The Minimal Zn-Symmetric Graphs that are Not Zn-Spherical

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The Minimal $\mathbb{Z}_n$-Symmetric Graphs That Are Not $\mathbb{Z}_n$-Spherical

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Abstract

Given a graph G equipped with faithful and fixed-point-free Γ-action (Γ a finite group) we define an orbit minor H of G to be a minor of G for which the deletion and contraction sets are closed under the Γ-action. The orbit minor H inherits a Γ-symmetry from G, and when the contraction set is acyclic the action inherited by H remains faithful and fixed-point free. When G embeds in the sphere and the Γ-action on G extends to a Γ-action on the entire sphere, we say that G is Γ-spherical. In this paper we determine for every odd value of $n \geq 3$ the orbit-minor-minimal graphs G with a faithful and free $\mathbb{Z}_n$-action that are not $\mathbb{Z}_n$-spherical. There are 11 infinite families of such graphs, each of the 11 having exactly one member for each n. For $n = 3$, another such graph is $K_{3,3}$. The remaining graphs are, essentially, the Cayley graphs for $\mathbb{Z}_n$ aside from the cycle of length n. The result for $n = 1$ is exactly Wagner's result from 1937 that the minor-minimal graphs that are not embeddable in the sphere are K_5 and $K_{3,3}$.

1 Introduction

A 1937-result of Wagner [10] tells us that the minor-minimal graphs that are not embeddable in the sphere are K_5 and $K_{3,3}$. Let Γ be a group that acts faithfully on the sphere. It would be interesting to find the analogous minimal graphs with faithful Γ-symmetry that do not embed in the sphere in a way that respects the Γ-action. In this paper we find these minimal freely and faithfully Γ-symmetric graphs for $\Gamma = \mathbb{Z}_{2k+1}$ (these being the only odd-order abelian groups that act faithfully on the sphere). For example, the graph G shown on the left in Figure 1.1 has an obvious 3-fold rotational symmetry which is free of fixed points on G and is not spherical because the three triads come together to form a $K_{3,3}$-minor. Furthermore, all connected minors of G that inherit this fixed-point-free rotational symmetry are spherically embeddable with respect to the 3-fold symmetry, as the following argument shows.

Each edge of the graph is in the orbit of one of the labeled edges a, b, c, x, y, z. When taking a minor of this graph, deletion or contraction of an edge must be taken over the entire orbit of that edge in order to preserve the rotational symmetry. Contracting the orbit of edge y results in the second graph shown; this graph does embed in the sphere with the rotational symmetry extending to the whole sphere. (Similarly for the edge x or z.) Deleting the orbit of the edge y (similarly x or z) disconnects the graph. Deleting the orbit of the edge b (similarly a or c) results in the third graph shown, which has the required rotationally symmetric embedding. Contracting the orbit of the edge a (similarly b or c) results in the fourth graph shown which has a fixed point under the rotational symmetry. It is worth noting that this fourth graph is still not spherical; however, we will not consider minors of freely symmetric graphs that inherit the symmetry but

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not fixed-point freeness. This could of course be addressed by future work; in this paper we address what we find to be the most basic of such minimality questions.

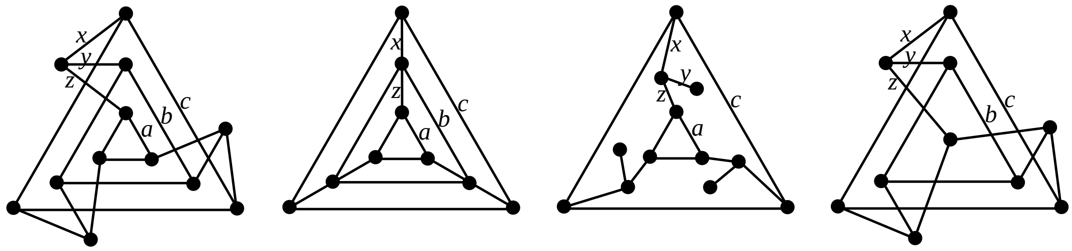


Figure 1.1.

An automorphism φ of a graph G may fix vertices, centers of edges, and/or entire edges. If there is a positive integer t such that $\varphi^t \not\equiv 1$ and $\varphi^t(x) = x$ where x is some vertex, edge, or midpoint of an edge of G, then call x a *pseudofixed point* of φ on G. Denote the set of pseudofixed points of φ by $\mathrm{fix}_G(\varphi)$. We say that φ is *free* on G when $\mathrm{fix}_G(\varphi) = \emptyset$. We say that φ is *pseudofree* on G when $\mathrm{fix}_G(\varphi)$ is a collection of topologically isolated points on G. This is equivalent to saying that none of $\varphi, \varphi^2, \ldots, \varphi^{|\varphi|-1}$ fixes an entire edge of G.

Given a group Γ, a Γ-*action* on G is a homomorphism $\mathfrak{a}\colon \Gamma \to \mathrm{Aut}(G)$ (where $\mathrm{Aut}(G)$ is the full automorphism group of the graph G). Let

$$\mathrm{fix}_G(\mathfrak{a}) = \bigcup_{\substack{g\in\Gamma \\ \mathfrak{a}(g)\not\equiv 1}} \mathrm{fix}_G(\mathfrak{a}(g))$$

and call $\mathrm{fix}_G(\mathfrak{a})$ the set of *pseudofixed points* of $\mathfrak{a}$ on G. The action $\mathfrak{a}$ is *faithful* when $\ker(\mathfrak{a})$ is trivial (*i.e.*, $\mathfrak{a}$ is injective), *free* when $\mathrm{fix}_G(\mathfrak{a}) = \emptyset$, and *pseudofree* when $\mathrm{fix}_G(\mathfrak{a})$ is a collection of topologically isolated points on G.

Given a graph G (not necessarily simple) and a subset of the edge set $X \subseteq E(G)$, by $G\backslash X$ we mean the graph obtained from G by deleting the edges in X along with any resulting isolated vertices and by G/X we mean the graph obtained from G by contracting the edges in X. We do not delete multiple edges or loops resulting from the contraction of X in G. Now given disjoint edge sets C and D in G, the graph $G/C\backslash D$ is called a *minor* of G. (It is well known that deletion and contraction of edges can be done in any order without affecting the resulting minor.)

A Γ-*symmetric graph* is a pair $(G, \mathfrak{a})$ where G is a graph and $\mathfrak{a}$ is a Γ-action on G. Given $(G, \mathfrak{a})$ and edge sets $C, D \subset E(G)$ such that $\mathsf{orbit}(C) \cap \mathsf{orbit}(D) = \emptyset$, consider the minor $G' = G/\mathsf{orbit}(C)\backslash\mathsf{orbit}(D)$ of G. Each automorphism $\mathfrak{a}(g)$ on G has an induced automorphism on G' as described in [1, Prop.2.10]. So we obtain a homomorphism $\mathfrak{m}\colon \mathrm{Im}(\mathfrak{a}) \to \mathrm{Aut}(G')$. Writing $\mathfrak{a}' = \mathfrak{m}\mathfrak{a}$, the pair $(G', \mathfrak{a}')$ is a Γ-symmetric graph that is called an *orbit minor* of $(G, \mathfrak{a})$.

In our context, dealing with freely and faithfully Γ-symmetric graphs is easier than with those that are not free or not faithful. Thus we will restrict our attention to $\mathbb{Z}_n$-symmetric graphs $(G, \mathfrak{a})$ and their orbit minors $(G', \mathfrak{a}')$ where the actions are both free and faithful. Proposition 2.1 gives us a way to convey freeness and faithfulness from the action on the primary graph to the induced action on the orbit minor; namely to contract on an acyclic set. Conversely, if the action on the primary graph is free and the induced action on the orbit minor is still free, then Proposition 2.3 tells us that the contraction set may as well have been acyclic. Therefore it is not restrictive to insist that orbit minors of freely and faithfully Γ-symmetric graphs be obtained without contractions of cycles. In fact, this provides an attractive way to characterize freely and faithfully symmetric orbit minors of freely and faithfully symmetric graphs.

An embedding of a graph G in a closed, compact surface S is said to be *cellular* when the complement of G in S is a disjoint union of open 2-cells. This necessarily requires that G is connected. A group Γ is said to *act cellularly* on closed, compact surface S when there is a faithfully Γ-symmetric graph $(G, \mathfrak{a})$ cellularly embedded in S such that the action $\mathfrak{a}$ takes facial-boundary walks of G in S to facial-boundary walks of G in S. If Γ acts cellularly on a closed surface S, then it may be that a Γ-symmetric graph $(G, \mathfrak{a})$ has an embedding on S for which the Γ-action on G takes facial boundary walks to facial boundary walks. In this case, we say that $(G, \mathfrak{a})$ is Γ-*embeddable* in S or sometimes we just say *embeddable* in S. When S is the sphere we say that $(G, \mathfrak{a})$ is Γ-*spherical* or just *spherical.* It is known that the set of pseudofixed points on a closed surface S under the cellular action of a single automorphism φ on S is a disjoint collection of simple closed curves and isolated points. Thus if $(G, \mathfrak{a})$ is freely Γ-symmetric and also Γ-embeddable on S, then the action of $\mathfrak{a}$ extended to all of S must be a pseudofree action on S.

The class of freely and faithfully Γ-symmetric graphs that are embeddable in S is closed under taking connected orbit minors [1, Prop. 2.11]. When Γ is the trivial group there is the additional property that for any graph G, there is some minor H of G that is embeddable in S. For general Γ, however, this is not the case as there are Cayley graphs for $\mathbb{Z}_n$ defined on minimal generating sets for $\mathbb{Z}_n$ that are not $\mathbb{Z}_n$-spherical even though $\mathbb{Z}_n$ acts cellularly on the sphere (see Propositions 3.2 and 3.4); furthermore, any orbit minor of these Cayley graphs is either not connected or not freely $\mathbb{Z}_n$-symmetric.

Thus, for a given group Γ that acts pseudofreely on S, it is natural to ask the following: for which freely and faithfully Γ-symmetric graphs $(G, \mathfrak{a})$ that are not Γ-embeddable in S is every proper connected orbit minor with free and faithful induced Γ-action Γ-embeddable in S? Call such a Γ-symmetric graph a *minimal free and faithful obstruction* for Γ-embedability in S. Of course, one could ask about minimal pseudofree and faithful obstructions or just minimal faithful obstructions, however, we will not do this here. For the trivial group and the sphere, the minimal obstructions are, of course, K_5 and $K_{3,3}$ (this is due to Wagner [10] which is a result very similar to Kuratowski's [5]). In this paper, we will determine all of the minimal free and faithful obstructions for $\mathbb{Z}_n$-sphericity when $n \geq 3$ is odd. These obstructions (and their quotient graphs) are catalogued in Section 3.1. The completeness of the catalogue is proven in Section 3.2. One point that some readers may find interesting is that a class of signed graphs first identified by Gerards [2, Ch. 3] in the context of matroid theory plays a central role. In Section 4 we discuss the difficulties that could be involved in extending this work to other groups and to non-free actions.

2 Preliminaries

2.1 Graphs

A graph G consists of a collection of vertices (*i.e.*, topological 0-cells), denoted by $V(G)$, along with a collection of edges (*i.e.*, topological 1-cells), denoted by $E(G)$, where an edge has two ends, each of which is attached to a vertex. A *link* is an edge that has its ends incident to distinct vertices and a *loop* is an edge that has both of its ends incident to the same vertex. A *cycle* is a connected, 2-regular graph (*i.e.,* a simple closed path).

Given a graph G, an *oriented edge* is an edge together with a direction along that edge. The tail vertex of oriented edge e is denoted $\mathbf{t}(e)$ and the head vertex is denoted $\mathbf{h}(e)$. Let $-e$ denote the reverse orientation of e and so $\mathbf{t}(-e) = \mathbf{h}(e)$ and $\mathbf{h}(-e) = \mathbf{t}(e)$. The collection of oriented edges of G is denoted $\vec{E}(G)$. Let $C_1(G)$ be the free abelian group $\langle e : e \in \vec{E}(G)\rangle$ in which $(-1)e = -e$. A *walk* is a sequence $w = e_1, \ldots, e_n$ of oriented edges for which the head of e_i is the tail of e_{i+1} for each $i \in \{1, \ldots, n-1\}$. If the tail of e_1 is u and the head of e_n is v, then the walk is called a uv-walk and if $u = v$, then the walk is called *closed.* The *reverse walk* is $-w = -e_n, \ldots, -e_1$. Given a closed walk $w = e_1, \ldots, e_n$, we misuse notation and also use w to denote $\sum_i e_i \in C_1(G)$. Let $Z_1(G)$ be the subgroup of $C_1(G)$ generated by the collection of closed walks in G. Given a cycle C in G, let $\vec{C}$ be a closed Eulerian walk around C. Of course $\vec{C}$ is only well defined up to choice of starting vertex and direction around C.

