Invariance Pressure for Control Systems

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Abstract Notions of invariance pressure for control systems are introduced based on weights for the control values. The equivalence is shown between inner invariance pressure based on spanning sets of controls and on invariant open covers, respectively. Furthermore, a number of properties of invariance pressure are derived and it is computed for a class of linear systems.

Keywords Invariance pressure · Invariance entropy · Control systems · Invariant covers · Feedbacks

1 Introduction

This paper extends the notion of invariance entropy for discrete-time and continuous-time control systems to a notion of invariance pressure and discusses some of its properties. Invariance entropy (and feedback invariance entropy) indicates the amount of "information" on controls necessary in order to make a subset of the state space invariant, and is closely related to minimal data rates. Basic references are the seminal paper Nair, Evans, Mareels and Moran [9] and the monograph Kawan [7]. Further studies of invariance entropy include Da Silva and Kawan [4] for hyperbolic control sets, Da Silva [3] for linear control systems on Lie groups and Colonius, Fukuoka and Santana [1] for topological semigroups. Huang and Zhong [5] present a version of invariance entropy as a dimension-like invariant in the sense of Carathéodory structures (cf. Pesin and Pitskel [11]). A problem closely related

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to controlled invariance occurs for observability, where, instead of the uncertainty of the controls, uncertainty of the state of the system is considered; cf., e.g., Savkin [13], Pogromsky and Matveev [12] and Liberzon and Mitra [8]; the latter reference introduces the notion of estimation entropy in this context. This illustrates that invariance entropy is part of the vast field of control over communication channels with data-rate constraints, cf. Nair, Fagnani, Zampieri and Evans [10] for a general survey.

Invariance entropy is modeled with some analogy to topological entropy of dynamical systems. A generalization of the latter notion is topological pressure of dynamical systems where a potential function gives weights to the points in the state space, cf., e.g., Walters [15], Viana and Oliveira [14] or Katok and Hasselblatt [6]. We will construct a notion of invariance pressure that is analogously based on weights for the control values.

The main result is the equivalence between the inner invariance pressure based on spanning sets of controls and feedback invariance pressure based on invariant open covers (see Theorem 11). Furthermore, a number of properties of invariance pressure are derived which are analogous to properties of topological pressure for dynamical systems. Here, however, no full analogy should be expected, since no notion of separated sets of controls is available. While inner invariance pressure, as discussed in detail here, is a generalization of inner invariance entropy, we indicate how also other notions of invariance entropy, in particular, outer invariance entropy, can be generalized. Furthermore, some properties of invariance entropy for continuous-time control systems are also derived and the invariance pressure for a class of linear systems is computed.

The contents of this paper is as follows. Section 2 constructs invariance pressure based on spanning sets of controls and on invariant open covers and shows that they are equivalent. Section 3 proves several properties of inner invariance pressure and indicates variants based on different technical conditions. Finally, Sect. 4 analyzes invariance pressure for continuous-time control systems and computes the invariance pressure for a class of linear systems.

2 Invariance Pressure for Discrete-Time Systems

In this section we introduce the notion of invariance pressure for discrete-time control systems. Then a feedback version is defined and it is shown that these two notions are equivalent.

The considered discrete-time control systems have the form

$$x_{k+1} = F(x_k, u_k), k \in \mathbb{N}_0 = \{0, 1, \ldots\}, \tag{1}$$

where $F : X \times U \to X$ and (X, d) is a metric space and U is a topological space. We assume that $F_u := F(\cdot, u)$ is continuous for every $u \in U$. Define $\mathcal{U} := U^{\mathbb{N}_0}$ as the set of all sequences $\omega = (u_k)_{k\in\mathbb{N}_0}$ of elements in the control range U. We endow the set $\mathcal{U}$ of control sequences with the product topology. Sometimes, we will assume that the set of control values U is a compact metric space implying that also $\mathcal{U}$ is compact metrizable. The shift θ on $\mathcal{U}$ is defined by $(\theta\omega)_k = u_{k+1}$, $k \in \mathbb{N}_0$. For $x_0 \in X$ and $\omega \in \mathcal{U}$ the corresponding solution of (1) will be denoted by

$$x_k = \varphi(k, x_0, \omega), k \in \mathbb{N}_0.$$

where convenient, we also write $\varphi_{k,\omega}(\cdot) := \varphi(k, \cdot, \omega)$. By induction, one sees that this map is continuous. Observe also that this is a cocycle associated with the dynamical system on $\mathcal{U} \times X$ given by

$$\Phi(k, \omega, x_0) = (\theta^k\omega, \varphi(k, x_0, \omega)), k \in \mathbb{N}_0, \omega \in \mathcal{U}, x_0 \in X.$$

We note the following property which is of independent interest (it is not used in the following).

Proposition 1 *The shift θ is continuous and, if $F : X \times U \to X$ is continuous, then Φ is a continuous dynamical system.*

Proof Continuity of θ follows since the sets of the form

$$W = W_0 \times W_1 \times \cdots \times W_N \times U \times \cdots \subset U^{\mathbb{N}_0}$$

with $W_i \subset U$ open for all i and $N \in \mathbb{N}$ form a subbasis of the product topology and the preimages

$$\theta^{-1} W = U \times W_0 \times W_1 \times \cdots \times W_N \times U \times \cdots$$

are open. If F is continuous, then induction shows that $\varphi(k, x_0, \omega)$ is continuous in $(x_0, \omega) \in X \times \mathcal{U}$ for all k. □

Throughout the text, we will consider a compact set $Q \subset X$ and denote by $C(U, \mathbb{R})$ the set of all continuous function $f : U \to \mathbb{R}$. We suppose that the set Q is strongly invariant in the sense that for all $x \in Q$ there is $u \in U$ with $F(x, u) \in \operatorname{int} Q$. Clearly, this means that for all $x \in Q$ there is $\omega \in \mathcal{U}$ with $\varphi(k, x, \omega) \in \operatorname{int} Q$ for all $k \geq 1$. We are interested in the minimal information to make Q strongly invariant.

