Innovation, Imitation and Policy Inaction

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Abstract

The paper deals with the controversial issue of intellectual property rights. We deal with an optimization problem to model the optimal government's behaviour in presence of dynamic uncertainty and intervention costs. More specifically, we search for the optimal strategies to be implemented by a policy maker to optimally balance the number of innovators and imitators. The problem is first tackled from a purely theoretical perspective and then by implementing extensive numerical simulations on the basis of empirical data. By the theoretical perspective, we obtain a rigorous proof that optimal strategies depend on the initial value of the number of imitators and not

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on the initial ratio between innovators and imitators, whereas the simulations provide us with intuitive insights from an economic point of view, along with a validation of the theoretical results. The results support the evidence that governments choose the possible widest bandwidth and minimize the size of interventions so as to curb intervention costs.

Keywords: imitation, innovation, intellectual property, inaction region, optimization model, numerical simulation.

JEL Classification: O31; O34; C61; C63

1 Introduction

The role of governments in the process of technological development and diffusion is generally crucial but varies across countries and over time. When an economy is far from the technological frontier, it can grow quickly by imitating. Because the technology already exists, the flow of ideas is relatively non problematic and the major task is to manage the flow of resources. However, when a country is on the frontier, it can grow quickly only by innovating, which requires managing the flow of ideas and resources equally (Hikino and Amsden, 1994).

There can be also other reasons that lead governments to change their policies. Conventional wisdom holds that granting intellectual property rights (henceforth IPR) will stimulate the incentive for firms to devote resources to innovative activity and encourage the disclosure of inventions so that others can use and build upon research results, stimulating economic growth (Denicolò and Franzoni, 2003). Therefore, the existence of IPR is essentially regarded as a social need. Put another way, even when the social benefits of inventions exceed the costs, potential innovators without patent protection, and more in general IPR protection, may decide against innovating altogether.

Recently, this classical rational for the existence of IPR has been challenged. Many scholars argue that IPR protection is not essential to appropriately rewarding the innovators and/or that the damages it provokes to society exceeds the benefits (Pollock, 2007; Boldrine and Levine, 2002). As a consequence, one of the main issues that policy makers have to deal with is the extent of innovation protection they should provide.

The objection to strong IPR protection, or even to the existence of IPR, originates from the fact that all monopolies introduce a distortion, a deadweight in the economy. This deadweight shows up in different ways, in different forms and has been modeled in very different setups. Sohn (2008) compares the payoffs accruing from innovation *versus* imitation activities and shows that, although imitation weakens the incentive to innovate, it can benefit society on the whole by leading to a larger number of innovations. Fershtman and Markovich (2010) find that imitation may be beneficial when firms have different $R\&D$ abilities in that it may provide higher consumers' surplus and higher value for firms than a strong patent protection regime. Bessen and Maskin (2009) argue that, although imitation reduces the current profit of the firms who innovate, it raises the probability of further innovation and thereby improves the prospect that firms will make another profitable discovery later on. A very similar result is obtained by Belleflamme and Picard (2007), studying the dynamic effect of piracy. When innovation is sequential and complementary, standard reasoning about patents and imitation is deceptive. Imitations becomes a spur to innovation while strong patents become an impediment.

It has been theoretically (Kolèda 2005; Furukawa, 2007) and empirically (Aghion et al. 2005) claimed that the rate of innovation is an inverse-U shaped function of imitation, which, in turn, is an inverse measure of IPR protection. The straightforward consequence of this finding is that relaxing IPR protection can be beneficial whenever its level is above the apex of the inverse-U function.

Summarizing the controversy, on the one hand, imitations (both legal and illegal) have a positive effect from the perspective of society, accidentally mitigating the damage introduced by monopolies and speeding up innovation diffusion. On the other hand, an excessively large number of infringements can kill innovation activity, thus damaging the production of knowledge. Moreover, the absence of innovation in turn can lead to the extinction of imitators, who are left with nothing to imitate. It follows that innovators and imitators must coexist. This endogenous relationship has been described in the literature through the well known Lotka-Volterra model, whose use in economics was first examined by Andersen (1994) and which has been applied with a specific focus on innovation in Bharagava (1989), Morris and Pratt (2003), Watanabe et al. (2003), Castiaux (2007), Lee et al. (2005), Kim et al. (2006), Michalakelis et al. (2012), Balaz and Williams (2012), Chang et al. (2014), Guidolin and Guseo (2015) to cite only a few. Recently, Cerqueti et al. (2015) have shown that assuming the Lotka-Volterra co-dynamics for innovators and imitators is consistent with economic theory. Surprisingly, despite the extensive use of this model, to the best of our knowledge, no efforts have been made to introduce the effects of public interventions. This paper aims at filling this gap and providing practitioners and academics with a modified random Lotka-Volterra scheme that complies with the above. The introduction of randomness captures the link between innovation dynamics and the uncertainty due to the *R&D* phase. This makes the model more complicated, but also much more realistic, hence leading to consistent formulations of prescriptions and forecasts.

Actually, the Lotka-Volterra setting has already been extended by mathematical biologists[1] in the context of impulse control problems (Liu and Rohlf, 1998; Liu et al, 2005, just to cite a few). By a purely mathematical point of view and in the context of impulse control, the optimal strategies found in these contributions do not contemplate the opportunity of endogenously identifying neither the size of interventions, nor the dates, which are indeed fixed. For instance, Zhang et al (2003, 2005) introduce controls exogenously by introducing a constant amount of predators into the population at periodic intervals of given length. Baek (2008) deals with constant and proportional controls in a deterministic framework, to capture the presence of biological and chemical controls in the environmental setting. Akman et al. (2015) consider a stochastic model but provide a control of the prey-predator paths by applying a deterministic rule. Differently, we add to the literature by including the entity and the time of the intervention as control variables of the problem. In so doing, we include further reasonable sources of complexity in our setting. We point out that our model is tailored to account for industrial economic features, and control stems from a rational maximizing behavior, perfectly in accordance with economic theory. This is far from the general idea of the Lotka-Volterra systems, which usually deals with species evolutions and biological themes. More specifically, the paper deals with the trade-off between private and social interests generated by IPR protection and the problem of optimal government interventions. The paper assumes the perspective of a policy maker who wishes to optimize the relationship between innovations and imitations, given that the latter is an inverse measure of IPR protection. A unbalance between the two "populations" is costly to society: too many imitations are a drawback for the economy, while too little reduce the

[1]The extended Lotka-Volterra model is used to study how to control for pests using a mix of biological control, i.e. predators, and proportional control, i.e. chemical pesticides.

rate of innovation diffusion. As a consequence, the governments should intervene whenever the ratio between innovations and imitations (IPR protection) is not optimal (according to predefined criteria). This seemingly straightforward task is made difficult by the fact that innovation is a dynamic phenomenon evolving in an uncertain environment and optimization needs to be carried out continuously, and this of course has a cost. Hence, the governments must intertemporally balance the costs and benefits of intervening. Unfortunately, public interventions are costly and the cost is largely sunk . The most remarkable consequence is that there will be a non-intervention region, the so called "inaction region". Whenever the ratio lies in this region the governments finds it optimal not to intervene even though IPR protection in not at its maximizing value, and the two populations coexist (Stokey, 2009). Moreover, after an intervention is carried out and IPR optimally reset, uncertainty will make future dynamics of innovation and imitation unpredictable in the sense that it is not possible to know a priori whether the ratio will go up or down with respect to the optimal point chosen by the governments. This occurrence generates unpredictability in the next intervention, both in terms of timing and optimal value, which in turn will depend upon altered market situations.

