalent to $\Sigma_1 \cup e_0$, $\Sigma_2 \cup e_0$, or $\Sigma_3 \cup e_0$ where e_0 is a joint attached to some vertex. One can easily check that $\Sigma_1 \cup e_0$, $\Sigma_2 \cup e_0$, and $\Sigma_3 \cup e_0$ each contain an Υ minor, no matter which vertex e_0 is attached to. Thus Σ has an Υ minor.

It remains only to prove that Γ is nonseparable. First, since the contraction of a loose edge or positive loop is the same as its deletion, the minimality of Γ implies that Γ has no loose edges and no positive loops. Second, since Σ_1, Σ_2, and Σ_3 are all connected, by minimality Γ must be connected. Third, the contraction of a joint in a connected signed graph leaves a separable signed graph. Thus Γ is connected, ordinary, and loopless. Last, the contraction of a link in a separable signed graph leaves a separable signed graph unless the link has an endpoint of degree one. Since Σ_1, Σ_2, and Σ_3 are all nonseparable, Σ must be nonseparable except perhaps for a link incident to a vertex of degree one. But deleting such a link (along with its endpoint vertex) will be the same as contracting the link. This contradicts the minimality of Γ. Thus Γ is nonseparable.

Part (3) As stated at the beginning of the proof of Part (2), $M^*(G) = M(\Sigma)$ is a connected matroid and Σ is connected after removing isolated vertices. Now by way of contradiction, assume that Σ has a separating vertex, call it v. Thus Σ is the one-vertex join of connected signed graphs $\Sigma_1, \ldots, \Sigma_k$ at v with $k \geq 2$. Since $M(\Sigma)$ is connected, each of $\Sigma_1, \ldots, \Sigma_k$ must be unbalanced. By Part (1), v is not a balancing vertex of Σ. Thus one of $\Sigma_1 \backslash v, \ldots, \Sigma_k \backslash v$ is unbalanced, say $\Sigma_1 \backslash v$. By Part (2), Σ does not contain half edges so $\Sigma_1 \backslash v$ is unbalanced because it contains a negative circle, call it N. But since Σ_2 is unbalanced and does not contain half edges, there is a negative circle N' in Σ_2 that is vertex-disjoint from N. This contradicts Part (2) which says that Σ does not contain two vertex-disjoint negative circles.

Part (4) Given some vertex v in Σ, let S be the set of edges incident to v. By Part (3), Σ is nonseparable so S is nonempty, S contains only links, and $\Sigma\backslash S$ consists of two components, $\Sigma\backslash v$ and the isolated vertex v. Since Σ has no balancing vertex (by Part (1)), $r(\Sigma\backslash S) = r(\Sigma) - 1$ and, for any $e \in S$, $r((\Sigma\backslash S) \cup e) = r(\Sigma)$. Thus S is a cocircuit of $M(\Sigma)$. □

Proof of Theorem 3. When G is planar, Part (1) of our theorem follows because any planar graph imbeds in the projective plane. So for the remainder of the proof we will assume that G is nonplanar.

Parts (1) and (2) Assume that $M^*(G) = M(\Sigma)$ for some signed graph Σ. We will use Theorem 4 and Corollary 5 to show that G and $\|\Sigma\|$ (the underlying graph of Σ) are dual graphs of a polyhedral imbedding in the projective plane, after removing any isolated vertices from both.

Let S be the set of edges meeting a vertex v in G. Because G is nonseparable, S does not contain any loops and S is a bond of G. So S is a cocircuit of $M(G)$, which is also a circuit of $M(\Sigma)$. Theorem 7(2) guarantees that all edges of Σ are links and Σ does not contain two vertex-disjoint negative circles. So $\Sigma{:}S$ is either a positive circle or the union of two negative circles that intersect in a single vertex. In each case S satisfies the conditions of Theorem 4.

If S is the set of edges meeting a vertex v in Σ, then Theorem 7(2)–(4) guarantee that S contains only links and is a cocircuit of $M(\Sigma)$. Thus S is also a circuit of $M(G)$, which makes $G{:}S$ a circle. Thus S satisfies the conditions of Theorem 4.

Since G is nonseparable, Theorem 7(3) guarantees that Σ is nonseparable as well. Thus G and Σ are both connected and since $r(M(G)) + r(M(\Sigma)) = |E(G)|$ when $M^*(G) = M(\Sigma)$, it follows that $|V(G)| - 1 + |V(\Sigma)| = |E(G)|$. Thus $|V(G)| - |E(G)| + |V(\Sigma)| = 1$. So by Theorem 4, Corollary 5, and the previous two paragraphs, G and $\|\Sigma\|$ are dual graphs of a polyhedral imbedding in a closed surface of Euler characteristic one. The only such surface is the projective plane.

Let (G^*, σ) be the projective-planar dual signed graph of G in the imbedding given above. Thus $\|\Sigma\| = G^*$ and Theorem 2 implies that $M^*(G) = M(G^*, \sigma)$. Thus $M(\Sigma) = M^*(G) = M(G^*, \sigma)$. Two signed-graphic matroids on the same underlying graph are equal iff they have the same list of positive circles. Two signed graphs with the same underlying graph have the same list of positive circles iff they are switching equivalent. The projective-planar dual signed graph is only well defined up to switching equivalence, so $\Sigma = (G^*, \sigma)$.

Part (3) Let S be the set of edges incident to a vertex $v \in V(G)$.