Remark 2 At the end of Sect. 3 we will comment on possibilities to relax the property of strong invariance.

2.1 Inner Invariance Pressure

The definition of inner invariance pressure will require the following notion from Kawan [7, p. 76].

Definition 3 Let $Q \subset X$ a compact set with nonempty interior and $n \in \mathbb{N}$. We say that a subset $\mathcal{S} \subset \mathcal{U}$ is a strongly (n, Q) ***-spanning set*** if for each $x \in Q$ there is $\omega \in \mathcal{S}$ such that $\varphi(i, x, \omega) \in \operatorname{int} Q$ for $i = 1, \ldots, n$.

The minimal cardinality of such a set is denoted by $r_{inv,int}(n, Q) \leq \infty$, and [7, p. 76] defines the ***inner invariance entropy*** of Q by

$$h_{inv,int}(Q) = \lim_{n \to \infty} \frac{1}{n} \log r_{inv,int}(n, Q).$$

In order to construct the inner invariance pressure of control systems let for $f \in C(U, \mathbb{R})$ and $n \in \mathbb{N}$

$$(S_n f)(\omega) := \sum_{i=0}^{n-1} f(u_i), \quad \omega = (u_i)_{i \in \mathbb{N}_0} \in \mathcal{U},$$

and

$$a_n(f, Q) := \inf \left\{ \sum_{\omega \in \mathcal{S}} e^{(S_n f)(\omega)};\ \mathcal{S} \text{ strongly } (n, Q)\text{-spanning} \right\}.$$

Definition 4 For a discrete-time control system of the form (1), a strongly invariant compact set $Q \subset X$ and $f \in C(U, \mathbb{R})$ consider

$$P_{int}(f, Q) = \lim_{n\to\infty} \frac{1}{n} \log a_n(f, Q). \tag{2}$$

The ***inner invariance pressure*** in Q is the map $P_{int}(\cdot, Q) : C(U, \mathbb{R}) \to \mathbb{R} \cup \{-\infty, \infty\}$.

This definition deserves several comments. First observe that $P_{int}(f, Q) \geq 0$ for $f \geq 0$. If $f = \mathbf{0}$ is the null function in $C(U, \mathbb{R})$, then $\sum_{\omega\in\mathcal{S}} e^{(S_n\mathbf{0})(\omega)} = \sum_{\omega\in\mathcal{S}} 1 = \#\mathcal{S}$, hence

$$\begin{aligned} a_n(\mathbf{0}, Q) &= \inf\left\{\sum_{\omega\in\mathcal{S}} e^{(S_n\mathbf{0})(\omega)};\ \mathcal{S} \text{ strongly } (n, Q)\text{-spanning}\right\} \\ &= \inf\{\#\mathcal{S};\ \mathcal{S} \text{ strongly } (n, Q)\text{-spanning}\} \\ &= r_{inv,int}(n, Q). \end{aligned} \tag{3}$$

Taking the logarithm, dividing by n and letting n tend to ∞ one finds that $P_{int}(\mathbf{0}, Q) = h_{inv,int}(Q)$. Hence the inner invariance pressure generalizes the inner invariance entropy.

Next we show that it is sufficient to consider finite spanning sets. More precisely, the following holds.

Proposition 5 *For a strongly invariant compact set Q and $f \in C(U, \mathbb{R})$ it suffices to taken in the definition of $a_n(f, Q)$ the infimum over all finite strongly (n, Q)-spanning sets.*

Proof First we show for a strongly (n, Q)-spanning set $\mathcal{S}$ there exists a finite strongly (n, Q)-spanning set $\mathcal{S}' \subset \mathcal{S}$. In fact, take an arbitrary $x \in Q$. Since $\mathcal{S}$ is strongly (n, Q)-spanning, there is $\omega_x \in \mathcal{S}$ with $y_j := \varphi(j, x, \omega_x) \in \operatorname{int}Q$ for $j = 1, \ldots, n$. By continuity, we find open neighborhoods $W_1, \ldots, W_n$ of x such that $\varphi(j, W_j, \omega_x) \subset \operatorname{int}Q$ for all $j = 1, \ldots, n$. The sets $W_x = \bigcap_{i=1}^n W_i$, $x \in Q$, form an open cover of Q. By compactness of Q there are finitely $x_1, \ldots, x_k \in Q$ such that $Q \subset \bigcup_{i=1}^k W_{x_i}$. Then $\mathcal{S}' = \{\omega_{x_1}, \ldots, \omega_{x_k}\} \subset \mathcal{S}$ is strongly (n, Q)-spanning.