The paper aims at modeling this complex problem by determining the optimal inaction region, which is defined in terms of an upper and a lower threshold of interventions and by return points. The latter are the points at which the ratio is set after the intervention, within the inaction region, while the former constitute the boundary of the inaction region itself. Given that resetting the value of the ratio from above or below entails different costs, it follows that the two thresholds are not necessarily symmetrical, but crucially depend on the cost of interventions. A relative increase in the number of innovators requires investing public resources to subsidize and foster $R\&D$, or to make the enforcement system more effective. In addition, fighting counterfeiting has a cost. This very complex reality is implemented in a model of dynamic uncertainty in which innovators and imitators interact and the governments wishes to maximize a public utility function, subject to intervention costs. In our model, the maximization model takes on the form of an optimal control problem in a stochastic setting.

The control problem is analyzed from two different perspectives. The first is purely theoretical and provides a rigorous description of the key elements of the problem, after which a formal solution of the model is provided. Two facts emerge from the theoretical study: i) the value function of the optimization problem can be regarded as the unique solution of the Hamilton-Jacobi-Bellman equation, ii) the optimal controls do not depend on the initial value of the ratio between innovators and imitators. Second, the problem is simulated and solved by implementing extensive numerical techniques on the basis of a consistent set of parameters. This allows us to obtain more intuitive insights from an economic point of view, along with a validation of the theoretical results. The interested reader can find a mixed theoretical simulation approach in control problems in He and Liu (2008), Castellano and Cerqueti (2012), Cerqueti (2012), Cerqueti and Quaranta (2012), Gollmann and Maurer (2014) and Cerqueti et al. (2016). The numerical optimization technique adopted is a Monte Carlo methodology with a grid search. For a proper description of the numerical procedures, refer to Rust (1996), He and Zhang (2013), Cai et al. (2015) and the monograph of Ensor and Glynn (1997). In particular, the numerical results show that the government optimal strategies consist in choosing the widest possible bandwidth and minimizing the size of interventions.

Given this premise, countries endowed with a more efficient enforcement system are expected to have a lower cost of interventions and, therefore, they are expected not to tolerate relatively low values of the ratio. This consideration may help in explaining different behaviors among apparently similar countries. For instance, the same laws on IPR apply in all EU countries, but actual interventions are quite different. Italy in particular is known for its high number of illegal imitations. To the best of our knowledge, this paper is the first one which deals with competition between innovators and imitators from a stochastic optimal control point of view with a target objective function. This approach is particularly effective in exploring how a policy maker may prevent one of the two populations from increasing too much.

The remainder of the paper is organized as follows. Section 2 contains the model. Section 3 presents the theoretical solution of the maximization problem. In Section 4 the numerical analysis of the optimization problem is provided along with a discussion of the optimal strategies. Section 5 discusses the economic implications of the model. Section 6 contains our final remarks as well as a subsection with outlines for possible future research. Some mathematical derivations are relegated in the Appendix.

2 The model

We assume the existence of an endogenous relationship between innovators and imitators with a cycle of flows of the following type: 'higher innovators population → more imitators → lower density of innovators → less imitators → higher innovators population' after which the cycle repeats itself. In the literature this relationship has been described by the Lotka-Volterra model and we build upon those contributions enriching the model by postulating the presence of a policy maker. Specifically, our model refers to a dynamic stochastic environment insofar as innovation dynamics is supposedly random, since $R\&D$ invariably entails uncertainty. On the other hand, imitation dynamics is assumed to follow a deterministic law, since it does not necessarily require the $R\&D$ phase.

Let us introduce two time-dependent processes, n_t and m_t, representing respectively the number of innovators and imitators at time t. To gain mathematical tractability without loss of generality we introduce the log of the ratio between innovators and imitators as follows:

$$X_t = \log\left[\frac{n_t}{m_t}\right], \qquad \forall\, t > 0, \tag{1}$$

that is equivalent to define

$$X_t = N_t - M_t, \qquad \forall\, t > 0, \tag{2}$$

with $N_t = \log[n_t]$ and $M_t = \log[m_t]$.

Thus, the cycle described above can be formalized by means of the following interrelated equations:

$$\begin{cases} dN_t = N_t(r - wN_t - fM_t)dt + \gamma d\eta_t, \\ \\ dM_t = M_t(g + cN_t - hM_t)dt. \end{cases} \tag{3}$$

where $r, w,\ f,\ h,\ c,\ g, \gamma \in [0, +\infty)$ and η_t is a standard 1-dimensional brownian motion. The first line of (3) describes the evolution of the innovators, while the second that of the imitators. Furthermore, equations in (3) state that in each period of time innovators (imitators) increase by a proportion of r (g) and, at the same time, die out by "natural death" by a proportion $-wN_t$ $(-hM_t)$. The model presents the relation between imitators and innovators as a relation between predators and preys. The last term captures the "hunting" rate of imitators, f, namely it evidences the rate at which imitators "kill" innovators by causing them to lose profits. The greater the number of preys the higher the possibility of hunting for predators, and the greater the number of predators the greater the number of victims. Symmetrically, the term c indicates that imitators thrive by hunting innovators.[2]

By (2) and (3) the dynamics of X_t can be rewritten as:

$$\begin{cases} dX_t = d(N_t - M_t) = [N_t(r - wN_t - fM_t) - M_t(g + cN_t - hM_t)]dt + \gamma N_t d\eta_t & \forall\ t > 0; \\ \\ X_0 = x, \end{cases} \tag{4}$$

where $x \in \mathbb{R}$.

By substituting $M_t = N_t - X_t$, and after simple algebra, we have

$$dX_t = [hX_t^2 + gX_t + (f + c - 2h)N_tX_t + (r - g)N_t + (h - c - w - f)N_t^2]dt + \gamma N_t d\eta_t. \tag{5}$$

The government aims at controlling the evolution of the log-ratio X_t. Intervening involves a cost of C, therefore an optimal policy requires exercising control only occasionally. In this respect the model, although in continuous time, manages to capture discrete policy interventions. The presence of costs generates a lower threshold $b \in \mathbb{R}$ and an upper threshold $B \in \mathbb{R}$ to the fluctuations in X_t. When X_t reaches b (B), then it is pushed upward (downward) at the return point q (Q). The costs are related to the gap between the barrier and the return points, which are also optimally derived. This double line of interventions from above or below captures the fact that the policy maker is aware of the trade-off between knowledge production and diffusion, i.e. innovation and imitation, and intervenes in both directions. Thus, the presence of the barriers can be though of as a metaphor of the social need to protect both sides of the coin.

