3-2011

Projective-Planar Graphs With No K3,4-Minor

John Maharry

Dan Slilaty

Wright State University - Main Campus, daniel.slilaty@wright.edu

Repository Citation

Maharry, J., & Slilaty, D. (2011). Projective-Planar Graphs With No K3,4-Minor. *Journal of Graph Theory, 70* (2), 121-134.

https://corescholar.libraries.wright.edu/math/315

Projective-planar graphs with no $K_{3,4}$-minor

John Maharry * **Daniel Slilaty** †

December 28, 2010

Abstract

An exact structure is described to classify the projective-planar graphs that do not contain a $K_{3,4}$-minor.

1 Introduction

There are several graphs H for which the precise structure of graphs that do not contain a minor isomorphic to H is known. In particular, such structure theorems are known for K_5 [13], V_8 [8] and [7], the cube [3], the octahedron [4], and several others. Such characterizations can often be very useful, e.g., Hadwiger's conjecture for $k = 4$ is verified by using the structure for K_5-free graphs, and the structure theorem for V_8-free graphs is used to characterize how projective-planar graphs may be re-embedded in the projective plane [5].

Characterizations of K_6-free graphs and Petersen-free graphs are highly sought-after results, mostly due to their connections with Hadwiger's conjecture and Tutte's 4-flow conjecture. Such characterizations seem to be very difficult. The Petersen graph and K_6 belong to a collection of seven graphs known as the Petersen Family of graphs (see [9]). They are all graphs obtained by sequences of $Y\Delta$ and ΔY operations on the Petersen graph.

The difficulty of characterizing $K_{3,4}$-free graphs seems to lie between the characterizations of H-free graphs mentioned in the first paragraph and H-free graphs for H in the Petersen Family. In this paper we give an exact structure for projective-planar graphs that are $K_{3,4}$-free (Theorem 3.4 along with Propositions 3.1 and 3.2). The authors hope that this might be a first step in a complete structure theorem of $K_{3,4}$-free graphs. The non-projective-planar $K_{3,4}$-free graphs might be characterized using the known list of 35 minor-minimal non-projective planar graphs in [1] and [2].

Another possible point of interest for characterizing $K_{3,4}$-free graphs might be the following. A k-separation (B_1, B_2) in a graph G is called *flat* if the subgraph of G induced by some B_i along with a vertex of degree k attached to the k vertices of $V(B_1) \cap V(B_2)$ is a planar graph. In Section 3 we will see that a 3-connected graph G is $K_{3,4}$-free iff every 3-separation in every 3-connected minor of G is flat.

*Dept. of Mathematics, The Ohio State University, Columbus OH 43210, `maharry@math.ohio-state.edu`
†Dept. of Mathematics and Statistics, Wright State University, Dayton OH 45435, `daniel.slilaty@wright.edu`

2 Preliminaries

The *representativity* (or *face width*) of an embedding of a graph in a surface is the minimum number of intersection points of the graph and a noncontractible curve in the surface. An embedding with representativity k is called a *k-representative embedding.*

Suppose that a graph G has a 2-representative embedding on the projective plane. Then some noncontractible curve γ is entirely contained in two faces and intersects the graph in precisely two vertices, say A and B, as shown in Figure 1.

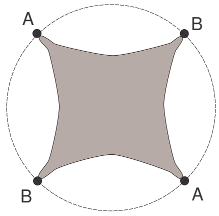


Figure 1.
A 2-representative embedding in the projective plane.

Given a simple graph H, a graph G is said to *contain an H-minor* if for each vertex of $v \in V(H)$ there corresponds a connected subgraph $\nu(v) \subseteq V(G)$ such that the subgraphs $\nu(v)$ are pairwise disjoint and for any u and v that are adjacent in H, there exists $u' \in \nu(u)$ and $v' \in \nu(v)$ that are adjacent in G. If G does not contain an H-minor, we will say that G is *H-free*. Given $N = \{v_1, v_2, \ldots, v_k\} \subset V(H)$ and a set $M = \{v'_1, v'_2, \ldots, v'_k\} \subset V(G)$, we will say that G contains an *H-minor with N rooted on M* if G contains an H-minor such that each $v'_i \in \nu(v_i)$.

Given a subgraph H of a graph G an *H-bridge* is either an edge outside of H whose endpoints are both in H or a connected component of $G \setminus V(H)$ along with the edges that connect that component to H. Given an H-bridge B the *vertices of attachment* are the vertices of B that are in H. The edges of B incident to the vertices of attachment are called *legs*. A bridge with n vertices of attachment is also called an *n-bridge*.

If S is a subdivision of a graph G where G has no vertices of degree 2, then a path in S that corresponds to an edge in G is called a *branch* of S. A vertex of S corresponding to a vertex of G is called a *branch vertex* of S.

Given a set of edges X in G, let $V(X)$ denote the vertices of G incident to edges in X. For $k \geq 0$, a *k-separation* in G is a bipartition (A_1, A_2) of the edges of G with nonempty parts such that each $|A_i| \geq k$ and and $|V(A_1) \cap V(A_2)| = k$. The k-separation is called *vertical* when $V(A_1) \setminus V(A_2) \neq \emptyset$ and $V(A_2) \setminus V(A_1) \neq \emptyset$. A graph on at least $k+1$ vertices is called *vertically k-connected* when every vertical t-separation has $t \geq k$. We use vertical k-connectivity rather than Tutte-k-connectivity to allow for loops and multiple edges. A vertically 3-connected graph G is *almost 4-connected* when any vertical 3-separation (A_1, A_2) has some $|V(A_i)| = 4$. This is similar to but weaker than the usual notion of internal 4-connectivity.

