Public debt management and tax evasion

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Abstract

This paper deals with the optimal management of the public debt to GDP ratio. We specifically focus on a contrasting tax evasion-based strategy for controlling the debt to GDP ratio. Two devices can be employed by the policymaker: by one side, the tax rate to be applied to the tax payers; by the other side, the monitoring activity to be performed in order to detect the evaded taxes. To pursue our scopes, a stochastic control problem is developed and solved. Some numerical experiments validate the theoretical proposal and lead to an intuitive discussion of the obtained findings.

Keywords: public debt to GDP ratio; tax evasion; tax rate; evasion monitoring; stochastic control, dynamic programming.

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JEL Classification: C61; H26; K34

1 Introduction

Public debt management is a relevant issue for countries with a high public debt to GDP ratio in the presence of interest rates that exceed the GDP growth rates. Such a scenario, when not coupled with primary surpluses linearly increasing in the public debt to GDP ratio (see e.g. Bohn, 1998; Greiner and Fincke, 2009) or with an accomodating monetary policy, determines an unsustainable public debt.

In a context characterized by low GDP growth rates and high volatility on public bond yields, as for some European countries during the recent economic downturn, such as Italy and Greece, the increase of primary surplus remains the relevant strategy for public debt stabilization. This objective may be pursued both through a reduction of public expenditure and an increase of taxation. Yet, the realization of these aims could be made difficult by some socio-economic features of a country, such as the high cost of a national public health service and a *pay-as-you-go* pension scheme with a low birth-rate population, and the high level of tax evasion. Hence, in order to reduce the public debt to GDP ratio, a rationalization of both public expenditure and taxation is required, so that wasteful expenses are cut and the theoretical tax revenues coincide with the effective ones, i.e. tax evasion is removed. This latter can be seen as a rational optimal choice by a taxpayer who compares revenues and costs of tax compliance with those deriving from fraudulent behaviors (Becker, 1968).

In particular, in the presence of tax evasion the disposable incomes are always higher than in a tax compliance scenario, whereas the magnitude of the costs depends crucially on the probability of being detected by the fiscal authorities. These latter, on their turn, decide to undertake a tax inspection after a comparison of the expected revenues, deriving from the discovered tax evasion, with the costs of the audit activity.

Since the theoretical work of Allingham and Sadmo (1972) the role of enforcement policies in lowering tax evasion has been widely deepened by the economic literature both theoretically (see e.g. Andreoni et al. 1998, Cerqueti and Coppier, 2011, Orsi et al. 2014, Bovi and Cerqueti, 2014, Argentiero and Bollino, 2015) and empirically (see e.g. Carfora et al. 2017); on the other hand, an increase in the tax rates may reduce tax compliance (see e.g. Gutmann 1977, Clotfelter 1983, Myles and Naylor 1996). Thus, following the pioneering theoretical intuition of Buchanan and Lee (1982) and Ireland (1994) together with a consolidated empirical literature (see e.g. Trabandt and Uhlig, 2011; Novales and Ruiz, 2012; Strulik and Trimborn, 2012; Busato and Chiarini, 2013; Orsi et al. 2014), we build a dynamic economic model where the underground economy leads to stochastic tax revenues and such that there exists a Laffer curve for income taxation. Specifically, there is a Laffer threshold for the tax rate such that the tax revenues present a inverse U-shape with respect to the tax rates, so that a tax-cut generates an increase of tax revenues. The growth of these latter reduces the public debt to GDP ratio through the rise of primary surplus component.

The aim of this paper is to explore a strategy for minimizing the public debt to GDP ratio. In particular, our contribution bridges the two strands of literature aforementioned above: the former focuses on the role of tax enforcement in reducing tax evasion, which has a negative impact on the public debt to GDP ratio, whereas the latter emphasizes the role of tax-cuts in enhancing tax base and tax compliance. We specifically focus on the reduction of tax evasion as a mean for letting the costs related to the public debt to GDP ratio be lower. Tax evasion is here contrasted through two devices: by one side, one can reduce the tax rates in order to incentive people to pay taxes; by the other side, one can increase the level of the monitoring activity for disincentivizing people to evade.

The economic problem of managing the public debt to GDP ratio is here assessed through a stochastic optimal control model (for a survey on this field, refer to Fleming and Soner, 1993 and Yong and Zhou, 1999). In particular, the tax rate and the monitoring level are taken as random dynamic *control variables*, which can be effectively managed for minimizing the entity of tax evasion. The policymaker implements strategies to increase the level of detected evasion and/or to decrease the amount of evaded taxes. The final target is the reduction of the costs, which are assumed to be related to the public debt to GDP ratio. Thus, the *objective function* of the problem is given by the expected aggregated discounted costs generated by the public debt to GDP ratio. The optimized objective function is the so-called *value function* of the problem. The *state variable* of the problem is represented by the public debt to GDP ratio, and it is reasonably assumed to exhibit dependence on the tax rate and on the monitoring level in both its deterministic and stochastic terms. Specifically, we here consider that the deterministic side of the public debt to GDP ratio grows with respect to the probability of being detected in evading taxes and is U-shaped with respect to the tax rate. In so doing, we adopt the point of view of having a bidirectional relationship between tax rate and public debt to GDP ratio: a too high tax rate leads to a pervasive evasion while a too small tax rate is associated to a low level of tax incomes. In both of cases, we have an increase of the public debt to GDP ratio. Furthermore, we assume that the state variable follows a stochastic differential equation driven by a Brownian motion. Finally, the *horizon* of the problem is random, in that there is no reason to restrict the analysis into a bounded time range but one needs also to consider the event of the failure of the observed country or the complete repaid of the debt. The unlucky case of failure appears when the public debt to GDP ratio goes above a prefixed threshold[1]. The stochastic time horizon leads to a nonstandard optimal control problem, as we will see.

To solve the optimal control problem, we adopt a Dynamic Programming Principle (DPP) approach. From this result, we are able to check that the value function has a representation as a classical solution of a second-order ordinary differential equation called Hamilton Jacobi Bellman (HJB) equation. The optimal strategies can be derived accordingly through a Verification Theorem.

Some numerical experiments support the theoretical findings and give an intuitive interpretation of the results of the problem. We present three different scenarios. The first one is associated to independent control processes, so that there is no mutual influence of one of them on the other. In the second scenario, the level of tax rate is assumed to constrain the entity of monitoring activity. In so doing, we implicitly include in the analysis the costs of monitoring, so that inspector activity can be sustained only by applying a sufficiently high tax rate. The third scenario discusses the situation of monitoring constraining tax rates, to assume that if monitoring is of high level, then a high enough tax rate needs to be applied.

The paper proceeds as follows. Section 2 describes the economic framework we deal with. Section 3 presents the stochastic optimal control model. Section 3 contains the solution strategy of the problem through DPP approach. Section 4 is dedicated to numerical experiments in the context of the three scenarios described above. In section 5, we discuss the empirical results and derive some policy implications. Section 6 concludes and traces lines of future research.