Intervention costs are assumed to be composed of two parts: one is a function of the stock of innovators, and another depends linearly on the size of the intervention. From a more

[2]For an in-depth interpretation of the parameters of the system under different scenarios we refer the reader to Castiaux (2007 p. 40-41)

intuitive economic perspective, the first component can be viewed as the cost of protecting existing innovators, e.g., through the legal system and/or the patent office. The second component is the unit cost of adding to the stock of innovators with respect to that of imitators. To provide an example of the second cost component, think of the intervention put into place in Italy at the end of 2012 by means of the decree 179/2012 nicknamed "Growth Act 2.0". Pursuing the declared scope of increasing the number of innovators, the act establishes supports, in different forms, to any new entrant, conditioned on the fulfillment of some eligibility criteria apt to characterize the entrant as an innovator. These supports clearly entail a cost which linearly increases as the number of innovators adds to the existing stock, for any given number of existing imitators. Broadly speaking, subsidizing policies, typically (but not strictly necessarily) focused on the $R\&D$ phase fall within this second part of the cost structure. Thus, the cost function is supposed to be a function of both the overall number of innovators and the relative composition of the two species at intervention points.

Accordingly, we formalize intervention costs at time t as follows:

$$C(N_t, z) = \begin{cases} C_0(N_t) + Kz, & \text{if } z > 0; \\ 0, & \text{if } z = 0; \\ c_0(N_t) + kz, & \text{if } z < 0, \end{cases} \tag{6}$$

where c_0, C_0 are increasing functions of the number N_t of innovators at time t, such that $c_0(N_t), C_0(N_t) > 0$ and $k \leq 0 \leq K$. The variable z represents the gap between the barrier and the return point, i.e.: z may assume values $B - Q$ or $b - q$.

Both components of adjustment costs can differ for upward and downward adjustments. In the above example, a relative increase in the number of innovators requires investing public resources to make the legal and the enforcement system more effective and/or to foster

R&D. Besides, fighting counterfeiting implies a higher cost than lessening the fight. This asymmetry can easily be captured by assuming for the first component $c_0(N_t) > C_0(N_t)$, for each N_t, and $|k| > K$ for the second component. The value $k \leq 0$ is the unit cost of adding to the relative stock of innovators. In terms of the Italian "Growth Act 2.0" it represents the amount of support received by any new entrant. Analogously, $K \geq 0$ is the unit cost of adding to the relative stock of imitators.

The Government's objective is to choose b, B, q, Q such to maximize the expected discounted utility function $U : \mathbb{R} \to \mathbb{R}$ as a function of X_t. At this purpose, the utility function is assumed to satisfy the following condition.

Assumption 1. *$U \in C^1(\mathbb{R} - \{0\}) \cap C^0(\mathbb{R})$, it is strictly increasing in $(-\infty, 0)$ and strictly decreasing in $(0, +\infty)$, strictly convex in $(-\infty, 0)$ and in $(0, +\infty)$, $U(0) = 0$ and the Inada condition is satisfied, i.e.*

$$\lim_{x \to 0^+} U'(x) = -\infty; \qquad \lim_{x \to 0^-} U'(x) = +\infty; \qquad \lim_{|x| \to +\infty} U'(x) = 0. \tag{7}$$

It is worth noting that the Inada condition (7), jointly with $U(0) = 0$, implies that U is bounded in $\mathbb{R}$. Furthermore, the rationale behind the assumed behavior of U in the semilines $(-\infty, 0)$ and $(0, +\infty)$ is as follows: the utility function increases when the difference between the number of innovators and imitators grows. This is the optimization rule to be followed by the policy maker, who should maintain these two quantities roughly at the same level. Hence, an excessive growth of the ratio between innovators and imitators should be avoided, as well as an excessive reduction of it.

The presence of the thresholds b and B entails the inaction to stop when the barriers are reached. Therefore, we need to introduce a stopping time for the dynamics X_t, which represents the horizon of the problem. We denote such a stopping time as τ, and it is defined as follows:

$$\tau = \inf\{t \geq 0 \,|\, X_t \notin (b, B)\}. \tag{8}$$

The problem can be formalized through the definition of the objective function $J : \mathbb{R}^5 \to \mathbb{R}$ such that

$$J(x; b, B, q, Q) = \mathbb{E}_x \left[\int_0^\tau e^{-\delta t} U(X_t) dt - e^{-\delta \tau} C(N_\tau, X_\tau - Q) \mathbf{1}_{\{X_\tau = B\}} - e^{-\delta \tau} C(N_\tau, X_\tau - q) \mathbf{1}_{\{X_\tau = b\}} \right], \tag{9}$$

where $\mathbb{E}_x$ denotes the expectation conditioned to $X_0 = x$. The corresponding value function $V : \mathbb{R} \to \mathbb{R}$ is

$$V(x) = \max_{(b,B,q,Q) \in \mathbb{R}^4} J(x; b, B, q, Q). \tag{10}$$

3 Theoretical discussion of the optimization problem

The search of the solution of the optimal control problem outlined in the previous section moves from the Hamilton-Jacobi-Bellman equation. This section is devoted to the statement of the main theoretical results.

Theorem 1. *Suppose that Assumption 1 holds and consider the optimized value function V.*

(b^*, B^*, q^*, Q^*) is an optimal policy if and only if the following conditions hold:

- *V is the classical solution of:*

$$\delta V(x) = U(x) + \frac{\gamma^2 N^2}{2} V''(x) +$$

$$+ [hx^2 + gx + (c + f - 2h)Nx + (r - g)N + (h - c - w - f)N^2] V'(x), \qquad x \in (b^*, B^*) \tag{11}$$

with

$$\begin{cases} V(x) = h_b + k(N)(b^* - x), & x \leq b^*; \\ \\ V(x) = h_B + K(N)(B^* - x), & x \geq B^*. \end{cases} \tag{12}$$

- *V and V' are continuous at b^*:*

$$\lim_{x \to (b^*)^+} V(x) = h_b \qquad \text{and} \qquad \lim_{x \to (b^*)^+} V'(x) = -k(N). \tag{13}$$

- *V and V' are continuous at B^*:*

$$\lim_{x \to (B^*)^-} V(x) = h_B \qquad \text{and} \qquad \lim_{x \to (B^*)^-} V'(x) = -K(N). \tag{14}$$

- *q^* and Q^* satisfy:*

$$V'(q^*) = -k(N) \qquad \text{and} \qquad V'(Q^*) = -K(N). \tag{15}$$

where $N = N_0$ and h_b and h_B are the unique scalars which satisfy the following condition:

$$\begin{cases} h_b = C_0(N) + k(N)(q^* - b^*) + \varphi_1(q^*) + \varphi_2(q^*)h_b + \varphi_3(q^*)h_B, \\ \\ h_B = c_0(N) + K(N)(Q^* - B^*) + \varphi_1(Q^*) + \varphi_2(Q^*)h_b + \varphi_3(Q^*)h_B, \end{cases} \tag{16}$$

with

$$\begin{cases} \varphi_1(x) = \mathbb{E}_x\left[\int_0^\tau e^{-\delta t} U(X_t) dt\right], \\ \\ \varphi_2(x) = \mathbb{E}_x\left[e^{-\delta \tau} \,\middle|\, X_\tau = b\right] \mathbb{P}[X_\tau = b], \\ \\ \varphi_3(x) = \mathbb{E}_x\left[e^{-\delta \tau} \,\middle|\, X_\tau = B\right] \mathbb{P}[X_\tau = B]. \end{cases} \tag{17}$$

For the proof of Theorem 1, see the Appendix.

Notice that the first relations in (13) and (14) represent the value matching conditions, while the second are the smooth pasting conditions.