For two graphs G and H an *isomorphism* $\iota\colon G \to H$ is a bijection $\iota\colon (V(G) \sqcup \vec{E}(G)) \to (V(H) \sqcup \vec{E}(H))$ where $\iota(V(G)) = V(H)$, $\iota(\vec{E}(G)) = \vec{E}(H)$, $\iota\mathbf{h} = \mathbf{h}\iota|_{\vec{E}(G)}$, and $\iota\mathbf{t} = \mathbf{t}\iota|_{\vec{E}(G)}$. We will reserve the letter ι for graph isomorphisms. Two Γ symmetric graphs $(G, \mathfrak{a})$ and $(H, \mathfrak{b})$ are *equivariantly isomorphic* when there is an isomorphism $\iota\colon G \to H$ such that for any $g \in \Gamma$, $\iota\mathfrak{a}(g) = \mathfrak{b}(g)\iota$. In this case, we call ι an *equivariant isomorphism.*

If $X \subseteq E(G)$, then we denote the subgraph of G consisting of the edges in X and all vertices incident to an edge in X by $G{:}X$. The collection of vertices in $G{:}X$ is denoted by $V(X)$. For $k \geq 1$, a *k-separation* of a graph is a bipartition (A, B) of the edges of G such that $|A| \geq k$, $|B| \geq k$, $|V(A) \cap V(B)| = k$, $V(A) \setminus V(B) \neq \emptyset$, and $V(B) \setminus V(A) \neq \emptyset$. A connected graph on at least $k+2$ vertices is said to be *k-connected* when there is no r-separation for $r < k$. A graph on $k+1$ vertices is said to be k-connected when it has a spanning complete subgraph. This type of k-connectivity is often referred to as "vertical k-connectivity", especially in the study of matroids coming from graphs. Here there is no need for the adjective "vertical" as there is no other type of connectivity under consideration.

Given a subgraph H of G, an *H-bridge* is either an edge not in H whose endpoint(s) are both in H or a connected component C of $G \setminus H$ along with the links between C and H. For a given H-bridge B of G, a *foot* of B is a link of B with an endpoint in H, a *vertex of attachment* of B is a vertex in H that is an endpoint of a foot of B. If G' is a subdivision of a graph G with minimum degree three, then a *branch vertex* of G' is a vertex of degree at least three in G' and a *branch* is a path in G' corresponding to an edge in G.

We now state and prove Propositions 2.1 and 2.3 which were referred to in the introduction. Proposition 2.1 requires Lemma 2.2.

Proposition 2.1. *If $(G, \mathfrak{a})$ is freely Γ-symmetric, $\mathsf{orbit}(C) \cap \mathsf{orbit}(D) = \emptyset$, and $\mathsf{orbit}(C)$ is the edge set of a forest in G, then the orbit minor $(G/\mathsf{orbit}(C)\backslash\mathsf{orbit}(D), \mathfrak{a}')$ is also freely Γ-symmetric. Furthermore, if $(G, \mathfrak{a})$ is in fact faithfully Γ-symmetric, then $(G/\mathsf{orbit}(C)\backslash\mathsf{orbit}(D), \mathfrak{a}')$ is also faithfully Γ-symmetric.*

Lemma 2.2. *Let G be a graph and φ a nontrivial and free automorphism of G. If v is a vertex in G and γ is a $v\varphi(v)$-path in G, then the closed walk $\gamma\varphi(\gamma)\varphi^2(\gamma)\cdots\varphi^{|\varphi|-1}(\gamma)$ contains a cycle.*

Proof. We proceed by induction on the length of γ. It cannot be that γ has length zero because the freeness of φ implies that $v \neq \varphi(v)$. Now select a vertex v in G that produces a shortest possible path γ that contradicts our result. Since the walk $w = \gamma\varphi(\gamma)\varphi^2(\gamma)\cdots\varphi^{|\varphi|-1}(\gamma)$ does not contain a cycle, its edges form a tree T. If e is a pendant edge of T, then without loss of generality the walk w must contain somewhere within it the walk $e, -e$ (*i.e.*, traverse e in one direction and then immediately traverse e in the opposite direction). Since γ is a path and φ is an automorphism of G, it must be that $\varphi^k(\gamma)$ is also a path in G for any k. Thus $e, -e$ does not occur within some path $\varphi^k(\gamma)$ and so e is the final edge of $\varphi^k(\gamma)$ for some k and $-e$ is the initial edge of $\varphi^{k+1}(\gamma)$. These imply that $\varphi(e') = -e''$ where e' is the first edge of γ and e'' is the last edge of γ. It cannot be that $e' = e''$ because then φ would fix the midpoint of e'. So now the path γ' obtained from γ by deleting e' and e'' is a $v'\varphi(v')$-path in G where v' is the head of e'. By the minimality of the length of γ, the walk $w' = \gamma'\varphi(\gamma')\varphi^2(\gamma')\cdots\varphi^{|\varphi|-1}(\gamma')$ contains a cycle; however, the edges in the walk w' are contained in the tree T, a contradiction. $\square$

Proof of Proposition 2.1. By assumption, an automorphism $\mathfrak{a}(g) \not\equiv 1$ fixes no edge or vertex of $G\backslash\mathsf{orbit}(D) \subseteq G$. Therefore, no edge of $G' = G/\mathsf{orbit}(C)\backslash\mathsf{orbit}(D)$ is fixed by any $\mathfrak{a}'(g)$. Assume by way of contradiction that a vertex v' of G' is fixed by an automorphism $\varphi' = \mathfrak{a}'(g)$. Thus for any automorphism φ on G whose induced automorphism on G' is φ', any vertex v in G that projects down to v' in G' yields a nontrivial $v\varphi(v)$-path γ whose edges are in $\mathsf{orbit}(C)$. By Lemma 2.2 the closed walk $\gamma\varphi(\gamma)\varphi^2(\gamma)\cdots\varphi^{|\varphi|-1}(\gamma)$ contains a cycle whose edges are in $\mathsf{orbit}(C)$, a contradiction. If $\mathfrak{a}$ is faithful, then $\mathfrak{a}(g) \not\equiv 1$ iff $g \neq 1$. That $\mathfrak{a}'(g)$ fixes no edge of G' when $\mathfrak{a}(g) \not\equiv 1$ implies that $\mathfrak{a}'(g) \not\equiv 1$. $\square$

Proposition 2.3. *If $(G, \mathfrak{a})$ is freely Γ-symmetric and an orbit minor $(G\backslash\mathsf{orbit}(D)/\mathsf{orbit}(C), \mathfrak{a}')$ is also freely Γ-symmetric, then there is an acyclic $\mathsf{orbit}(C') \subseteq \mathsf{orbit}(C)$ and an $\mathsf{orbit}(D') \supseteq \mathsf{orbit}(D)$ such that*

$$(G\backslash\mathsf{orbit}(D')/\mathsf{orbit}(C'), \mathfrak{a}'') = (G\backslash\mathsf{orbit}(D)/\mathsf{orbit}(C), \mathfrak{a}').$$

Proof. Consider the edge set of some connected component C_0 of the subgraph $G{:}\mathsf{orbit}(C)$; the freeness of $\mathfrak{a}'$ on the minor G' implies that $G{:}\mathsf{orbit}(C)$ is a vertex-disjoint union of copies of $G{:}C_0$, say $G{:}C_0, G{:}C_1, \ldots, G{:}C_n$. Let T_0 be the edges of a spanning tree of $G{:}C_0$ and so $\mathsf{orbit}(T_0) = T_0 \cup T_1 \cup \cdots \cup T_n$ where $G{:}T_i$ is a spanning tree of $G{:}C_i$. So now $G\backslash\mathsf{orbit}(D \cup (C_0\backslash T_0))/\mathsf{orbit}(T_0) = G\backslash\mathsf{orbit}(D)/\mathsf{orbit}(C)$ with the same induced Γ-action from $\mathfrak{a}$. $\square$

2.2 Voltage graphs

Given an abelian group Γ, a Γ-*voltage graph* (sometimes just referred to as a *voltage graph*) is a pair (G, σ) where $\sigma\colon \vec{E}(G) \to \Gamma$ satisfies $\sigma(-e) = -\sigma(e)$. In general, voltage graphs may be defined using nonabelian groups, but their applications require a few more details involving fundamental groups of graphs (which we will not need in this paper). The Γ-voltage function σ extends to a walk $w = e_1, \ldots, e_n$ additively and this naturally induces a homomorphism $\sigma_*\colon Z_1(G) \to \Gamma$. We say that a closed walk w is a *zero* walk in (G, σ) when $\sigma_*(w) = 0$. We say that a cycle C is a *zero* cycle in (G, σ) when $\sigma_*(\vec{C}) = 0$. Two Γ-voltage functions σ and ψ on G are said to be *switching equivalent* when $\sigma_* = \psi_*$ and *equivalent* when there is an automorphism α of Γ for which $\sigma_* = \alpha\psi_*$. Two Γ-voltage graphs (G, σ) and (H, ψ) are *isomorphic* when there is a graph isomorphism $\iota\colon G \to H$ such that $\psi\iota$ is equivalent to σ. Given a maximal forest F of G, a Γ-voltage function σ on G is said to be F-*normalized* when $\sigma(e) = 0$ for all $e \in \vec{E}(F)$. Proposition 2.4 is well known; we call σ_F in Proposition 2.4 the F-*normalization* of σ.

Proposition 2.4. *Given any maximal forest F in G, any Γ-voltage function σ on G is switching equivalent to a unique Γ-voltage function σ_F on G that is F-normalized.*

Proof. Set $\sigma_F(e) = 0$ for all $e \in \vec{E}(F)$ and then, for any $e \in E(G)\backslash E(F)$, let C_e be the unique cycle in $F \cup e$. Given $\vec{C}_e$, say that e is oriented in the direction of $\vec{C}_e$. Now set $\sigma_F(e) = \sigma_*(\vec{C}_e)$. Since $\vec{C}_{e_1}, \ldots, \vec{C}_{e_n}$ (where $\{e_1, \ldots, e_n\} = E(G)\backslash E(F)$) is a $\mathbb{Z}$-basis for $Z_1(G)$, our result follows. $\square$

Given a Γ-voltage function σ on G, a *switching function* is a function $\eta\colon V(G) \to \Gamma$. Define Γ-voltage function σ^η on G by $\sigma^\eta(e) = \eta(\mathbf{h}(e)) + \sigma(e) - \eta(\mathbf{t}(e))$.

Proposition 2.5. *Two Γ-voltage functions σ and ψ on a graph G are switching equivalent iff there is η such that $\sigma^\eta = \psi$.*

Proof. Certainly σ and σ^η are switching equivalent, *i.e.*, $(\sigma^\eta)_* = \sigma_*$ for any η. Conversely if $\sigma_* = \psi_*$, then pick a maximal forest F in G and let σ_F be the unique F-normalized Γ-voltage function σ_F that is switching equivalent to σ_* and ψ_*.

