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Algebraic characterizations of graph imbeddability in surfaces and pseudosurfaces

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Abstract

Given a finite connected graph G and specifications for a closed, connected pseudosurface, we characterize when G can be imbedded in a closed, connected pseudosurface with the given specifications. The specifications for the pseudosurface are: the number of face-connected components, the number of pinches, the number of crosscaps and handles, and the dimension of the first $\mathbb{Z}_2$-homology group. The characterizations are formulated in terms of the existence of a dual graph G^* on the same set of edges as G which satisfies algebraic conditions inspired by homology groups and their intersection products.

1 Introduction

All graphs in this paper are finite. Theorem 1.1 from [17] is a classic result of H. Whitney characterizing planar graphs in terms of the existence of what Whitney calls a *dual graph.* Let $c(G)$, $e(G)$, and $v(G)$ be the number of components, edges, and vertices, respectively, of a graph G. A *dual graph* of a graph G is a graph G' on the same edge set as G such that, for any $H \subseteq G$, the complementary edge set of H forms a subgraph $H' \subseteq G'$ satisfying

$$e(H) - v(H) + c(H) = v(G') - c(G') - v(H') + c(H')$$

We will refer to this notion of a dual graph as a *combinatorial dual graph.*

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Theorem 1.1 (Whitney). *A graph is planar if and only if it has a combinatorial dual graph. Furthermore, if G is connected and imbedded in the plane, then the topological dual graph of G is also a combinatorial dual graph of G.*

Similar to Whitney's result is that of S. MacLane from [12]. There he uses the idea of a 2-*basis* to algebraically characterize planar graphs. Let V be a $\mathbb{Z}_2$-vector space with basis $e_1, \ldots, e_k$ and let W be a subspace of V. A 2-*basis* of W relative to $e_1, \ldots, e_k$ is a basis B of W for which each e_i is a summand of at most two of the vectors in B.

Theorem 1.2 (MacLane). *A graph G is planar if and only if the cycle space of G has a 2-basis relative to the edges of G.*

Whitney's and MacLane's theorems are related in that a 2-basis for the cycle space of G can be used as the face boundaries (for the closed faces) of an imbedding of G in the plane, in which case the 2-basis can also be viewed as the vertex set of a combinatorial dual graph of G. Thus both theorems classify planarity in terms of the existence of a dual graph.

There are two notable generalizations of the theorems of Whitney and MacLane: one by S. Lefschetz in [11] and another by J. Edmonds in [7]. The result of Lefschetz uses the combinatorial language of rotation systems (which Lefschetz calls "umbrellas") of graphs imbedded in surfaces, and extends MacLane's criterion to any orientable surface. Edmonds' Theorems (shown below) are elegant and simply-stated combinatorial results using the idea of dual graphs.

Theorem 1.3 (Edmonds). *A one-to-one correspondence between the edges of two connected graphs is a duality with respect to some surface S if and only if, for each vertex v of each graph, the edges which meet v in the graph of v form in the other graph a subgraph which is connected and has an even number of edge ends to each of its vertices (where if an edge meets v at both ends, its image in H is counted twice).*

Theorem 1.4 (Edmonds). *A necessary and sufficient condition for a connected graph G to have a polyhedral surface imbedding in a surface S of Euler characteristic $\chi(S)$ is that it has an edge correspondence with another graph G^* for which*

(1) the conditions of Theorem 1.3 are satisfied and

(2) $v(G) - e(G) + v(G^*) = \chi(S)$.

Edmonds comments in the conclusion of [7] that his theorems cannot be generalized using oriented edges in order to distinguish between orientable and nonorientable surfaces. (He does not comment on characterizing imbeddability in pseudosurfaces.)

In [1] the authors of this paper presented an algebraic generalization of Whitney's theorem to the projective plane. In this paper we build on the work in [1] in two ways: a more topological approach is used and the results here cover all surfaces and pseudosurfaces, rather than just the projective plane.

Imbeddability of graphs in surfaces has also been discussed from other algebraic perspectives. Archdeacon, Bonnington, and Little provide a novel characterization of planarity in [2] in terms of diagonals, which are a particular kind of walk double cover. The topological content of that result is clarified by the approach of Richter and Keir in [15] and [9]. The latter approach essentially deals with homology and the former with cohomology, although neither is explicitly stated in these terms.

