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Dan Slilaty
Wright State University - Main Campus, daniel.slilaty@wright.edu

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Integer functions on the cycle space and edges of a graph

Daniel C. Slilaty*

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Abstract

A directed graph has a natural $\mathbb{Z}$-module homomorphism from the underlying graph's cycle space to $\mathbb{Z}$ where the image of an oriented cycle is the number of forward edges minus the number of backward edges. Such a homomorphism preserves the parity of the length of a cycle and the image of a cycle is bounded by the length of that cycle. Pretzel and Youngs [1] showed that any $\mathbb{Z}$-module homomorphism of a graph's cycle space to $\mathbb{Z}$ that satisfies these two properties for all cycles must be such a map induced from an edge direction on the graph. In this paper we will prove a generalization of this theorem and an analogue as well.

1 Introduction

We begin with two paragraphs of definitions that we need to state our main results. These definitions and a few more in the next section will be all definitions necessary to read the paper aside from a basic familiarity with graph-theory terminology. All graphs are finite and by the term *homomorphism* we mean a $\mathbb{Z}$-module homomorphism. An oriented edge e in a graph is an edge of the graph with a given orientation. The reverse orientation of e is denoted $-e$. The $\mathbb{Z}$-module of 1-*chains* of G is $C_1(G) = \langle e : e \text{ is an oriented edge in } G\rangle$ in which we can say $-e = (-1)e$. The usual notion of a directed graph with underlying graph G can be regarded as a homomorphism $\varphi : C_1(G) \to \mathbb{Z}$ in which each $\varphi(e) \in \{-1, +1\}$. The usual notion of a mixed graph with underlying graph G can be regarded as a homomorphism $\varphi : C_1(G) \to \mathbb{Z}$ in which each $\varphi(e) \in \{-1, 0, +1\}$. We define a k-*direction* of G as a homomorphism $\varphi : C_1(G) \to \mathbb{Z}$ in which each $|\varphi(e)| \leq k$. We say that a k-direction φ is *odd* when k is odd and each $\varphi(e)$ is odd.

A *walk* W in G is a sequence of oriented edges $e_1, \dots, e_n$ where for each $i \in \{1, \dots, n-1\}$, the head of e_i is the tail of e_{i+1}. When the tail of e_1 is u and the head of e_n is v, we call W a uv-walk. When $u = v$, then we say W is a *closed* walk. The reverse walk $-W = -e_n, \dots, -e_1$. By $|W|$ we mean the length of the walk W which is n and we misuse notation and also write $W = \sum_i e_i \in C_1(G)$. The submodule $Z_1(G) = \langle W : W \text{ is a closed walk in } G\rangle$ of $C_1(G)$ is often called the *cycle space* of G or the space of 1-*cycles* of G. It is well known that $Z_1(G)$ is generated by the closed walks that correspond to the cycles (i.e., 2-regular connected subgraphs) of G. We denote $\varphi : C_1(G) \to \mathbb{Z}$ restricted to the domain $Z_1(G)$ by $\widehat{\varphi}$. When $\varphi : C_1(G) \to \mathbb{Z}$ and $\delta : C_1(G) \to \mathbb{Z}$ satisfy $\widehat{\varphi} = \widehat{\delta}$ we say that φ and δ are *equivalent*.

Theorem 1.2 is a generalization of Theorem 1.1 and Theorem 1.3 is an analogue of Theorem 1.2 for arbitrary positive integers. To prove Theorems 1.2 and 1.3 we simply adapt the techniques of the Pretzel and Young's proof in [1] to our more general setting and then, perhaps surprisingly, the details of the proof are nearly the same. Theorems 1.2 and 1.3 may be of some interest to those who study integer gain graphs. See [2] for an introduction to gain graphs.

Theorem 1.1 (Pretzel and Youngs [1])**.** *Let G be a graph.*

*Department of Mathematics and Statistics, Wright State University, Dayton, OH 45435. Email: daniel.slilaty@wright.edu.
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(1) *If φ is an odd 1-direction of G, then for each walk W in G*

   (a) $|\varphi(W)| \leq |W|$ *and*

   (b) $\varphi(W) \equiv |W| \mod 2$.