2 The economic framework

This section is devoted to the description of the economic context we deal with. Specifically, we develop a simple discrete-time framework to provide a microfoundation of the more complex continuous-time model treated in Section 3. In so doing, we aim at assisting the intuition of the reader and giving the details of the reference economic environment.

All the quantities introduced below are time-dependent. Therefore, we will denote them by employing an time-index $t \geq 0$.

2.1 The firms

The supply-side of our economy is composed of representative firms whose production, $Y_t = f(N_t)$, makes use of a technology based only on labor N_t and f is a given production function. For the sake of simplicity, we assume that labor demand is given with an exogenous hourly wage (as in Goerke, 2013).

Moreover, the following standard market clearing condition holds:

$$Y_t = A_t + G_t \tag{1}$$

where A_t is private consumption and G_t is public consumption.

2.2 The households

The economy is populated by infinitely-lived representative households with preferences defined over private consumption, A_t, sunlight labor services (namely, those ones subject to regular taxation, see e.g. Argentiero and Bollino, 2015), N_t^s, underground labor services (namely, those ones evading taxation), N_t^u, and public consumption, G_t. Households maximize the expected discounted value of an intertemporal utility function U_t, i.e.:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} e^{-\delta t} U_t \left(A_t, N_t^s, N_t^u, G_t\right) \tag{2}$$

with $\mathbb{E}_0$ is the expected value operator given the situation at time $t = 0$, $\delta > 0$ and $e^{-\delta}$ is a continuous uniperiodal discount factor.

Let us assume that the utility function has the following form:

$$U_t = \nu \frac{(A_t)^{1-q}}{1-q} + (1-\nu) G_t - \frac{(N_t^s)^{1+\psi}}{1+\psi} - B^u \frac{(N_t^u)^{1+\omega}}{1+\omega} \tag{3}$$

where q is the constant relative risk aversion parameter, ν is the share of utility related to private consumption, ψ and ω are the inverse Frisch elasticities for sunlight and underground labors and B^u is the idiosyncratic cost of working in the underground sector, following Busato and Chiarini (2004), Orsi et al. (2014) and Argentiero and Bollino (2015).

The maximization of (3) is subject to the following budget constraint[2]:

$$A_t \leq (1-\gamma_t) W_t N_t^s + (1-p_t) W_t N_t^u + p_t \left[(1-\gamma_t-\theta) W_t N_t^u\right] \tag{4}$$

where W_t is the exogenously given hourly wage (as in Goerke, 2013), γ_t is the tax rate on wage, p_t is the probability of being detected evading that is a positive function of α_t, i.e. the effort undertaken by the tax auditors in the enforcement activity associated to the detection of the tax evasion. We can think at α_t as the average labor productivity of tax auditors, i.e. the degree of efficiency in the audit activity, following Argentiero and Bollino (2015). The parameter θ indicates a penalty factor paid by the workers detected evading in addition to the regular taxation.

2.3 The Government

The Government collects tax revenues on sunlight labor (R_t^s) and expected value obtained by detected underground labor $\mathbb{E}(R_t^u)$ in order to obtain total expected tax revenues $\mathbb{E}(R_t)$:

$$\mathbb{E}(R_t) = R_t^s + \mathbb{E}(R_t^u) = \gamma_t W_t N_t^s + \underbrace{p_t\left[(\gamma_t + \theta) W_t N_t^u\right]}_{\mathbb{E}(R_t^u)} \tag{5}$$

and uses these resources to finance public expenditures (E_t) that include the interest rates paid on the stock of public debt at time $t-1$ $(r_t D_{t-1})$, the cost of audit activity proportional to the underground labor (kN_t^u) and public consumption G_t :

$$\underbrace{\gamma_t W_t N_t^s}_{R_t^s} + \underbrace{p_t\left[(\gamma_t + \theta) W_t N_t^u\right]}_{\mathbb{E}(R_t^u)} = \underbrace{r_t D_{t-1} + kN_t^u + G_t}_{E_t} \tag{6}$$

The stock of public debt corresponds to the expected discounted sum of deficits, given by the difference between public expenditures and expected tax revenues:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} e^{-\delta t}\left[r_t D_{t-1} + kN_t^u + E_t - \gamma_t W_t N_t^s - p_t\left[(\gamma_t + \theta) W_t N_t^u\right]\right] = D_t \tag{7}$$

The public debt to GDP ratio reads as

$$X_t = \frac{D_t}{Y_t} \tag{8}$$

2.4 The equilibrium characterization

The first order optimal conditions characterizing the households' constrained discounted expected utility maximization problem are the following:

$$\begin{cases} \nu\,(A_t)^{-q} = \lambda_t \\ (N_t^s)^{\psi} = \lambda_t\,(1-\gamma_t)\,W_t \\ B^u\,(N_t^u)^{\omega} = \lambda_t\,[(1-p_t)\,W_t + p_t\,[(1-\gamma_t-\theta)\,W_t]] \end{cases} \tag{9}$$

where λ_t is the Lagrange multiplier.

By combining the first with the second equation of (9) and the first with the third equation of (9), the expressions for sunlight and underground labors are derived:

$$\begin{cases} \frac{\nu(A_t)^{-q}}{(N_t^s)^{\psi}} = \frac{1}{(1-\gamma_t)W_t} \text{ which implies } N_t^s = \left(\frac{\nu(1-\gamma_t)W_t}{(A_t)^q}\right)^{\frac{1}{\psi}} \\ \frac{\nu(A_t)^{-q}}{B^u(N_t^u)^{\omega}} = \frac{1}{[(1-p_t)W_t+p_t[(1-\gamma_t-\theta)W_t]]} \text{ which implies } N_t^u = \left[\frac{\nu(W_t-p_t\gamma_tW_t-p_t\theta W_t)}{B^u(A_t)^q}\right]^{\frac{1}{\omega}} \end{cases} \tag{10}$$

Note that the decrease of the underground labor, which in this case is the evaded tax base, depends crucially on the complementary action of tax rate, γ_t, the audit activity embedded in the probability of being detected evading, p_t, the penalty factor, θ and the idiosyncratic cost of working in the underground sector, B^u.

We now show that an endogenous Laffer tax rate threshold can be derived from the analysis of the revenues.

First, we consider the sunlight revenues, R_t^s, as a function of γ_t, and study the sign of the derivative $\frac{\partial R_t^s}{\partial \gamma_t}$.

The sunlight revenues can be written as

$$R_t^s = \gamma_t W_t N_t^s. \tag{11}$$

Therefore, by the first equation in (10), one has

$$\frac{\partial R_t^s}{\partial \gamma_t} = W_t \left[\frac{\nu (1-\gamma_t) W_t}{(A_t)^q} \right]^{\frac{1}{\psi}} - \nu \gamma_t \frac{W_t}{(A_t)^q} \frac{1}{\psi} \left[\frac{\nu (1-\gamma_t) W_t}{(A_t)^q} \right]^{\frac{1}{\psi}-1} =$$

$$= \left[\frac{\nu (1-\gamma_t) W_t}{(A_t)^q} \right]^{\frac{1}{\psi}-1} \left[\frac{\nu (1-\gamma_t) W_t}{(A_t)^q} - \frac{\nu \gamma_t}{\psi} \frac{W_t}{(A_t)^q} \right] = \left[\frac{\nu (1-\gamma_t) W_t}{(A_t)^q} \right]^{\frac{1}{\psi}-1} \frac{\nu W_t}{(A_t)^q} \left[1 - \gamma_t \left(1 + \frac{1}{\psi} \right) \right],$$

so that $\frac{\partial R_t^s}{\partial \gamma_t} > 0$ if and only if $\gamma_t < \frac{1}{1+\frac{1}{\psi}}$.