Two classical approaches to graph imbedding, although unrelated to the work here, require mention in any discussion of imbeddability. Kuratowski's planarity criterion [10] states that a graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor. This has given rise to a large study of imbeddability in surfaces under the title "graph minors" which is described in [13]. Another approach to imbedding grows out of the notion of voltage graphs (*i.e.*, directed graphs with edges labelled by elements of a group) and their derived graphs, which is a combinatorial version of the theory of covering spaces. A good reference for these ideas is [8].

A closed, connected *pseudosurface* P is a topological space obtained from a disjoint union of surfaces via a finite number of point identifications (henceforth *pinches*). We say that a graph G *properly imbeds* in a pseudosurface P when G imbeds in P, subdivides P into 2-cells, and pinch-points in P correspond to vertices in G. When we say in this paper that "G imbeds in P" we mean that G properly imbeds in P. Imbeddability for graphs in pseudosurfaces has received much less attention than imbeddability in surfaces, although the literature does contain some results. For instance, [3] describes a criterion for imbeddability in the pseudosurface B_n obtained from n spheres by identifying all north poles and identifying all south poles, and [4] contains a characterization of graphs of connectivity 1 or 2 which are minimally nonimbeddable (under edge-deletion and contraction of edges not in a triangle) in B_2.

In [14] some of the basic theorems about imbeddings in surfaces are extended to pseudosurfaces, and there is also a determination of the maximum pseudocharacteristic pseudosurface for various classes of complete multipartite graphs. (The *pseudocharacteristic* of a pseudosurface P is defined to be the Euler characteristic of any 2-cell decomposition of the pseudosurface.) Some calculations of maximum pseudocharacteristic for other classes of graphs appear in [16].

Our purpose in this paper is to characterize imbeddability of a connected graph G in a closed, connected pseudosurface with a given set of specifications: the number of face-connected components, the number of pinches, the number of crosscaps and handles, and the dimension of the first $\mathbb{Z}_2$-homology group. The characterizations are in terms of the existence of a type of dual graph G^* on the same set of edges as G satisfying certain algebraic conditions inspired by homology groups, intersection products, and Whitney's planarity theorem. Of note is that our results distinguish between imbeddability of a graph G in the orientable surface of even Euler characteristic from imbeddability of G in the nonorientable surface of the same characteristic (a problem left open at the end of [7]).

In Section 2 we review some definitions and notation for graphs and cellular complexes. We then show how a graph and a combinatorial dual graph produce a chain complex whose homology is 0. We then define an algebraic dual graph (introduced previously in [1]): An *algebraic dual* of a graph G is a graph G^* with $E(G^*) = E(G)$ whose coboundary space is a subspace of the cycle space of G. An algebraic dual graph will also yield a chain complex, but the homology of that complex is not necessarily 0.

Section 3 describes how to use an algebraic dual of a graph G to construct a pseudosurface in which G is imbedded and whose first homology group is that of the chain complex from Section 2. In [1] we studied algebraic and topological duals of planar and projective-planar graphs. In those cases it is easy to exclude loops from dual graphs, but in the more general context of this work, loops in duals cannot be avoided. (Actually, the existence of a loopless algebraic dual G^* for a given a graph G is equivalent to the existence of a cycle double cover for G; this is a famous open question in graph theory, see [18].) Thus we adapt the idea of weighted Eulerian walks from [7] to modify the construction in [1] in order to handle algebraic duals with loops.

Homology groups alone do not provide sufficient information to characterize the pseudosurface in which a particular graph has been imbedded. So in Section 4 we discuss intersection forms for curves imbedded on surfaces and an extension of these forms to pseudosurfaces. This will provide the additional information necessary for our main results about imbeddability.