We now define switching functions η_1 and η_2 such that $\sigma^{\eta_1} = \sigma_F$ and $\psi^{\eta_2} = \sigma_F$. For each connected component T in F, pick a root vertex r, let $\eta_1(r) = 0$ and $\eta_2(r) = 0$, and consider all of the edges of T to be oriented towards r. Assuming that $\eta_1(v)$ and $\eta_2(v)$ are already defined for each vertex v in T at distance d from r, we define $\eta_1(u)$ and $\eta_2(u)$ for a vertex u in T at distance $d+1$ from r with parent vertex v as $\eta_1(u) = \eta_1(v) + \sigma(e)$ and $\eta_2(u) = \eta_2(v) + \psi(e)$ where e is the unique edge in T with $\mathbf{t}(e) = u$ and $\mathbf{h}(e) = v$. Now σ^{η_1} and ψ^{η_2} are both F-normalized and so $\sigma^{\eta_1} = \sigma_F = \psi^{\eta_2}$ by Proposition 2.4. Thus $\sigma = \psi^{\eta_2 - \eta_1}$. $\square$

Given a $\mathbb{Z}_n$-voltage graph (G, σ), we say that (G, σ) is *balanced* when all its cycles are zero cycles. Given a subgraph H of G, we usually denote $(H, \sigma|_{\vec{E}(H)})$ by (H, σ). This does not cause any ambiguity in this paper. A vertex $v \in G$ is a *balancing vertex* when $(G\backslash v, \sigma)$ is balanced. The Γ-voltage function σ can also be defined on contractions of G by acyclic sets of edges, up to switching, as follows. If F is a forest of G and F_0 a maximal forest of G containing F, then consider the F_0-normalization σ_{F_0} and define $(G, \sigma)/F = (G/F, \sigma_{F_0}|_{E(G)\backslash E(F)})$. We usually denote $(G/F, \sigma_{F_0}|_{E(G)\backslash E(F)})$ by $(G/F, \sigma)$. (We sometimes call $\sigma|_{E(G)\backslash E(F)}$ the *induced* Γ-*voltage function*.) We consider $(G/F, \sigma')$ for any Γ-voltage function σ' switching equivalent to σ to be the contraction of (G, σ) by F. In this sense, contractions of acyclic sets of edges in (G, σ) are well defined up to switching equivalence of Γ-voltage functions. Any Γ-voltage graph $(G\backslash D/K, \sigma)$ obtained from (G, σ) where $G{:}K$ is acyclic is called a *link minor* of (G, σ). (Isolated vertices are normally deleted as they occur. In fact, they can always be avoided by an appropriate choice of edges to contract.) If edges are deleted and contracted one at a time, a link minor can be thought of as a minor obtained without contraction of any loop. If (H, ψ) is isomorphic to $(G\backslash D/K, \sigma)$, then if it does not cause confusion we say that (H, ψ) is also a *link minor* of (G, σ) rather than "isomorphic to a link minor".

A k-*split* of a loopless signed graph (G, σ) is defined when (G, σ) is connected and does not have a balancing vertex. If (A_1, A_2) is a 1-separation of G, then the two signed graphs $(G{:}A_1, \sigma)$ and $(G{:}A_2, \sigma)$ comprise a 1-*split* of (G, σ).

Let (A_1, A_2) be a bipartition of $E(G)$ with $V(A_1) \cap V(A_2) = \{u, v\}$, $|A_i| \geq 2$, and $|A_i| \geq 3$ when $(G{:}A_i, \sigma)$ is unbalanced. (Note that (A_1, A_2) need not be a 2-separation of G.) When (G, σ) is unbalanced, the stipulation that (G, σ) does not have a balancing vertex can be used to show that one of $(G{:}A_1, \sigma)$ and $(G{:}A_2, \sigma)$ is unbalanced. When (G, σ) is unbalanced, a 2-*split* of (G, σ) is defined for two separate cases. First, if both $(G{:}A_1, \sigma)$ and $(G{:}A_2, \sigma)$ are unbalanced, then define (G_i, σ_i) to be $(G{:}A_i, \sigma)$ taken with two new uv-links, say e and f, with $\sigma_i(e) = 0$ and $\sigma_i(f) = 1$. The signed graphs (G_1, σ_1) and (G_2, σ_2) comprise the 2-split of (G, σ). Second, if $(G{:}A_1, \sigma)$ is unbalanced and $(G{:}A_2, \sigma)$ is balanced, then there is $x \in \mathbb{Z}_2$ such that every uv-walk w in A_2 has $\sigma_*(w) = x$. Define (G_i, σ_i) to be $(G{:}A_i, \sigma)$ taken with one new uv-link, say e, with $\sigma_i(e) = x$ and so now (G_1, σ_1) is unbalanced and (G_2, σ_2) is balanced. Again, the signed graphs (G_1, σ_1) and (G_2, σ_2) comprise the 2-split of (G, σ).

Let (A_1, A_2) be a bipartition of $E(G)$ with $|V(A_1) \cap V(A_2)| = 3$, $(G{:}A_1, \sigma)$ unbalanced, $(G{:}A_2, \sigma)$ balanced, and $|A_2| \geq 4$. (Again note that (A_1, A_2) need not be a 3-separation of G.) Let F_2 be a maximal forest of $G{:}A_2$ and extend F_2 to a maximal forest F for all of G. Assuming that σ is F-normalized, we get that $\sigma(e) = 0$ for all $e \in \vec{A}_2$. Define (G_i, σ_i) to be $(G{:}A_i, \sigma)$ taken with a new vertex v_i of degree three attached, with links having voltage 0, to the three vertices of $V(A_1) \cap V(A_2)$. So now (G_1, σ_1) is unbalanced and (G_2, σ_2) is balanced and these two signed graphs comprise a 3-*split* of (G, σ).

Our drawings of $\mathbb{Z}_n$-voltage graphs for $n \geq 3$ utilize the following conventions. An edge with voltage 0 is drawn without decoration and an edge with voltage $k \in \{1, 2, \ldots, n-1\}$ in one direction is drawn with k arrowheads in that direction. A $\mathbb{Z}_2$-voltage graph is often referred to as a *signed graph*. When drawing signed graphs, edges with voltage 0 are drawn as solid lines and edges with voltage 1 are drawn as dashed lines.

A *Gerards* signed graph is a connected signed graph (G, σ) that has no loops and which has one of the following structures.

1. (G, σ) is balanced.
2. (G, σ) has a balancing vertex.
3. (G, σ) embeds in the plane so that exactly two faces are bounded by nonzero walks.
4. (G, σ) is isomorphic to the signed graph, call it T_6, in Figure 2.6.
5. (G, σ) has a 1-split or 2-split into (G_1, σ_1) and (G_2, σ_2) where each (G_i, σ_i) is a Gerards signed graph.
6. (G, σ) has a 3-split into (G_1, σ_1) and (G_2, σ_2) where (G_1, σ_1) is a Gerards signed graph and (G_2, σ_2) is balanced.

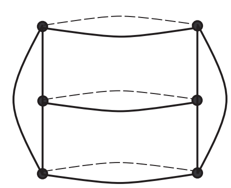


Figure 2.6. *The voltage graph T_6.*

Theorem 2.7 is proven using the Decomposition Theorem of Seymour [7] although it is not an immediate corollary of Seymour's result.

Theorem 2.7 (Gerards [2, Thm. 3.2.3])**.** *If (G, σ) is a connected and loopless signed graph, then (G, σ) is a Gerards signed graph iff (G, σ) does not contain a link minor isomorphic to one of the two signed graphs of Figure 2.8.*

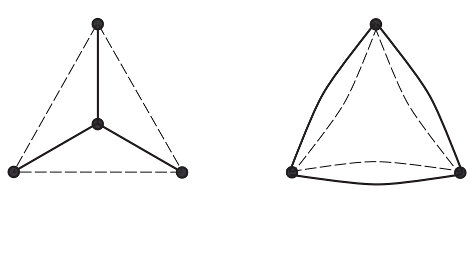


Figure 2.8.

Given a graph G, a *theta subgraph* of G is a subgraph Θ consisting of two vertices u and v taken with three internally disjoint uv-paths. Note that a theta subgraph contains exactly three cycles C_1, C_2, and C_3 where $\vec{C}_1$, $\vec{C}_2$, and $\vec{C}_3$ can be chosen so that $\vec{C}_1+\vec{C}_2=\vec{C}_3$. Parts (1) and (2) of Proposition 2.9 are immediate and Part (3) is by Zaslavsky [11, Thm. 2].

Proposition 2.9. *Let Θ be a theta subgraph of G and let σ be a Γ-voltage assignment on G.*

1. *There are either zero, one, or three zero cycles in Θ (i.e., there are not exactly two zero cycles in Θ).*
2. *If $\Gamma \cong \mathbb{Z}_2$, then there are either one or three zero cycles in Θ.*
3. *If every theta subgraph of (G,σ) has exactly one or three zero cycles, then there is a $\mathbb{Z}_2$-voltage function ψ on G such that the zero cycles of (G,ψ) are exactly the zero cycles of (G,σ).*

Theorem 2.10. *Let G be a connected and loopless graph, Γ an abelian group, σ a Γ-voltage assignment on G, and ψ a $\mathbb{Z}_2$-voltage assignment on G with the same zero cycles as σ.*

(1) For each block B of G, there is $a_B \in \Gamma$ such that for any cycle C in B, $\sigma_*(\vec{C}) \in \{0, a_B, -a_B\}$.

(2) If Γ has odd order, then (G,ψ) is a Gerards signed graph.

Proof. Since any signed graph constructed by identifying any two Gerards signed graphs along a vertex is still a Gerards signed graph, we need only prove our theorem for the case that G is a block.
(1) Certainly this is true if (G,σ) is balanced. So assuming that (G,σ) is unbalanced, take some arbitrary nonzero cycle C_0 in (G,σ) and let $a=\sigma(\vec{C}_0)$. Given any other nonzero cycle C in (G,σ), by [9, 4.34] there are nonzero cycles $C_0, C_1, \ldots, C_n$ such that $C_n = C$ and for each $i \in \{0,\ldots,n-1\}$, $C_i \cup C_{i+1}$ is a theta graph. If C_i' is the third cycle in the theta graph $C_i \cup C_{i+1}$, take $\vec{C}_0, \vec{C}_1, \ldots, \vec{C}_n$ and $\vec{C}_0', \ldots, \vec{C}_{n-1}'$ so that $\vec{C}_i + \vec{C}_{i+1} = \vec{C}_i'$. Since every theta graph of (G,ψ) contains at least one zero cycle (Proposition 2.9(2)) and (G,σ) has the same zero cycles as (G,ψ), we have that $\sigma_*(\vec{C}_i')=0$ for each i which makes $\sigma_*(\vec{C}_i)=(-1)^i a$.
(2) By way of contradiction, say that (G,ψ) is not a Gerards signed graph. Theorem 2.7 says that (G,ψ) contains a link minor from Figure 2.8. Let C and D be disjoint edge sets of G whose contraction and deletion, respectively, in (G,ψ) obtains one of the minors from Figure 2.8. Now the induced voltage functions σ and ψ on the link minor $G/C\backslash D$ still have the same zero and non-zero cycles. Normalize σ and ψ on a spanning tree T of $G/C\backslash D$ as in Figure 2.8. This affects neither σ_* nor ψ_*, but we now have the property that $\sigma(e)=0$ iff $\psi(e)=0$. Since Γ is of odd order, it easy to check that it is impossible to define σ on $G/C\backslash D$ to have the same zero cycles as ψ on $G/C\backslash D$. ☐

2.3 Derived graphs

We assume familiarity with derived graphs of voltage graphs as in Gross and Tucker [3] but we briefly touch on some of their main points here. Given an abelian group Γ and a Γ-voltage assignment σ on graph G, the *derived graph* G^σ is constructed as follows: $V(G^\sigma) = V(G)\times\Gamma$ and $\vec{E}(G^\sigma) = \vec{E}(G)\times\Gamma$ where $\mathbf{t}(e,g) = (\mathbf{t}(e), g)$, $\mathbf{h}(e,g) = (\mathbf{h}(e), g+\sigma(e))$, and $-(e,g) = (-e, g+\sigma(e))$.

If σ is a Γ-voltage assignment on a graph G, then of course G^σ has a canonical free and faithful Γ-action $\mathfrak{b}_\sigma$ defined by translation. Call this the *basic action* on G^σ. Conversely, if $(H,\mathfrak{a})$ is freely Γ-symmetric, then the quotient space $H/\mathfrak{a}$ is again a graph and there is a natural Γ-voltage assignment κ on $H/\mathfrak{a}$ whose derived graph $(H/\mathfrak{a})^\kappa$ with basic Γ-action $\mathfrak{b}_\kappa$ is equivariantly isomorphic to $(H, \mathfrak{a}\alpha)$ for some automorphism α of Γ. (We will reserve the letter κ for voltage assignments of this sort and usually use σ for other voltage assignments.)