To conclude the proof, set

$$\widetilde{a}_n(f, Q) = \inf\left\{\sum_{\omega\in\mathcal{S}} e^{(S_n f)(\omega)};\ \mathcal{S} \text{ is a finite strongly } (n, Q)\text{-spanning set}\right\}.$$

It is clear that $a_n(f, Q) \leq \widetilde{a}_n(f, Q)$. For the reverse inequality, let $\mathcal{S}$ be strongly (n, Q)-spanning. Then, as shown above, there is a finite strongly (n, Q)-spanning subset $\mathcal{S}' \subset \mathcal{S}$. Hence

$$\sum_{\omega\in\mathcal{S}'} e^{(S_n f)(\omega)} \leq \sum_{\omega\in\mathcal{S}} e^{(S_n f)(\omega)},$$

implying that $\widetilde{a}_n(f, Q) \leq a_n(f, Q)$ and then equality is proved. □

Based on this result, in the following we will only consider finite spanning sets. We still have to show that the limit in (2) actually exists.

Proposition 6 *For $f \in C(U, \mathbb{R})$, the following limit exists and satisfies*

$$\lim_{n\to\infty} \frac{1}{n} \log a_n(f, Q) = \inf_{n\geq 1} \frac{1}{n} \log a_n(f, Q).$$

Proof This follows by a standard lemma in this context (cf., e.g., Walters [15, Theorem 4.9] or Kawan [7, Lemma B.3]), if we can show that the sequence $\log a_n(f, Q)$, $n \in \mathbb{N}$, is subadditive. Let $\mathcal{S}_1$ be a strongly (n, Q)-spanning set and $\mathcal{S}_2$ a strongly (k, Q)-spanning set. Then define control sequences of length $n + k$ by

$$\omega := (u_0, \ldots, u_{n-1}, v_0, \ldots, v_{k-1}) \in U^{n+k}.$$

for each $\omega_1 = (u_0, \ldots, u_{n-1}) \in \mathcal{S}_1$ and $\omega_2 = (v_0, \ldots, v_{k-1}) \in \mathcal{S}_2$. We claim that the set $\mathcal{S}$ of these control sequences is strongly $(n + k, Q)$-spanning. In fact, for $x \in Q$ there exist $\omega_1 \in \mathcal{S}_1$ such that

$$\varphi(j, x, \omega) = \varphi(j, x, \omega_1) \in \operatorname{int} Q,\ j = 1, \ldots, n.$$

Since $\varphi(n, x, \omega_1) \in \operatorname{int} Q \subset Q$ and $\mathcal{S}_2$ is strongly (k, Q)-spanning, there is a $\omega_2 \in \mathcal{S}_2$ such that

$$\varphi(n + j, x, \omega) = \varphi(j, \varphi(n, x, \omega_1), \omega_2) \in \operatorname{int} Q,\ j = 1, \ldots, k.$$

This shows the claim. Furthermore, for all $\mathcal{S}_1$ and $\mathcal{S}_2$

$$\sum_{\omega \in \mathcal{S}} e^{(S_{n+k} f)(\omega)} = \sum_{\omega \in \mathcal{S}} e^{(S_n f)(\omega_1)} e^{(S_k f)(\omega_2)} \leq \sum_{\omega_1 \in \mathcal{S}_1} e^{(S_n f)(\omega_1)} \sum_{\omega_2 \in \mathcal{S}_2} e^{(S_k f)(\omega_2)}.$$

Hence $a_{n+k}(f, Q) \leq a_n(f, Q) a_k(f, Q)$ and the subadditivity property follows proving the assertion. □

The following example illustrates the definition of invariance pressure in a simple case.

Example 7 Assume that $f \in C(U, \mathbb{R})$ is bounded below (which, naturally, holds, if U is compact) and that $F(Q, U) \subset \operatorname{int} Q$, that is, the system always enters the interior of Q when starting in Q. We show that $P_{int}(f, Q) = \inf f$. Since for every strongly (n, Q)-spanning set $\mathcal{S}$ the estimate

$$\sum_{\omega \in \mathcal{S}} e^{(S_n f)(\omega)} \geq e^{n \inf f} \cdot \#\mathcal{S} \geq e^{n \inf f}$$

holds, it follows that $P_{int}(f, Q) \geq \inf f$. Conversely, our assumption implies that for $\varepsilon > 0$ there exists $u \in U$ with

$$f(u) \leq \inf f + \varepsilon.$$

Then the one-point set $\mathcal{S} = \{\omega\}$, where $\omega = (u, u, \ldots)$, is strongly (n, Q)-spanning and

$$\sum_{\omega \in \mathcal{S}} e^{(S_n f)(\omega)} = e^{(S_n f)(\omega)} = e^{n f(u)} \leq e^{n \inf f + n\varepsilon}.$$

Taking the infimum over all strongly (n, Q)-spanning sets one finds that the invariance pressure satisfies

$$P_{int}(f, Q) = \lim_{n \to \infty} \frac{1}{n} \log a_n(f, Q) \leq \lim_{n \to \infty} \frac{1}{n} \log e^{n \inf f} + \varepsilon = \inf f + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $P_{int}(f, Q) \leq \inf f$.

2.2 Invariance Feedback Pressure

Next we introduce a notion of invariance pressure based on feedbacks and show that it coincides with the invariance pressure defined above.

Open covers in entropy theory of dynamical systems are replaced in case of control systems by invariant open covers, introduced in Nair et al. [9]. For control systems of the form (1) they have the following form.

Definition 8 For a compact subset $Q \subset X$ an invariant open cover $\mathcal{C} = (\mathcal{A}, \tau, G)$ is given by $\tau \in \mathbb{N}$, a finite open cover $\mathcal{A}$ of Q and a map $G : \mathcal{A} \to U^{\tau}$ assigning to each set A in $\mathcal{A}$ a control function such that $\varphi(k, A, G(A)) \subset Q$ for all $k \in \{1, \ldots, \tau\}$.