(2) *If a homomorphism $f : Z_1(G) \to \mathbb{Z}$ satisfies conditions (a) and (b) above for each closed walk W in G, then there is an odd 1-direction φ on G for which $\widehat{\varphi} = f$.*

Theorem 1.2. *Let G be a graph and k be an odd positive integer.*

(1) *If φ is an odd k-direction of G, then for each walk W in G*

   (a) $|\varphi(W)| \leq k|W|$ *and*

   (b) $\varphi(W) \equiv |W| \mod 2$.

(2) *If a homomorphism $f : Z_1(G) \to \mathbb{Z}$ satisfies conditions (a) and (b) above for each closed walk W in G, then there is an odd k-direction φ on G for which $\widehat{\varphi} = f$.*

Theorem 1.3. *Let G be a graph and k be a positive integer.*

(1) *If φ is a k-direction of G, then for each walk W in G, $|\varphi(W)| \leq k|W|$.*

(2) *If a homomorphism $f : Z_1(G) \to \mathbb{Z}$ satisfies $|\varphi(W)| \leq k|W|$ for each closed walk W in G, then there is a k-direction φ on G for which $\widehat{\varphi} = f$.*

2 Lemmas and proofs

Let φ be a k-direction of G and v a vertex with incident links $e_1, \dots, e_n$ (a *link* is an edge that is not a loop) all oriented away from v. Say that v is *pushable for φ* if each $\varphi(e_i) < k$. When v is pushable for φ we define the *pushdown* δ of φ at v as the k-direction on G defined by $\delta(e_i) = \varphi(e_i) + 1$ for each $i \in \{1, \dots, n\}$ and $\delta(e) = \varphi(e)$ when $e \notin \{\pm e_1, \dots, \pm e_n\}$. Evidently δ is a k-direction equivalent to φ. We say that a walk W is *directed* for φ when $\varphi(W) = k|W|$. Of course, W is a directed walk iff each oriented edge e in W has $\varphi(e) = k$. When φ has no directed closed walks, we say φ is *acyclic*. Note that when φ is acyclic, there must exist a pushable vertex, because if an arbitrarily chosen vertex u_1 is not pushable, then there is a link f_1 from u_1 to u_2 with $\varphi(f_1) = k$. If u_2 is not pushable, then there is a link f_2 from u_2 to u_3 with $\varphi(f_2) = k$. Since G is finite, this process repeats until either a pushable vertex is found or a directed closed walk is found. But since G is acyclic, we will eventually find a pushable vertex.

Lemma 2.1. *If φ is a k-direction of G, e an oriented link with tail u and head v, $\varphi(e) < k$, and φ has no directed uv-walk, then there is a k-direction δ equivalent to φ such that $\delta(e) = \varphi(e) + 1$.*

Proof. Let C be the collection of edges of G that appear as oriented edges in directed closed walks. If $X \subseteq C$ is the edge set of a connected component of the subgraph of G corresponding to C, then there is a directed closed walk W whose edges traversed are exactly X. This is because any two directed closed walks that intersect at some vertex can be concatenated into one directed closed walk. Thus φ restricted to G/C (by G/C we mean the graph obtained from G by contracting the edges of C) is acyclic because any directed closed walk in G/C could be lifted to a directed closed walk of φ in G by again concatenating walks. Furthermore, since there is no directed uv-walk for φ in G, e is not a loop in G/C. Call the endpoints of e in G/C corresponding to u and v, respectively, u' and v'. Since there is no directed uv-walk for φ in G, there can be no directed $u'v'$-walk for φ in G/C because any such walk can be lifted to a directed uv-walk in G.

Now since φ on G/C is acyclic, there must be a pushable vertex for φ on G/C. Furthermore, there must be a pushable vertex besides v' or else we can construct a directed $u'v'$-path in G/C by a similar argument as that preceding this lemma, which would make a contradiction. So choose pushable vertex $u_1 \neq v'$ and let φ_1 be the pushdown of φ at u_1 on G/C. So now $\widehat{\varphi}_1 = \widehat{\varphi}$ on G/C but we can lift φ_1 to G by reinstating the original directions of φ on C and get $\widehat{\varphi}_1 = \widehat{\varphi}$ on G. So if $u_1 = u'$, then we are done. If $u_1 \neq u'$, then we note that there can be no directed $u'v'$-walk for φ_1 because if W were such a walk, then because $\varphi_1(e) = \varphi(e)$ and $\widehat{\varphi}_1 = \widehat{\varphi}$ we get