Proposition 2.11. *If σ and ψ are equivalent Γ-voltage assignments on G, then there is an automorphism α of Γ such that $(G^\sigma, \mathfrak{b}_\sigma)$ is equivariantly isomorphic to $(G^\psi, \mathfrak{b}_\psi\alpha)$.*

Proof. Since σ and ψ are equivalent, there is a switching function η and automorphism α of Γ such that $\psi = \alpha\sigma^\eta$. Define $\iota\colon G^\sigma \to G^{\alpha\sigma^\eta}$ by $\iota(v,g) = (v, \alpha(g+\eta(v)))$ and $\iota(e,g) = (e, \alpha(g+\eta\mathbf{t}(e)))$. Being careful to keep track of which graph the head and tail functions are acting on (either G, G^σ, or $G^{\alpha\sigma^\eta}$), we now get that ι is an isomorphism because

$$\iota\mathbf{t}(e,g) = \iota(\mathbf{t}(e),g) = (\mathbf{t}(e),\alpha(g+\eta\mathbf{t}(e))) = \mathbf{t}(e,\alpha(g+\eta\mathbf{t}(e))) = \mathbf{t}\iota(e,g)$$

and

$$\begin{aligned}
\iota\mathbf{h}(e,g) &= \iota(\mathbf{h}(e),g+\sigma(e)) \\
&= (\mathbf{h}(e),\alpha(g+\sigma(e)+\eta\mathbf{h}(e))) \\
&= (\mathbf{h}(e),\alpha(g+\eta\mathbf{h}(e)+\sigma(e)-\eta\mathbf{t}(e)+\eta\mathbf{t}(e))) \\
&= (\mathbf{h}(e),\alpha(g+\eta\mathbf{t}(e))+\alpha\sigma^\eta(e)) \\
&= \mathbf{h}(e,\alpha(g+\eta\mathbf{t}(e))) \\
&= \mathbf{h}\iota(e,g).
\end{aligned}$$

To show that ι is an equivariant isomorphism, take $g\in\Gamma$, (v,a), and (e,a) where v is a vertex of G and e is an oriented edge of G. Now

$$\begin{aligned}
\iota\mathfrak{b}_\sigma(g)(v,a) &= \iota(v,a+g) \\
&= (v,\alpha(a+g+\eta(v))) \\
&= (v,\alpha(a+\eta(v))+\alpha(g)) \\
&= \mathfrak{b}_\psi(\alpha(g))(v,\alpha(a+\eta(v))) \\
&= \mathfrak{b}_\psi(\alpha(g))\iota(v,a).
\end{aligned}$$

The calculation that $\iota\mathfrak{b}_\sigma(g)(e,a) = b_\psi(\alpha(g))\iota(e,a)$ is the same aside from the use of $\eta\mathbf{t}$ rather than just η. □

Given a Γ-voltage graph (G,σ), it is important for us to know when its derived graph G^σ is connected. Certainly that G is connected is a necessary condition for G^σ to be connected. We say that (G,σ) *generates* Γ when the image of σ_* is all of Γ rather than a proper subgroup of Γ.

Proposition 2.12. *A connected Γ-voltage graph (G,σ) generates Γ iff for any spanning tree T of G the group*

$$\langle \sigma_T(e) : e\in \vec{E}(G)\backslash\vec{E}(T)\rangle$$

equals Γ, where σ_T is the normalization of σ with respect to T.

Proposition 2.13. *If (G,σ) is a Γ-voltage graph, then G^σ is connected iff (G,σ) is connected and generates Γ.*

Proof. Given that G is connected, take a spanning tree T in G and consider the T-normalization σ_T of σ. That $G^{\sigma_T}\cong G^\sigma$ is given by Proposition 2.11.

Suppose that σ_T generates Γ. Proposition 2.12 now implies that $(G/T)^{\sigma_T}$ is a connected Cayley graph for Γ. Decontracting T, we now get that G^{σ_T} is connected. Conversely, if G^{σ_T} is connected, then $(G/T)^{\sigma_T}$ is connected, and so is a Cayley graph for Γ and so σ_T generates Γ. □

3 $\mathbb{Z}_n$ acting on the sphere for n odd and $n\geq 3$

A $\mathbb{Z}_n$-voltage graph is said to be *2-branch-point spherical* when there is a cellular embedding of G in the sphere (which requires that G be connected) such that there are exactly two facial boundary walks (up to reversal) that are not in the kernel of σ_*. So now if (G,σ) is 2-branch-point spherical and $F_1,\dots,F_n$ are the faces of such an embedding of (G,σ) in the sphere (all oriented in the clockwise direction) then $\sum_i \sigma_*(\partial(F_i)) = 0$ and so the two faces, say F_1 and F_2, whose boundary walks are not in the kernel of σ_* satisfy $\sigma_*(\partial(F_1)) = -\sigma_*(\partial(F_2))$. The branch points of this 2-branch-point-spherical embedding are the central points of F_1 and F_2. (One can also think of a 2-branch-point spherical embedding of (G,σ) as an embedding of (G,σ) in the annulus where a cycle is a zero cycle iff it is topologically contractible in the embedding.) So now if C is a cycle in G that separates the branch points of the embedding, then $\sigma_*(\vec{C}) = \pm\sigma_*(\partial(F_1))$, whereas if C is a cycle that does not separate the branch points of the embedding, then $\sigma_*(\vec{C}) = 0$. Note that when (G, σ) is 2-branch-point spherical, σ generates $\mathbb{Z}_n$ iff the value $\sigma_*(\partial(F_1))$ is a generator for $\mathbb{Z}_n$.

Of course we are using the term "spherical" for both $\mathbb{Z}_n$-symmetric graphs and $\mathbb{Z}_n$-voltage graphs, but this is appropriate given Proposition 3.1.

Proposition 3.1. *If $n \geq 3$ is odd and $(G, \mathfrak{a})$ is a connected and freely and faithfully $\mathbb{Z}_n$-symmetric graph, then $(G, \mathfrak{a})$ is $\mathbb{Z}_n$-spherical iff $(G/\mathfrak{a}, \kappa)$ is 2-branch-point spherical and generates $\mathbb{Z}_n$.*

Proof. If $(G, \mathfrak{a})$ is connected and freely and faithfully $\mathbb{Z}_n$-symmetric, then since n is odd, [1, Theorem 4.1] implies that the action of $\mathfrak{a}$ on G in the sphere is rotation of odd order around two fixed points that are in two distinct faces of G. That $(G/\mathfrak{a}, \kappa)$ is 2-branch-point spherical and generates $\mathbb{Z}_n$ follows.

Conversely, if $(G/\mathfrak{a}, \kappa)$ is 2-branch-point spherical and generates $\mathbb{Z}_n$, then the Riemann-Hurwitz Equation gives us that the derived embedding of $(G/\mathfrak{a})^\kappa \cong G$ is in the surface whose Euler Characteristic is $2n - 2(n-1) = 2$, which is the sphere. Therefore $((G/\mathfrak{a})^\kappa, \mathfrak{b}_\kappa)$ is $\mathbb{Z}_n$-spherical and there is an automorphism α of $\mathbb{Z}_n$ such that $(G, \mathfrak{a})$ is equivariantly isomorphic to $((G/\mathfrak{a})^\kappa, \mathfrak{b}_\kappa \alpha)$. □

Now if $(G, \mathfrak{a})$ is freely and faithfully $\mathbb{Z}_n$-symmetric, then taking an orbit minor of $(G, \mathfrak{a})$ corresponds to taking a link minor in the quotient voltage graph $(G/\mathfrak{a}, \kappa)$. Therefore, Proposition 3.1 yields an equivalent formulation of the problem of finding the minimal free and faithful obstructions for $\mathbb{Z}_n$-sphericity among freely and faithfully $\mathbb{Z}_n$-symmetric graphs; that is, finding the link-minor-minimal $\mathbb{Z}_n$-voltage graphs that are not 2-branch-point spherical, are connected, and generate $\mathbb{Z}_n$. Using $\bigotimes_n$ (pronounced "*not spherical n*") to denote the collection of such $\mathbb{Z}_n$-voltage graphs, we are seeking to identify the link-minor-minimal members of $\bigotimes_n$. Note that if (G, σ) is a link-minor-minimal member of $\bigotimes_n$, then any proper link minor of (G, σ) is either not connected, does not generate $\mathbb{Z}_n$, or is 2-branch-point $\mathbb{Z}_n$-spherical.

3.1 A catalogue of link-minor-minimal members of $\bigotimes_n$

Minimal Generating Bouquets Let (G, σ) be a bouquet of $m \geq 2$ loops $\{l_1, \ldots, l_m\}$ for which $\{\sigma(l_1), \ldots, \sigma(l_n)\}$ is a minimal generating set for $\mathbb{Z}_n$. This requires that the odd integer n has at least two distinct prime factors, which makes $n \geq 15$. Conversely, any odd $n \geq 15$ with two distinct prime divisors will have a minimal generating bouquet with $m \geq 2$ loops. Given a minimal generating bouquet, we must have that $\sigma(l_i) \neq \pm\sigma(l_j)$ for each $i \neq j$ and so we have Proposition 3.2.

Proposition 3.2. *For any odd n with two distinct prime divisors, a minimal generating bouquet for $\mathbb{Z}_n$ is a link-minor-minimal member of $\bigotimes_n$.*

A freely $\mathbb{Z}_n$-symmetric graph derived from a minimal generating bouquet on m loops is, of course, a Cayley graph for $\mathbb{Z}_n$ and can be viewed with vertices on an m-dimensional array where the length of the i^{th} dimension of the array is the order of the i^{th} generator and edges are placed appropriately. The $\mathbb{Z}_n$-action may be hard to visualize for $m \geq 3$; however, for $m = 2$, the array and $\mathbb{Z}_n$-action on it can be viewed as a rotation on a "torus-grid" as explained in [1, §6.1]. See Figure 3.3.

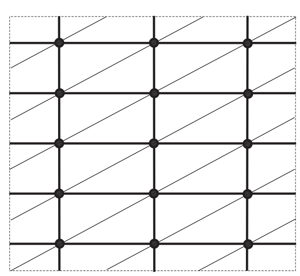


Figure 3.3.
Torus rotation for a 3×5 grid. The thin lines form a simple-closed curve along which the rotation occurs.

Spiral Bouquets Let (G, σ) be a bouquet of two loops l_1 and l_2 for which $\sigma(l_1)$ generates $\mathbb{Z}_n$ and $\sigma(l_2) \notin \{0, \sigma(l_1), -\sigma(l_1)\}$. This latter property requires $n \geq 4$, and any $\mathbb{Z}_n$ with $n \geq 4$ will have a spiral bouquet.

Proposition 3.4. *For any odd $n \geq 5$, a spiral bouquet for $\mathbb{Z}_n$ is a link-minor-minimal member of $\bigotimes_n$.*

A freely $\mathbb{Z}_n$-symmetric graph derived from a spiral bouquet is, again, a Cayley graph for $\mathbb{Z}_n$ and can be described as a "torus spiral" from [1, §6.1]. See Figure 3.5. The free $\mathbb{Z}_n$-action is rotation along the horizontal cycle in Figure 3.5.

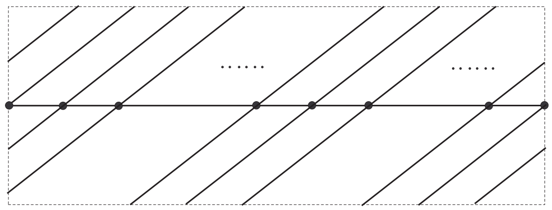


Figure 3.5.
A typical torus spiral (embedded in the torus) with the action being rotation along the horizontal cycle.

Up to voltage-graphic isomorphism there is only one spiral bouquet for $\mathbb{Z}_5$ and the freely $\mathbb{Z}_5$-symmetric graph derived from it is K_5. There is exactly one (again up to voltage-graphic isomorphism) spiral bouquet for $\mathbb{Z}_7$, there are exactly two spiral bouquets for each of $\mathbb{Z}_9$ and $\mathbb{Z}_{11}$, and there are exactly three spiral bouquets for $\mathbb{Z}_{13}$.

A $\mathbb{Z}_3$ theta graph Let (G, σ) be the $\mathbb{Z}_3$-voltage graph in Figure 3.6.

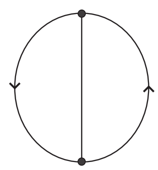


Figure 3.6.

Proposition 3.7. (G, σ) *is a link-minor-minimal member of* $\bigotimes_3$.

The derived $\mathbb{Z}_3$-symmetric graph $(G^\sigma, \mathfrak{b}_\sigma)$ is $K_{3,3}$ with the $\mathbb{Z}_3$-action being an order-3 rotation along a hexagon in $K_{3,3}$.

Gerrards-type obstructions For any $n \geq 2$, consider the eleven $\mathbb{Z}_n$-voltage graphs shown in Figure 3.8. The edges with an arrow represent an edge with voltage 1 in the direction of the arrow and the edges without an arrow have voltage 0. The zero cycles of these voltage graphs are independent of the choice of n.