Here $G(A)$ may be considered as a feedback when applied to the elements of A. Let $\mathcal{C} = (\mathcal{A}, \tau, G)$ be an invariant open cover. For any sequence $\alpha = (A_i)_{i \in \mathbb{N}_0} \in \mathcal{A}^{\mathbb{N}_0}$, we have the control sequence

$$\omega(\alpha) := (u_0, u_1, \ldots) \quad \text{with } (u_l)_{l=(i-1)\tau}^{i\tau-1} = G(A_{i-1}), \text{ for all } i \geq 1,$$

that is,

$$\omega(\alpha) = (\underbrace{u_0, \ldots, u_{\tau-1}}_{G(A_0)}, \underbrace{u_{\tau}, \ldots, u_{2\tau-1}}_{G(A_1)}, \ldots).$$

Then we can define, for each $n \in \mathbb{N}$, the set

$$B_n(\alpha) := \{x \in X;\ \varphi(i\tau, x, \omega(\alpha)) \in A_i \text{ for } i = 0, 1, \ldots, n-1\}. \tag{4}$$

Observe that $B_n(\alpha)$ is open in Q and that the control $\omega(\alpha)$ is uniquely determined by α, but not necessarily by the set $B_n(\alpha)$. For each $n \in \mathbb{N}$, letting α run through all sequences of elements in $\mathcal{A}$, the family

$$\mathcal{B}_n = \mathcal{B}_n(\mathcal{C}) := \{B_n(\alpha);\ \alpha \in \mathcal{A}^{\mathbb{N}_0}\}$$

is a finite open cover of Q. Here, and in the following, it is used tacitly that only the first n elements of α are relevant.

We say that a set of controls of the form

$$\mathcal{W}_n = \{\omega(\alpha_i); \alpha_i \in \mathcal{A}^{\mathbb{N}_0} \text{ for } i \in I\}$$

is a generating set of feedback controls (of length $n\tau$) for the invariant open cover $\mathcal{C}$, if the sets $B_n(\alpha_i)$, $i \in I$, form a subcover of $\mathcal{B}_n(\mathcal{C})$ which is minimal in the sense that none of its elements may be omitted in order to cover Q. (Its number of elements needs not be minimal among all subcovers.) Hence $Q = \bigcup_{i \in I} B_n(\alpha_i)$ and the number of elements $\#I$ in the index set I is bounded by $\#\mathcal{B}_n$.

Define for $\omega = (u_i)_{i \in \mathbb{N}_0} \in \mathcal{U}$

$$(S_{n\tau})(\omega) = \sum_{i=0}^{n\tau-1} f(u_i),$$

and set

$$q_n(f, Q, \mathcal{C}) = \inf \left\{ \sum_{\omega \in \mathcal{W}_n} e^{(S_{n\tau} f)(\omega)};\ \mathcal{W}_n \text{ generating for } \mathcal{C} \right\}.$$

Definition 9 Consider a discrete-time control system of the form (1), a strongly invariant compact set $Q \subset X$ and $f \in C(U, \mathbb{R})$. For an invariant open cover $\mathcal{C} = (\mathcal{A}, \tau, G)$, put

$$P_{fb}(f, Q, \mathcal{C}) = \lim_{n\to\infty} \frac{1}{n\tau} \log q_n(f, Q, \mathcal{C}) \tag{5}$$

and

$$P_{fb}(f, Q) = \inf\{P_{fb}(f, Q, \mathcal{C});\ \mathcal{C} \text{ is an invariant open cover of } Q\}.$$

The ***invariance feedback pressure*** is the map $P_{fb}(\cdot, Q) : C(U, \mathbb{R}) \to \mathbb{R} \cup \{-\infty, \infty\}$.

Here are several comments on this definition. If $f = \mathbf{0}$ is the null function in $C(U, \mathbb{R})$, then

$$\sum_{\omega\in\mathcal{W}_n} e^{(S_n\mathbf{0})(\omega)} = \sum_{\omega\in\mathcal{W}_n} 1 = \#\mathcal{W}_n,$$

hence

$$\begin{aligned} q_n(\mathbf{0}, Q, \mathcal{C}) &= \inf\left\{\sum_{\omega\in\mathcal{W}_n} e^{(S_{n\tau}0)(\omega)};\ \mathcal{W}_n \text{ generating for } \mathcal{C}\right\} \\ &= \inf\{\#\mathcal{B};\ \mathcal{B} \text{ a subcover of } \mathcal{B}_n\} = N(\mathcal{B}_n;\ Q), \end{aligned}$$

where $N(\mathcal{B}_n;\ Q)$ denotes the minimal number of elements in a subcover of $\mathcal{B}_n$.

Hence one finds that the strong topological feedback entropy $h_{fb}(\mathcal{C})$ of $\mathcal{C}$ (as defined in Kawan [7, p. 70]) is

$$h_{fb}(\mathcal{C}) := \lim_{n\to\infty} \frac{1}{n\tau} \log N(\mathcal{B}_n;\ Q) = \limsup_{n\to\infty} \frac{1}{n\tau} \log q_n(\mathbf{0}, Q, \mathcal{C}) = P_{fb}(\mathbf{0}, \mathcal{C}),$$

and so the strong topological feedback entropy of system (1) satisfies

$$\begin{aligned} h_{fb}(Q) &:= \inf\{h_{fb}(\mathcal{C});\ \ \mathcal{C} \text{ an invariant open cover of } Q\} \\ &= \inf\{P_{fb}(\mathbf{0}, \mathcal{C});\ \mathcal{C} \text{ an invariant open cover of } Q\} = P_{fb}(\mathbf{0}, Q). \end{aligned}$$

Hence the invariance feedback pressure is a generalization of the strong topological feedback entropy.