$$\begin{aligned} \varphi(W - e) &= \varphi_1(W - e) \\ \varphi(W) - \varphi(e) &= \varphi_1(W) - \varphi(e) \\ \varphi(W) &= k|W| \end{aligned}$$

which tells us that W is a directed $u'v'$-walk for φ, a contradiction. So again we have a pushable vertex $u_2 \neq v'$ for φ_1 and we let φ_2 be the pushdown of φ_1 at v_2. As before if $u_2 = u'$, then because $\widehat{\varphi}_2 = \widehat{\varphi}_1 = \widehat{\varphi}$ on G/C we are done. If not, then as before there is no directed $u'v'$-walk for φ_2 and we iterate this process again. This process will either halt with a pushdown at u' or we can keep pushing down vertices besides u' and v' indefinitely. The latter case does not happen by the following argument which completes our proof.

Let D_i be the vertices of G/C at a distance i from u'. Before any pushdown at u' there can be at most $2k$ pushdowns of each vertex in D_1 because we cannot exceed a value of k for any oriented edge. Thus only a finite number of pushdowns in D_1 are possible before a pushdown at u' can occur. Now given only a finite number of pushdowns possible on vertices in D_i before a pushdown at u', there can for the same reason only be a finite number of pushdowns at the vertices in D_{i+1} before a pushdown at u'. $\square$

Let φ be an odd k-direction of G and v a pushable vertex of φ. Since φ is odd we actually get that each link e incident to v and oriented away from v satisfies $\varphi(e) \leq k - 2$. We define the *double pushdown* δ of φ at v as the odd k-direction δ defined by $\delta(e) = \varphi(e) + 2$ for an oriented link with v as its tail and $\delta(e) = \varphi(e)$ for any link not incident to v and any loop. Evidently δ is an odd k-direction equivalent to φ. Lemma 2.2 follows by the same proof as Lemma 2.1 except that we use double pushdowns in place of pushdowns.

Lemma 2.2. *If φ is an odd k-direction of G, e an oriented link with tail u and head v, $\varphi(e) < k$, and φ has no directed uv-walk, then there is an odd k-direction δ equivalent to φ such that $\delta(e) = \varphi(e) + 2$.*

Lemma 2.3. *If φ is a k-direction of G and W is a uv-walk with $u \neq v$, then either*

(1) there is a directed uv-walk for φ or

(2) there is a k-direction δ equivalent to φ with $\delta(W) = \varphi(W) + 1$.

Proof. The proof is by induction on $|V(G)| + d_\varphi(W)$ where $d_\varphi(W) = k|W| - \varphi(W)$. If W contains only one distinct edge up to orientation, then our result follows by Lemma 2.1. So we can assume there are at least two distinct edges in W up to orientation. Choose an oriented edge e in W with $\varphi(e)$ as large as possible. Say that the head of e is h and the tail is t. Now φ restricted to G/e inductively satisfies our conclusion. If (2) holds in G/e, then δ can be lifted to a k-direction of G equivalent to φ by setting $\delta(e) = \varphi(e)$ and so (2) holds for G. So assume that (1) holds for G/e and let P be a directed u_1v_1-walk for φ in G/e where u_1 and v_1 are the images of u and v in G/e. Now either P lifts to a uv-walk in G in which case (1) holds for G or P lifts to two walks in G, say P_1 and P_2 where u is the initial vertex of P_1 and v is the terminal vertex of P_2. When P_1 has terminal vertex t and P_2 has initial vertex h call this Configuration A and when P_1 has terminal vertex h and P_2 has initial vertex t call this Configuration B. Let W_1 and W_2 be, respectively, the ut- and hv-walks in W In Case 1 say that $\varphi(e) = -k$, in Case 2 say that $-k < \varphi(e) < k$, and in Case 3 say that $\varphi(e) = k$.