In practice, the optimization model presented in the previous Section concerns only the case of $x \in [b^*, B^*]$. Therefore, we will not consider the case when x is outside this range of variation - i.e.: equations (12).

The uniqueness of V as a classical solution in $[b^*, B^*]$ of equation (11) with boundary conditions (13) or (14) is guaranteed by the second-order differential equation theory (see e.g. Wirkus and Swift, 2015, Theorem 3.1.1). In fact, equation (11) is a linear second order nonhomogeneous ordinary differential equation with continuous coefficients, which can be rewritten as follows:

$$V''(x) + G(x)V'(x) + S(x)V(x) = F(x), \tag{18}$$

where

$$G(x) = \frac{2[hx^2 + gx + (c + f - 2h)Nx + (r - g)N + (h - c - w - f)N^2]}{\gamma^2 N^2},$$

$$S(x) = -\frac{2\delta}{\gamma^2 N^2}, \qquad F(x) = -\frac{2U(x)}{\gamma^2 N^2}. \tag{19}$$

Unfortunately, a closed form solution in $[b^*, B^*]$ of equation (11) is not available. For some remarks on this, see the Appendix.

As Theorem 1 clearly states, the uniqueness of V as a classical solution in $[b^*, B^*]$ of equation (11) with conditions (12)-(15), guarantees the existence of the optimal strategies and theoretically identifies them (see also Stokey, 2009, Section 7.4 for a wider discussion on this).

The most important outcome of our theoretical analysis is that the optimal quadruple (b^*, B^*, q^*, Q^*) is dependent on N. In our specific context, this fact entails that a rational policy maker does not alter her/his optimal strategies as the state variable changes.

4 Monte Carlo simulation results

This section contains an illustrative example of the theoretical optimization model. At this aim, we perform simulations based on empirical data[3], with the specific target to compute numerically the values of b^*, B^*, q^* and Q^* maximizing the objective function $J(x, b, B, q, Q)$ in (9). In order to accomplish this task we have built a procedure based on the following steps.

- *Define a set of possible starting points of X.*

  $X_0 = x \in [-1, 1]$. The set $[-1, 1]$ is discretized by using a step of length 0.1, thus obtaining $s = 1, \ldots, 21$ possible starting points. According to (5), for each x_s we build via Monte Carlo simulation $i = 1, ..., I$ different dynamics of X_t, $t = 1, ..., T$, and we denote the i-th simulated dynamics with starting point x_s as $X_t^{x_s,i}$.

- *Take a series of N_t.*

  To the best of our knowledge, there are no available time series for the number of innovators for a given country. In order to anchor the dynamics of X_t to actual data and to make the simulation reproducible, we have based the construction of N_t on the OECD patents database for Italy over the period 1977 - 2007. The database contains the number of patent applications at the European Patent Office by inventor. The choice of Italy as a point of reference is particularly reasonable. Indeed, Italy is an interesting case because among the developed economies, it is know as a country in which illegal , especially infringements, are more tolerated.[4] The series of innovators

[3]Data and C code (Code::Blocks10.05) can be made available to the interested reader upon request.

[4]On this point see for instance Offeddu in Corriere della Sera 12 September 2008. *Falsi, Italia bancarella d'Europa* (Fakes, Italy the stall of Europe, our translation).

has been proxied by cumulating the applications over the nominal lifespan of patents, i.e. 20 years. Since patents do not necessarily survive until the final expiration date, given that an annual fee must be paid by the patent owner in order to keep patent alive, we have hypothesized a linear decay process to capture the drop out rate. Although rather simple, the linear hypothesis is consistent with the literature on patent renewals (Shankerman 1998; Lanjouw et al, 1998; Pakes and Simpson, 1989). Finally, annual data have been interpolated by an exponential approximation reproducing monthly data, in order to increase the number of time observations of N_t from 31 to 372. Consistently with the theoretical model the log of such a data has been taken as representing N_t.

- *Select appropriate cost and utility functions.*

  It is natural to assume that costs are proportional to the number of innovators populating the market. Therefore, we introduce two constants $c_0, C_0 > 0$ such that $c_0(N_t) = c_0 \cdot N_t$ and $C_0(N_t) = C_0 \cdot N_t$. In accord with the arguments put forth in Section 2, we also assume that $c_0 > C_0$.

  As a utility function, we select $U(x) = -\sqrt{|x|}$, which fulfils the requirements of Assumption 1.

- *Fix the values of the parameters.*

  We fix $I = 10,000; \gamma = 3 \times 10^{-2}; C_0 = 3 \times 10^{-2}; c_0 = 5 \times 10^{-2}; K = 1; k = -2; \delta = 3 \times 10^{-2}$ and refer to the following reasonable parameters of the system $g = 2 \times 10^{-4}$; $h = 8 \times 10^{-4}$; $c = 4 \times 10^{-4}$ $r = 3 \times 10^{-4}$; $w = 10^{-4}$; $f = 3 \times 10^{-4}$.

The values of the parameters set for the simulation are taken from Bischi et al. (2004, p. 194) who simulate a Lotka-Volterra system, and some calibrations have been made in order to avoid the dynamics to stop at the very first iterate.

At this point the Brownian Motion can be discretized, as usual: $dB(t) = \Lambda * \sqrt{\Delta t}$, where Λ is a random number drawn from a standard normal distribution and $\Delta t = 1$. Notice that C_0, c_0, K and k have been set such that the cost of intervening is higher when interventions occur from below, i.e. when X_t reaches b.

- *Execute a grid-search.*

  To find the maximizing value of J in (9) a grid-search is carried out over the quadruples (b, B, q, Q). In particular, the following intervals of each element of the quadruple have been set: $b \in [-4; -3]$ and $B \in [3; 4]$ both with step size equal to 0.1. The variation ranges of the return points q and Q depend on the corresponding thresholds b and B, respectively. Therefore, we consistently assume the following variation ranges: $q \in [b+0.5; b+0.99]$ and $Q \in [B-0.99; B-0.5]$ and we have discretized such intervals with step size of 0.049.

  For each combination of b and B and for each simulated dynamics $X_t^{x_s,i}$ the exit time $\tau^{x_s,i}$ from (b, B) has been calculated according to (8).

- *Find the maximum of the objective function (9) summing up the discretized version of its two components.*

The integral part of (9) can be discretized as follows:

$$\mathbb{E}\left[-\int_0^{\tau} e^{-\delta t}\sqrt{|X_t|}dt\right] = -\frac{1}{I}\sum_{i=1}^{I}\left\{\frac{1}{\tau^{x_s,i}}\sum_{v=1}^{\tau^{x_s,i}} e^{-\delta v}\sqrt{|X_v^{x_s,i}|}\right\}, \tag{20}$$

while, the second part becomes:

$$\begin{aligned}&\mathbb{E}\left[e^{-\delta\tau}\{C_0\cdot N_\tau + K\cdot(X_\tau - Q)\}\mathbf{1}_{\{X_\tau=B\}} + e^{-\delta\tau}\{c_0\cdot N_\tau + k\cdot(X_\tau - q)\}\mathbf{1}_{\{X_\tau=b\}}\right] = \\ &= \frac{1}{I}\sum_{i=1}^{I} e^{-\delta\tau^{x_s,i}}\left[\{C_0\cdot N_{\tau^{x_s,i}} + K\cdot(X_{\tau^{x_s,i}}^{x_s,i} - Q)\}\mathbf{1}_{\{X_{\tau^{x_s,i}}^{x_s,i}=B\}} + \{c_0\cdot N_{\tau^{x_s,i}} + k\cdot(X_{\tau^{x_s,i}}^{x_s,i} - q)\}\mathbf{1}_{\{X_{\tau^{x_s,i}}^{x_s,i}=b\}}\right],\end{aligned} \tag{21}$$

for each $s = 1, \ldots, 21$.