3 Patch graphs and $K_{3,4}$-free graphs

Summing and $K_{3,4}$-free graphs Given two graphs G_1 and G_2, a 1-sum $G_1 \oplus_1 G_2$ is the identification of G_1 and G_2 along some specified vertex and a 2-sum $G_1 \oplus_2 G_2$ is obtained by identifying G_1 and G_2 along some specified link and then deleting that link. (A *link* is an edge that is not a loop). If G_1 and G_2 both contain a 3-valent vertex, then a *Y-sum* $G_1 \oplus_Y G_2$ is obtained by identifying the neighbors of these 3-valent vertices in some specified ordering and then removing the 3-valent vertices. Propositions 3.1 and 3.2 reduce the problem of classifying $K_{3,4}$-free graphs to classifying almost-4-connected $K_{3,4}$-free graphs. The proof of the first is easy.

Proposition 3.1. *For each $k \in \{1,2\}$, G and H are both vertically k-connected and $K_{3,4}$-free if and only if $G \oplus_k H$ is vertically k-connected and $K_{3,4}$-free.*

Proposition 3.2.

(1) If G is simple, vertically 3-connected, and $K_{3,4}$-free, then G is obtained by taking one simple, almost 4-connected, $K_{3,4}$-free graph and then taking Y-sums with planar graphs with possible subdivisions of edges before each sum.

(2) If G is $K_{3,4}$-free and P is planar, then $G \oplus_Y P$ is $K_{3,4}$-free.

Proof. (1) If G is $K_{3,4}$-free and almost 4-connected, then our result follows. So suppose that G is $K_{3,4}$-free and vertically 3-connected. Thus $G = H_1 \oplus_Y H_2$ for some graphs H_1 and H_2 on at least 5 vertices each. Since G is vertically 3-connected and H_i has at least 5 vertices, it can be shown that H_i is vertically 3-connected except possibly for some 2-valent vertices adjacent to the 3-valent vertex along which the Y-sum is taken. Using Theorem 3.3 shown below, it cannot be that H_1 and H_2 are both nonplanar. So $G = H \oplus_Y P$ where P is planar.

Now if H is planar, then evidently G is planar as well. If H is not planar, then H must be $K_{3,4}$-free because otherwise any minimal subgraph H' of H that contracts to $K_{3,4}$ would by the vertical 3-connectivity of P have a corresponding subgraph H'' of $G = H \oplus_Y P$ that contracts to $K_{3,4}$, a contradiction. We may now smooth degree-2 vertices and iterate this process on H to get our result.

(2) Suppose that G is $K_{3,4}$-free but $G \oplus_Y P$ has a $K_{3,4}$-minor. If H is a minimal subgraph of $G \oplus_Y P$ that contracts to $K_{3,4}$, then one can show that $H \cap P$ is either a path or a subdivision of $K_{1,3}$. Thus we can then get a corresponding subgraph H' of G, that contracts to $K_{3,4}$, a contradiction. □

Theorem 3.3 (Truemper [11, 10.3.9]). *Let v be a 3-valent vertex of a vertically 3-connected non-planar graph. Then there is a $K_{3,3}$-subdivision in G that contains v as a branch vertex.*

Patch graphs Given a graph G embedded in the projective plane, certain faces with boundary cycles of length four will be designated as *patches*. Patches are drawn as shaded regions in the interior of their faces. A *patch graph* is a pair $(G, \mathcal{P})$ where G is a embedding of a graph in the projective plane with designated vertices A and B on a 2-representative cut and $\mathcal{P}$ is a collection of patches (possibly empty) which together are constructed iteratively as follows. We call the patch graph $(4K_2, \{P_0\})$ shown on the left in Figure 2 the *initial patch graph*.

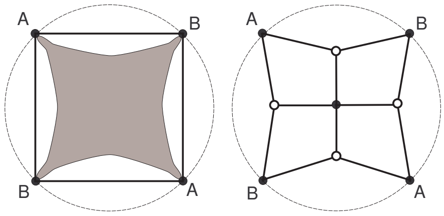


Figure 2.
The initial patch graph and the topologically unique embedding of $K_{3,4}$ on the projective plane.

Now all patch graphs are constructed from $(4K_2, \{P_0\})$ by applying sequences of operations H, Y, X, and I. Each of these operations replaces a patch with the respective configuration shown in Figure 3 or a configuration obtained by rotating or flipping the interior of the patch while leaving the boundary fixed. Since operations X and I remove a patch and do not introduce any new ones we call them *terminal patching operations*.

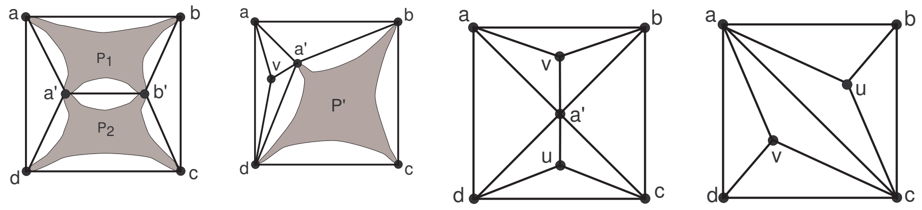


Figure 3.
Operations H, Y, X, and I, respectively.

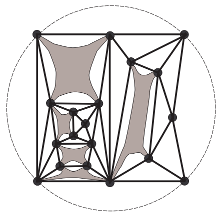


Figure 4.
An example of a subgraph of a patch graph.