Case 1: In Configuration B, we get that $P_1 - e + P_2$ is a directed uv-walk. So assume we have Configuration A and note that each $-W_i$ is a directed walk because otherwise we would have chosen e so that $\varphi(e) > -k$. Now if there is a directed th-walk Q for φ, then $P_1 + Q + P_2$ is a directed uv-walk for φ, otherwise we apply Lemma 2.1 to e to get k-direction δ equivalent φ with $\delta(e) = \varphi(e) + 1$. Now each $W_i - P_i$ is a directed closed walk for φ and so again is a directed closed walk for δ. Thus

$$\delta(W) = \delta(W_1 + e + W_2) = \delta(W_1) + \delta(e) + \delta(W_2) = -k|W_1| + \varphi(e) + 1 - k|W_2| = \varphi(W) + 1,$$

as required.

Case 2: In Case 2.1 we consider Configuration A and in Case 2.2 we consider configuration B. Furthermore in Case 2.2 we add the assumption that $-k < \varphi(e) \leq k$ rather than just $-k < \varphi(e) < k$.

Case 2.1: Since $|W| \geq 2$, $\varphi(e) < k$ is a maximum for oriented edges in W, and P is a directed walk for φ on G/e, we get that $d_\varphi(P_1 + e + P_2) < d_\varphi(W)$. So inductively there is either a directed uv-walk for φ in G (and so we are done) or there is k-direction δ equivalent to φ such that $\delta(P_1 + e + P_2) = \varphi(P_1 + e + P_2) + 1$. So now since δ and φ are equivalent we have

$$\begin{aligned} \delta(W - (P_2 + e + P_1)) &= \varphi(W - (P_2 + e + P_1)) \\ \delta(W) - \delta(P_2 + e + P_1) &= \varphi(W) - \varphi(P_2 + e + P_1) \\ \delta(W) - \varphi(P_2 + e + P_1) - 1 &= \varphi(W) - \varphi(P_2 + e + P_1) \\ \delta(W) &= \varphi(W) + 1, \end{aligned}$$

which satisfies (2).

Case 2.2: If $d_\varphi(W_1) = 0$, then $W_1 + P_2$ is a directed uv-walk. So now if $d_\varphi(W_1) > 0$, then $d_\varphi(P_1 + W_2) < d_\varphi(W)$ and so by induction there is either a directed uv-walk for φ (and so we are done) or there is k-direction δ equivalent to φ such that $\delta(P_1 + W_2) = \varphi(P_1 + W_2) + 1$. By equivalence we now get

$$\begin{aligned} \delta(W_1 + P_2 - W_2 - P_1) &= \varphi(W_1 + P_2 - W_2 - P_1) \\ \delta(W_1 + P_2) - \delta(W_2 + P_1) &= \varphi(W_1 + P_2) - \varphi(W_2 + P_1) \\ \delta(W_1 + P_2) - \varphi(W_2 + P_1) - 1 &= \varphi(W_1 + P_2) - \varphi(W_2 + P_1) \\ \delta(W_1 + P_2) &= \varphi(W_1 + P_2) + 1. \end{aligned}$$

and so now for each $\{i, j\} = \{1, 2\}$ we have

$$\begin{aligned} \delta(W_j + P_i) &= \varphi(W_j + P_i) + 1 \\ \delta(W_j) + \delta(P_i) &= \varphi(W_j) + \varphi(P_i) + 1 \\ \delta(W_j) &= \varphi(W_j) + k|P_i| - \delta(P_i) + 1 \\ \delta(W_j) &= \varphi(W_j) + d_\delta(P_i) + 1. \end{aligned}$$

Now using equivalence and the above calculation we get

$$\begin{aligned} \delta(W - P_2 + e - P_1) &= \varphi(W - P_2 + e - P_1) \\ \delta(W_1 + W_2) - \delta(P_1 + P_2) + 2\delta(e) &= \varphi(W_1 + W_2) - \varphi(P_1 + P_2) + 2\varphi(e) \\ \varphi(W_1 + W_2) + d_\delta(P_1 + P_2) + 2 - \delta(P_1 + P_2) + 2\delta(e) &= \varphi(W_1 + W_2) - k|P_1 + P_2| + 2\varphi(e) \\ 2\delta(e) &= 2\varphi(e) - k|P_1 + P_2| + \delta(P_1 + P_2) - d_\delta(P_1 + P_2) - 2 \\ 2\delta(e) &= 2\varphi(e) - 2d_\delta(P_1 + P_2) - 2 \\ \delta(e) &= \varphi(e) - d_\delta(P_1 + P_2) - 1 \end{aligned}$$

and so then finally we have

$$\begin{aligned} \delta(W) &= \delta(W_1 + W_2) + \delta(e) \\ &= \varphi(W_1 + W_2) + d_\delta(P_1 + P_2) + 2 + \varphi(e) - d_\delta(P_1 + P_2) - 1 \\ &= \varphi(W_1 + W_2) + \varphi(e) + 1 \\ &= \varphi(W) + 1, \end{aligned}$$

as required.