As a result, we obtain that independently of the starting points, i.e. of the index s, the maximum value of (9) is always reached with the quadruple $b^* = -4$, $B^* = 4$, $q^* = b^* + 0.5 = -3.5$ and $Q^* = B^* - 0.5 = 3.5$.

The value of the optimal quadruple sheds light on the government maximizing behavior. To this end, it is of great help to analyze the government's strategy separately in order to set the optimal bandwidth and the return points.

As far as the optimal bandwidth is concerned, $[b^*, B^*]$, the government's behavior is to select the lowest admissible value for interventions from below, b^*, and the highest possible for interventions from above, B^*, making the inaction region the widest possible.

With reference to the optimal return point, q^* (Q^*), is such that the distance of X_t from b^* (B^*) is the smallest possible. These two aspects together make clearly emerge the duality of the maximization problem in (9). Indeed, they indicate that the government seeks to delay the intervention time, by setting the widest band and, at the same time, to minimize the cost of each intervention by minimizing their size by choosing the closest value of the return points to the barriers.

As a robustness check of the quality of the simulation and the economic intuition behind the government strategy we have elaborated some indicators and studied their behavior. In particular, two parameters representing the *ex-post* probability of the exit from below and above have been elaborated, respectively $p_{b^*}^{(s)}$ and $p_{B^*}^{(s)}$. Formally, they are defined as:

$$\begin{cases} p_{b^*}^{(s)} = \frac{\sum_{i=1}^{I} \mathbf{1}(X^{x_s,i}_{\tau^{x_s,i}}=b^*)}{I} \\ \\ p_{B^*}^{(s)} = \frac{\sum_{i=1}^{I} \mathbf{1}(X^{x_s,i}_{\tau^{x_s,i}}=B^*)}{I}. \end{cases}$$

Substantially, $p_{b^*}^{(s)}$ and $p_{B^*}^{(s)}$ measure the frequency of simulated dynamics starting from x_s and hitting for the first time the threshold b^* or B^*, respectively.

Similarly, we have defined as τ_s the average exit time as:

$$\tau_s = \frac{1}{I} \sum_{i=1}^{I} \tau^{x_s,i} \tag{22}$$

A graphical view of the behavior of $p_{b^*}^{(s)}$, $p_{B^*}^{(s)}$ and τ_s, along with that of $V(x)$ in (10) as functions of the starting points, reveals other interesting results.

The values of τ_s range in a rather narrow interval, from 115 to 120 time periods, showing an inverse U-shape (Figure 1). For low values of x_s (from -1 to -0.2), τ_s increases reaching its maximum for $x_s \in [-0.2; 0.1]$ and then decreases.

The inverse U-shape is due to the fact that for low (high) values of x_s the dynamics is more likely to hit b^* (B^*) while for central values of x_s the probability of hitting a barrier is lower and this is reflected in higher average exit time.

INSERT FIGURE 1 ABOUT HERE

Caption: Exit time as a function of the starting points

Figure 2 shows that as x_s increases, the probability of hitting the barrier b^* (B^*) decreases (increases). Interestingly, the intersection point (when $x_s = -0.2$) is not placed around the middle values, but it is slightly shifted on the left hand at lower values of x_s, implying that the probability of interventions from below are minimized, being more expensive than those from above.

INSERT FIGURE 2 ABOUT HERE

Caption: Probabilities of hitting barriers as a function of the starting points

Finally, Figure 3 plots the value function (10) as a function of the starting points. The values of $V(x)$ show an inverse U-shape: for low values of x_s (from -1 to -0.1), the value function increases, at $x_s = 0$, consistently with the functional form of $U(x)$, it reaches its maximum and then decreases. From another vantage point, this pattern can be interpreted as claiming that an equal proportion between innovators and imitators make the interventions less likely, hence making also the value function higher.

INSERT FIGURE 3 ABOUT HERE

Caption: Value Function as a function of the starting points

5 Implications of our analysis

Among the aspects highlighted by the model two in particular have important implications. First, the presence of the barriers brings to our attention the fact that the dynamics of the Lotka-Volterra must be considered as bounded. Second, the presence of the return points implies that the values of the dynamics are discretionarily defined by the policy maker at some points in time, according to its objective function. It follows that considering uncontrolled dynamics can lead to wrong prescriptions and biased estimates and forecasts. Given that, to some extent, markets are usually regulated in almost all countries, the model provides a way to consider public interventions without abandoning the Lotka-Volterra model.

In Lee et al. (2005) the dynamics of competing financial stock markets are analyzed in terms of unregulated Lotka-Volterra dynamics, but financial markets are regulated markets, thus the ensuing prescriptions and forecasts may be biased, generating great losses. Yet, as another example Michalakelis et al. (2012) apply the same unbounded dynamics to telecommunications market, which again is a regulated one. Our model is flexible enough to be tailored to different market conditions with little effort, by simply defining different cost schedules and welfare functions. Intuitively, we do not have any *a-priori* to set an asymmetric cost structure in telecommunications and financial markets, in the sense that interventions from above or below should entail symmetric costs.

6 Concluding remarks

The debate about IPR is fraught with controversy. Many theoretical and empirical works highlight the presence of an inverse U-shape relationship between innovation and imitation, suggesting that optimal interventions should keep the ratio between innovations and imitations at its maximum. We have argued that in practice the complex economic environment makes this simple strategy unfeasible for a number of reasons. To put it briefly, innovation is a dynamic phenomenon, hence interventions should occur continuously, but they are costly and the economic environment is fraught with uncertainty. The model developed in the present work manages to capture all these real features in a relatively simple fashion. Its outcome shows that optimal interventions can occur only at random points in time defined by the cost structure and the intrinsic characteristic of the innovation-imitation process. The economic intuition is confirmed by numerical simulations showing that the government optimal strategy consists in setting the widest possible bandwidth and minimizing the entity of interventions. The model evidence how different intervention costs may help in explaining why countries with the same IPR regulations, such as the EU Member States, may enact very different intervention policies.