Main result Our main result is Theorem 3.4 which along with Propositions 3.1 and 3.2 give a complete characterization of $K_{3,4}$-free projective-planar graphs.

Theorem 3.4 (Main Result)**.** *If H is a simple, nonplanar, and almost-4-connected projective-planar graph that is $K_{3,4}$-free, then $H \cong K_6$ or H is a subgraph of a patch graph. Furthermore, all patch graphs are $K_{3,4}$-free.*

In Section 3.1 we show that every 2-representative, almost 4-connected, $K_{3,4}$-free graph in the projective plane is a subgraph of a patch graph and in Section 3.2 we show that all patch graphs are $K_{3,4}$-free. For a 3-representative or higher almost 4-connected $K_{3,4}$-free graph in the projective plane we have Theorem 3.5.

Theorem 3.5. *If G is vertically 3-connected, simple, and $K_{3,4}$-free and G has an embedding in the projective plane with representativity at least 3, then $G \cong K_6$.*

Proof. It is shown by Vitray [12] and also by Randby [6], that if G embeds in the projective plane with representativity at least 3, then G contains as a minor one of the six projective-planar graphs from the Petersen Family (see [9] for a listing of these seven graphs). One can easily check that each of these seven graphs except for K_6 has a $K_{3,4}$-minor. Thus G has a K_6-minor. Now there is no vertically 3-connected and simple single-edge extension of K_6 and only one vertically 3-connected and simple single-edge decontraction of K_6 (up to isomorphism). This decontraction clearly contains a $K_{3,4}$-minor and so by the Splitter Theorem [10] we get that $G \cong K_6$. □

3.1 $K_{3,4}$-free graphs and patch graphs

Theorem 3.6 along with Theorem 3.5 prove the first part of Theorem 3.4.

Theorem 3.6. *If R is almost 4-connected, simple, and $K_{3,4}$-free and R has a 2-representative embedding on the projective plane, then R is planar or a subgraph of a patch graph.*

Proposition 3.7. *If $(G, \mathcal{P})$ is a patch graph and $P \in \mathcal{P}$, then there are four disjoint (some possibly trivial) paths in G from the four corners of the patch P to the vertices A and B on the boundary.*

Proof. Our statement is true for the initial patch graph $(4K_2, \{P\})$. Now assume that the statement holds for some arbitrary patch graph $(G, \mathcal{P})$ and that the patch graph $(G', \mathcal{P}')$ is obtained from $(G, \mathcal{P})$ by patching operation Y or H. (We need not consider the terminal patching operations as they do not introduce any new patches or alter the graph outside the patch.) For each of these operations, there are clearly paths from the vertices of any new patch to the vertices of the original patch and these paths can be extended to paths to the vertices on the boundary. We need not consider any old patches as the patching operation does not alter the graph outside of the patch. $\square$

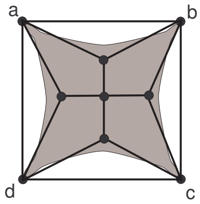


Figure 5.

Corollary 3.8. *If $(G, \mathcal{P})$ is a patch graph and H is obtained from G by placing the configuration of Figure 5 in some patch P, then H contains a $K_{3,4}$-minor.*

Proof. If H is the subgraph consisting of the configuration inside P from Figure 5 along with the four paths from Proposition 3.7, then contracting the four paths yields $K_{3,4}$. $\square$

Proof of Theorem 3.6. Given R as in the hypothesis of the theorem, we will iteratively construct a patch graph $(G, \mathcal{P})$ such that $R \subseteq G$. Consider $(G_0, \mathcal{P}_0) = (4K_2, \{P_0\})$ being the initial patch graph on the left of Figure 2. Let R_0 be the graph obtained from R by adding whichever of the four AB-edges of $(4K_2, \{P_0\})$ that may be missing. Thus we have a patch graph $(G_i, \mathcal{P}_i)$ and supergraph $R_i \supseteq R$ such that $G_i \subsetneq R_i$ and all G_i-bridges in R_i are subgraphs of R and are contained in the patches $\mathcal{P}_i$. We will now show that there is $(G_{i+1}, \mathcal{P}_{i+1})$ with $G_i \subsetneq G_{i+1}$ such that either $R_i \subseteq G_{i+1}$ (in which case we have our desired conclusion) or we will define $R_{i+1} \supseteq R_i \supseteq R$ such that $G_{i+1} \subsetneq R_{i+1}$ and all G_{i+1}-bridges in R_{i+1} are subgraphs of R and are contained in the patches $\mathcal{P}_{i+1}$. In this latter case we can iterate the process again and by the finiteness of R and the fact that $G_i \subsetneq G_{i+1}$ this process will eventually halt with our desired patch graph containing R.

Because $G_i \subsetneq R_i$, there is $P \in \mathcal{P}_i$ such that R_i contains G_i-bridges in the face P. Let $\mathcal{B}_P$ be the collection of these G_i-bridges. In Case 1 say that $|\mathcal{B}_P| > 1$ and in Case 2 say that $|\mathcal{B}_P| = 1$.

Case 1: Here we must have that any G_i-bridge in $\mathcal{B}_P$ has at most three attachments among a, b, c, d. So by the almost 4-connectivity of R each such bridge is either a single link or a triad and so the bridges of $\mathcal{B}_P$ in P are a subconfiguration of the configuration for patch operation I. So now apply the terminal patch operation I to patch P in $(G_i, \mathcal{P}_i)$ to obtain $(G_{i+1}, \mathcal{P}_{i+1})$ where $\mathcal{P}_{i+1} = \mathcal{P}_i \setminus P$. If $R_i \subseteq G_{i+1}$, then we are done. Otherwise let R_{i+1} be obtained from R_i by placing all of the edges of patch operation I into the quadrilateral P of R_i and we have that $G_{i+1} \subset R_{i+1}$ and the G_{i+1}-bridges in R_{i+1} are subgraphs of R and are contained in the patches $\mathcal{P}_{i+1}$.