In Section 5 we describe our main results characterizing imbeddability. The first is: given a connected graph G, $n \geq 0$, $f \geq 1$, and $p \geq f-1$, we characterize when G may be imbedded in a connected pseudosurface with f face-connected components, p pinches, and $\mathbb{Z}_2$-homology group of dimension n. The second is: given n, p, h, and $c \geq 0$ we characterize when a connected graph G imbeds in a face-connected pseudosurface with p pinches, h handles, c crosscaps, and $\mathbb{Z}_2$-homology group of dimension n. The last characterizes when G can be imbedded in the orientable surface with $\mathbb{Z}_2$-homology group of dimension n (for $n \geq 0$ and even) and when G can be imbedded in the nonorientable surface with $\mathbb{Z}_2$-homology group of dimension n (for any $n \geq 1$).

2 Definitions and basic information

Given a graph G we denote the vertex set by $V(G)$ and the edge set by $E(G)$. Each edge has two ends and each end is attached to a vertex. If an edge has both ends attached to the same vertex, then we call the edge a *loop*. If an edge has its ends attached to two different vertices, then we call the edge a *link*. A *circle* in G is a simple-closed path in G. A *bond* in G is a minimal set of edges whose removal increases the number of components of G. Given $v \in V(G)$, by $\mathrm{star}_G(v)$ we mean the collection of links in G incident to v. This is called the *vertex star* of v in G. By $\mathrm{star}^+_G(v)$ we mean the collection of all edges in G incident to v. This is called the *augmented vertex star* of v in G. If $X \subseteq E(G)$, then by $G{:}X$ we mean the subgraph of G consisting of the edges in X and the vertices in G incident to edges in X.

Let $C_0(G)$ and $C_1(G)$ denote the $\mathbb{Z}_2$-vector spaces of formal linear combinations of elements of $V(G)$ and $E(G)$, respectively. Note that addition of vectors in these spaces amounts to symmetric difference of the support of those vectors, so for $c \in C_i(G)$, we let c also denote its own support. Let

$$Z_1(G) = \langle c \in C_1(G) : c \text{ is the edge set of a circle in } G\rangle.$$

(Here $\langle v_1, \ldots, v_t\rangle$ denotes the subspace generated by $v_1, \ldots, v_t$.) The subspace $Z_1(G)$ is the *cycle space* of G and we call its elements *cycles.* This subspace is exactly the kernel of the linear map $\partial\colon C_1(G) \to C_0(G)$ defined by $\partial(e) = u + v$ where u and v are the vertices to which the ends of the edge e are attached. It is easy to show that $z \in Z_1(G)$ if and only if $G{:}z$ has an even number of edge ends attached to every vertex. Let

$$B^1(G) = \langle b \subseteq E(G) : b \text{ is a bond of } G\rangle.$$

The subspace $B^1(G)$ is the *coboundary space* of G and we call its elements *coboundaries.* (In parts of the graph-theory literature, *e.g.*, [5], these are called "cocycles.") It is easy to show that the orthogonal complement of $Z_1(G)$ in $C_1(G)$ is $B^1(G)$.

Let $C_0^*(G)$ and $C_1^*(G)$ denote the vector-space duals of $C_0(G)$ and $C_1(G)$, respectively. Since we have chosen bases for $C_0(G)$ and $C_1(G)$, namely $V(G)$ and $E(G)$, respectively, we may identify each $v \in V(G)$ with the corresponding dual element $v^* \in C_0^*(G)$, and each $e \in E(G)$ with the corresponding $e^* \in C_1^*(G)$. The space $B^1(G)$ may thus be viewed as the image of the adjoint map $\partial^*\colon C_0^*(G) \to C_1^*(G)$ which is characterized by $\partial^*(v) = \mathrm{star}_G(v)$ for each vertex v.

Whitney's combinatorial dual graphs are characterized in the above language in Proposition 2.1.

Proposition 2.1. *A graph G' on the same edge set as a graph G is a combinatorial dual graph of G exactly when, for each $S \subseteq E(G) = E(G')$, the subgraph $G{:}S$ is a circle if and only if $G'{:}S$ is a bond.*

Given a graph G with a combinatorial dual graph G', we can construct a chain complex as follows:

$$C_0(G) \quad \xleftarrow{\partial} \quad C_1(G) = C_1(G') \cong C_1^*(G') \quad \xleftarrow{\partial^*} \quad C_0^*(G') \qquad (1)$$

Because of the circle/bond relationship between G and G', the composition $\partial \circ \partial^*$ (suppressing the isomorphism in the middle) is the zero map, and moreover the homology of this complex is 0.