6.1 Future research and extensions

Nonlinear models have been extensively used to study and forecast the trajectories of innovation, in particular the Bass model (Bass et al., 1994) and its variations and the Lotka-Volterra family. Although the former is preferable because of its greater flexibility in accounting for external perturbations, such as regulatory interventions, a better forecasting ability of the latter has been proved by Guidolin and Guseo (2015) and Chang et al. (2014). It follows that an extension of the Lotka-Volterra, as proposed in this work, now able to capture external perturbations makes it more flexible and attractive to fit actual data and is quite likely beneficial for improving further its forecasting ability. Therefore, a further extension of our contribution consists in comparing the performances of both the non regulated and regulated Lotka-Volterra dynamics. In particular, it would be quite interesting to re-estimate the Lotka-Volterra parameters as in Lee et al. (2005) and Michalakelis et al. (2012), by considering also the presence of bounds defined by appropriate cost and welfare functions and to reformulate prescriptions and forecasts accordingly.
Under the perspective of the optimization algorithm, the computation of the optimal values b^*, B^*, q^* and Q^* may be explored also by implementing a robust optimization of the objective function J in (9) (see e.g.: Ben-Tal and Nemirovski, 2002; Ben-Tal et al., 2009; Quaranta and Zaffaroni, 2008). The adoption of this methodology offers the opportunity to include the presence of uncertainty in the definition of the optimization parameters, which represents a remarkable generalization of our setting. The solution method of many robust programs involves creating a deterministic equivalent, called the robust counterpart. Actu-

ally, the practical difficulty of a robust program depends on whether its robust counterpart is computationally tractable. In other words, the feasibility of a robust optimization-type procedure depends on the possibility to make the robust counterpart of the original programming problem tractable from a mathematical point of view. In our specific case, such a tractability is rather questionable, for the presence of four intersecting variation ranges of the parameters involved in the optimization. This complexity leads to a challenging problem also from a theoretical and computational point of view, which deserves an *ad hoc* contribution.

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Appendix

Proof of Theorem 1

First of all, we need to prove that V satisfies equation (11) with boundary conditions (12). In order to do this, we adapt to our case an important result.

Theorem 2 (Dynamic Programming Principle)**.** *Let us consider a stopping time $\rho \in \mathcal{T}$. Then*

$$V(x) = \max_{(b,B,q,Q)\in\mathbb{R}^4} \mathbb{E}_x \left[\int_0^{\tau\wedge\rho} e^{-\delta t} U(X_t) dt + V\left(X_{\tau\wedge\rho}\right) \right]. \qquad (23)$$

For the details of the proof of Theorem 2 we refer to Cerqueti (2009), where a Dynamic Programming Principle more general than ours is presented.

Formula (23) implies that, if $(b, B, q, Q) \in \mathbb{R}^4$ and $s > 0$, then

$$V(x) \leq \mathbb{E}_x \left[\int_0^{s} e^{-\delta t} U(X_t) dt + V\left(X_s\right) \right].$$

Therefore:

$$0 \geq -\frac{\mathbb{E}_x \left[V\left(X_s\right) - V(x) \right]}{s} - \frac{1}{s} \cdot \mathbb{E}_x \left[\int_0^{s} e^{-\delta t} U(X_t) dt \right] =$$

$$= -\frac{\mathbb{E}_x\left[\int_0^s dV(X_u)\right]}{s} - \frac{1}{s}\cdot\mathbb{E}_x\left[\int_0^s e^{-\delta t}U(X_t)dt\right]. \tag{24}$$

By applying Ito's Lemma and by (5), we can rewrite the last term of (24) as:

$$-\frac{1}{s}\cdot\mathbb{E}_x\Big[\int_0^s \{V'(X_u)[h(X_u)^2 + gX_u + (c+f-2h)NX_u + (r-g)N + (h-c-w-f)N^2] +$$

$$+\frac{\gamma^2 N^2}{2}V''(X_u)\}du\Big] - \frac{1}{s}\cdot\mathbb{E}_x\left[\int_0^s e^{-\delta t}U(X_t)dt\right]. \tag{25}$$

By setting a limit for $s \to 0$ and taking the max over $(b, B, q, Q) \in \mathbb{R}^4$, we obtain

$$0 \geq \delta V(x) - U(x) - \frac{\gamma^2 N^2}{2}V''(x) - [hx^2 + gx + (c+f-2h)Nx + (r-g)N + (h-c-w-f)N^2]V'(x). \tag{26}$$

We then need to show that the converse inequality of (26) to obtain that V satisfies (11).

Given $\epsilon > 0$ and $s > 0$ small enough, there exists a quadruple $(b_\epsilon, B_\epsilon, q_\epsilon, Q_\epsilon) \in \mathbb{R}^4$ such that

$$V(x) + s\epsilon \geq \mathbb{E}_x\left[\int_0^s e^{-\delta t}U(X_t)dt + V\left(X_s\right)\right].$$

Ito's Lemma and (5) give:

$$-\epsilon \leq -\frac{\mathbb{E}_x\left[V\left(X_s\right) - V(x)\right]}{s} - \frac{1}{s}\cdot\mathbb{E}_x\left[\int_0^s e^{-\delta t}U(X_t)dt\right] =$$

$$= -\frac{1}{s}\cdot\mathbb{E}_x\Big[\int_0^s \{V'(X_u)[h(X_u)^2 + gX_u + (c+f-2h)NX_u + (r-g)N + (h-c-w-f)N^2] +$$

$$+\frac{\gamma^2 N^2}{2}V''(X_u)\}du\Big] - \frac{1}{s}\cdot\mathbb{E}_x\left[\int_0^s e^{-\delta t}U(X_t)dt\right] \leq$$

$$\leq -\frac{1}{s}\cdot \max_{(b,B,q,Q)\in\mathbb{R}^4}\mathbb{E}_x\Big[\int_0^s \{V'(X_u)[h(X_u)^2 + gX_u + (c+f-2h)NX_u + (r-g)N + (h-c-w-f)N^2] +$$

$$+\frac{\gamma^2 N^2}{2}V''(X_u)\}du\Big] - \frac{1}{s}\cdot\mathbb{E}_x\left[\int_0^s e^{-\delta t}U(X_t)dt\right], \tag{27}$$

being the last inequality a consequence of have taking the maximum. Now, by setting a limit for $s \to 0$, the last term of (27) becomes:

$$0 \leq \delta V(x) - U(x) - \frac{\gamma^2 N^2}{2}V''(x) - [hx^2 + gx + (c+f-2h)Nx + (r-g)N + (h-c-w-f)N^2]V'(x). \tag{28}$$

Formulas (26) and (28) assures that V satisfies (11).

The validity of conditions (12)-(15) – which assure that (b^*, B^*, q^*, Q^*) is an optimal quadruple for the control problem – comes out from Proposition 7.5 in Stokey (2009).

Remark on the closed form solution of equation (11)

Equation (11) is a linear second order ordinary differential equation with continuous coefficients, see formulas (18-19).

A particular solution of equation (18) can be written in an explicit way, even if the presence of the utility function U with generic shape avoids a closed form expression.