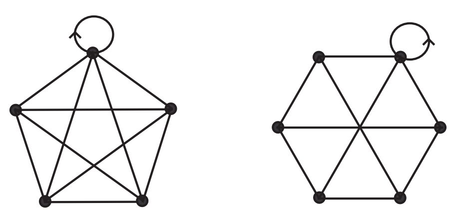
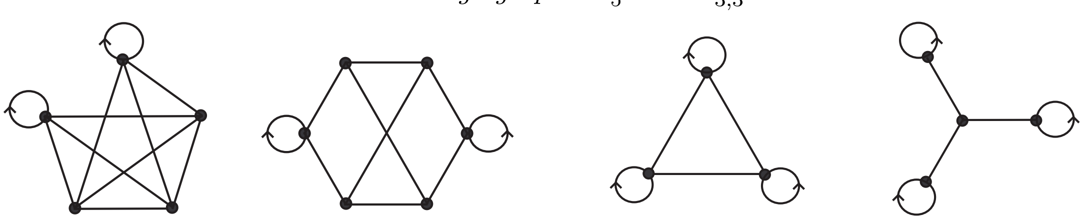
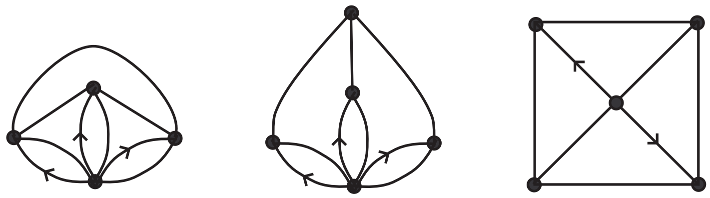


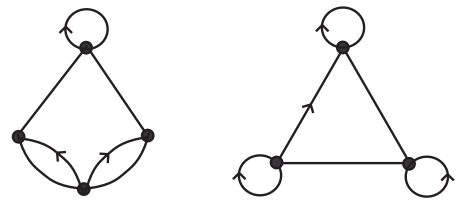
Figure 3.8.
The voltage graphs K_5^ℓ and $K_{3,3}^\ell$



The voltage graphs $K_5^{2\ell}$ and $K_{3,3}^{2\ell}$ *The voltage graphs $\Delta^{3\ell}$ and $Y^{3\ell}$.*



The voltage graphs $\widehat{K}_5/\hat{e}$, $(K_{3,3} \cup \hat{e})/\hat{e}$, $\widehat{K}_{3,3}/\hat{e}$.



The voltage graphs $\widehat{V}^{\ell}$ and $\widehat{\Delta}^{3\ell}$.

Proposition 3.9. *For any $n \geq 2$, each $\mathbb{Z}_n$-voltage graph in Figure 3.8 is a link-minor minimal member of $\bigotimes_n$.*

Sketch of Proof. Each voltage graph of Figure 3.8 clearly generates $\mathbb{Z}_n$ and is connected. For each voltage graph we must show that: it is not 2-branch-point spherical and any proper link minor that is still connected and generates $\mathbb{Z}_n$ is 2-branch-point spherical. This is obvious for K_5^{ℓ} and $K_{3,3}^{\ell}$. We will not show this for the all of the remaining 9 voltage graphs, but just for $K_5^{2\ell}$ and $(K_{3,3} \cup \hat{e})/\hat{e}$. The remaining cases are dealt with in a similar manner.

Figure 3.10 shows all of the single-edge deletions of $K_5^{2\ell}$ up to isomorphism and 2-branch-point spherical embeddings of them. These embeddings are unique save for the placement of the loop in the first embedding. One can now check that the deleted edge cannot be added to any of the three embeddings. Thus $K_5^{2\ell}$ is not 2-branch-point spherical and every single-element deletion is. Each single-link contraction of $K_5^{2\ell}$ is also 2-branch-point spherical because any single-link contraction will contain a pair of parallel links that form a zero cycle. These parallel links have no effect on embeddability in the 2-branch-point sphere and so one of these two links could have been deleted before contraction without affecting embeddability; hence returning us to the deletion case.

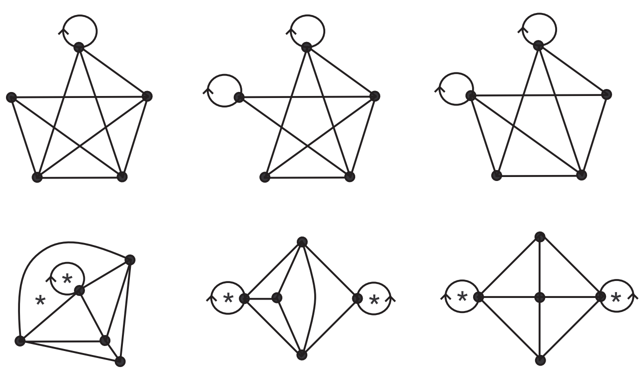


Figure 3.10.

Figure 3.11 shows all of the single-link contractions of $(K_{3,3} \cup \hat{e})/\hat{e}$ up to isomorphism and and 2-branch-point spherical embeddings of them. These embeddings are unique. One can now check that the contracted edge cannot be restored to either of these embeddings. Thus $(K_{3,3} \cup \hat{e})/\hat{e}$ is not 2-branch-point spherical and every single-link contraction is. Each single-edge deletion of $(K_{3,3} \cup \hat{e})/\hat{e}$ is also 2-branch-point spherical because any single-edge deletion will contain a subdivision of a subgraph of one of the single-link contractions. Subdivisions have no effect on embeddability in the 2-branch-point sphere and so one of the two series edges could have been contracted before deletion without affecting embeddability; hence returning us to the contraction case.

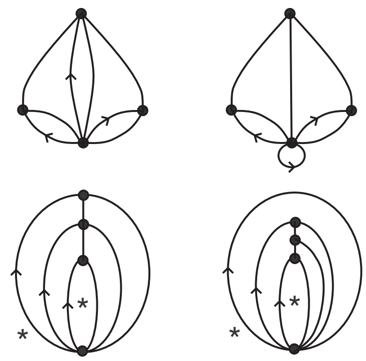
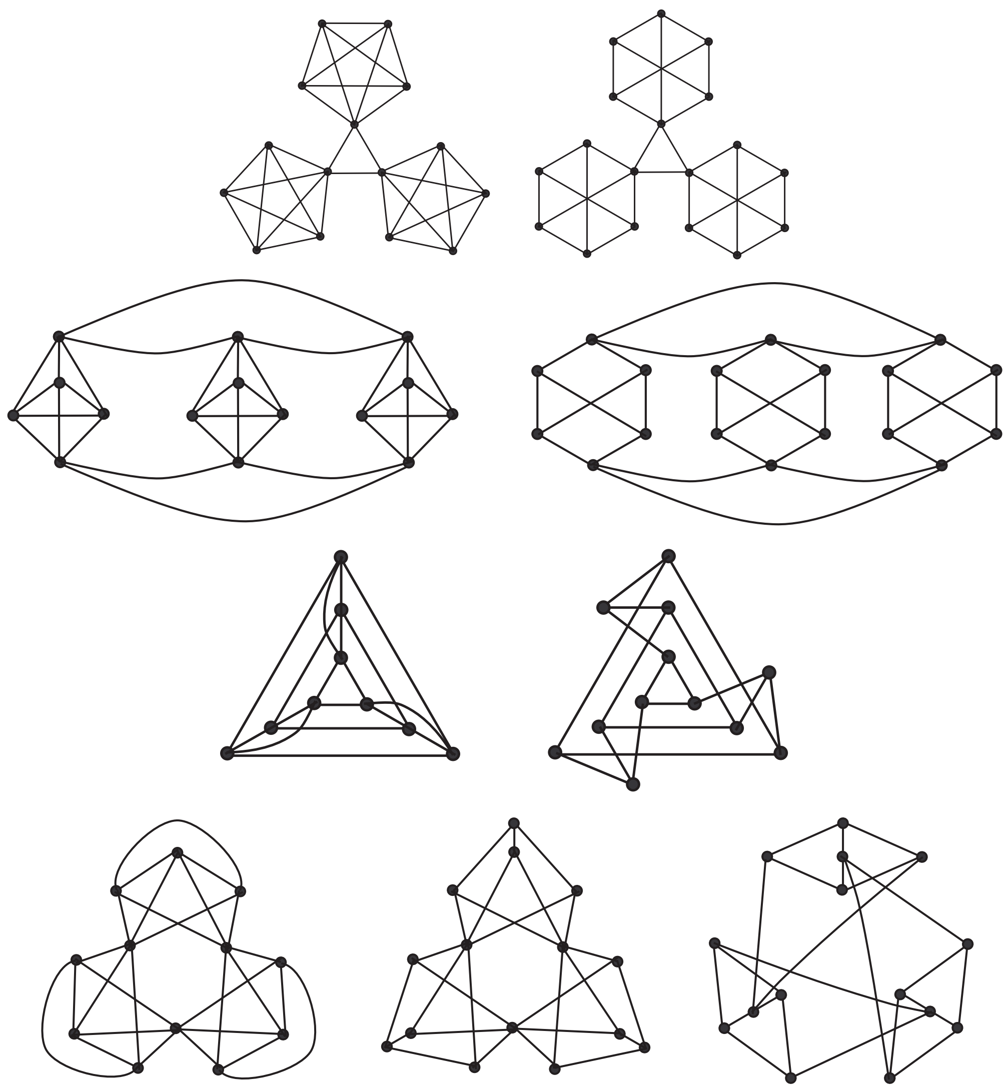


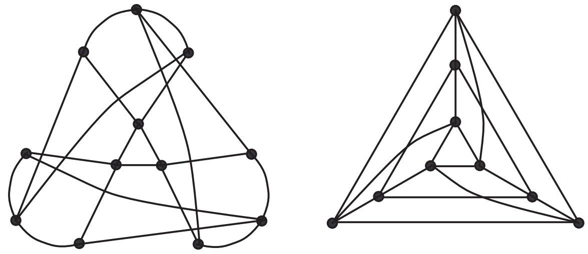
Figure 3.11.

□

Figure 3.12 shows the 11 freely $\mathbb{Z}_3$-symmetric graphs derived, respectively, from the voltage graphs in Figure 3.8 with $\mathbb{Z}_3$-voltages. These are minimal free obstructions for $\mathbb{Z}_3$-sphericity by Propositions 3.1 and 3.9. The renderings of the $\mathbb{Z}_3$-symmetric graphs in Figure 3.12 clearly show the free $\mathbb{Z}_3$-action as a rotation. Therefore we can derive the analogous freely $\mathbb{Z}_n$-symmetric graphs for other n in the obvious fashion.

Figure 3.12. *The freely $\mathbb{Z}_3$-symmetric graphs derived, respectively, from the voltage graphs in Figure 3.8 with $\mathbb{Z}_3$-voltages. These are minimal free obstructions for $\mathbb{Z}_3$-sphericity.*





3.2 Completeness of the catalogue

In this section we will show that the catalogue of link-minor-minimal members of $\bigotimes_n$ presented in Section 3.1 is complete. For n odd and $n \geq 3$ assume that (G, σ) is a link-minor-minimal member of $\bigotimes_n$. We will show that (G, σ) is isomorphic to one of the voltage graphs from Section 3.1.

Claim 1. *If (G, σ) has one vertex, then (G, σ) is a spiral bouquet or a minimal generating bouquet.*

Proof of Claim: If (G, σ) has only one vertex v, then (G, σ) is a bouquet of loops $l_1, \ldots, l_m$. Since (G, σ) generates $\mathbb{Z}_n$, the group $\langle \sigma(l_1), \ldots, \sigma(l_m) \rangle$ equals $\mathbb{Z}_n$ and since (G, σ) is not 2-branch-point-spherical, $m \geq 2$. If this set of voltages is a minimal generating set for $\mathbb{Z}_n$, then we are done. If not, then there is some l_i (say l_1) such that $\langle \sigma(l_2), \ldots, \sigma(l_m) \rangle = \mathbb{Z}_n$. So since (G, σ) is a link-minor-minimal member of $\bigotimes_n$, $(G \backslash l_1, \sigma)$ is 2-branch-point spherical. So up to automorphism of $\mathbb{Z}_n$, $\{\sigma(l_2), \ldots, \sigma(l_m)\} = \{1, -1\}$. Since (G, σ) is not 2-branch-point spherical, we must have that $\sigma(l_1) \neq \pm\sigma(l_2)$. So now (G, σ) contains a spiral bouquet, which is not 2-branch-point spherical and generates $\mathbb{Z}_n$. By minimality we then get that (G, σ) is a spiral bouquet. ♣

Claim 2. *If (G, σ) has more than one vertex and contains a theta subgraph Θ whose three cycles are all nonzero under σ, then $\mathbb{Z}_n = \mathbb{Z}_3$ and (G, σ) is isomorphic to the $\mathbb{Z}_3$-voltage graph shown in Figure 3.6.*

Proof of Claim: If (G, σ) contains a theta subgraph Θ in which all three cycles are nonzero under σ, then when $n = 3$, (Θ, σ) is a subdivision of a $\mathbb{Z}_3$-voltage graph isomorphic to the one in Figure 3.6. This $\mathbb{Z}_3$-voltage graph is a link-minor-minimal member of $\bigotimes_n$ and thus, by minimality, (G, σ) is the $\mathbb{Z}_3$-voltage graph shown.