The property that $\partial \circ \partial^* = 0$ for complex (1) of course holds if and only if $\mathrm{im}\, \partial^* \subseteq \ker \partial$. This gives rise to the following generalization of combinatorial dual: An *algebraic dual* to a graph G is a graph G^* with $E(G^*) = E(G)$ and $B^1(G^*) \subseteq Z_1(G)$.

In [1] the authors of the present article offer the following reformulation of Whitney's planarity criterion in terms of vector spaces associated with graphs as well as (1). Its proof also follows as a corollary to the results in Section 5.

Theorem 2.2 (Whitney). *A graph G is planar if and only if there exists an algebraic dual G^* satisfying $Z_1(G) = B^1(G^*)$. Furthermore, if G is planar, then the topological dual graph $G^\perp$ satisfies $Z_1(G) = B^1(G^\perp)$.*

If K is a 2-dimensional cellular complex (or 2-complex, for brevity), write $V(K)$, $E(K)$, and $F(K)$ for its sets of vertices (*i.e.*, 0-cells), edges (*i.e.*, 1-cells) and faces (*i.e.*, 2-cells), respectively. Let $|K|$ denote the geometric realization of K. A 2-complex is 2-*regular* if each edge is either attached to exactly two faces or is attached to one face twice, *i.e.*, an edge of K either appears in two distinct boundary walks once, or twice in one boundary walk. We say that K is *face connected* if for any two faces f and $f' \in F(K)$ there is a sequence of faces $f = f_1, \ldots, f_n = f'$ such that, for each i, f_i and f_{i-1} share a common boundary edge.

Given a 2-regular 2-complex K with 1-skeleton G, there exists a topological dual graph $G^\perp$ constructed as follows: Let $V(G^\perp) = F(K)$ and $E(G^\perp) = E(G)$. An edge e connects distinct vertices f_1 and f_2 in $G^\perp$ (*i.e.*, e is a link) exactly when e is an edge in the boundary walks of distinct faces f_1 and f_2 in G. An edge e is a loop on vertex f in $G^\perp$ exactly when e appears twice in the boundary walk of face f in G. When the imbedding of a graph G in a complex K is understood, we sometimes write $F(G)$ for $F(K)$. When viewing G and $G^\perp$ as subsets of $|K|$, we will presume that each point corresponding to a vertex of $G^\perp$ lies in the interior of the appropriate face of G, and that the two curves in $|K|$ corresponding to an edge of G and $G^\perp$, respectively, cross transversely and only at a single point. Finally, when K is a surface, it is well known that $(G^\perp)^\perp = G$. Proposition 2.3 is easy to prove.

Proposition 2.3. *Given a 2-regular 2-complex K with 1-skeleton G, the topological dual graph $G^\perp$ is an algebraic dual of G.*

An algebraic dual G^* of a graph G is called *component-split* if for each $v \in V(G^*)$, the subgraph $G{:}\mathrm{star}^+_{G^*}(v)$ is connected. If G^* fails to be component-split, it is not difficult to alter G^* so as to produce a component-split algebraic dual: If vertex v of G^* is such that $G{:}\mathrm{star}^+_{G^*}(v)$ has components $G_1, \ldots, G_m$, then replace v by vertices $v_1, \ldots, v_m \notin V(G^*)$ where, for each i, the edge ends incident to v_i are the ends of the edges in $E(G_i)$ incident to v.

For K and G as above, let $C_2(K)$ denote the $\mathbb{Z}_2$-vector space of formal linear combinations of elements of $F(K)$. We also write $C_1(K)$ for $C_1(G)$. Define the *boundary map* $\partial_2 \colon C_2(K) \to C_1(K)$ by mapping each face to the sum of the edges in its boundary walk, and extending by linearity. Write $B_1(K)$ for im ∂. Using the isomorphisms $C_0^*(G^\perp) \cong C_0(G^\perp) = C_2(K)$ and $C_1^*(G^\perp) \cong C_1(G^\perp) = C_1(K)$, we may view ∂_2 and $\partial^* \colon C_0^*(G^\perp) \to C_1^*(G^\perp)$ as having the same domain and codomain. In this view, each vertex of $G^\perp$ is a face of K, and each vertex star of $G^\perp$ is the set of links in the boundary walk of the corresponding face of K. Since ∂_2 is characterized by sending faces to boundary walks and ∂^* is characterized by sending vertices to vertex-stars, we may view $B_1(K)$ and $B^1(G^\perp)$ as equal.