Fix x and suppose that u_1 and u_2 are two linearly independent solutions of the homogeneous equation:

$$V''(x) + G(x)V'(x) + S(x)V(x) = 0. \tag{29}$$

We conjecture that the solution of (18) takes the form

$$u(x) = \gamma_1(x)u_1(x) + \gamma_2(x)u_2(x), \tag{30}$$

where γ_1 and γ_2 are differentiable functions such that

$$\gamma_1'(x)u_1(x) + \gamma_2'(x)u_2(x) = 0, \qquad \forall x. \tag{31}$$

Condition (31) gives that:

$$\begin{cases} u'(x) = \gamma_1(x)u_1'(x) + \gamma_2(x)u_2'(x), \\ \\ u''(x) = \gamma_1(x)u_1''(x) + \gamma_2(x)u_2''(x) + \gamma_1'(x)u_1'(x) + \gamma_2'(x)u_2'(x). \end{cases} \tag{32}$$

By replacing V with u and substituting the terms in (32) into (18) we obtain:

$$\sum_{i=1}^{2} \left[\gamma_i(x)u_i''(x) + \gamma_i'(x)u_i'(x) + G(x)\gamma_i(x)u_i'(x) + S(x)\gamma_i(x)u_i(x)\right] = F(x) \tag{33}$$

Since u_1 and u_2 satisfy the homogeneous equation (29), several terms in (33) cancel out, and we obtain:

$$\sum_{i=1}^{2} \gamma_i'(x) u_i'(x) = F(x). \tag{34}$$

Therefore, the conjecture of u of the form (30) with the condition (31) is fulfilled if one can find γ_1 and γ_2 such that:

$$\begin{pmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{pmatrix} \begin{pmatrix} \gamma_1'(x) \\ \gamma_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ F(x) \end{pmatrix} \tag{35}$$

so that

$$\gamma_1'(x) = -\frac{u_2(x)F(x)}{W(u_1,u_2)(x)}, \qquad \gamma_2'(x) = \frac{u_1(x)F(x)}{W(u_1,u_2)(x)}, \tag{36}$$

being $W(u_1,u_2)(x) = u_1(x)u_2'(x) - u_2(x)u_1'(x) \neq 0$ the Wronskian determinant at x. The fact that the Wronskian is nonnull is due to the assumption that u_1 and u_2 are linearly independent.

By integrating, we have

$$\gamma_1(x) = -\int \frac{u_2(x)F(x)}{W(u_1,u_2)(x)} dx, \qquad \gamma_2(x) = \int \frac{u_1(x)F(x)}{W(u_1,u_2)(x)} dx, \tag{37}$$

By plugging (37) into the expression of u, we obtain:

$$u(x) = \left[-\int \frac{u_2(x)F(x)}{W(u_1,u_2)(x)} dx + C_1 \right] u_1(x) + \left[\int \frac{u_1(x)F(x)}{W(u_1,u_2)(x)} dx + C_2 \right] u_2(x), \tag{38}$$

where the constants C_1 and C_2 depends on the boundary conditions.

The problems of the identification of the linearly independent solutions u_1 and u_2 of the homogeneous equation and of the direct computation of the integrals leading to γ_1 and γ_2, as in formula (37), remain open. This fact is due to the nonstandard polynomial shape of the coefficients G and S and the generic form of the utility function in F[5]

[5]For an idea of the difficulties in providing the closed form solution of (18), see Mejlbro (2008).

To conclude, even in presence of an explicit method for determining a particular solution of equation (11) with boundary conditions (13) or (14), the identification of the closed form solution of it is not available in the present setting.