Case 2: Suppose $\mathcal{B}_P = \{B\}$. Now either B is a k-bridge for $k \leq 3$ or B is a 4-bridge. Let these be Cases 2.1 and 2.2, respectively.

Case 2.1: By the almost 4-connectivity of R, B is either a triad or a single edge. Thus we are done as in Case 1.

Case 2.2: If there is one vertex of attachment, say a, with only one leg of B incident to it, then the G_i-bridge B in the face P is contained within the configuration for patch operation Y. So we define $(G_{i+1}, \mathcal{P}_{i+1})$ and R_{i+1} using patch operation Y and we are done as in Case 1. So say that each of a, b, c, d has at least two legs of B incident to it. So now consider the four facial cycles inside of P that include the four boundary edges of P. Call these faces $F_{a,b}, F_{b,c}, F_{c,d}$ and $F_{d,a}$ and call the corresponding paths obtained after removing the four boundary edges of the patch $P_{a,b}, P_{b,c}, P_{c,d}$ and $P_{d,a}$. By the vertical 3-connectivity of R and restrictions on parallel edges, two of these paths that are consecutive around P (e.g., $P_{a,b}$ and $P_{b,c}$) must be internally disjoint. Also two of these paths that are antipodal around P (e.g., $P_{b,c}$ and $P_{d,a}$) are either internally disjoint or intersect in a path of length 0 or 1. So without loss of generality we can split the remainder of the proof into the following three cases. In Case 2.2.1 $P_{b,c}$ and $P_{d,a}$ intersect in a path of length one, in Case 2.2.2 $P_{b,c}$ and $P_{d,a}$ intersect in a path of length zero, and in Case 2.2.3 $P_{b,c}$ and $P_{d,a}$ are disjoint.

Case 2.2.1: Let the endpoints of the intersecting path of $P_{b,c}$ and $P_{d,a}$ be a' and a'' where a' is above a'' (see the leftmost graph in Figure 6). So now there are 3-separations of R_i at a, a', b. This separation has the interior vertices of $P_{a,b}$ on one side and so by the almost 4-connectivity of R, $P_{a,b}$ has exactly one interior vertex and so one can show that B_P is exactly as shown on the left of Figure 6.

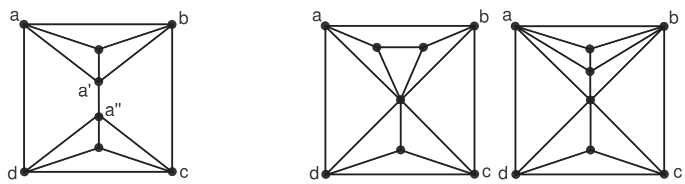


Figure 6.

But now there is a 3-separation of R at either a, a'', b or d, a', c that contradicts the almost 4-connectivity of R unless $\{a, b, c, d\} = \{A, B\}$, i.e., $(G_i, \mathcal{P}_i)$ is the initial patch graph and R is contained in the graph of Figure 6. But this graph is planar when $A = a = c$ and $B = b = d$.

Case 2.2.2: Here B contains a subdivision of the configuration for patch operation X after removing vertices u and v. Using a similar argument to Case 2.2.1 we get that either B is contained within the configuration for patch operation X or $A = a = c$ and $B = b = d$ and R is a subgraph of one of the two graphs shown on the right in Figure 6. In the latter case R is planar and in the former case we define $(G_{i+1}, \mathcal{P}_{i+1})$ from $(G_i, \mathcal{P}_i)$ using patch operation X and we are done as in Case 1.

Case 2.2.3: In this case we have $|V(P_{a,b} \cup P_{b,c} \cup P_{c,d} \cup P_{d,a})| \geq 6$ and so $|V(R_i)| \geq 7$ unless $\{a, b, c, d\} = \{A, B\}$ and by the simplicity of R, R is a subgraph of the graph in Figure 7 which is a subgraph of a patch graph constructed using the sequence of patching operations H, X, X.

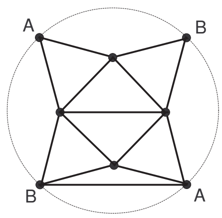


Figure 7.

If there are paths P_1 and P_2 (which must be of non-zero length in this case) from the interior of $P_{a,b}$ to the interior of $P_{c,d}$ and from the interior of $P_{b,c}$ to the interior of $P_{a,d}$, then P_1 and P_2 must intersect. This leads to a minor rooted on the corners of the patch as in Figure 5, which cannot happen by

Corollary 3.8. So, without loss of generality, we can assume that no path exists from the interior of $P_{a,b}$ to the interior of $P_{c,d}$. Then there must exist a face, f, of the embedding as an obstruction that is incident with $P_{b,c}$ and $P_{d,a}$. By the vertical 3-connectivity of R and the restrictions on parallel edges, the boundary cycle of f intersects $P_{b,c}$ in a path of length 0 or 1 and similarly so for $P_{d,a}$. Call these paths b' and a', respectively. Again because of vertical 3-connectivity we can say without loss of generality one of the following cases occurs: $(a' \cup b') \cap \{a, b, c, d\} = \emptyset$, a' is incident with d and $b' \cap \{a, b, c, d\} = \emptyset$, or a' is incident with d and b' is incident with b (see Figure 8).