Since G is nonseparable, S is a bond in G and so is cocircuit of $M(G)$. Theorem 2 implies that $M^*(G) = M(\Sigma)$ and so S is a circuit of $M(\Sigma)$. Since G is nonseparable and nonplanar, Theorem 7(2) implies that $\Sigma{:}S$ is either a positive circle (our desired conclusion) or a union of two negative circles that intersect in a single vertex, call it w. By way of contradiction, assume the latter is true. By the definition of a topological dual graph, S is the set of edges of the face boundary walk in Σ corresponding to $v \in V(G)$. Now let T be the set of edges incident to w. Similarly, T is the set of edges of the face boundary walk in G corresponding to $w \in V(\Sigma)$. But since $\Sigma{:}S$ is a union of two circles intersecting in a single vertex, the face boundary walk in G of the face corresponding to $w \in V(\Sigma)$ has a repeated vertex, a contradiction of Lemma 10. $\square$

4 Concluding Remarks and Conjectures

Given a signed graph Σ there are two matroids defined on Σ other than the signed-graphic matroid of this paper: the lift matroid (denoted $L(\Sigma)$) and the complete lift matroid (denoted $L_0(\Sigma)$). [10, §3] is a good introduction to these two matroids. The complete lift matroid's element set is $E(\Sigma_0)$ where Σ_0 is Σ along with a new vertex and a new joint attached to it. Call the new joint e_0. The rank of an edge set $X \subseteq E(\Sigma_0)$ is $|V(\Sigma_0{:}X)| - c(\Sigma_0{:}X) + \epsilon_X$ where $c(\Sigma_0{:}X)$ is the number of components of $\Sigma_0{:}X$ and $\epsilon_X = 0$ when $\Sigma_0{:}X$ is balanced and 1 when it is unbalanced. The lift matroid $L(\Sigma)$ is equal to $L_0(\Sigma)\backslash e_0$. One may check that when all half edges of Σ are replaced by negative loops $M(\Sigma) = L(\Sigma)$ iff Σ does not contain two vertex-disjoint negative circles. So, if Σ is projective planar, then $M(\Sigma) = L(\Sigma)$.

Perhaps one would now believe that a result similar to Theorem 3 holds when $M^*(G) = L(\Sigma)$ for some signed graph Σ. Such a result, however, is not true. For example, $M^*(K_{2,2,2,1}) = L(\Sigma)$ for some signed graph Σ but $K_{2,2,2,1}$ does not imbed in the projective plane. ($K_{2,2,2,1}$ is one of the 29 nonseparable forbidden minors of projective-planar graphs). We conjecture, however, that $M^*(G) = L(\Sigma)$ for some signed graph Σ iff G imbeds in a connected pseudosurface of Euler characteristic one.

A result like Conjecture 13, if true, would also be of interest. Suppose that $M(\Sigma_1)$ and $M(\Sigma_2)$ are connected signed-graphic matroids satisfying $M^*(\Sigma_1) = M(\Sigma_2)$. In [5, Section 4] it is proven that this is true when Σ_1 and Σ_2 are topological dual signed graphs in some imbedding in the torus (See [5, Section 4] for our definition of imbedding a signed graph in the torus.) There are also notions of imbedding signed graphs in other surfaces of small Euler characteristic that give similar results. Such results are unpublished and are due to T. Zaslavsky, L. Lovász, and the author. Given these notions of imbedding signed graphs, we conjecture the following.

Conjecture 13. *If $M(\Sigma_1)$ is a connected signed-graphic matroid and $M^*(\Sigma_1) = M(\Sigma_2)$ for some other signed graph Σ_2, then Σ_1 imbeds in the torus, Klein bottle, annulus, projective plane, or plane.*

Part of the interest of such a result would be that it gives more insight into the list of forbidden minors for the class of signed-graphic matroids. That is, if $M(\Sigma)$ is connected and Σ is a minor-minimal signed graph that does not imbed in any one of the torus, Klein bottle, annulus, projective plane, and plane, then $M^*(\Sigma)$ would a forbidden minor for the class of signed-graphic matroids.

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References

[1] Dan Archdeacon, *A Kuratowski theorem for the projective plane*, J. Graph Theory **5** (1981), no. 3, 243–246.

[2] Jack Edmonds, *On the surface duality of linear graphs*, J. Res. Nat. Bur. Standards Sect. B **69B** (1965), 121–123.

[3] James G. Oxley, *Matroid theory*, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1992.

[4] Hongxun Qin and Thomas Dowling, *Excluded minors for classes of cographic matroids*, Discrete Math. (2001), to appear.

[5] Daniel C. Slilaty, *Matroid duality from topological duality in surfaces of nonnegative Euler characteristic*, Combin. Probab. Comput. **11** (2002), no. 5, 515–528.

[6] Hassler Whitney, *Planar graphs*, Fund. Math. **21** (1933), 73–84.

[7] Thomas Zaslavsky, *Signed graphs*, Discrete Appl. Math. **4** (1982), no. 1, 47–74.

[8] ———, *Biased graphs. I. Bias, balance, and gains*, J. Combin. Theory Ser. B **47** (1989), no. 1, 32–52.

[9] ———, *Biased graphs whose matroids are special binary matroids*, Graphs Combin. **6** (1990), no. 1, 77–93.

[10] ———, *Biased graphs. II. The three matroids*, J. Combin. Theory Ser. B **51** (1991), no. 1, 46–72.

[11] ———, *Is there a matroid theory of signed graph embedding?*, Ars Combin. **45** (1997), 129–141.