The following lemma provides the remaining proof that the limit in (5) actually exists.

Lemma 10 *If $f \in C(U, \mathbb{R})$ and $\mathcal{C} = (\mathcal{A}, \tau, G)$ is an invariant open cover of Q, then the following limit exists and satisfies*

$$\lim_{n\to\infty} \frac{1}{n} \log q_n(f, Q, \mathcal{C}) = \inf_{n\geq 1} \frac{1}{n} \log q_n(f, Q, \mathcal{C}).$$

Proof The assertions will follow from Walters [15, Theorem 4.9] if the sequence $\log q_n(f, Q, \mathcal{C}), n \in \mathbb{N}$, is subadditive. This will be shown by constructing a generating set $\mathcal{W}_{n+k}$ from generating sets $\mathcal{W}_n$ and $\mathcal{W}_k$ with the desired properties.

Let $\mathcal{W}_n = \{\omega(\alpha_{i_1}), \ldots, \omega(\alpha_{i_M})\}$ and $\mathcal{W}_k = \{\omega(\beta_{i_1}), \ldots, \omega(\beta_{i_K})\}$ be generating sets of feedback controls. Here α_i and β_j are given by sequences of sets in $\mathcal{A}$ in the form $\alpha_i = \left(A_\sigma^{\alpha_i}\right)_\sigma$ and $\beta_j = \left(A_\sigma^{\beta_i}\right)_\sigma$. Then define for all i and j sequences in $\mathcal{A}$ by

$$\alpha_i\beta_j = \left(A_0^{\alpha_i}, \ldots, A_{n-1}^{\alpha_i}, A_0^{\beta_j}, \ldots, A_{k-1}^{\beta_j}, \ldots\right).$$

If we denote by $A_\sigma^{\alpha_i\beta_j}$ the σth element of $\alpha_i\beta_j$, then

$$A_\sigma^{\alpha_i\beta_j} = \begin{cases} A_\sigma^{\alpha_i}, \text{ if } 0 \le \sigma \le n-1 \\ A_{\sigma-n}^{\beta_j}, \text{ if } \sigma \ge n. \end{cases}$$

Claim: The set

$$\left\{\omega(\alpha_i\beta_j);\, i \in \{i_1,\dots,i_M\},\, j \in \{j_1,\dots,j_K\}\right\} \tag{6}$$

contains a generating set of feedback controls.

First note that by the cocycle property one finds for $\sigma = 0,\dots,k$

$$\varphi_{(\sigma+n)\tau,\omega(\alpha_i\beta_j)} = \varphi_{\sigma\tau,(\theta^{n\tau}\omega(\alpha_i\beta_j))} \circ \varphi_{n\tau,\omega(\alpha_i\beta_j)} = \varphi_{\sigma\tau,\omega(\beta_j)} \circ \varphi_{n\tau,\omega(\alpha_i)},$$

and hence

$$\varphi^{-1}_{(\sigma+n)\tau,\omega(\alpha_i\beta_j)} = \varphi^{-1}_{n\tau,\omega(\alpha_i)} \circ \varphi^{-1}_{\sigma\tau,\omega(\beta_j)}.$$

Thus for all i and j

$$B_{n+k}(\alpha_i\beta_j) = B_n(\alpha_i) \cap \varphi^{-1}_{n\tau,\omega(\alpha_i\beta_j)} B_k(\beta_j). \tag{7}$$

In fact,

$$\begin{aligned} B_{n+k}(\alpha_i\beta_j) &= \bigcap_{\sigma=0}^{n+k-1} \varphi^{-1}_{\sigma\tau,\omega(\alpha_i\beta_j)}(A_\sigma^{\alpha_i\beta_j}) \\ &= \bigcap_{\sigma=0}^{n-1} \varphi^{-1}_{\sigma\tau,\omega(\alpha_i\beta_j)}(A_\sigma^{\alpha_i\beta_j}) \cap \varphi^{-1}_{n\tau,\omega(\alpha_i)}\left[\bigcap_{\sigma=0}^{k-1} \varphi^{-1}_{\sigma\tau,\omega(\beta_j)}(A_{\sigma+n}^{\alpha_i\beta_j})\right] \\ &= \bigcap_{\sigma=0}^{n-1} \varphi^{-1}_{\sigma\tau,\omega(\alpha_i)}(A_\sigma^{\alpha_i}) \cap \varphi^{-1}_{n\tau,\omega(\alpha_i)}\left[\bigcap_{\sigma=0}^{k-1} \varphi^{-1}_{\sigma\tau,\omega(\beta_j)}(A_\sigma^{\beta_j})\right] \\ &= B_n(\alpha_i) \cap \varphi^{-1}_{n\tau,\omega(\alpha_i)} B_k(\beta_j). \end{aligned}$$

Clearly the sets $B_{n+k}(\alpha_i\beta_j)$ are elements of $\mathcal{B}_{n+k}(\mathcal{C})$. It follows from (7) that they cover Q, since this is valid for the families $\{B_n(\alpha_i);\, i \in \{i_1,\dots,i_M\}\}$ and $\left\{B_n(\beta_j);\, j \in \{j_1,\dots,j_K\}\right\}$. Hence the collection in (6) is a subcover of $\mathcal{B}_{n+k}(\mathcal{C})$ and one finds in the family (6) an associated generating set of feedback controls which we denote by $\mathcal{W}_{n+k}$. Thus the **Claim** is proved.