For the remainder of the proof take $n \geq 5$ and say that C_1, C_2, and C_3 are the cycles of Θ with $\vec{C}_1 + \vec{C}_2 = \vec{C}_3$. Write $\sigma_*(\vec{C}_1) = a$ and $\sigma_*(\vec{C}_2) = b$ so that $\sigma_*(\vec{C}_3) = a + b \neq 0$.

If (Θ, σ) generates $\mathbb{Z}_n$, then either there is one element of $\{a, b, a+b\}$ that generates $\mathbb{Z}_n$ (without loss of generality assume a is a generator) or any pair of elements from $\{a, b, a+b\}$ is a minimal generating set for $\mathbb{Z}_n$. In the first case, (Θ, σ) has a bouquet of two loops as a proper link minor where we may choose the voltages on the two loops to be either a and b (if $a \neq b$) or a and $a + b$ (if $a = b$). If $a \neq b$, then the fact that $a \neq -b$ implies that the two loops form a spiral bouquet. If $a = b$, then the facts that $a \neq -b$ and $n \geq 5$ imply that the bouquet of two loops with voltages a and $2a \neq \pm a$ form a spiral bouquet. These both contradict the minimality of (G, σ). In the second case, the link minor that is the bouquet of two loops with voltages a and b is a minimal generating bouquet, a contradiction to the minimality of (G, σ).

If (Θ, σ) does not generate $\mathbb{Z}_n$, then take a spanning tree T' of Θ and extend it to a spanning tree $T \supset T'$ of G. (Recall that the link-minor-minimal members of $\bigotimes_n$ must be connected.) Because (G, σ) does generate $\mathbb{Z}_n$, Proposition 2.12 guarantees us that the group $\langle \sigma_T(e) : e \in \vec{E}(G) \backslash \vec{E}(T) \rangle$ equals $\mathbb{Z}_n$. Now $(G/T, \sigma_T)$ is a proper link minor of (G, σ_T) which generates $\mathbb{Z}_n$ which (by the minimality of (G, σ)) must be 2-branch-point spherical. Thus there is $x \in \mathbb{Z}_n$ such that $\sigma_T(e) \in \{0, x, -x\}$ for all $e \in \vec{E}(G) \backslash \vec{E}(T)$. Thus $\langle x \rangle = \mathbb{Z}_n$ and $\pm x \in \{a, b, a+b\}$, which implies that $\{a, b, a+b\}$ is a generating set for $\mathbb{Z}_n$, contradicting the assumption that (Θ, σ) does not generate $\mathbb{Z}_n$. ♣

For the remainder of this section we may now assume that every theta subgraph of (G, σ) has at least one zero cycle. By Proposition 2.9(2) there is a signed graph (G, ψ) with the same zero cycles as (G, σ). Let L_G be the set of loops in G and so by Theorem 2.10 $(G \backslash L_G, \psi)$ is a Gerards signed graph. Claim 3 completes the proof of the completeness of the catalogue of voltage graphs presented in Section 3.1.

Claim 3. *If $(G\backslash L_G,\psi)$ is a Gerards signed graph, then (G,σ) is isomorphic to one of the eleven $\mathbb{Z}_n$-voltage graphs in Figure 3.8.*

Proof of Claim: In Case 1 assume that $(G\backslash L_G,\psi)$ is balanced. In Case 2, assume that $(G\backslash L_G,\psi)$ is unbalanced and not 2-connected. In Case 3, assume that $(G\backslash L_G,\psi)$ is unbalanced and 2-connected and $L_G\neq\emptyset$. In the remaining cases we assume that $(G\backslash L_G,\psi)$ is unbalanced, 2-connected, but $L_G=\emptyset$. In Case 4, assume that (G,ψ) is not 3-connected. So in the remaining cases we may assume that (G,ψ) is 3-connected and so the structure of Gerrards signed graphs gives us the following four remaining cases. In Case 5, assume that (G,ψ) is 3-connected and has a balancing vertex. In Case 6, assume that (G,ψ) is 3-connected and has a planar embedding with exactly two nonzero facial boundary cycles. In Case 7, assume that $(G,\psi)\cong T_6$. In Case 8, assume that (G,ψ) has a 3-split.

Case 1 Since $(G\backslash L_G,\psi)$ is balanced, so is $(G\backslash L_G,\sigma)$. Since $(G\backslash L_G,\sigma)$ is balanced, we may normalize σ on any spanning tree of G and get $\sigma(e)=0$ for any $e\notin\vec{L}_G$. So since (G,σ) generates $\mathbb{Z}_n$, the loop voltages generate $\mathbb{Z}_n$. However, by minimality, (G,σ) cannot contain as a link minor a minimal generating bouquet nor a spiral bouquet. So up to automorphism of $\mathbb{Z}_n$, $\sigma(e)=\pm1$ for each $e\in\vec{L}_G$. In Case 1.1 assume that $|L_G|=1$, in Case 1.2 assume that $|L_G|=2$, and in Case 1.3 assume that $|L_G|\geq 3$.

Case 1.1 Write $L_G=\{\ell\}$. It cannot be that $G\backslash\ell$ is planar, because then (G,σ) is 2-branch-point spherical. So by minimality (G,σ) is K_5^{ℓ} or $K_{3,3}^{\ell}$.

Case 1.2 Write $L_G=\{\ell_1,\ell_2\}$. Since $(G\backslash\ell_1,\sigma)$ still generates $\mathbb{Z}_n$, it must be that $(G\backslash\ell_1,\sigma)$ is 2-branch-point spherical while (G,σ) is not. This tells us there is no planar embedding of $(G\backslash\{\ell_1,\ell_2\},\sigma)$ with the endpoints of ℓ_1 and ℓ_2 on the same face, and so the graph obtained from $G\backslash\{\ell_1,\ell_2\}$ by adding an edge connecting the endpoints of ℓ_1 and ℓ_2 is not planar and is minor minimally so, which can only be K_5 or $K_{3,3}$. Thus (G,σ) is $K_5^{2\ell}$ or $K_{3,3}^{2\ell}$.

Case 1.3 Write $L_G=\{\ell_1,\ldots,\ell_n\}$ where ℓ_i has endpoint v_i. By minimality, we must have that $v_i\neq v_j$ for $i\neq j$. As G is connected, let T be a spanning tree of G and take a path p of longest length in T connecting two loop endpoints. If there is a third loop in G whose endpoint is not on p, then the maximality of p implies that (G,σ) will contain $Y^{3\ell}$ as a link minor. So, by minimality, (G,σ) is $Y^{3\ell}$.

So now assume that all loop endpoints of G appear on p. Reindex so that $v_1,\ldots,v_n$ are arranged in order along the path. We claim that there is not some v_i with $1<i<n$ that is a cut vertex of G. By way of contradiction, if v_i with $1<i<n$ is a cut vertex of G, then (G,σ) has a 1-split at v_i into (G_1,σ_1) and (G_2,σ_2) where $\ell_1\in G_1$ and $\ell_n\in G_2$. Say without loss of generality that $\ell_i\in G_1$ and add a copy of ℓ_i to v_i in (G_2,σ_2) to get (G_2',σ_2'). Both (G_1,σ_1) and (G_2',σ_2') are proper link minors of (G,σ) that generate $\mathbb{Z}_n$ and so each is 2-branch-point spherical. As $(G\backslash L_G,\sigma)$ is balanced, the embedding of (G_1,σ_1) must be as shown in Figure 3.13 (the embedding of (G_2',σ_2') is similar).

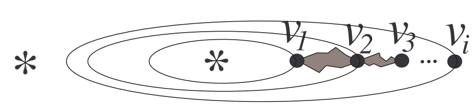


Figure 3.13.

Glue the embeddings of (G_1,σ_1) and (G_2',σ_2') together along ℓ_i to obtain a 2-branch-point spherical embedding of (G,σ), a contradiction. So there is some path p_0 of G that is internally disjoint from p such that $p\cup p_0$ contains a v_1v_n-path bypassing some v_i with $1<i<n$. Thus (G,σ) contains $\Delta^{3\ell}$ as a link minor and so, by minimality, (G,σ) is $\Delta^{3\ell}$.

Case 2 Since $(G\backslash L_G,\psi)$ is connected but not 2-connected, consider the blocks $B_1,\ldots,B_m$ of $G\backslash L_G$. Without loss of generality block B_1 contains a vertex x such that $(E(B_1),E(B_2)\cup\cdots\cup E(B_n))$ is a 1-separation of G at vertex x with $B_2\cup\cdots\cup B_m$ connected (*i.e.*, B_1 is an "end block" of G). In Case 2.1, assume that (B_1,σ) is balanced and in Case 2.2 assume that (B_1,σ) is unbalanced.

Case 2.1 If (B_1,σ) is balanced, then either there is a loop ℓ of G attached to a vertex $x'\neq x$ in B_1 or there is not. Let these be Cases 2.1.1 and 2.1.2.

Case 2.1.1 If B_1 is not planar, then (G, σ) contains a proper K_5^ℓ- or $K_{3,3}^\ell$-link minor, a contradiction. If B_1 is planar, then either there is an embedding of B_1 in the plane with x and x' on the same face or there is not such an embedding. If not, then (as in Case 1.2) (G, σ) contains a proper $K_5^{2\ell}$- or $K_{3,3}^{2\ell}$-link minor, a contradiction. So now assume that B_1 has a planar embedding with x and x' on the same face. Since (B_1, σ) is balanced, $(G/E(B_1), \sigma)$ still generates $\mathbb{Z}_n$ and so, by minimality, must be 2-branch-point-spherical. Let b be the endpoint of ℓ in $(G/E(B_1), \sigma)$. Either there is a 2-branch-point-spherical embedding of $(G/E(B_1), \sigma)$ with b on the face bounding a branch point, or there is not.

If there is such an embedding, then we can reembed ℓ so as to bound a face containing one of the branch points. We can now decontract B_1 to obtain an embedding of (G, σ) in the 2-branch-point sphere, a contradiction.

If there is no such embedding, then any embedding of $(G/E(B_1), \sigma)$ has ℓ separating two vertex-disjoint nonzero cycles C_1 and C_2. Take paths γ_1 and γ_2 in $G/E(B_1)$ from b to C_1 and b to C_2, respectively. Now $B_1 \cup \ell \cup \gamma_1 \cup \gamma_2 \cup C_1 \cup C_2$ contains $Y^{3\ell}$ as a link minor. Thus $(G, \sigma) \cong Y^{3\ell}$.

Case 2.1.2 Here $(G \backslash E(B_1), \sigma)$ is connected (up to removal of isolated vertices) and generates $\mathbb{Z}_n$, and so by minimality must be 2-branch-point spherical. So now since (B_1, σ) is balanced and (G, σ) is not 2-branch-point spherical, we must have that B_1 is not planar. Thus (G, σ) has a proper K_5^ℓ- or $K_{3,3}^\ell$-link minor, a contradiction.

Case 2.2 By Theorem 2.10, each block (B_i, σ) has $a_i \in \Gamma$ such that each cycle C in B_i has voltage from $\{0, a_i, -a_i\}$. Since (B_1, σ) is unbalanced $a_1 \neq 0$. By the following argument we now get that there is a nonzero element $a \in \Gamma$ such that each $a_i \in \{0, a, -a\}$ and each loop of G has voltage $\pm a$. Take a spanning tree T of G and note that T restricted to a block of G is a spanning tree of that block. As such, each $e \in \vec{E}(G) \backslash \vec{E}(T)$ in block B_i has $\sigma_T(e) \in \{0, a_i, -a_i\}$. Since (G, σ) generates $\mathbb{Z}_n$, the bouquet of loops $(G, \sigma)/E(T)$ generates $\mathbb{Z}_n$ and so must be 2-branch-point spherical, which implies that each nonzero loop in this bouquet has the same voltage. Up to automorphism of $\mathbb{Z}_n$ we may assume that this voltage is ± 1.