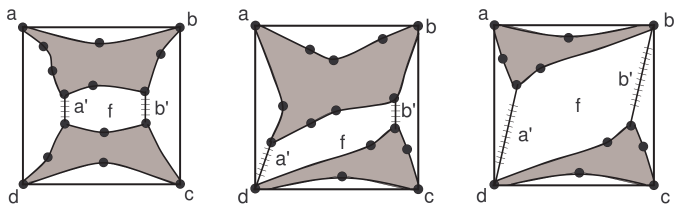


Figure 8.
A crosshatched path has length one or zero.

In the first case B is contained in the configuration for patch operation H. In the second case there is a 3-separation of R_i and R at vertices c, d, b'' where b'' is the lower endpoint of b'. Because of the almost 4-connectivity of R, there is at most one vertex in B separated by c, d, b'' from the rest of R. Thus B is contained in the configuration for patch operation Y. In the third case, there are 3-separations of R_i and R at b, c, d and a, b, d. So by the almost 4-connectivity of R, B is contained in the configuration for patch operation I. In each of these three cases we define $(G_{i+1}, \mathcal{P}_{i+1})$ and R_{i+1} as in Case 1 using the appropriate patching operation and we are done. $\square$

3.2 Patch graphs are $K_{3,4}$-free

This section is devoted to proving that no patch graph contains a $K_{3,4}$-minor (Theorem 3.11). We begin by proving Propositions 3.9 and Proposition 3.10. Let H_1 and H_2 be the projective-planar graphs with the specified quadrilateral face P as shown in Figure 9. We say that a patch graph $(G, \mathcal{P})$ has an H_1- or H_2-minor when G contains the minor shown rooted on A and B and the corners of some patch $P \in \mathcal{P}$.

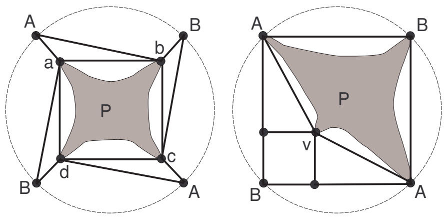


Figure 9.
H_1- and H_2-minors rooted on the corners of a patch P and on A and B

Proposition 3.9. *If $(G, \mathcal{P})$ is a patch graph, then $(G, \mathcal{P})$ does not contain an H_1-minor rooted on any patch $P \in \mathcal{P}$.*

Proof. Of course, the initial patch graph $(4K_2, \{P\})$ has no rooted H_1-minor on the patch P. So by way of contradiction, say that $(G, \mathcal{P})$ has a rooted H_1-minor on some patch Q. Furthermore assume that the number of patching operations done to obtain $(G, \mathcal{P})$ from $(4K_2, \{P\})$ is a the minimum number of operations necessary to obtain such a minor. The patch Q in $(G, \mathcal{P})$ is created by some patching operation on patch Q' with vertex set $\{a, b, c, d\}$ in some intermediate patch graph $(G', \mathcal{P}')$ in the construction of $(G, \mathcal{P})$ from $(4K_2, \{P\})$ as shown on the left of Figure 10. Let the vertex set of patch Q be $\{a', b', c, d\}$ as in Figure 10 as well. Since $\{a, b, c, d\}$ separates Q from A and B and since we have an H_1-minor rooted on P, we must then have one of the rooted minors shown on the right of Figure 10. Thus there is an H_1-minor rooted on Q' in $(G', \mathcal{P}')$, a contradiction of the minimality of $(G, \mathcal{P})$. $\square$

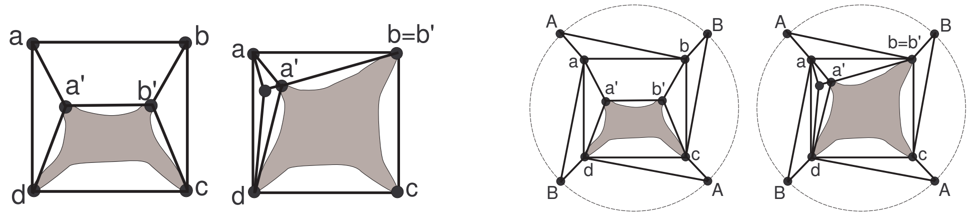


Figure 10.

Proposition 3.10. *If $(G, \mathcal{P})$ is a patch graph, then $(G, \mathcal{P})$ does not contain an H_2-minor rooted on any patch $P \in \mathcal{P}$.*

Proof. Let $(G, \mathcal{P})$, $Q \in \mathcal{P}$, $(G', \mathcal{P}')$, and $Q' \in \mathcal{P}'$ be as in the proof of Proposition 3.9 for an H_2-minor rooted on Q in $(G, \mathcal{P})$. Thus we again have the one of the configurations on the left of Figure 10. If $\overline{H}_2$ is a minimal subgraph in $(G, \mathcal{P})$ that contracts to the H_2-minor, then let C be the cycle in $\overline{H}_2$ that contracts to the corner quadrilateral in H_2. By the argument in the following paragraph, C does not intersect the interior of Q'.

If C does intersect the interior of Q', then possible placements for C in $(G, \mathcal{P})$ are shown in Figure 11. In the first configuration of Figure 11, there must be four disjoint paths (some possibly trivial) starting at C with one such path to a' and the other three to A and B on the boundary. Furthermore no one of these four paths may intersect two vertices of Q. However, in our configuration any four such paths have one that intersects two vertices of Q, a contradiction. In the last two configurations of Figure 11, at most two disjoint paths from $\{a', b', c, d\}$ to A and B can be disjoint from C. However, in $\overline{H}_2$, there are three such paths disjoint from the image of C (i.e., the corner quadrilateral), a contradiction.