In order to show subadditivity of the sequence $\log q_n(f, Q, \mathcal{C})$, $n \in \mathbb{N}$, note that for all $n, k \in \mathbb{N}$

$$\begin{aligned} \sum_{\omega\in\mathcal{W}_{n+k}} e^{(S_{(n+k)\tau} f)(\omega)} &= \sum_{\omega\in\mathcal{W}_{n+k}} e^{(S_{n\tau} f)(\omega)} e^{(S_{k\tau} f)(\theta^{n\tau}\omega)} \\ &\le \sum_{\omega\in\mathcal{W}_n} e^{(S_{n\tau} f)(\omega)} \sum_{\omega\in\mathcal{W}_k} e^{(S_{k\tau} f)(\omega)}. \end{aligned}$$

Since $\mathcal{W}_n$ and $\mathcal{W}_k$ are arbitrary it follows that $q_{n+k}(f, Q, \mathcal{C}) \le q_n(f, Q, \mathcal{C}) \cdot q_k(f, Q, \mathcal{C})$. This implies the required subadditivity concluding the proof. □

Next we show that this feedback invariance pressure coincides with the inner invariance pressure introduced in Definition 4. This generalizes a result for invariance entropy from Colonius, Kawan and Nair [2].

Theorem 11 *Let Q be a strongly invariant compact subset of X. Then for every $f \in C(U, \mathbb{R})$*

$$P_{int}(f, Q) = P_{fb}(f, Q).$$

Proof First we prove the inequality $P_{int}(f, Q) \leq P_{fb}(f, Q)$. Let $\mathcal{C} = (\mathcal{A}, \tau, G)$ be an invariant open cover. Then for $n \in \mathbb{N}$, every generating set $\mathcal{W}_n$ of controls for $\mathcal{C}$ is a strongly $(n\tau, Q)$-spanning set and hence

$$a_{n\tau}(f, Q) = \inf_{\mathcal{S}} \sum_{\omega \in \mathcal{S}} e^{(S_{n\tau} f)(\omega)} \leq \sum_{\omega \in \mathcal{W}_n} e^{(S_{n\tau} f)(\omega)},$$

where the infimum is taken over all strongly $(n\tau, Q)$-spanning set $\mathcal{S}$. It follows that $a_{n\tau}(f, Q) \leq q_n(f, Q, \mathcal{C})$ and therefore

$$P_{int}(f, Q) = \lim_{n\to\infty} \frac{1}{n\tau} \log a_{n\tau}(f, Q) \leq \lim_{n\to\infty} \frac{1}{n\tau} \log q_n(f, Q, \mathcal{C}) = P_{fb}(f, Q, \mathcal{C}).$$

Since this holds for every invariant open cover $\mathcal{C}$, we conclude

$$P_{int}(f, Q) \leq \inf_{\mathcal{C}} P_{fb}(f, Q, \mathcal{C}) = P_{fb}(f, Q),$$

where the infimum is taken over all invariant open covers $\mathcal{C}$ of Q.

To show that $P_{fb}(f, Q) \leq P_{int}(f, Q)$ consider a strongly (τ, Q)-spanning set $\mathcal{S}$ with $\tau \in \mathbb{N}$. We will construct an invariant open cover. For each $\omega \in \mathcal{S}$ define

$$A(\omega) = \{x \in Q;\ \varphi(j, x, \omega) \in \operatorname{int} Q \text{ for } j = 1, \ldots, \tau\}.$$

The set $\mathcal{A} = \{A(\omega);\ \omega \in \mathcal{S}\}$ forms a finite open cover of Q. Now define a map $G : \mathcal{A} \to U^\tau$ by

$$G(A(\omega)) = (\omega_0, \ldots, \omega_{\tau-1}).$$

Clearly, $\mathcal{C} := (\mathcal{A}, \tau, G)$ is an invariant open cover of Q.

Recall that $\alpha \in \mathcal{A}^{\mathbb{N}_0}$ defines a control $\omega(\alpha)$ and for $n \in \mathbb{N}$ the set $B_n(\alpha)$ is given by (4),

$$B_n(\alpha) = \{x \in X;\ \varphi(i\tau, x, \omega(\alpha)) \in A_i \text{ for } i = 0, 1, \ldots, n-1\}.$$

These sets form on open cover $\mathcal{B}_n = \mathcal{B}_n(\mathcal{C})$ of Q. Consider a generating set of feedback controls of the form

$$\mathcal{W}_n = \{\omega(\alpha_i); \alpha_i \in \mathcal{A}^{\mathbb{N}_0} \text{ for } i \in I\},$$

hence the sets $B_n(\alpha_i)$, $i \in I$, form a subcover of $\mathcal{B}_n(\mathcal{C})$ which is minimal. Therefore

$$\begin{aligned}
\sum_{\omega \in \mathcal{W}_n} e^{(S_{n\tau} f)(\omega)} &= \sum_{\omega \in \mathcal{W}_n} e^{(S_\tau f)(\omega)} e^{(S_\tau f)(\theta^\tau \omega)} \cdots e^{(S_\tau f)(\theta^{(n-1)\tau} \omega)} \\
&\leq \left(\sum_{\omega \in \mathcal{B}_n} e^{(S_\tau f)(\omega)}\right) \left(\sum_{\omega \in \mathcal{B}_n} e^{(S_\tau f)(\theta^\tau \omega)}\right) \cdots \left(\sum_{\omega \in \mathcal{B}_n} e^{(S_\tau f)(\theta^{(n-1)\tau} \omega)}\right) \\
&\leq \left(\sum_{\omega \in \mathcal{S}} e^{(S_\tau f)(\omega)}\right)^n .
\end{aligned}$$