We find that when $\gamma_t < \frac{1}{1+\frac{1}{\psi}}$ the sunlight revenues increase, that is the increasing part of the Laffer curve, whereas for $\gamma_t > \frac{1}{1+\frac{1}{\psi}}$, the sunlight revenues decrease and we are in the presence of the decreasing part of the Laffer curve. Note that the endogenous threshold tax rate $\bar{\gamma}_t^{(1)} = \frac{1}{1+\frac{1}{\psi}}$ is a negative function of the Frisch elasticity of sunlight labor supply $\left(\frac{1}{\psi}\right)$. Hence, an increasing Frisch elasticity of labor supply, which indicates a greater elasticity of labor offer to the wage, reduces the Laffer threshold rate $\bar{\gamma}_t^{(1)}$.

Let us now take the expected value obtained by detected underground labor $\mathbb{E}(R_t^u)$ as a function of γ_t. A simple computation gives:

$$\frac{\partial \mathbb{E}(R_t^u)}{\partial \gamma_t} = \frac{\nu p_t (W_t)^2}{(A_t)^q B^u} \left[\frac{\nu \left[(1-p_t) W_t + p_t \left[(1-\gamma_t-\theta) W_t \right] \right]}{\left[(A_t)^q \right] B^u} \right]^{\frac{1}{\omega}-1} \cdot \left[1 - p_t \left(1 + \frac{1}{\omega} \right) (\gamma_t + \theta) \right],$$

so that $\frac{\partial \mathbb{E}(R_t^u)}{\partial \gamma_t} > 0$ if and only if $\gamma_t < \bar{\gamma}_t^{(2)} := \frac{1}{p_t\left(1+\frac{1}{\omega}\right)} - \theta$.

Therefore, in the general case of revenues due to (5), we have that

$$\gamma_t < \min \left\{ \bar{\gamma}_t^{(1)}; \bar{\gamma}_t^{(2)} \right\} \text{ implies that } \frac{\partial E(R_t)}{\partial \gamma_t} > 0$$

and

$$\gamma_t > \max\left\{\bar{\gamma}_t^{(1)}; \bar{\gamma}_t^{(2)}\right\} \text{ implies that } \frac{\partial E\left(R_t\right)}{\partial \gamma_t} < 0.$$

In general, one can reasonably assume that the behavior of the total revenues of the State is of Laffer type, with a threshold $\min\left\{\bar{\gamma}_t^{(1)}; \bar{\gamma}_t^{(2)}\right\} \leq \bar{\gamma} \leq \max\left\{\bar{\gamma}_t^{(1)}; \bar{\gamma}_t^{(2)}\right\}$. A trivial condition on p_t or on θ could be imposed to have $\bar{\gamma}_t^{(1)} = \bar{\gamma}_t^{(2)} =: \bar{\gamma}$, so that $\bar{\gamma}$ is the explicit form of the Laffer threshold of the revenues.

3 The stochastic optimal control problem

This section is devoted to the development and solution of the proposed dynamic stochastic optimization problem related to the public debt to GDP ratio management. We proceed by dealing with a continuous-time setting, which is more general than the discrete-time framework. The microfoundation presented in Section 2 – which plays the important role of giving the economic intuition of the mathematical optimization problem and, moreover, describes in details the economic context we are dealing with – can be suitably adapted to the presented continuous-time setting by implementing a standard time-discretization procedure of the optimal control problem.

We consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ which satisfies all the usual conditions and where we define a standard Brownian motion W with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ under the probability measure P.

We also denote by $X = \{X_t\}_{t\geq 0}$ a stochastic process which models the time-dependent public debt to GDP ratio. The stochastic process X is the state variable of the problem.

X is assumed to evolve according to a controlled stochastic differential equation as follows:

$$dX_t = X_t \mu(\gamma_t, \alpha_t)dt + \sigma(\gamma_t, \alpha_t, X_t)dW_t \qquad \forall\ t > 0; \quad X_0 = x, \tag{12}$$

$\gamma = \{\gamma_t\}_{t>0}$ and $\alpha = \{\alpha_t\}_{t>0}$ are stochastic processes with support in $[0, 1]$ and $[0, +\infty)$, respectively, which are $\mathcal{F}_t$-progressively measurable, for each $t > 0$. As in the previous Section, the former process represents the time-dependent tax rate, while the latter one formalizes the effort undertaken by the tax auditors in the enforcement activity associated to the detection of the tax evasion. Differently with the model presented in Section 2, we enrich the analysis by considering γ and α non deterministic.

$\mu : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is the contribution of the tax rate and the audit activity to the deterministic growth rate of the public debt to GDP ratio. It is assumed to be decreasing with respect to α_t – i.e. a strong action against evasion leads to a reduction of the public debt to GDP ratio – and U-shaped with respect to the tax rate – i.e. high tax rates push towards evasion while low tax rates lead to low incomes for the State, and for us this leads to a high public debt to GDP ratio. The underling idea is the existence of a Laffer curve for tax evasion, according to which there is a threshold, $\bar{\gamma}$, of the tax rate beyond which tax revenues decrease due to tax avoidance and tax evasion. Such a condition is strongly supported in the economic setting we are dealing with (see the previous Section). As a consequence, if $\gamma > \bar{\gamma}$ the public debt to GDP ratio increases, thus determining the U-shaped form of μ. $x \in (0, +\infty)$ is a constant representing the initial observation of the public debt to GDP ratio. We are aware that the standard representation of (12) includes the real interest rate, the growth of GDP in real terms and the primary deficit. Yet, in this framework we aim at focusing on the role of tax evasion and enforcement as two devices for increasing primary surplus and thus reducing the public debt to GDP ratio[3] (see e.g. Ferrari, 2018).

$\sigma : [0,1] \times [0,+\infty)^2 \to [0,+\infty)$ is the volatility term of the public debt to GDP ratio. It is continuous with respect to X. We also reasonably assume that $\sigma(\bullet,\bullet,X) \neq 0$ when $X \neq 0$.

Hereafter, we consider that the *usual conditions* hold true, i.e. all the regularity conditions on function μ and σ leading to the existence and uniqueness of the solution of equation (12) are satisfied.