Let B_2° be $B_2 \cup \cdots \cup B_m$ along with the loops of L_G that are attached to vertices in $B_2 \cup \cdots \cup B_m$. Let B_1° be B_1 along with the loops of L_G that are attached to vertices in B_1. Now (B_2°, σ) generates $\mathbb{Z}_n$ or is balanced. In the latter case we can rechoose B_1 and revert back to Case 2.1. So now there are proper link minors (G_1, σ_1) and (G_2, σ_2) of (G, σ) where (G_i, σ_i) is (B_i°, σ) along with a new loop ℓ_i added at x with voltage $\sigma_i(\ell_i) = 1$. Furthermore, each of these link minors generates $\mathbb{Z}_n$ and so is 2-branch-point spherical. Because B_1 is a block, the embedding of (G_1, σ_1) has ℓ_1 bounding a face containing one of the branch points. Now either the loop ℓ_2 in the embedding of (G_2, σ_2) may be reembedded so that it bounds a face containing one of the branch points or it cannot. If it can be, then we can identify the embeddings of (G_1, σ_1) and (G_2, σ_2) along ℓ_1 and $\pm\ell_2$ to obtain a 2-branch-point spherical embedding of (G, σ), a contradiction. So assume that ℓ_2 cannot be reembedded so that it bounds a face containing a branch point. Now if x is a balancing vertex of (G_1, σ_1), then again we can identify the embeddings of (G_1, σ_1) and (G_2, σ_2) along ℓ_1 and $\pm\ell_2$ to obtain a 2-branch-point spherical embedding of (G, σ), a contradiction. If x is not a balancing vertex of (G_1, σ_1), then there is a nonzero cycle C_1 in $(G_1, \sigma) \backslash x$. Furthermore, as in Case 2.1.1, ℓ_2 separates two vertex-disjoint nonzero cycles in $(G_2, \sigma_2) \backslash x$, call them C_2 and C_3. Taking C_1, C_2, and C_3 along with a path from each C_i to x, we find $Y^{3\ell}$ as a link minor of (G, σ), making $(G, \sigma) \cong Y^{3\ell}$.

Case 3 Since $(G \backslash L_G, \psi)$ is unbalanced and 2-connected, Theorem 2.10 implies that there is $a \in \mathbb{Z}_n$ such that $\sigma_*(C) \in \{0, -a, a\}$ for all cycles C in $G \backslash L_G$. If a does not generate $\mathbb{Z}_n$, then there are loops in L_G whose voltages along with a do generate $\mathbb{Z}_n$, in which case (G, σ) contains a minimal generating bouquet or a spiral bouquet as a proper link minor, a contradiction to the minimality of (G, σ). Thus a does generate $\mathbb{Z}_n$, so up to automorphism of $\mathbb{Z}_n$ we can assume that $a = 1$. Now if there is a loop $l \in L_G$ such that $\sigma(l) \neq \pm 1$, then (G, σ) contains a spiral bouquet as a proper link minor, a contradiction to the minimality of (G, σ). Thus each $l \in L_G$ satisfies $\sigma(l) = \pm 1$. Since there is an $l \in L_G$ and since $(G \backslash l, \sigma)$ still generates $\mathbb{Z}_n$, minimality implies that $(G \backslash l, \sigma)$ is 2-branch-point spherical. Now consider a 2-branch-point spherical embedding of $(G \backslash l, \sigma)$ where C_1 and C_2 are the facial-boundary cycles around the two branch points and u is the endpoint of l. Since (G, σ) is not 2-branch-point spherical, it cannot be that u is a vertex of $C_1 \cup C_2$. In Case 3.1, say that C_1 and C_2 are vertex disjoint, in Case 3.2 say that C_1 and C_2 intersect in a single vertex, and in Case 3.3 say that C_1 and C_2 intersect in several vertices and/or paths.

Case 3.1 Either there are two paths, say γ_1' and γ_2', where γ_i' goes from u to C_i and γ_i' does not intersect C_j for $i \neq j$, or such paths do not exist.

In the first case, there are three internally disjoint paths $\gamma_1, \gamma_2, \gamma \subseteq (\gamma_1' \cup \gamma_2')$ and vertex $x \in (\gamma_1' \cup \gamma_2')$ such that γ is from u to x (γ of length zero is possible), γ_1 is from x to C_1 and γ_2 is from x to C_2. Since G is 2-connected, $G\backslash x$ is still connected and so there is a path γ_3' in $G\backslash x$ connecting C_1 and C_2. Contract C_1 and C_2 down to loops. The path γ_3' shares an endpoint with each γ_i, and furthermore, there is a subpath $\gamma_3 \subset \gamma_3'$ that is internally disjoint from γ_1 and γ_2 and such that one endpoint of γ_3 is in $\gamma_1\backslash\gamma_2$ and the other endpoint is in $\gamma_2\backslash\gamma_1$ (and γ_3 avoids x, of course). Now starting from u, let $\hat{x}$ be the first vertex of γ that is in $\gamma_1 \cup \gamma_2 \cup \gamma_3$. Necessarily either $\hat{x} = x$ or $\hat{x}$ is in the interior of γ_3. In either case, (G, σ) contains either $\Delta^{3\ell}$ or $\widehat{\Delta}^{3\ell}$ as a link minor. So by minimality, (G, σ) is either $\Delta^{3\ell}$ or $\widehat{\Delta}^{3\ell}$.

In the latter case (where γ_1' and γ_2' do not exist), without loss of generality, we may suppose that every path from u to C_2 must first intersect C_1. Let P_u be the union of all minimal paths from u to C_1; the subgraph $C_1 \cup P_u$ is therefore vertex disjoint from C_2. The induced embedding of $C_1 \cup P_u$ has a face in which C_2 is embedded. Since we cannot have a path from C_2 to P_u that does not first intersect C_1, this face contains on its boundary a unique maximal proper subpath γ of C_1 whose two endpoints, u_1 and u_2, separate u from C_2. Now let B_u be the $\{u_1, u_2\}$-bridge of G that contains u in its interior. We can detach B_u from the embedding of $G\backslash l$ and reattach it inside the face bounded by C_1. Either there is a reembedding of B_u such that u, u_1, u_2 are all on the same face or there is no such reembedding. In the former case, this common face is the face with the branch point in it and so we can add in the loop l to this embedding, a contradiction. In the latter case, attach a new vertex x to B_u that is adjacent to u, u_1, u_2, so that $B_u \cup x$ is not planar while B_u is planar. Thus there is a K_5- or $K_{3,3}$-subdivision, call it H, in $B_u \cup x$ that uses x. Let γ be a u_1u_2-subpath of C_1 that is internally disjoint from B_u and such that $B_u \cup \gamma$ contains a nonzero cycle.

If H is a subdivision of K_5, then because K_5 has no vertices of degree 3, x appears in the middle of a branch of H and H contains only two edges incident to x. If H contains the xu_1- and xu_2-edges, then $B_u \cup C_1$ is nonplanar, a contradiction. If (without loss of generality) H contains the xu_1- and xu-edges, then take l and a path γ' in G from u_1 to C_2 to obtain $K_5^{2\ell}$ as a proper link minor of (G, σ), a contradiction to the minimality of (G, σ).

If H is a subdivision of $K_{3,3}$, then either x is in the interior of a branch of H or is a branch vertex of H. The former case is similar to the previous paragraph. If x is a branch vertex of H, then $B_u \cup \gamma \cup l$ contains $\widehat{V}^{\ell}$ as a proper link minor, a contradiction to the minimality of (G, σ).

Case 3.2 Write $v = C_1 \cap C_2$. Either there are two paths, say γ_1' and γ_2', where γ_i' goes from u to $C_i\backslash v$ and γ_i' does not intersect C_j for $i \neq j$, or such paths do not exist. If they do exist, then using reasoning nearly the same as in Case 3.1 we get that $(G, \sigma) \cong \widehat{V}^{\ell}$. If the two paths do not exist, then (without loss of generality) every path from u to C_2 must first intersect C_1 and we finish as in Case 3.1.

Case 3.3 Since C_1 and C_2 intersect, we have a configuration as in Figure 3.14 excluding the vertex x. As in the second part of Case 3.1, we can either reembed the $\{u_1, u_2\}$-bridge B_u of G containing u so that l can be added to the embedding or attach new vertex x so that $B_u \cup x$ is nonplanar and then finish as before.

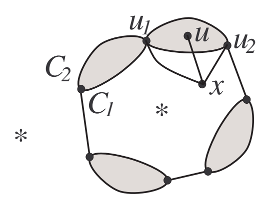


Figure 3.14.

Case 4 Since G is 2-connected and loopless, Theorem 2.10 implies that (up to automorphism of $\mathbb{Z}_n$) $\sigma_*(C) \in \{0, -1, 1\}$ for all cycles C in G. Say that G has a 2-separation at vertices x_1 and x_2. Let $B_1, \ldots, B_m, E_1, \ldots, E_k$ be the $\{x_1, x_2\}$-bridges of G where each B_i contains more vertices that just x_1 and x_2 and each E_i is a single edge on $\{x_1, x_2\}$. Note that the minimality and looplessness of (G, σ) implies that $k \leq 2$ and since G is not 3-connected, $m \geq 2$.

First, it cannot be that some (B_i, σ) is balanced. Suppose for the sake of contradiction, and without loss of generality, that (B_1, σ) is balanced. Then there is $a \in \mathbb{Z}_n$ such that every x_1x_2-path γ in B_1 has total voltage a, and (G, σ) has a link minor (G_2, σ_2) obtained from (G, σ) by replacing B_1 with a single x_1x_2-link e having voltage a. Minimality now implies that (G_2, σ_2) is 2-branch-point spherical. If B_1 has a planar embedding with x_1 and x_2 on the same face, then in an embedding of (G_2, σ_2) we can replace e with B_1 to obtain a 2-branch-point-spherical embedding of (G, σ), a contradiction. So suppose that B_1 does not have a planar embedding with x_1 and x_2 on the same face. Since (G, σ) is unbalanced, it has a proper link minor $(B_1 \cup \{e', e''\}, \sigma)$ where $\sigma(e') = 1$ and $\sigma(e'') = 0$. Without loss of generality, we may now assume that $(B_1 \cup e', \sigma)$ is unbalanced. Furthermore, we have that $B_1 \cup e'$ is 2-connected and nonplanar. Thus $(B_1 \cup e', \sigma)$ contains a link minor H with $e' \in H$ and $H \cong K_5$, $K_{3,3}$, or $K_{3,3} \cup e'$ (see [6]). Thus $(H, \sigma)/e'$ is a proper link minor of (G, σ) and is one of $\widehat{K}_5/\hat{e}$, $(K_{3,3} \cup \hat{e})/\hat{e}$, or $\widehat{K}_{3,3}/\hat{e}$, a contradiction to the minimality of (G, σ).

Second, it cannot be that both x_1 and x_2 are balancing vertices of some (B_i, σ). If this were the case, then there would be a 2-separation (A, B) of B_i at x_1 and x_2 (see [12]). Since B_i has no x_1x_2-links, this 2-separation would be a vertical 2-separation of B_i, a contradiction to the fact that B_i is an $\{x_1, x_2\}$-bridge. Since one of x_1 and x_2 is not a balancing vertex of B_i (say x_1 is not a balancing vertex) then there is an nonzero cycle C in $(B_i \backslash x_1, \sigma)$. Because (G, σ) is 2-connected, there are internally disjoint paths connecting C to x_1 and x_2 and so (B_i, σ) contains the link minor shown in Figure 3.15 rooted at $\{x', x''\} = \{x_1, x_2\}$.

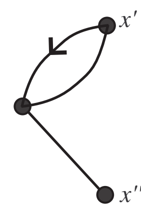


Figure 3.15.

If (G, σ) contains at least three B_i's, then (G, σ) contains either $(K_{3,3} \cup \hat{e})/\hat{e}$ as a link minor or $\widehat{V}^{\ell}$ as a proper link minor. Minimality now implies that $(G, \sigma) \cong (K_{3,3} \cup \hat{e})/\hat{e}$.