Case 3: In Case 2.2 we included the possibility that $\varphi(e) = k$ in Configuration B and so we have Configuration A with $\varphi(e) = k$ which has $P_1 + e + P_2$ as a directed uv-walk. □

Lemma 2.4 has the analogous proof to Lemma 2.3 with Lemma 2.2 cited in the place of Lemma 2.1.

Lemma 2.4. *If φ is an odd k-direction of G and W is a uv-walk with $u \neq v$, then either*

(1) there is a directed uv-walk for φ or

(2) there is an odd k-direction δ equivalent to φ with $\delta(W) = \varphi(W) + 2$.

Proof of Theorem 1.3. We may assume that G is connected. The proof will be by induction on the number of edges with the base case being when G is a spanning tree for which our result immediately follows. So now take any edge e in G and inductively, there is a k-direction φ on $G\backslash e$ (by $G\backslash e$ we mean G with the edge e deleted) for which $\widehat{\varphi} = f$ on $G\backslash e$. If e is a loop in G, then φ extends to all of G because $|f(e)| \leq k$ for any oriented loop e is part of our hypothesis. So say e is a link with head endpoint h and tail endpoint t. If W and X are any ht-walks in $G\backslash e$, then because $W - X$ is a closed walk we get

$$\begin{aligned} f(W - X) &= \varphi(W - X) \\ f(W + e - e - X) &= \varphi(W - X) \\ f(W + e) - f(X + e) &= \varphi(W) - \varphi(X) \\ f(W + e) - \varphi(W) &= f(X + e) - \varphi(X). \end{aligned}$$

Thus is well defined to set the *discrepancy* for e with respect to φ as $d_{f,\varphi}(e) = f(W + e) - \varphi(W)$ where W is any ht-walk in $G\backslash e$. Note that such a walk must exist because the base case is for a spanning tree. So now if $d_{f,\varphi}(e) \in \{-k, \ldots, k\}$, then we extend φ to all of G by setting $\varphi(e) = d_{f,\varphi}(e)$ and we get that $\widehat{\varphi} = f$ on all of G.

If $d_{f,\varphi}(e) > k$, then we note that there can be no directed ht-walk for φ in $G\backslash e$ because any such directed walk P makes

$$d_{f,\varphi}(e) = f(P + e) - \varphi(P) = f(P + e) - k|P| \leq k(|P| + 1) - k|P| = k,$$

a contradiction. So then if we choose any ht-walk W in $G\backslash e$, then we can apply Lemma 2.3 to get k-direction φ_1 equivalent to φ on $G\backslash e$ for which

$$d_{f,\varphi_1}(e) = f(P + e) - \varphi_1(P) = f(P + e) - \varphi(P) - 1 = d_{f,\varphi}(e) - 1.$$

So now if $d_{f,\varphi_1}(e) \leq k$, then we are done. If not, then as before there can be no directed ht-walk for φ_1 and so we can keep repeating this process until we get φ_n equivalent to φ on $G\backslash e$ with $d_{f,\varphi_n}(e) \leq k$.

If $d_{f,\varphi}(e) < -k$, then $d_{f,\varphi}(-e) = f(-P - e) - \varphi(-P) = -d_{f,\varphi}(e) > k$ and so we obtain our result as in the previous paragraph. □

Proof of Theorem 1.2. We set up our induction as in the proof of Theorem 1.3 and then note that $d_{f,\varphi}(e) = f(W+e) - \varphi(W)$ must be odd. Thus we can follow the analogous proof to that for Theorem 1.3 using Lemma 2.4 in the place of Lemma 2.3. □

References

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[2] Thomas Zaslavsky, *Biased graphs. I. Bias, balance, and gains*, J. Combin. Theory Ser. B **47** (1989), no. 1, 32–52.