Again there must be four disjoint paths (some possibly trivial) in $\overline{H}_2$ starting at C with one such path to $v \in \{a', b', c, d\}$ and the other three to A and B on the boundary. Furthermore these four paths cannot intersect all the vertices of Q. Because C does not intersect the interior of Q', we must have in $(G, \mathcal{P})$ one of the rooted minors shown in Figure 12. But each gives us an H_2-minor in $(G', \mathcal{P}')$, a contradiction to the minimality of $(G, \mathcal{P})$. $\square$

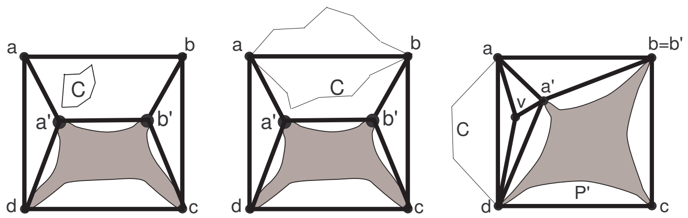


Figure 11.

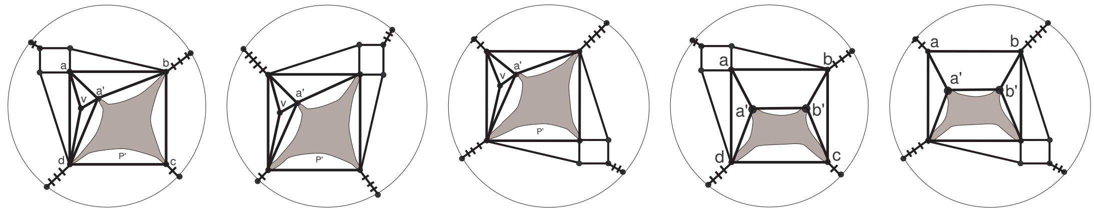


Figure 12.

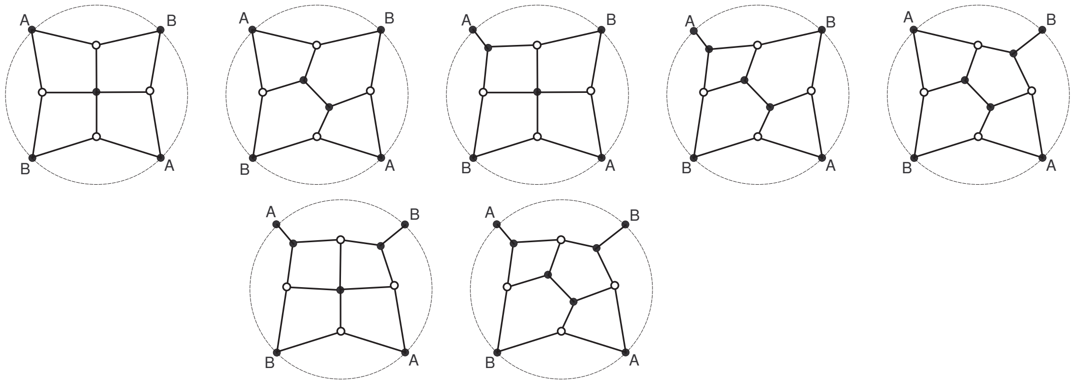
Here a crosshatched edge represents a path that may have length zero.

Theorem 3.11. *No patch graph contains a $K_{3,4}$-minor.*

Proof. Evidently, the initial patch graph $(4K_2, \{P\})$ does not contain a $K_{3,4}$-minor, so we need only show that none of the four patching operations can introduce a $K_{3,4}$-minor to a patch graph. So by way of contradiction say that $(G', \mathcal{P}')$ is obtained from $(G, \mathcal{P})$ by some patch operation on $P \in \mathcal{P}$ and say that $(G, \mathcal{P})$ is $K_{3,4}$-free while $(G', \mathcal{P}')$ is not.

The unique embedding of $K_{3,4}$ in the projective plane is the first graph shown in Figure 13. Furthermore, up to isomorphism, then the only essential 2-cut is the one shown in Figure 13. Given this, one can check that the only proper splits[1] of $K_{3,4}$ up to isomorphism whose embeddings are 2-representative are the seven embedded graphs shown in Figure 13. Let $\mathcal{M} = \{M_1, \ldots, M_7\}$ be the set of these embeddings numbered respectively from left to right. (Note that $M_1 = K_{3,4}$.) Thus G' contains a subdivision, call it S_M, of some $M \in \{M_1, \ldots, M_7\}$ while G contains none of $M_1, \ldots, M_7$ as a minor. Thus at least one edge of $E(G') \setminus E(G)$ must be contained in a branch of S_M. Denote the vertices of P in cyclic order by a, b, c, d and note that P divides the projective plane into an interior disk and an exterior Möbius band given the imbedding of $(G', \mathcal{P}')$. Let M^{Int} be the open interior of the disk.

Figure 13.



The embeddings of $K_{3,4}$ and its seven proper splits.

We will examine all such placements of P on the graphs in $\mathcal{M}$ and find a contradiction in each case. The contradictions will be obtained by either finding a rooted H_i-minor or a $K_{3,4}$-minor in $(G, \mathcal{P})$. The analysis falls into three cases. In Case 1, say that M^{Int} contains none of the branch vertices of S_M. In Case 2, say that M^{Int} contains a single 3-valent branch vertex of S_M. In Case 3, say M^{Int} contains a 4-valent branch vertex of S_M or several branch vertices.