The analysis goes on till the possible "failure" of the State – here represented by the crossing of a prefixed threshold $\bar{X} \in \mathbb{R}$ by the public debt to GDP ratio – or till the repaid of the debt – i.e., when the public debt to GDP ratio becomes null. In so doing, we can restrict reasonably the analysis to an initial value of the public debt to GDP ratio lying in the *meaningful range*, i.e. $x \in (0,\bar{X})$. Thus, the timing of the economic problem is unknown and depends on the stochastic process X described in (12). We introduce the set of the stopping times in $[0,+\infty]$ as follows:

$$\mathcal{T} := \{\eta : \Omega \to [0,+\infty] \,:\, \{\eta \le t\} \in \mathcal{F}_t,\ \forall\, t \in [0,+\infty)\},$$

and the exit time τ of X from $(0,\bar{X})$ is $\tau := \inf \left\{ t \in [0,+\infty] \,:\, X_t \notin (0,\bar{X}) \right\}$. The hypotheses on $\{\mathcal{F}_t\}_{t\ge 0}$ assure that $\tau \in \mathcal{T}$.

The control variables (γ,α) live in the so-called *admissible region*, which is a functional space denoted hereafter by $\mathcal{A}$.

The policymaker minimizes the expected aggregated discounted costs of public debt to GDP ratio. At this aim, we define

$$C : [0,\bar{X}] \to [0,+\infty) \qquad : \qquad X \mapsto C(X),$$

where $C(X)$ is the costs associated to a public debt to GDP ratio whose level is X. Notice that it is not necessary to define the costs for $X \notin [0, \bar{X}]$, because the region is in the case not of interest for the problem. According to empirical evidence (see e.g. Woo and Kumar, 2015), the costs increase with the level of public debt to GDP ratio. Moreover, C is assumed to be twice differentiable and and it is convex in $(0, \bar{X})$, in accord to well-established economic literature (see e.g. Woodford, 1990; Checherita-Westphal and Rother, 2012; Ferrari, 2018). We also assume that C satisfies a growth condition with respect to X, in the sense that there exists $c_s > 0$ such that

$$|C(x)| \leq c_s(1 + x), \qquad \forall\, x \in (0, \bar{X}). \tag{13}$$

Condition (13) will turn out to be useful in the theoretical solution procedure of the control problem. It is not restrictive in our context. Indeed, under a practical point of view, (13) states that costs exhibit a controlled way to grow or have to be bounded with by a large enough scalar in $(0, \bar{X})$.

The optimal tax rate and monitoring parameter $(\gamma^\star, \alpha^\star) \in \mathcal{A}$ is then the result of the minimization of the objective function J, defined as follows: $J : [0, \bar{X}] \times \mathcal{A} \to \mathbb{R}$ such that

$$J(x, \gamma, \alpha) = \mathbb{E}_x \left\{ \int_0^\tau e^{-\delta t} C(X_t) dt + \Lambda(X_\tau) e^{-\delta \tau} \right\}, \tag{14}$$

where $\Lambda(\bar{X}) = M > 0$ is the terminal fixed cost of public debt to GDP ratio at the (possible) moment of the failure of the State, while $\Lambda(0) = 0$ stands for null costs when the public debt becomes null and the State solves its public debt problems. $\mathbb{E}_x$ is the expected value operator conditioned on $X(0) = x$, $\delta > 0$ and $e^{-\delta}$ is a continuous uniperiodal discount factor defined in the previous Section.

The value function $V : [0, \bar{X}] \to \mathbb{R}$ is

$$V(x) := \inf_{(\gamma,\alpha)\in\mathcal{A}} J(x, \gamma, \alpha). \tag{15}$$

3.1 Solution of the optimal control problem: a dynamic programming approach

The optimal control problem in Section 3 is here solved by employing a dynamic programming method. By using (13), we can adopt to our framework the general form of the Dynamic Programming Principle (DPP) for stochastic optimal control problems with random horizon in Cerqueti (2009), which is based on measurable selection. For other approaches, see the classical Fleming and Soner (1993).

Theorem 1 (DPP)**.** *For each $\eta \in \mathcal{T}$, we have*

$$V(x) = \inf_{(\gamma,\alpha)\in\mathcal{A}} \mathbb{E}_x \left\{ \int_0^{\tau\wedge\eta} e^{-\delta t} C(X_t) dt + e^{-\delta(\tau\wedge\eta)} V(X_{\tau\wedge\eta}) \right\}. \tag{16}$$

Theorem 1 has a direct consequence. Indeed, it leads to state that the value function is the solution of a second order ODE – the so-called Hamilton Jacobi Bellman (HJB) equation – whenever the appropriate regularity conditions for V are satisfied. Formally, we have:

Theorem 2 (HJB Equation)**.** *Assume that $V \in C^0([0, \bar{X}]) \cap C^2((0, \bar{X}))$. Then one has*

$$\delta V(x) = \inf_{(g,a)\in[0,1]\times[0,+\infty)} \left\{ \frac{\sigma(g,a,x)^2}{2} V''(x) + x\mu(g,a)V'(x) \right\} + C(x), \qquad \forall\, x \in (0, \bar{X}), \tag{17}$$

with the relaxed boundary conditions for $x \in \{0, \bar{X}\}$:

$$\min\left\{ \delta V(x) - \inf_{(g,a)\in[0,1]\times[0,+\infty)} \left\{ \frac{\sigma(g,a,x)^2}{2} V''(x) + x\mu(g,a)V'(x) \right\} - C(x), V(x) - \Lambda(x) \right\} \leq 0; \tag{18}$$

$$\max\left\{\delta V(x) - \inf_{(g,a)\in[0,1]\times[0,+\infty)}\left\{\frac{\sigma(g,a,x)^2}{2}V''(x) + x\mu(g,a)V'(x)\right\} - C(x), V(x) - \Lambda(x)\right\} \geq 0. \tag{19}$$

Notice that Theorem 2 requires that V is twice differentiable in $(0, \bar{X})$, and it can be extended continuously to $[0, \bar{X}]$. In the proposed setting we can prove such a regularity condition, which implies also that V is the unique classical solution of the HJB equation (17) with boundary conditions (18)-(19).

We also point out that the boundary conditions presented in Theorem 2 are relaxed boundary conditions. As we will see, such a concept of boundary conditions in a weak sense is needed for proving that the value function is regular enough and is the unique solution of the HJB equation (and satisfies the boundary conditions). Indeed, the strategy to prove that V is the unique (classical) solution of the HJB equation with boundary conditions is firstly to check that it is its unique viscosity solution, and for this reason also the boundary conditions have to be presented in a viscosity weak sense. We address the reader to Barles and Burdeau (1995) and Barles and Rouy (1998) for some details on this point.

Theorem 3. *One has $V \in C^0([0, \bar{X}]) \cap C^2((0, \bar{X}))$, and the value function in 15 is the unique solution of equation (17) with boundary conditions (18)-(19).*

The proof of Theorem 3 can be written by adapting to our context the strategy used in Cerqueti (2012) and Cerqueti et al. (2016).

Proof. First of all, Barles and Burdeau (1995) and Barles and Rouy (1998) guarantee that V is the unique solution which is continuous in $[0, \bar{X}]$ of equation (17) with boundary conditions (18)-(19) in a weak sense, i.e. in terms of viscosity solution (for the concept of viscosity solution of a second order ODE, refer to Fleming and Soner, 1993 and the famous User's Guide by Crandall et al., 1992).