3 Constructing a cellular complex from G and an algebraic dual

Consider a graph G and an algebraic dual G^*. We now describe a method for constructing a 2-regular 2-complex $K(G, G^*)$ which has 1-skeleton G and topological dual graph that is the component split of G^*. Similar to what is done in [7], to each edge in $E(G) = E(G^*)$ assign weight 1 if the edge is a link in G^* and assign weight 2 if the edge is a loop in G^*. A subset $S \subseteq E(G)$ is *weighted Eulerian* if $G{:}S$ is connected and each vertex of $G{:}S$ has even weighted degree, that is, the sum of the weights of the edge ends meeting any vertex in $G{:}S$ is even.

A *walk* in G is a sequence $v_1 e_1 v_2 e_2 \dots e_n v_{n+1}$ where for each i, v_i is a vertex and e_i is an edge with end-point(s) v_i and v_{i+1}. The walk is called *closed* if $v_1 = v_{n+1}$. When S is weighted Eulerian, $G{:}S$ has a closed walk covering all edges of weight 1 exactly once and all edges of weight 2 exactly twice (just as would be the case for a walk along a face boundary of a graph properly imbedded in a pseudosurface). Call such a walk a *weighted Eulerian walk*. Note that if $S \in Z_1(G)$, then each connected component of $G{:}S$ is Eulerian in the usual sense because there is an even number of edge ends incident to each vertex in $G{:}S$.

Let v be a vertex in G^*. Since $\mathrm{star}_{G^*}(v) \in B^1(G^*) \subseteq Z_1(G)$, the connected components of $G{:}\mathrm{star}_{G^*}(v)$ are Eulerian. Since the difference between $\mathrm{star}^+_{G^*}(v)$ and $\mathrm{star}_{G^*}(v)$ consists of the edges that are loops in G^* incident to v, it must be that the connected components of $G{:}\mathrm{star}^+_{G^*}(v)$ are weighted Eulerian.

Construction 3.1. This construction uses as input a graph G and an algebraic dual G^*, and gives as output a 2-regular 2-complex $K(G, G^*)$. Take the 1-skeleton of $K(G, G^*)$ to be the graph G. Write $V(G) = \{v_1, \dots, v_n\}$. Each component of $G{:}\mathrm{star}^+_{G^*}(v_i)$ is weighted Eulerian. Identify the boundaries of 2-cells $F^i_1, \dots, F^i_{k_i}$ with some choice of weighted Eulerian walks in the connected components $C^i_1, \dots, C^i_{k_i}$ of $G{:}\mathrm{star}^+_{G^*}(v_i)$, and let the faces of $K(G, G^*)$ be the cells F^i_j glued to the edges of G.

Proposition 3.2 follows readily follows from Construction 3.1, the definition of a topological dual graph, and the definition of component split.

Proposition 3.2. *If G^* is an algebraic dual graph of G, then G is the 1-skeleton of $K(G, G^*)$ and the topological dual graph of G in $K(G, G^*)$ is the component split of G^*.*

4 An intersection form for curves in pseudosurfaces

Suppose that K has $f(K)$ face-connected components. Then K may be obtained via a sequence of 2-regular 2-complexes $K_0, \dots, K_p = K$ where K_0 is a disjoint union of $f(K)$ closed surfaces and K_i is obtained from K_{i-1} by identifying two distinct points of K_{i-1} to one point (*i.e.*, making a new pinch). Note that the order of pinching is immaterial; since exactly $f(K) - 1$ pinches are required to connect up the $f(K)$ components of

K_0, we may require that $K_{f(K)-1}$ is a connected pseudosurface. By the definition of a 2-regular 2-complex, all pinches are made at vertices of K. Let $G_i^\perp$ denote the topological dual of the 1-skeleton of K_i. Since pinching does not change the incidence of faces with edges, we have that $G_0^\perp = \cdots = G_p^\perp$.