Since the previous inequality holds for all finite strongly (τ, Q)-spanning sets $\mathcal{S}$, it follows that $q_n(f, Q, \mathcal{C}) \leq [a_\tau(f, Q)]^n$ for all $n \in \mathbb{N}$. Hence

$$P_{fb}(f, Q, \mathcal{C}) = \lim_{n\to\infty} \frac{1}{n\tau} \log q_n(f, Q, \mathcal{C}) \leq \lim_{n\to\infty} \frac{1}{n\tau} \log [a_\tau(f, Q)]^n = \frac{1}{\tau} \log a_\tau(f, Q).$$

Using Proposition 6 we conclude that

$$P_{fb}(f, Q) = \inf_{\mathcal{C}} P_{fb}(f, Q, \mathcal{C}) \le \inf_{\tau \in \mathbb{N}} \frac{1}{\tau} \log a_\tau(f, Q) = P_{int}(f, Q).$$

□

3 Properties of the Invariance Pressure

In this section, we collect several properties of invariance pressure which are analogous to properties of topological pressure for dynamical systems. Furthermore, we discuss some alternative versions of invariance pressure.

We start with the following elementary lemma which will be used in the proof of Proposition 13.

Lemma 12 *Let $a_i \ge 0, b_i > 0, i = 1, \dots, n \in \mathbb{N}$, be real numbers. Then*

$$\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i} \ge \min_{i=1,\dots,n} \left(\frac{a_i}{b_i} \right).$$

Proof Let $n = 2$. Then we may assume that $\frac{a_1}{b_1} \le \frac{a_2}{b_2}$. Dividing numerator and denominator by b_1 one can further assume that $b_1 = 1$, hence the assumption takes the form $a_1 \le \frac{a_2}{b_2}$ and the assertion reduces to $\frac{a_1+a_2}{1+b_2} \ge a_1$. This is equivalent to

$$a_1 + a_2 \ge a_1 + a_1 b_2, \text{ i.e., } a_2 \ge a_1 b_2,$$

which is our assumption. The induction step from n to $n + 1$ follows since

$$\frac{\sum_{i=1}^{n+1} a_i}{\sum_{i=1}^{n+1} b_i} = \frac{\sum_{i=1}^n a_i + a_{n+1}}{\sum_{i=1}^n b_i + b_{n+1}} \ge \min \left(\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i}, \frac{a_{n+1}}{b_{n+1}} \right) \ge \min_{i=1,\dots,n+1} \left(\frac{a_i}{b_i} \right).$$

□

Proposition 13 *Consider a discrete-time control system of the form (1), let Q be a compact strongly invariant subset of X and let $f, g \in C(U, \mathbb{R})$ and $c \in \mathbb{R}$. Then the following assertions hold:*

(i) *If $f \le g$, then $P_{int}(f, Q) \le P_{int}(g, Q)$.*
(ii) *$P_{int}(f + c, Q) = P_{int}(f, Q) + c$.*
(iii) *If U is compact, then $|P_{int}(f, Q) - P_{int}(g, Q)| \le \|f - g\|_\infty$.*

Proof (i) If $f \le g$, it follows that $\sum_{\omega \in \mathcal{S}} e^{(S_n f)(\omega)} \le \sum_{\omega \in \mathcal{S}} e^{(S_n g)(\omega)}$ for all (n, Q)-spanning sets $\mathcal{S}$, because the exponential function is increasing. Hence $a_n(f, Q) \le a_n(g, Q)$ and so $P_{int}(f, Q) \le P_{int}(g, Q)$.

(ii) One finds that

$$\begin{aligned} a_n(f + c, Q) &= \inf \left\{ \sum_{\omega \in \mathcal{S}} e^{(S_n(f+c))(\omega)};\ \mathcal{S}\ (n, Q)\text{-spanning} \right\} \\ &= \inf \left\{ e^{nc} \sum_{\omega \in \mathcal{S}} e^{(S_n f)(\omega)};\ \mathcal{S}\ (n, Q)\text{-spanning} \right\} \\ &= e^{nc} a_n(f, Q), \end{aligned}$$

hence

$$P_{int}(f+c,Q)=\lim_{n\to\infty}\frac{1}{n}\log a_n(f+c,Q)=\lim_{n\to\infty}\frac{1}{n}\log\left(e^{nc}a_n(f,Q)\right)$$
$$=c+P_{int}(f,Q).$$