Thus (G, σ) contains exactly two B_i's which are both unbalanced. Now (G, ψ) has 2-split (G_1, ψ_1) and (G_2, ψ_2) which are both proper link minors of (G, ψ). Thus we get the corresponding proper link minors (G_1, σ_1) and (G_2, σ_2) of (G, σ) where $G_i = B_i \cup \{e_1, e_2\}$, $\sigma_i(e_1) = 0$, $\sigma_i(e_2) = 1$, and $\sigma_i|_{B_i} = \sigma|_{B_i}$. Since each (G_i, σ_i) generates $\mathbb{Z}_n$, each (G_i, σ_i) is 2-branch-point spherical. A 2-branch-point spherical embedding of (G_i, σ_i) would be as shown on either the left or right of Figure 3.16; however, the embedding must be as shown on the right because the embedding on the left shows a vertical 2-separation of B_i at $\{x_1, x_2\}$, a contradiction to the fact that B_i is a $\{x_1, x_2\}$-bridge. We can now combine the embeddings of (G_1, σ_1) and (G_2, σ_2) to obtain an embedding of (G, σ), a contradiction.

Figure 3.16.

Case 5 Say that v is a balancing vertex of (G, ψ) and that (G, σ) is 3-connected. Thus v must also be a balancing vertex of (G, σ). Choose a spanning tree T of G where v is a leaf of T. If we normalize σ on T, then any oriented edge e without v as one of its endpoints has $\sigma(e) = 0$. Because (G, σ) is 3-connected and generates $\mathbb{Z}_n$, Theorem 2.10 implies that, up to automorphism of $\mathbb{Z}_n$, $\sigma(e) \in \{0, 1\}$ for each oriented edge e with v as its tail endpoint. In Case 5.1 say that G is nonplanar and in Case 5.2 say that G is planar.

Case 5.1 Hall's Theorem [4] states that a 3-connected and simple nonplanar graph is either isomorphic to K_5 or contains a $K_{3,3}$-subdivision. Now G is 3-connected and simple up to removal of parallel edges. If the underlying simple graph of G is K_5, then, up to equivalence and deletion of parallel edges, (G, σ) contains

(K_5, σ) with either one nonzero edge incident to v or two nonzero edges incident to v. In the case of one nonzero edge, (K_5, σ) contains $\widehat{K}_5/\hat{e}$ as a proper link minor and in the case of two nonzero edges, (K_5, σ) contains $\widehat{K}_{3,3}/\hat{e}$ as a proper link minor; each possibility contradicts the minimality of (G, σ). If G contains a $K_{3,3}$-subdivision H, then either (H, σ) has a balancing vertex or is already balanced. If there is a balancing vertex, then (H, σ) has (up to switching) one nonzero edge. In this case, (H, σ) contains $\widehat{K}_{3,3}/\hat{e}$ as a proper link minor, a contradiction to the minimality of (G, σ). If (H, σ) is balanced, then append a path γ to H that runs through v so that $(H \cup \gamma, \sigma)$ is unbalanced. Thus $(H \cup \gamma, \sigma)$ contains either $\widehat{K}_{3,3}/\hat{e}$ or $(K_{3,3} \cup \hat{e})/\hat{e}$ as a proper link minor, a contradiction to the minimality of (G, σ).

Case 5.2 Since G is 3-connected and loopless, there is a unique embedding of G in the plane up to exchanging places of edges within pairs of parallel edges. Furthermore, uniqueness implies that parallel edges in G must be embedded so as to bound faces of length two, and 3-connectivity ensures that each face of the embedding of G is bounded by a cycle in G. Assume that the parallel edges are embedded in such a way as to minimize the number of nonzero facial boundary cycles. Since the sum of the facial boundary cycles in $Z_1(G)$ is zero and since each facial boundary cycle has total voltage in $\{0, 1, -1\}$, the number of nonzero facial boundary cycles must be even. Since (G, σ) is not 2-branch-point spherical, this number of nonzero facial boundary cycles is at least 4.

The 3-connectivity of G implies that the union of the facial boundary cycles that contain the balancing vertex v form a wheel-like subgraph G_v consisting of a *rim* cycle R not containing v along with *spoke* edges connecting v to R. Let $e_1, \ldots, e_n$ be the spoke edges of G_v in cyclic order around v and let v_i be the endpoint of e_i on R. Without loss of generality we may assume that $\sigma(e_i) \in \{0, 1\}$ where e_i is oriented away from v. If there are three or more parallel pairs of edges among $e_1, \ldots, e_n$, then (G_v, σ) contains $\widehat{K}_5/\hat{e}$ as a link minor and so the minimality of (G, σ) implies that (G, σ) is $\widehat{K}_5/\hat{e}$. So now we may assume that there are at most two parallel pairs of edges among $e_1, \ldots, e_n$. We will explain the proof for two parallel pairs of edges; the proof for one or zero parallel pairs uses detail similar (and easier) to Case 5.2.1 below.

If there are two parallel pairs of edges, say a_1, a_2 and b_1, b_2, then we may assume that neither of the two parallel pairs is adjacent to two nonzero faces; otherwise, we could exchange the places of the parallel pair of edges to make them adjacent to two zero faces. This would reduce the number of zero faces in an embedding of (G_v, σ), a contradiction to the fact that we chose an embedding with the minimum possible number of nonzero faces. Now, either a_1, a_2 and b_1, b_2 do not appear in consecutive order around v or they do. Let these be Cases 5.2.1 and 5.2.2, respectively.

Case 5.2.1 Say without loss of generality that a_1 is on a zero face, so that the induced embedding of $(G_v \backslash a_1, \sigma)$ has the same number of nonzero faces as the embedding of (G_v, σ); furthermore, since a_1, a_2 and b_1, b_2 are not ordered consecutively around v, we cannot reduce this number of nonzero faces in the embedding of $(G_v \backslash a_1, \sigma)$ by exchanging b_1 and b_2. Similarly and without loss of generality the induced embedding of $(G_v \backslash \{a_1, b_1\}, \sigma)$ has the same number of nonzero faces as the embedding of (G_v, σ) (which is at least 4). Additionally $(G_v \backslash \{a_1, b_1\}, \sigma)$ is a subdivision of a wheel and so its planar embedding is unique. Since the number of nonzero faces in this embedding is at least 4, there are four spokes $e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}$ in cyclic order around v whose σ-values alternate $0, 1, 0, 1$. Thus (G_v, σ) contains $\widehat{K}_{3,3}/\hat{e}$ as a proper link minor, a contradiction to minimality.

Case 5.2.2 If there is a nonzero face, call it F, of the embedding of (G_v, σ) that is adjacent to neither a_1, a_2 nor b_1, b_2, then we can contract the edges $R' \subset R$ along F to get a proper link minor $(G_v/R', \sigma)$ of (G_v, σ) with three parallel pairs of edges among $e_1, \ldots, e_n$. As before at the beginning of Case 5.2, $(G_v/R', \sigma)$ contains $\widehat{K}_5/\hat{e}$ as a link minor, a contradiction to the minimality of (G, σ). So now assume that there is no such nonzero face in the embedding of (G_v, σ). Write $D_1, F, D_2, F_1, \ldots, F_n$ for the faces of (G_v, σ) in cyclic order around v where D_1 and D_2 are of length two, F and each F_i are of length at least three, and $F_2, \ldots, F_{n-1}$ are all zero faces. By switching places of a_1 and a_2 (if necessary) we may assume that F is a zero face without affecting the total number of zero faces of the embedding. So now $F_1 \neq F_n$ and F_1 and F_n are both nonzero faces. Now switching the places of a_1 and a_2 and also of b_1 and b_2 yields an embedding of (G, σ) with only two nonzero faces, which implies that (G, σ) is 2-branch-point spherical, a contradiction.

Case 6 The fact that (G, ψ) is planar with exactly two nonzero facial boundary cycles implies that (G, σ) is

2-branch-point spherical, a contradiction.

Case 7 That $(G, \psi) \cong T_6$ implies that (G, σ) contains $\widehat{K}_5/\hat{e}$ as a proper link minor, a contradiction to the minimality of (G, σ).

Case 8 Let (G_1, ψ_1) be the unbalanced part of the 3-split and (G_2, ψ_2) the balanced part of the 3-split. Let v be the trivalent vertex of summation in G_1 and G_2 and let $\{x, y, z\}$ be the three neighbors of v. Since $G_2\backslash v$ has at least four edges we get that $V(G_2)\backslash\{x, y, z\} \neq \emptyset$ because otherwise we would have two parallel edges with the same voltage in (G, ψ), contradicting minimality. Furthermore, since (G, ψ) does not have a balancing vertex and (G, ψ) contains neither of the signed graphs of Figure 2.8 as a link minor, it must be that $V(G_1)\backslash\{x, y, z\} \neq \emptyset$, as well. Now the 3-connectivity of G implies that (G_1, ψ_1) is a proper link minor of (G, ψ). By the minimality of (G, σ), the corresponding link minor (G_1, σ_1) is 2-branch-point spherical. In Case 8.1, say that G_2 is planar and in Case 8.2, say that G_2 is not planar.

Case 8.1 A planar embedding of G_2 and a 2-branch-point spherical embedding of (G_1, σ_1) can be summed along their common triads to obtain a 2-branch-point-spherical embedding of (G, σ), a contradiction.

Case 8.2 We will find one of the following as a link minor of (G, σ): (G_2, σ_2') which is unbalanced with balancing vertex v, or $(G_2 \cup e, \sigma_2')$ where e is a link on $\{x, y, z\}$ and all nonzero cycles of $(G_2 \cup e, \sigma_2')$ contain e. Let A_i be the edges of $G_i\backslash v$. Take a vertex v' in $V(G_1)\backslash\{x, y, z\} \neq \emptyset$ and three internally disjoint paths $\gamma_x, \gamma_y, \gamma_z$ in $G_1\backslash v$ linking v' to $\{x, y, z\}$. Now $G_2'' = (G{:}A_2) \cup \gamma_x \cup \gamma_y \cup \gamma_z$ is a subdivision of G_2 and either (G_2'', σ) is unbalanced or not. If it is unbalanced, then because $(G{:}A_2, \sigma)$ is balanced, v' is a balancing vertex of (G_2'', σ) and we have our desired minor. If (G_2'', σ) is balanced, then extend $\gamma_x \cup \gamma_y \cup \gamma_z$ to a spanning tree T of $G{:}A_1$. Because $(G{:}A_1, \sigma)$ is unbalanced, there is some $e \in A_1$ such that the single cycle $T \cup e$ is unbalanced. That G is 3-connected now allows us to get one of our desired link minors of (G, σ).

Now G_2 is 3-connected, possibly after smoothing degree-2 vertices from $\{x, y, z\}$. So, using [8, 10.3.9] and the fact that G_2 is nonplanar, there is a $K_{3,3}$ subdivision H in G_2 with v as a branch vertex of H. Now we have either $\widehat{K}_{3,3}/\hat{e}$ or $(K_{3,3} \cup \hat{e})/\hat{e}$ as a proper link minor of (G, σ), a contradiction to the minimality of (G, σ). ♣

4 Future Directions

There are several significant difficulties for trying to extend the techniques in this paper to other groups that act faithfully on the sphere, or to non-freely Γ-symmetric graphs.

First, the investigation of the minimal free and faithful obstructions for $\mathbb{Z}_{2k+1}$-sphericity begins with the correspondence of these obstructions with the link-minor-minimal members of $\bigotimes_n$ given by Proposition 3.1. Using any other group Γ that acts faithfully on the sphere introduces other quotient spaces in addition to the 2-branch-point sphere. For example, pseudofree cellular actions of $\mathbb{Z}_{2k}$ for $k \geq 2$ on the sphere have quotient space that is either the 2-branch-point sphere or the single-branch-point projective plane.

Second, in Section 3.2 our proof relies heavily on the tightly defined structure of Gerards signed graphs. We reduce our proof to the class of Gerards signed graphs by applying Theorem 2.10. If Γ acts pseudofreely on the sphere but $|\Gamma|$ is even, then we cannot apply Theorem 2.10.

Third, investigating non-freely symmetric graphs is complicated by the fact that their quotients need not be graphs. Even when their quotients are graphs, derived graphs of voltage graphs do not suffice to reconstruct the original symmetric graph from the quotient.

All this being said, we imagine that finding the minimal free and faithful obstructions for $\mathbb{Z}_2$-sphericity should not be too hard.

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