As described in [6, p.221], there is a nondegenerate bilinear form $\langle\, ,\rangle_0$ on the $\mathbb{Z}_2$-homology group $H_1(K_0)$ which, after choosing an appropriate basis for $H_1(K_0)$, corresponds to the block-diagonal matrix

$$\left(\begin{array}{c|c} I_c & 0 \\ \hline 0 & J_{2h} \end{array}\right).$$

Here, c is the number of crosscaps in one realization (up to homeomorphism) for K_0, h is the corresponding number of handles, I_c is the $c \times c$ identity matrix, and J_{2h} is a block-diagonal matrix with h blocks of the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We may apply this intersection form to the cycles in $Z_1(G_0^\perp) = \cdots = Z_1(G_p^\perp)$. Now, the quotient group

$$\frac{Z_1(G_0^\perp)}{B^1(G_0)}$$

is the homology group $H_1(K_0)$ and, because of the requirement that $K_{f(K)-1}$ be connected,

$$\frac{Z_1(G_i^\perp)}{B^1(G_i)} \cong H_1(K_0) \text{ for } 0 \leq i \leq f(K)-1.$$

So now we have that

$$\frac{Z_1(G_i^\perp)}{B^1(G_i)} \cong H_1(K_0) \times \mathbb{Z}_2^{i-f(K)+1} \text{ for } f(K) \leq i \leq p,$$

and the new generators introduced by the pinches are circles in $G_i^\perp = G_0^\perp$ that contract to the pinchpoints (*i.e.*, circles that orbit the pinchpoints). Because all pinchpoints occur at the vertices of G, pinching does not affect the intersections of cycles in $G^\perp$. Thus, extending the basis for $H_1(K_0) = Z_1(G_0^\perp)/B^1(G_0)$ to a basis for $Z_1(G_i^\perp)/B^1(G_i)$ by including linearly independent circles which contract to pinchpoints yields the intersection form $\langle\, ,\rangle_i$ for $Z_1(G_i^\perp)/B^1(G_i)$ with block-diagonal matrix

$$\left(\begin{array}{c|c|c} I_c & 0 & 0 \\ \hline 0 & J_{2h} & 0 \\ \hline 0 & 0 & 0_m \end{array}\right) \quad \text{with} \quad m = i - f(K) + 1.$$

Here 0_m is the $m \times m$ matrix of zeros.

In light of Proposition 3.2, we may apply the intersection form $\langle\, ,\rangle_{K(G,G^*)}$ to the cycle space $Z_1(G^*)$ to obtain topological information about intersections of cycles in G^* on $K(G,G^*)$. In fact, it is possible to define the desired intersection form $\langle\, ,\rangle_{K(G,G^*)}$ on $Z_1(G^*)$ without actually referring to the 2-complex $K(G,G^*)$ because the choice of Eulerian walks in G is equivalent to choosing a rotation system on the vertices of G^*. We will not, however, explicitly make this construction here.

Proposition 4.1. *Let $\langle , \rangle$ be a symmetric bilinear form on a $\mathbb{Z}_2$-vector space V. Then the sets*

$$\begin{aligned} D(V) &:= \{v \in V \mid \forall (x \in V) \langle v, x \rangle = 0\} \\ O(V) &:= \{v \in V \mid \langle v, v \rangle = 0\} \end{aligned}$$

are subspaces of V with $D(V) \subseteq O(V)$.

In light of this proposition, for any symmetric bilinear form $\langle , \rangle$ on a $\mathbb{Z}_2$-vector space V we define the *degeneracy* of $\langle , \rangle$ to be the dimension of $D(V)$ and the *self-orthogonality* to be the dimension of $O(V)$.

Proposition 4.2. *Let K be a connected, 2-regular 2-complex with p pinches whose face-connected components may be expressed, up to homeomorphism, with h handles and c crosscaps.*

(1) The dimension of $Z_1(G^\perp)/B^1(G)$ is $2h + c + p - f(K) + 1$.

(2) The degeneracy and self-orthogonality of the intersection form $\langle\, , \rangle$ for $Z_1(G^\perp)/B^1(G)$ are $p - f(K) + 1$ and $2h + c + p - f(K) + 1 - \min(1, c)$, respectively.