Figure 1

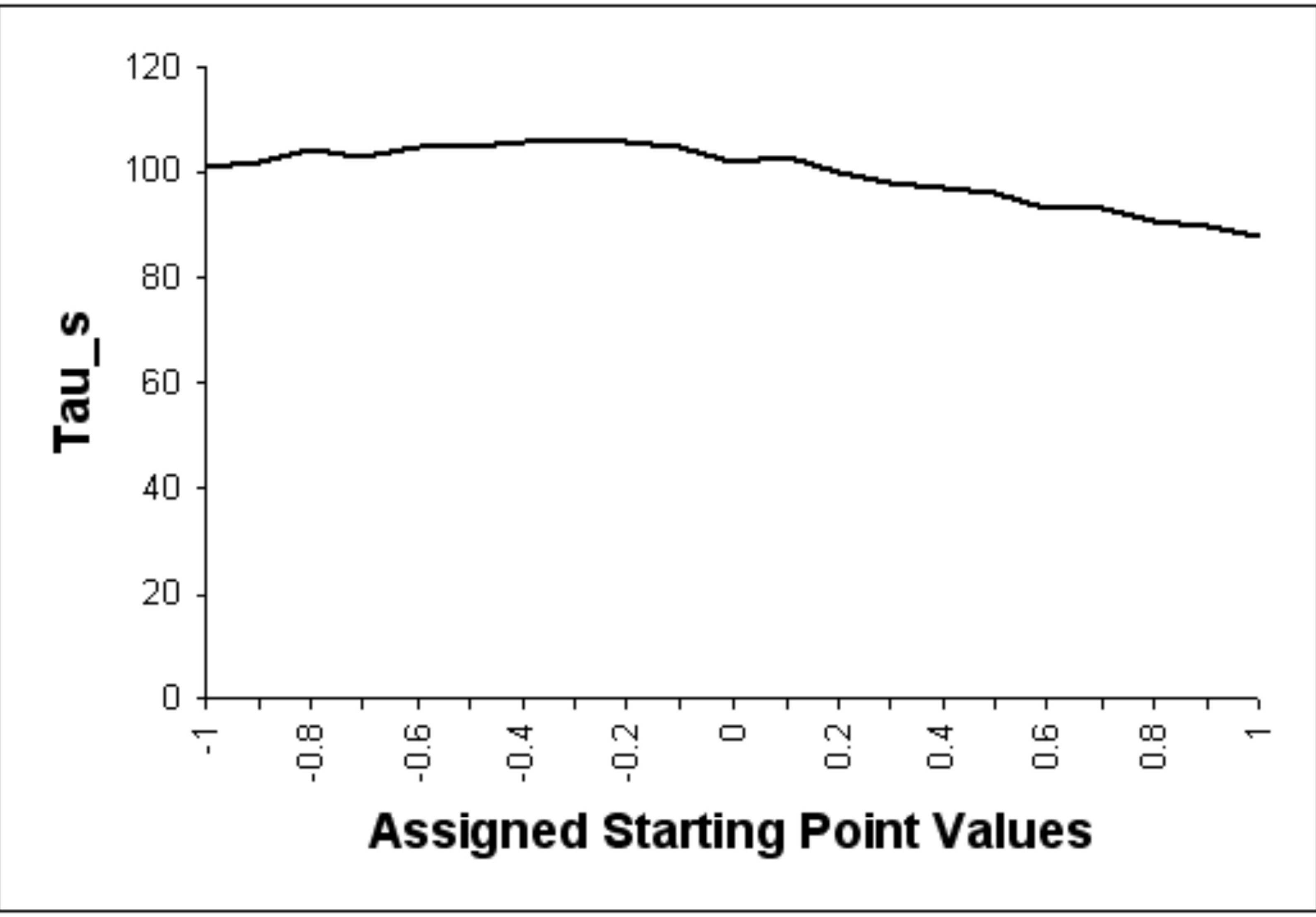
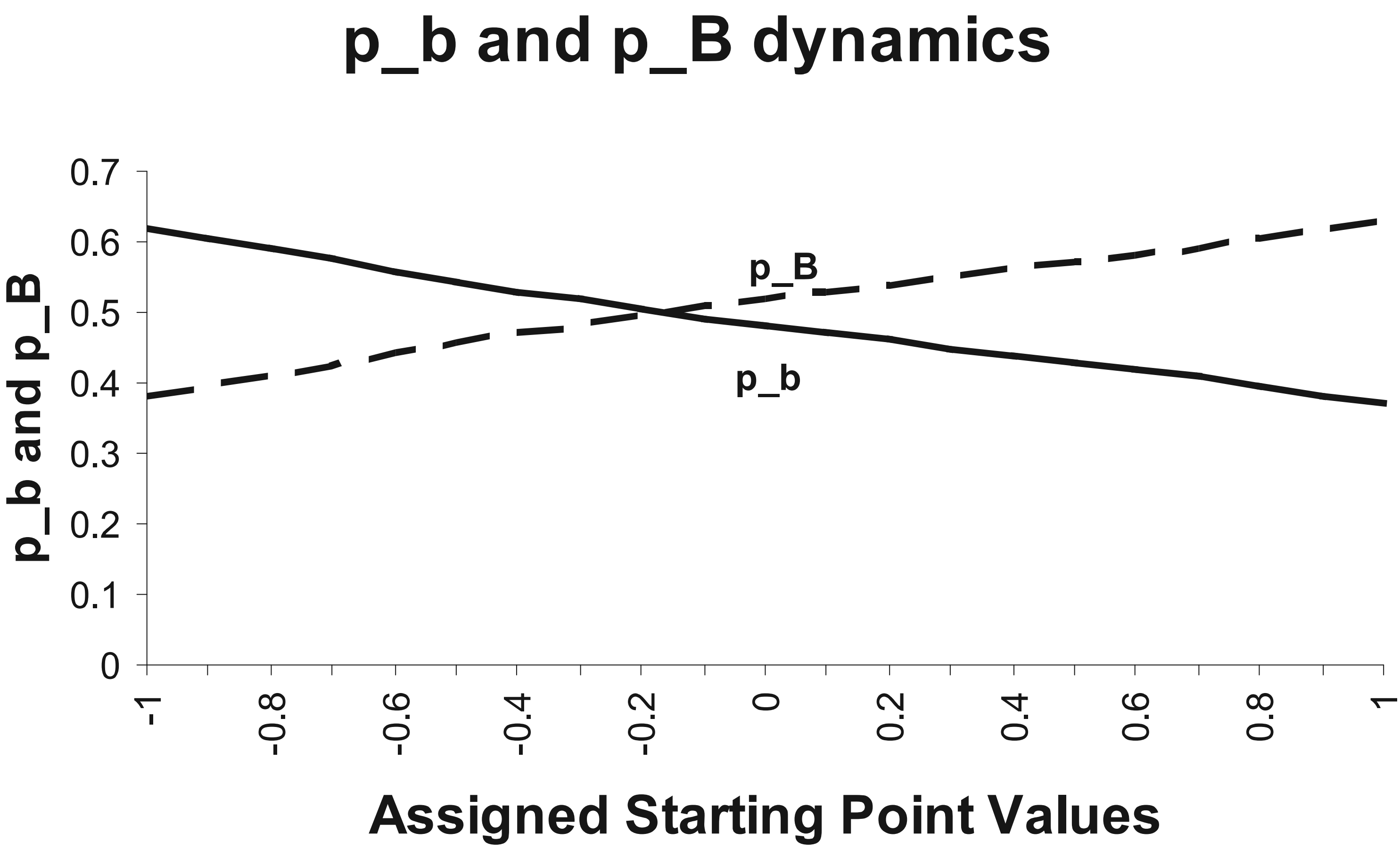
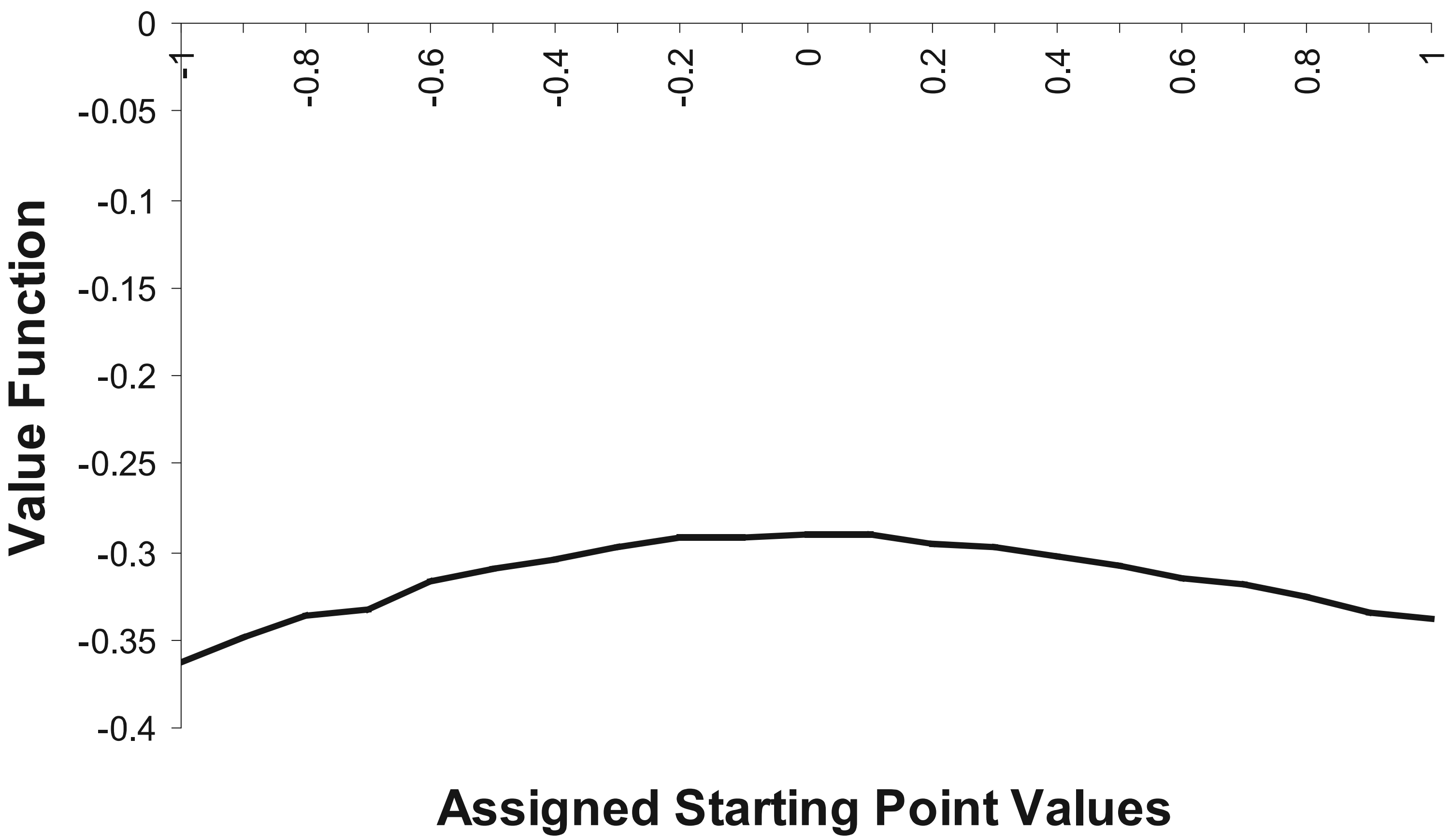

Tau_s
Assigned Starting Point Values
0
20
40
60
80
100
120
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Figure 2



p_b and p_B dynamics
p_b and p_B
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
p_B
p_b
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Assigned Starting Point Values

Figure 3



Value Function
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Assigned Starting Point Values