After this preliminary step, by using the convexity of the cost function C, one can check that V is convex by adopting a result due to Alvarez et al. (1997) on the convex envelop of the value function and its identifiability with the unique viscosity solution of equation (17) with boundary conditions (18)-(19) (see Cerqueti, 2012 and Cerqueti et al., 2016 for such technical details).

The convexity is then used to finalize the proof that V is twice differentiable in $(0, \bar{X})$. In fact, define $I_\epsilon = [\epsilon, \bar{X} - \epsilon]$, where $\epsilon \in (0, \bar{X}/2)$. Equation (17) is then uniformly elliptic in I_ϵ. The convexity and the continuity of V in I_ϵ leads to the existence of the second order derivative of V in I_ϵ a.e., in virtue of Alexandrov's Theorem (see e.g. Fleming and Soner, 1993). Since $\sigma(g, a, x) \neq 0$ when $x \neq 0$, a simple computation gives that V is an element of the Sobolev space $W^{2,\chi}(I_\epsilon)$, with $\chi \in [1, +\infty]$. Thus, one can use Sobolev's Embedding Theorem (see Gilbarg and Trudinger, 1977), which leads to $V \in C^m(I_\epsilon)$, with $m \in [0, 2 - \frac{1}{k})$, $\forall\, k \in (1, +\infty)$. Therefore, $V^{'}$ is a continuous function. By rearranging the terms of (17), one obtains that $V'' \in C^0(I_\epsilon)$. Therefore, $V \in C^{2,\alpha}(I_\epsilon)$, with $\alpha = 1 - \frac{1}{k}$, $k \in (1, +\infty)$. The twice differentiability of V in $(0, \bar{X})$ comes from setting $\epsilon \to 0^+$. ■

Theorems 2 and 3 contribute jointly to assure that V is the unique classical solution of the HJB equation. This is a fundamental property of V, and represents the key ingredient for the formalization of the optimal strategies and trajectories associated to the optimal control problem.

Next result assures the existence of the optimal strategies of the stochastic control problem under investigation.

Theorem 4 (Verification Theorem). *Assume that* $u \in C^0([0,\bar{X}]) \cap C^2((0,\bar{X}))$ *is a classical solution of (17) with boundary conditions (18)-(19). Then one has:*

(a) $u(x) \leq J(x,\alpha,\gamma), \qquad \forall\, x \in [0,\bar{X}]$ *and* $(\alpha,\gamma) \in \mathcal{A}$.

(b) Let us consider (α^*,γ^*,X^*) *an admissible triple at the initial value* x *such that*

$$(\alpha_t^*,\gamma_t^*) \in \arg\inf_{a,g}\left\{\frac{\sigma(g,a,X_t^*)^2}{2}V''(X_t^*) + \mu(g,a,X_t^*)V'(X_t^*)\right\}. \tag{20}$$

Then (α^*,γ^*,X^*) *is optimal at* x *if and only if* $u(x) = J(x,\alpha^*,\gamma^*), \qquad \forall\, x \in [0,\bar{X}]$.

Theorem 4 can be checked through long and tedious computations. The proof is based on the application of Ito's Lemma and of the Dominated Convergence Theorem, by adapting the arguments proposed in Cerqueti (2012).

We are now in the position of formalizing the optimal strategies of the problem through the so-called closed loop equation.

Theorem 5. *Let us consider* (α^*,γ^*) *as in (20) and* μ *and* γ *in (12) such that* $\underline{X}$ *is the unique solution of the closed loop equation:*

$$d\underline{X}_t = \underline{X}_t\mu(\gamma^*(\underline{X}_t),\alpha^*(\underline{X}_t))dt + \sigma(\gamma^*(\underline{X}_t),\alpha^*(\underline{X}_t),\underline{X}_t)dW_t \qquad \forall\ t>0; \qquad \underline{X}_0 = x, \tag{21}$$

Then, by taking $\underline{\alpha}_t := \alpha^*(\underline{X}_t)$ *and* $\underline{\gamma}_t := \gamma^*(\underline{X}_t)$, *we have* $J(x,\underline{\alpha}_\cdot,\underline{\gamma}_\cdot) = V(x)$ *and the triple* $(\underline{\alpha},\underline{\gamma},\underline{X})$ *is optimal for the control problem.*

Theorem 5 contains the form of the optimal strategies for the considered stochastic control problem and concludes the related theoretical solution procedure. Such a result is based on the fact that the closed loop equation (21) admits a unique solution. In the proposed general setting, such a statement is far from being trivial and depends on the assumptions on the parameters of the model. However, one can take the quite general case $\sigma(g,a,x) = \sigma_1(x)\sigma_2(g,a)$ and properly select functions μ, σ_1 and σ_2 whose regularity is such that

$$\begin{cases} \gamma^*(\underline{X}_t) = h_a^{-1}\left(\frac{-2\underline{X}_t V'(\underline{X}_t)}{\sigma_1(\underline{X}_t)V''(\underline{X}_t)}\right); \\ \alpha^*(\underline{X}_t) = h_g^{-1}\left(\frac{-2\underline{X}_t V'(\underline{X}_t)}{\sigma_1(\underline{X}_t)V''(\underline{X}_t)}\right), \end{cases} \tag{22}$$

where

- $h_a : [0,1] \to \mathbb{R}$ such that $g \mapsto h_a(g) = \frac{\partial \sigma_2^2(g,a)}{\partial g}\left(\frac{\partial \mu(g,a)}{\partial g}\right)^{-1}$, for $a \in [0,+\infty)$;
- $h_g : [0,+\infty) \to \mathbb{R}$ such that $a \mapsto h_g(a) = \frac{\partial \sigma_2^2(g,a)}{\partial a}\left(\frac{\partial \mu(g,a)}{\partial a}\right)^{-1}$, for $g \in [0,1]$,

and the μ and σ with γ^* and α^* given by (22) satisfy the *usual conditions*, so that the closed loop equation (21) admits unique strong solution (see e.g. Karatzas and Shreve, 1991).

4 Numerical experiments

The theoretical derivation of the solution of the optimal control problem is now validated through some numerical experiments.

We implement an optimization algorithm by adopting a combined grid search and Monte Carlo procedure. In so doing, by one side we remove the complexity arising when implementing simulations for solving a stochastic optimal control problem; by the other side, we propose an intuitive reading of the optimal strategies along with clear policy implications.

For the sake of completeness, three scenarios are presented: the former relies to the situation in which the controls are not in contrast, in the sense that the policymaker has the opportunity of pursuing public debt to GDP ratio reduction by considering α and γ as independent control variables; the latter two ones present the case of mutual interaction between the control variables, where the selection of one of them implies constraints on the other one.

4.1 Parameter set

This section contains the definition of the parameters and quantities used in the numerical experiments.

The costs related to the public debt to GDP ratio are assumed to be normalized and defined by

$$C(X) = \xi X^2, \qquad for\ X \in [0, \bar{X}], \tag{23}$$

where $\xi > 0$, so that they are increasing and convex in $[0, \bar{X}]$.