Note that the dimension of $H_1(K(G, G^*))$ is uniquely determined by the graph G and the number of vertices and components of the component split of algebraic dual G^*. The intersection form $\langle\, , \rangle_{K(G,G^*)}$, however, depends on the choice of weighted Eulerian walks in the construction of $K(G, G^*)$. Given G and G^*, there may be different choices possible for the face walks in the construction of $K(G, G^*)$. When and how these variations occur is a topic of interest for future investigations.

5 Imbeddability criteria

Given a pseudosurface P, we define $f(P)$ to be the number of face-connected components in any 2-regular 2-complex K with $P \cong |K|$. This number is well defined as it is actually the rank of the second homology group $H_2(P)$.

Theorem 5.1. *A connected graph G properly imbeds in a closed, connected pseudosurface P with p pinches, $f(P)$ face-connected components, and $\dim(H_1(P)) = n$ if and only if G has a component-split algebraic dual G^* such that*

(1) G^* has $f(P)$ components,

(2) $\dim\left(\dfrac{Z_1(G)}{B^1(G^*)}\right) = n$, and

(3) Eulerian walks can be chosen in the construction of $K(G, G^*)$ so that $\langle\, , \rangle_{K(G,G^*)}$ has degeneracy $d = p - f(P) + 1$.

Proof. Suppose that G properly imbeds in pseudosurface P with p pinches, $f(P)$ face-connected components, and $\dim(H_1(P)) = n$. Then let K be the cellular complex resulting from subdividing P by G, and let $G^* = G^\perp$ (as in Proposition 3.2). First, by the definition of a topological dual, G^* has $f(P)$ components, satisfying property (1). Second, because $B^1(G^*) =$

$B_1(K)$ we have that $\dim\left(\frac{Z_1(G)}{B^1(G^*)}\right) = \dim\left(\frac{Z_1(G)}{B_1(K)}\right) = \dim(H_1(P)) = n$, satisfying property (2). Last, by choosing weighted Eulerian walks in the construction of $K(G, G^*)$ to match the face walks of K, we have $\langle\,,\rangle_{K(G,G^*)} = \langle\,,\rangle_K$ with degeneracy $d = p - f(P) + 1$ (by Proposition 4.2), satisfying property (3).

Conversely, suppose that G has a component-split algebraic dual G^* satisfying (1)–(3). Using Construction 3.1 we construct $K = K(G, G^*)$. By the construction and the fact that G^* has $f(P)$ components, we get that $K(G, G^*)$ has $f(P)$ face-connected components, and because $B_1(K) = B^1(G^*)$ we get that $\dim(H_1(K)) = \dim\left(\frac{Z_1(G)}{B_1(K)}\right) = \dim\left(\frac{Z_1(G)}{B^1(G^*)}\right)$ $= n$. Since $|K|$ is a pseudosurface, it has an intersection form $\langle\,,\rangle$ as described in Section 4. We know that the degeneracy of $\langle\,,\rangle$ is due exclusively to generators of $H_1(K)$ arising from pinches; since we know that K has $f(P)$ face-connected components and the degeneracy of $\langle\,,\rangle$ is d, we deduce from Proposition 4.2 that K has $d + f(P) - 1 = p$ pinches. $\square$

In Theorem 5.1, it is difficult to be more specific about the distribution of handles and crosscaps among the face-connected components of P. We can, however, analyze the algebraic dual G^* componentwise. These components correspond to the face-connected components of $K(G, G^*)$, and Theorem 5.2 tells us more about the handles and crosscaps within a face-connected component.

Theorem 5.2. *A connected graph G properly imbeds in a closed, face-connected pseudosurface P with p pinches, h handles, and c crosscaps if and only if G has a component-split algebraic dual G^* such that*

(1) G^* is connected,

(2) $\dim\left(\dfrac{Z_1(G)}{B^1(G^*)}\right) = 2h + c + p$ with $c, h, p \geq 0$, and

(3) Eulerian walks can be chosen in the construction of $K(G, G^*)$ so that $\langle\,,\rangle_{K(G,G^*)}$ has degeneracy p and self-orthogonality $2h+p+c-\min(1, c)$.