Case 1: We must have that the intersection of S_M with M^{Int} contains one or two paths because G is $K_{3,4}$-free. Furthermore, we can assume that any such path is an ac- or bd-path because we could reroute an ab-, bc-, cd-, or da-path in M^{Int} on the quadrilateral boundary of P to get another M-subdivision. So now since G is $K_{3,4}$-free we get without loss of generality that M^{Int} intersects S_M in an ac-path. Again we could reroute this path on the quadrilateral boundary of P to get an M-subdivision in G (a contradiction) unless both b and d are also vertices in S_M.

$\underline{M = M_1}$: We first, if necessary, contract subdivided edges in S_M so that a, b, c, d all lie on the branch vertices of the contraction of S_M, call it S_M as well. When making these contractions we always choose not to contract onto the boundary vertices A and B whenever possible. Now there are, up to isomorphism, only a five ways that $\{a, b, c, d\}$ can be positioned in S_M, see Figure 14. In the first graph there is an H_1-rooted minor. In the second graph, there there is an H_2-minor obtained by contracting the vertical edge from the lower left boundary vertex and the horizontal edge from the

[1]A *split of a graph G is a decontraction that does not create any new vertices of degree 1 or 2.*

upper left boundary vertex. The third graph would make a violation of Proposition 3.7 in $(G, \mathcal{P})$ because we assume that we do not contract onto A and B whenever possible. The fourth graph contains an H_2-minor. Lastly, for the fifth graph, note that we must have contracted the vertex of the quadrilateral onto the center vertex of S_M because otherwise we would have the fourth graph as a minor. So now Proposition 3.7 implies the existence of an extra path in G as shown in the figure that allows us to find an H_2-minor.

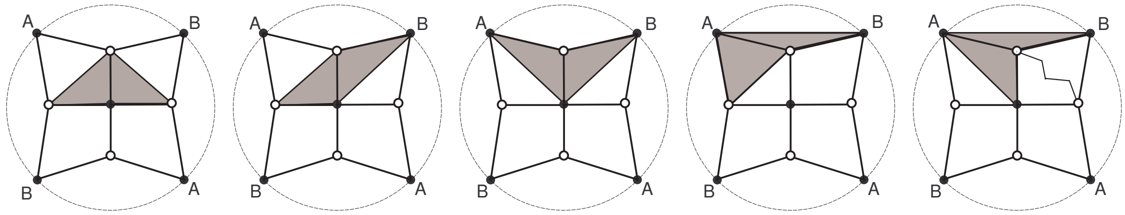
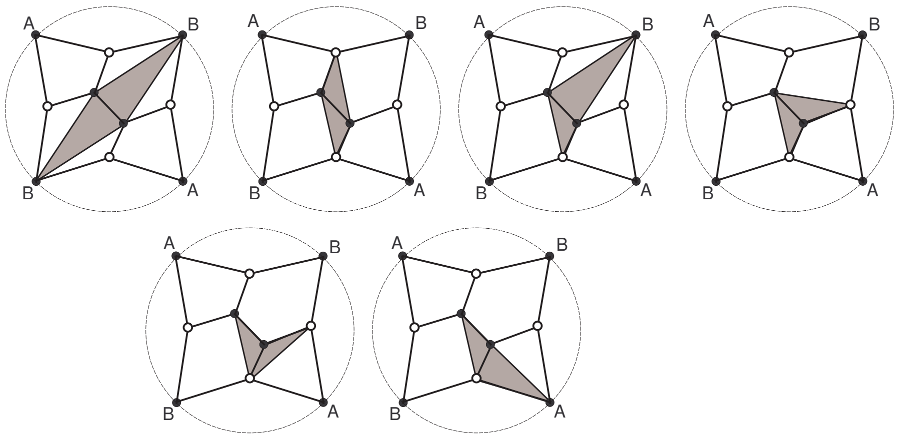


Figure 14.

$\underline{M = M_2}$: Let (x, y) be the edge that was decontracted to create M_2 from M_1. If the ac-path of S_M in M^{Int} does not intersect the xy-branch or an incident branch of S_M, then we fall back into one of the cases for $M = M_1$. As in the case for $M = M_1$ we contract subdivided edges in S_M so that a, b, c, d all lie on the branch vertices of the contraction of S_M, call it S_M as well. The graphs of Figure 15 show all possible configurations for a, b, c, d on S_M. In the first and third configurations there is an H_2-minor, in the second case there is an H_1-minor, and in the fourth through sixth cases there is a $K_{3,4}$-minor in G, a contradiction in each case.

Figure 15.



$\underline{M \in \{M_3, \ldots, M_7\}}$: Again contract subdivided edges in S_M to get branch vertices on a, b, c, d. For M_3, there is only one case where the decontracted edge is in the interior of the patch. In this case, the first graph in Figure 16, the original G contains a $K_{3,4}$-minor. The second and third graphs in the figure are the only cases where the decontracted edge is one of the edges of the patch. In both cases, the original graph G already contains a $K_{3,4}$-minor. Each of the graphs M_4, M_5, M_6 and M_7 are obtained by decontracting at least two edges from M_1. In each case, for any possible position of $\mathcal{P}$, at least one of those edges along with at least one of its endpoints would be in the open exterior of $\mathcal{P}$. Hence contracting that edge would reduce to one of the previous case with $M \in \{M_1, M_2, M_3\}$.