We parameterize our model for Italy, a country within the European Monetary Union, where tax evasion is high[4].

We have implemented the analysis by selecting two different initial values of the dynamics: $x = 1$ and $x = 1.32$. The latter value is in agreement with the value of the public debt to GDP ratio in Italy in 2016 (Source: Ameco, 2018), whereas $x = 1$ describes the situation of equality between the stock of public debt and GDP, and it is exceeded only by five countries belonging to the European Monetary Union (Italy, Greece, Cyprus, Belgium and Portugal).

Moreover, the value of β is assumed to be 0.108, which corresponds to the percentage standard deviation of debt to GDP ratio from 1995 to 2016 (source: Ameco, 2018).

The critical Laffer value $\bar{\gamma}$ is set to $\frac{1}{1+1/\psi}$, where $1/\psi$ is the Frisch elasticity of labor supply.

This value is the one found in the endogeneization procedure proposed in Section 2 for the specific case of absence of detected underground labor. Such a selection is consistent with the negligibility of the detected tax evasion with respect to the tax revenues coming from the official economy. We propose two different values for the Frisch elasticity: $1/\psi$=0.3 and $1/\psi$=1. The former value is consistent with the Frisch elasticity of regular labor supply in Busato and Chiarini (2004); the latter value, $1/\psi$=1, is suitable for our purposes since it allows to evaluate a scenario with a greater labor market flexibility. We notice that the Laffer thresholds become 0.5 for $1/\psi$=1 – which is consistent with the no underground case for income taxation of Orsi et al., (2014) – and 0.77 for $1/\psi$=0.3 – which represents a stressed situation of excessive tax burden. The failure threshold for public debt to GDP ratio is set to $\bar{X} = 1.4$ and the corresponding terminal cost associated to the failure of the State is $M = 0.75$. Indeed, the choice of the failure value for public debt to GPD ratio represents an historical threshold beyond which the Argentine Government declared the *default* in 2001 and Greece needed a debt restructuring in 2014. The discount factor parameter is set to $\delta = 0.03$, while ξ is set to 0.4.

The grid search procedure is implemented on the control variables α and γ. We consider α and γ varying within two bands. The former term is assumed to vary in $[0, \bar{\alpha}]$, where $\bar{\alpha}$ is the maximum effort that the State is able to do for contrasting tax evasion through monitoring activity. As we will see below, in the first and third scenarios we set a constant $\bar{\alpha}$, while in the second one it is a function of γ. The latter one is assumed to range in $[0, 1]$ in the first and second scenarios, where 0 means *no taxes* while 1 stands for *total wage to taxes.* Of course, 0 and 1 represent only hypothetical bounds with no practical reliability. In the third scenario, $\gamma \in [0, \bar{\gamma}]$ where $\bar{\gamma}$ is a function of α[5].

4.2 First scenario: absence of conflict between the control variables

This is the case in which the control variables can be taken without stating any mutual interaction between them.

The term dependent from the controls in the drift of the state equation (12) is assumed to be

$$\mu(\gamma_t, \alpha_t) = \frac{(\gamma_t - \bar{\gamma})^2}{\alpha_t + 1}. \tag{24}$$

According to (24), all the conditions required for μ are fulfilled.

The diffusion term in (12) is

$$\sigma(\gamma_t, \alpha_t, X_t) = \frac{\beta X_t \gamma_t}{\alpha_t + 1}, \tag{25}$$

where $\beta \in (0, +\infty)$ is the instantaneous volatility of public debt to GDP ratio.

4.3 Second scenario: tax rate constraining monitoring activity

In this case the value of the tax rate represents a constraint for the level of monitoring activity. The drift term of the state equation (12) is assumed to be as in (24) but $\alpha \in [0, \bar{\alpha}^*(\gamma)]$, where $\bar{\alpha}^* : [0, 1] \to [0, \bar{\alpha}]$ is increasing and $\bar{\alpha}^*(0) = 0$. The considered assumptions capture the idea that the monitoring activity is more funded in the presence of high tax rates, and the extreme case of null tax rate is associated to the absence of monitoring activity. In this experiment, we set $\bar{\alpha}^*(\gamma) = 10\gamma$. This selection is in agreement with our position of $\bar{\alpha} = 10$ in the first unconstrained scenario.

The diffusion term is as in (25) with α varying in $[0, \bar{\alpha}^*(\gamma)]$.

4.4 Third scenario: monitoring activity constrained by tax rate

We here set the assumption on this scenario for having a constraint on the monitoring induced by the level of tax rate.

Function μ is as in (24) but $\gamma \in [0, \bar{\gamma}^*(\alpha)]$, where $\bar{\gamma}^* : [0, \bar{\alpha}] \to [0, 1]$ increases with respect to γ and $\bar{\gamma}^*(0) = 0$. The definition of $\bar{\gamma}^*$ models that a high level of monitoring activity can be funded only by applying a high level of tax rate, consistently also with the statement presented in the second scenario. We here specifically set $\bar{\gamma}^*(\alpha) = \alpha/10$, so that $\bar{\gamma}^*(\alpha) \in [0, 1]$ for each $\alpha \in [0, \bar{\alpha}]$.

Function σ is defined as in (25) with $\gamma \in [0, \bar{\gamma}^*(\alpha)]$.

4.5 Monte Carlo procedure

The Monte Carlo procedure insists on the random term of the state equation, and it is implemented according to the following steps.

- The differential of the Brownian Motion is discretized: $dW_t = \mathcal{N} * \sqrt{\Delta t}$, where $\mathcal{N}$ is a random sampling from a standard normal distribution, while $\Delta t = 1/365$ (so that t measures years and Δt is one day).

- The variation ranges of the variables α and γ are discretized with steps 1 and 0.1, respectively. Each discretization step is denoted as α_s and γ_r, with $s = 1, \ldots, \bar{S}$ and $r = 1, \ldots, \bar{R}$. The bounds $\bar{S}$ and $\bar{R}$ are selected consistently to the considered scenario.

- Time variations are measured through days and we take $1,000$ points, so that trajectories cover about three years.

- For each α_s and γ_r, we build 1000 trajectories by considering sampling $\mathcal{N}$ at each time t, according to the discretization of (12):

$$X_{t+\Delta t} = X_t + X_t \mu(\gamma_r, \alpha_s)\Delta t + \sigma(\gamma_t, \alpha_t, X_t)\mathcal{N} * \sqrt{\Delta t}, \tag{26}$$

  with $t = 0/365, 1/365, \ldots, 1000/365$ and $X_0 = x$.

  We denote the trajectories by $X_j^{(\alpha_s,\gamma_r)}$, where $j = 1, \ldots, 1000$.

- for each $X_j^{(\alpha_s,\gamma_r)}$ we identify the exit time $\tau_j^{(\alpha_s,\gamma_r)}$ where the trajectory $X_j^{(\alpha_s,\gamma_r)}$ goes out of the range $[0, \bar{X}]$ for the first time.