Proof. Suppose that G properly imbeds in a face-connected pseudosurface P with h handles, c crosscaps, and p pinches. Then let K be the cellular complex resulting from subdividing P by G, and let $G^* = G^{\perp}$ (as in Proposition 3.2). That K satisfies conditions (1)–(3) follows by arguments analogous to those in the first paragraph of the proof of Theorem 5.1.

Conversely, suppose that G has a connected component-split algebraic dual satisfying conditions (1)–(3). As in the proof of Theorem 5.1 we construct $K = K(G, G^*)$ and, because G^* is connected, $K(G, G^*)$ is face-connected. Thus $\dim(H_1(K)) = \dim(Z_1(G)/B^1(G^*)) = 2h + c + p$ with $c, h, p \geq 0$. By assumption, the intersection form $\langle\,,\rangle_K$ corresponding to K has degeneracy p. Since G^* is connected and K is face-connected, Proposition 4.2 implies that the degeneracy of $\langle\,,\rangle_K$ equals the number of pinches in K. Thus K has p pinches.

By assumption, the self-orthogonality of $\langle\,,\rangle_K$ is $2h+p+c-\min(1, c)$. Let K' be the 2-regular 2-complex obtained after releasing the p pinches of K; since K is face connected, K' is a surface. We have $\dim(H_1(K')) =$

$2h+c$ and $\langle\,,\rangle_{K'}$ has self-orthogonality $s = 2h + c - \min(1, c)$. This self-orthogonality either equals $\dim(H_1(K'))$ (iff $c = 0$) or equals $\dim(H_1(K'))-1$ (iff $c \geq 1$). When $c = 0$ we know that $|K'|$ is orientable and must have h handles. In the case that $c \geq 1$ we know that $|K'|$ is nonorientable and may be represented up to homeomorphism by $2h + c$ crosscaps. We can then use the standard homeomorphism to exchange three crosscaps for a handle and crosscap, showing that K' will have h handles and c crosscaps. It follows that we know, up to homeomorphism, exactly the number of crosscaps and handles of K', and thus of K as well. □

Even in Theorem 5.2 we are not able to distinguish between imbeddability in homotopically equivalent pseudosurfaces with different configurations of pinches (e.g., a sphere with a triple pinch and a sphere with two double pinches); however, different configurations of pinches do actually affect imbeddability. For example, in [14, p.32–33] the author displays an imbedding of K_6 in the sphere with two double pinches but argues that K_6 does not imbed in the sphere with one triple pinch because releasing the triple pinch in some assumed imbedding of K_6 would yield an imbedding of K_5 in the sphere, which is impossible. Whether or not algebraic techniques could distinguish between such cases is unknown.

Corollaries 5.3 and 5.4 of Theorem 5.2 characterize imbeddability in orientable and nonorientable surfaces. They may also be used to distinguish between imbeddability in the orientable surface of even Euler characteristic $2k$ from imbeddability in the nonorientable surface with the same characteristic. This is because the intersection form for an orientable surface Σ has self-orthogonality $\dim(H_1(\Sigma))$ and the intersection form for a nonorientable surface Σ' has self-orthogonality $\dim(H_1(\Sigma'))-1$.

Corollary 5.3. *A connected graph G properly imbeds in the closed, orientable surface with h handles if and only if G has a component-split algebraic dual G^* such that*

(1) G^* is connected,

(2) $\dim\left(\dfrac{Z_1(G)}{B^1(G^*)}\right) = 2h$ and

(3) Eulerian walks can be chosen in the construction of $K(G, G^*)$ so that $\langle\,,\rangle_{K(G,G^*)}$ has self-orthogonality $2h$.

Corollary 5.4. *A connected graph G properly imbeds in the closed, nonorientable surface with c crosscaps if and only if G has a component-split algebraic dual G^* such that*

(1) G^* is connected,

(2) $\dim\left(\dfrac{Z_1(G)}{B^1(G^*)}\right) = c$ and

(3) Eulerian walks can be chosen in the construction of $K(G, G^*)$ so that $\langle\,,\rangle_{K(G,G^*)}$ has self-orthogonality $c-1$.

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