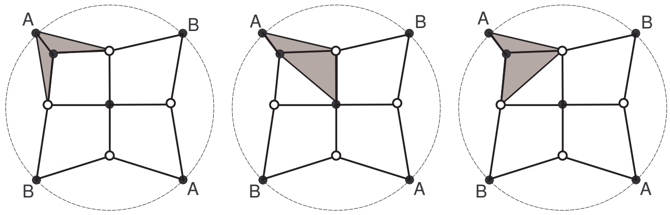


Figure 16.

Case 2: Now consider the case where M^{Int} contains a single 3-valent vertex of S_M. So now we can contract subdivided edges of S_M on the three branches incident to this 3-valent vertex so that at least three vertices from a, b, c, d lie on the other endpoints of the branches of the 3-valent vertex. The fourth vertex of a, b, c, d may or may not be on S_M. If so we contract subdivided edges in S_M to get it onto the branch vertices of S_M. If not, then Proposition 3.7 will yield a path in G off of S_M connecting this fourth vertex to S_M. (See the first graph in Figure 17.) This path can be contracted to get the fourth vertex of a, b, c, d on S_M. After contracting, all of the possible configurations of a, b, c, d on our contraction of S_M are shown in Figure 17. The first, second, third and fifth graphs all contain an H_2-minor and the fourth yields a $K_{3,4}$-minor in G, a contradiction in each case.

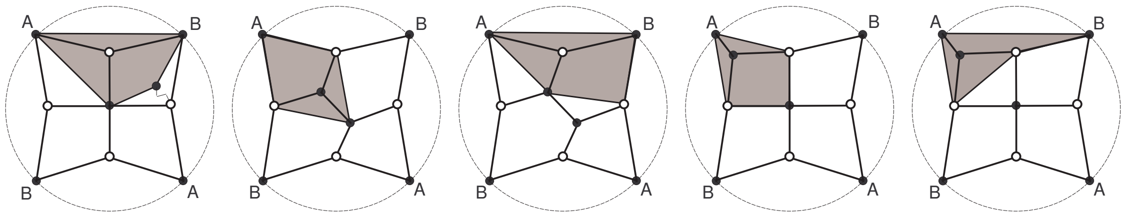


Figure 17.

Case 3: Here all of a, b, c, d are in S_M and so we contract subdivided edges in S_M to get a, b, c, d onto the branch vertices of the contraction of S_M, call it S_M as well. If M^{Int} contains the central 4-valent vertex of S_M or the two 3-valent vertices obtained by splitting the central 4-valent vertex, then we have one of the first two configurations of Figure 18. The first configuration contains an H_1-minor. The second configuration is not possible because the patching operations introduce at most 3 new vertices and the second configuration has at least five vertices in M^{Int}. If M^{Int} contains neither the central 4-valent vertex of S_M nor both 3-valent vertices obtained by splitting the central 4-valent vertex but yet does contain two branch vertices of S_M, then we we obtain one of the second through fourth configurations of Figure 18. For the second one we already know that it is not possible and the third and fourth each contain an H_2-minor. □

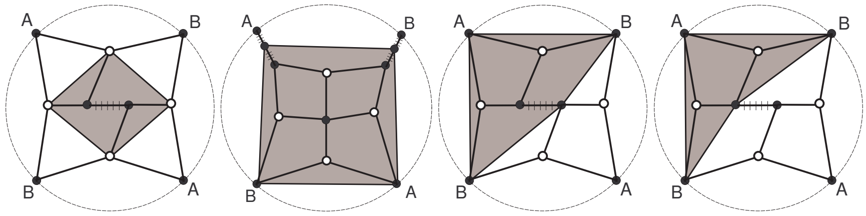


Figure 18.

Acknowledgement

The authors would like to thank the anonymous referees for their careful reading and helpful suggestions. We would also like to thank the public library of London, Ohio for their hospitality in providing a nice meeting place for us to work together.

References

[1] D. Archdeacon, A Kuratowski theorem for the projective plane, *J. Graph Theory*, 5 (1981) 243–246.

[2] H. Glover, J. Huneke, C.S. Wang, 103 graphs that are irreducible for the projective plane, *J. Combin. Theory Ser.B.*, 27 (1979), no.3, 332–370.

[3] J. Maharry, A characterization of graphs with no cube minor, *J. Combin. Theory Ser.B.*, 80 (2000), no. 2, 179–201.

[4] J. Maharry, An excluded minor theorem for the octahedron, *J. Graph Theory*, 31 (1999), no. 2, 95–100.

[5] J. Maharry, N. Robertson, V. Sivaraman, D. Slilaty, The flexibilities of graphs embedded in the projective plane, in preparation.

[6] S. Randby, Minimal embeddings in the projective plane, *J. Graph Theory*, 25 (1998), no. 2, 153–163.

[7] N. Robertson, personal communication.

[8] A. K. Kelmans, personal communication.

[9] N. Robertson, P. Seymour, R. Thomas, Petersen family minors, *J. Combin. Theory Ser. B*, 64 (1995), no. 2, 155–184.

[10] P.D. Seymour, Decomposition of regular matroids, *J. Combin. Theory Ser. B*, 28 (1980), no. 3, 305–359.

[11] K. Truemper, *Matroid Decomposition,* Academic Press, Boston (USA) 1992.

[12] R. Vitray, The 2- and 3-representative projective planar embeddings, *J. Combin. Theory Ser. B*, 54 (1992), no. 1, 1–12.

[13] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, *Math. Ann.*, 114 (1937) 570–590.