We now consider the number of times in which the trajectories labeled by the j's exits from 0 and $\bar{X}$. We denote the former quantity by $n_0^{(\alpha_s,\gamma_r)}$ and the latter by $n_{\bar{X}}^{(\alpha_s,\gamma_r)}$. It is clear that, in general, $n_0^{(\alpha_s,\gamma_r)} + n_{\bar{X}}^{(\alpha_s,\gamma_r)} < 1000$. In fact, some trajectory can vary within the range $(0, \bar{X})$ for all the time $t = 1/365, \ldots, 1000/365$. In this case, we set null terminal costs in our simulations, according to the theoretical definition of J in (14)[6].

$$P(X_j^{(\alpha_s,\gamma_r)}(\tau_j^{(\alpha_s,\gamma_r)}) = 0) = \frac{n_0^{(\alpha_s,\gamma_r)}}{1000}, \qquad P(X_j^{(\alpha_s,\gamma_r)}(\tau_j^{(\alpha_s,\gamma_r)}) = \bar{X}) = \frac{n_{\bar{X}}^{(\alpha_s,\gamma_r)}}{1000}. \tag{27}$$

Fix $s = 1, \ldots, \bar{S}$ and $r = 1, \ldots, \bar{R}$. By (14) and by using the arithmetic mean for the expected value operator, one has

$$J(x, \alpha_s, \gamma_r) = \frac{1}{1000} \sum_{j=1}^{1000} \left\{ \left[\sum_{t=1}^{1000} e^{-\delta t} C(X_j^{(\alpha_s,\gamma_r)}(t))\Delta t \right] + M e^{-\delta \tau_j^{(\alpha_s,\gamma_r)}} \cdot \frac{n_{\bar{X}}^{(\alpha_s,\gamma_r)}}{1000} \right\}. \tag{28}$$

The value function in (15) is then obtained by minimizing J in (28) with respect to α_s and γ_r. Since the admissible region contains a finite number of elements, such a minimum exists, i.e. there exists $s^* \in \{1, \ldots, \bar{S}\}$ and $r^* \in \{1, \ldots, \bar{R}\}$ such that

$$V(x) = \sup_{s=1,\ldots,\bar{S},\ r=1,\ldots,\bar{R}} J(x, \alpha_s, \gamma_r) = J(x, \alpha_{s^*}, \gamma_{r^*}). \tag{29}$$

5 Results and discussion

The results of the simulations run under the first, second and third scenario hypothesis are represented in Figures 1, 2 and 3, respectively. Moreover, Table 1 shows the values of V in all the presented situations.

Table 1: Value function $V(x)$ in the three scenarios and in all the cases of x and ξ.

	$x = 1$; $1/\psi = 0.3$	$x = 1$; $\frac{1}{\psi} = 1$	$x = 1.32$; $1/\psi = 0.3$	$x = 1.32$; $1/\psi = 1$
First scenario	613.21	613.27	704.00	704.20
Second scenario	613.21	613.27	704.00	704.20
Third scenario	613.41	613.45	704.39	704.42

The three scenarios show some common patterns. First, *ceteris paribus*, the value function grows as the value of the starting point of the public debt to GDP ratio increases or the Frisch elasticity decreases (see Table 1). This is not surprising since the cost related to the public debt to GDP ratio is a convex function and a high Laffer threshold $\bar{\gamma}$ means high tax revenues. Second, the qualitative behavior of the objective function does not show remarkable differences in terms of the controls as the starting points of the public debt to GDP ratio vary (see Figures 1,2 and 3). This is quite reasonable, in that the proposed model and the nature of the employed controls leads to a rank of the strategies – to be intended in terms of their effectiveness in managing the cost – which is universal. In particular, the actions to be implemented for cost minimization should be independent from the distance of the public debt to GDP ratio to the default barrier $\bar{X}$, even though a small value of $|x - \bar{X}|$ suggests to be rapid in the implementation (see the comment in the next Subsection for a discussion on this point). Third, the worst case of highest costs for the public debt to GDP ratio occurs always in the presence of tax rates far from the Laffer threshold and low level of monitoring activity effort.

The results of the simulations run under the first scenario hypothesis, i.e. in the absence of conflict between the control variables, α and γ, are represented in Figure 1.

INSERT FIGURE 1 ABOUT HERE

Caption: First scenario: on the x-axis the rank of the costs in increasing order, so that the public debt to GDP ratio increases as the value of x increases. The "o" represent the value of α while the "*" is the value of γ. For a better visualization, we have rescaled α and presented its linear transformation $1 + \alpha/10$. Panels a), b), c) and d) correspond to the cases of $x = 1$ and $1/\psi = 0.3$; $x = 1$ and $\frac{1}{\psi} = 1$; $x = 1.32$ and $1/\psi = 0.3$ and $x = 1.32$ and $1/\psi = 1$, respectively.

In all the cases, the costs associated to the public debt to GDP ratio are minimized when the tax rate is taken as close as possible to the Laffer threshold $\bar{\gamma}$, whereas tax enforcement is set to the minimum value.

Then, the cost of the public debt to GDP ratio becomes bigger when the tax rate is maintained on the Laffer threshold and for an increasing value of the audit effort. This behavior can be justified under an economic point of view. In fact, when the Laffer tax rate is achieved, then the public debt to GDP ratio is mainly driven by the diffusion term σ while its deterministic trend is quite negligible. In this circumstance, the control due to the monitoring activity becomes an inefficient and costly device for managing the public debt to GDP ratio, and thus it has to be reduced at its minimum level.

The second best strategies behave under the following idea: one has to maintain the tax rate as close as possible to the Laffer threshold and – if not exactly in the Laffer case – implement strong audit effort. Figure 1 suggests that tax rates drives the optimality in the cost management, in that it is broadly better to optimize them with respect to the optimal selection of the monitoring effort. More in details, it is better to maintain the tax rate close to the Laffer level than to implement a strong audit activity if one wants to manage the cost of the public debt to GDP ratio.

This said, some discrepancies emerge once the Laffer threshold is no longer taken. When the Frisch elasticity is 0.3, then the second best strategies move from taking the tax rate immediately below the Laffer level and maximize decreasingly the audit effort, in order to stimulate tax compliance (Orsi et al., 2014). Then, one has to jump to the tax rate immediately above the Laffer threshold and continue to take the highest possible effort of the monitoring activity. This behavior becomes then less clear, since one has to consider only the minimum distance from the Laffer tax rate and a high level of effort for audit activity. We notice that, once the tax rate is far from the Laffer value, the role of the effort for audit activity becomes more relevant. Indeed, when the tax rate is far from Laffer, one observes that the sequential strategies in terms of cost minimization may proceed also by augmenting the audit effort at the cost of increasing the distance of the tax rate from the Laffer value. When the Frisch elasticity is 1, then the sequentiality of the strategies in optimizing the cost build a symmetric behavior of the tax rate around the Laffer threshold, along with the maximum possible level of the audit effort. In particular, we notice that the same level of monitoring activity is associated to the same distance from the Laffer value. Once the tax rate becomes quite far from the Laffer threshold, the audit effort becomes more relevant as in the Frisch elasticity equal to 0.3 case.

The simulations under the second scenario are summarized in Figure 2.

INSERT FIGURE 2 ABOUT HERE

Caption: Second scenario: as in the previous case, on the x-axis the rank of the costs in increasing order, so that the public debt to GDP ratio increases as the value of x increases. The "o" represent the value of α while the "*" is the value of γ. For a better visualization, we have rescaled α and presented its linear transformation $1 + \alpha/10$. Panels a), b), c) and d) correspond to the cases of $x = 1$ and $1/\psi = 0.3$; $x = 1$ and $\frac{1}{\psi} = 1$; $x = 1.32$ and $1/\psi = 0.3$ and $x = 1.32$ and $1/\psi = 1$, respectively.

In the second scenario, the level of tax audit activity depends on the tax rate, which synthesizes the amount of tax revenues collected by the Government. The obtained findings are quite similar to those derived for the first scenario (see also Figure 1). The minimum level of costs associated to a public debt to GDP ratio is realized when tax revenues are maximized, i.e. when $\gamma = \bar{\gamma}$, with α at its minimum level, which is in agreement with what the unconstrained case suggests. As in the first scenario, we observe that the Laffer threshold is associated to optimality of the cost, with an increasing cost associated to increasing effort of audit. Moreover, the cases of Frisch elasticity equal to 0.3 is coupled to an initial preference for a value of the tax rate immediately below the Laffer value – which is then followed by the value immediately above it – and maximum possible monitoring activity effort. Then, the situation becomes less straightforward, with an increasing role of the audit effort in determining the level of the cost of public debt to GDP ratio. Differently, when the Frisch elasticity is 1, we observe the symmetry around the tax rate found in the first scenario along with the selection of the maximum available level of audit effort.

Yet, it is worth to mention that in this scenario, unlike the first one, high levels of audit activity are more expensive. A comparison of the Figures 1 and 2 shows, for example, that the maximum level of enforcement is associated with higher costs for public debt ratio in the second scenario than in the first one. This result is very interesting since it shows that with a binding Government budget constraint for monitoring activity, the best fiscal policy response to tax evasion is to keep the tax rate as close as possible to the threshold $\bar{\gamma}$ with a minimum effort in tax enforcement.

Finally, the results of the third scenario are shown in Figure 3.

INSERT FIGURE 3 ABOUT HERE

Caption: Third scenario: also in this case, on the x-axis the rank of the costs in increasing order, so that the public debt to GDP ratio increases as the value of x increases. The "o" represent the value of α while the "*" is the value of γ. For a better visualization, we have rescaled α and presented its linear transformation $1 + \alpha/10$. Panels a), b), c) and d) correspond to the cases of $x = 1$ and $1/\psi = 0.3$; $x = 1$ and $\frac{1}{\psi} = 1$; $x = 1.32$ and $1/\psi = 0.3$ and $x = 1.32$ and $1/\psi = 1$, respectively.

In this case, taxation is constrained by tax enforcement. The results of the simulations show that a higher levels of minimal available effort in the monitoring activity than in the other scenarios – constrained by the optimal critical value of the Laffer curve and together with it – are able to minimize the costs associated to the public debt to GDP ratio. In particular, the lower is the value of the Frisch elasticity (high values of $\bar{\gamma}$) the higher is the enforcement. As in the previous scenarios, the case of Frisch elasticity equal to 1 is coupled to symmetry around the tax rate and maximum available level of audit effort in the sequential optimizing strategies. The case of Frisch elasticity equal to 0.3 presents the preference for low values of the tax rate and still high audit effort activity for minimizing the cost of the public debt to GDP ratio.

The policy implications of this case is that the level of tax rate must be next to the critical Laffer value, in order to finance an expensive audit activity. In so doing, the Government minimizes the costs of the monitoring activity, through the reduction of the cost of the public debt to GDP ratio. The choice of tax rates far from the Laffer threshold, instead, increases such a cost, mainly for the presence of an increase of tax evasion.

5.1 Some further remarks on the model

For the sake of completeness, we here present some suggestions on the development of the theoretical model and on the parameters involved in the economic problem we are dealing with.

In particular, we discuss the role of the interest rate and of the default value $\bar{X}$.

The increase of the exogenous interest rate generates a raise of the public debt. Furthermore, in the presence of a GDP growth rate less than the interest rate, debt unsustainability may occur. In general, in the absence of an accomodating monetary policy, a growth of the interest rate pushes the Government to procyclical fiscal policies, based on an increase of taxation and/or reduction of public expenditures (including the expenses for the audit effort), thus generating a decline of GDP. Consequently, a high interest rate is expected to lead to high public debt to GDP ratio. In turn, since costs are increasing with respect to public debt to GDP ratio, they are expected to increase with respect to the interest rate.

For what concerns the default value $\bar{X}$, a discussion can be carried out by considering a fixed value of the initial observation x. In this case, we notice that increasing the value of $\bar{X}$ means an increasing distance of the starting point of the public debt to GDP ratio from the default barrier. Thus, intuitively, the strategies to be adopted for minimizing the cost of the public debt to GDP ratio have to be implemented in a more rapid and effective way when $\bar{X}$ is small, while they can be relaxed for large values of such a default threshold.

6 Conclusions

This paper presents a public debt management model in a stochastic framework, where the optimization procedure is grounded on contrasting tax evasion. The control variables are tax rate and monitoring activity level, and the policymaker is assumed to select them with the scope of minimizing the expected aggregated public debt to GDP ratio cost.

The stochastic optimal control problem is grounded on the development of the economic framework we deal with. Specifically, we support the existence of a Laffer-type relationship between tax revenues and tax rate.

The stochastic optimal control problem is developed and solved through a dynamic programming approach. However, extensive numerical instances are carried out for grasping an intuitive take home message. Basically, the presented framework suggests that a fair Laffer tax rate and the minimum level of control represent the optimal strategies. In analyzing the behavior of the costs with respect to the control variables, it seems that a more relevant role is played by tax rate than monitoring activity in the light of controlling public debt to GDP ratio. Such a general outcome is confirmed also when inspection level and tax rate are assumed to be mutually dependent.

A deep exploration of the individual terms which contribute to the debt to GDP ratio and of their dependence on tax rate and monitoring level would lead to a more specific strategy for managing public debt by contrasting tax evasion. Such a theme is challenging and is already in our research agenda.

Notes

[1]This condition is verified in the presence of a primary deficit or small primary surplus when the GDP growth rate is less than the interest rate paid on public debt.

[2]The consumer price index is normalized to one.

[3]The increase of primary surpluses acts also on the real interest rate due to the reduction of government bond yields.

[4]In Italy, a country where tax evasion amounts almost at 90 billions of euros per year from 2010 to 2014, in 2014 the share of taxpayers declaring an income above 200,000 euros is 0.25% of the total (source: Italian Ministry of Economy).

[5]The results of numerical experiments are also robust to a different parametrization. In order to save space, we do not report this sensitivity analysis, which is available upon request.

[6]For the sake of notation, we rewrite the temporal index t as (t) when such a substitution let formulas be clearer.

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