(iii) Recall that for $a_n(f,Q)$ and $a_n(g,Q)$ the infimum is taken over all strongly (n,Q)-spanning sets $\mathcal{S}$. Thus, using Lemma 12 for the second inequality below, one finds

$$\frac{a_n(g,Q)}{a_n(f,Q)}=\frac{\inf_{\mathcal{S}}\left\{\sum_{\omega\in\mathcal{S}}e^{(S_ng)(\omega)}\right\}}{\inf_{\mathcal{S}}\left\{\sum_{\omega\in\mathcal{S}}e^{(S_nf)(\omega)}\right\}}\geq\inf_{\mathcal{S}}\left\{\frac{\sum_{\omega\in\mathcal{S}}e^{(S_ng)(\omega)}}{\sum_{\omega\in\mathcal{S}}e^{(S_nf)(\omega)}}\right\}$$
$$\geq\inf_{\mathcal{S}}\left\{\min_{\omega\in\mathcal{S}}\frac{e^{(S_ng)(\omega)}}{e^{(S_nf)(\omega)}}\right\}\geq e^{-n\|f-g\|_\infty}.$$

Therefore $\frac{a_n(f,Q)}{a_n(g,Q)}\leq e^{n\|f-g\|_\infty}$ and so

$$P_{int}(f,Q)-P_{int}(g,Q)=\lim_{n\to\infty}\frac{1}{n}\log\frac{a_n(f,Q)}{a_n(g,Q)}\leq\lim_{n\to\infty}\frac{1}{n}\log e^{n\|f-g\|_\infty}$$
$$=\|f-g\|_\infty.$$

Interchanging the roles of f and g one finds assertion (iii). □

Next we discuss changes in the considered set Q.

Proposition 14 *Let $f\in C(U,\mathbb{R})$ and $Q\subset X$ a compact strongly invariant set. Assume that $Q=\bigcup_{i=1}^N Q_i$ with compact strongly invariant sets $Q_1,\dots,Q_N$. Then*

$$P_{int}(f,Q)\leq\max_{1\leq i\leq N}P_{int}(f,Q_i).$$

Proof For every $i\in\{1,\dots,N\}$, let $\mathcal{S}_i$ a strongly (n,Q_i)-spanning set and define $\mathcal{S}=\bigcup_{i=1}^N\mathcal{S}_i$. Then $\mathcal{S}$ is a strongly (n,Q)-spanning set with

$$\sum_{\omega\in\mathcal{S}}e^{(S_nf)(\omega)}\leq\sum_{i=1}^N\sum_{\omega\in\mathcal{S}_i}e^{(S_nf)(\omega)}.$$

With

$$a_n(f,Q_i)=\inf\left\{\sum_{\omega\in\mathcal{S}_i}e^{(S_nf)(x,\omega)};\ \mathcal{S}_i\text{ strongly }(n,Q_i)\text{-spanning}\right\},$$

we have $a_n(f,Q)\leq\sum_{i=1}^N a_n(f,Q_i)$. Now Kawan [7, Lemma 2.1] implies that

$$P_{int}(f,Q)=\lim_{n\to\infty}\frac{1}{n}\log a_n(f,Q)\leq\limsup_{n\to\infty}\frac{1}{n}\log\sum_{i=1}^N a_n(f,Q_i)$$
$$\leq\max_{1\leq i\leq N}\limsup_{n\to\infty}\frac{1}{n}\log a_n(f,Q_i)=\max_{1\leq i\leq N}P_{int}(f,Q_i).$$

□

Consider two control systems of the form (1) given by

$$x_{k+1} = F_1(x_k, u_k) \text{ and } y_{k+1} = F_2(y_k, v_k) \tag{8}$$

in X_1 and X_2 with corresponding solutions $\varphi_1(n, x, \omega_1)$ and $\varphi_2(n, y, \omega_2)$ and control spaces $\mathcal{U}_1$ and $\mathcal{U}_2$ corresponding to control ranges U_1 and U_2, respectively. Then

$$z_{k+1} = F(z_k, w_k),$$

with $z_k = (x_k, y_k)$, $w_k = (u_k, v_k)$, $F = (F_1, F_2)$, again is a control system of the form (1) in $X_1 \times X_2$ with control space $\mathcal{U}_1 \times \mathcal{U}_2$ and solution $\varphi_1 \times \varphi_2 : \mathbb{N}_0 \times (X_1 \times X_2) \times (\mathcal{U}_1 \times \mathcal{U}_2) \to X_1 \times X_2$,

$$(\varphi_1 \times \varphi_2)(n, z, \omega) = (\varphi_1 \times \varphi_2)(n, (x, y), (\omega_1, \omega_2)) = (\varphi_1(n, x, \omega_1), \varphi_2(n, y, \omega_2)).$$

Proposition 15 *Let $f_i \in C(U_i, \mathbb{R})$ and let $Q_i \subset X_i$ be compact strongly invariant sets for the control systems in (8), $i = 1, 2$. Then*

$$P_{int}(f_1 \times f_2, Q_1 \times Q_2) = P_{int}(f_1, Q_1) + P_{int}(f_2, Q_2),$$

where $f_1 \times f_2 \in C(U_1 \times U_2, \mathbb{R})$ is defined by $(f_1 \times f_2)(u, v) = f_1(u) + f_2(v)$.

Proof Note that $Q_1 \times Q_2 \subset X_1 \times X_2$ is a compact strongly invariant set. Furthermore, if $\mathcal{S}_i$ is a strongly (n, Q_i)-spanning set for Q_i, $i = 1, 2$, then $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \subset \mathcal{U}_1 \times \mathcal{U}_2$ is a strongly $(n, Q_1 \times Q_2)$-spanning set and

$$\begin{aligned}\sum_{\omega \in \mathcal{S}} e^{(S_n(f_1 \times f_2))(\omega)} &= \sum_{(\omega_1, \omega_2) \in \mathcal{S}_1 \times \mathcal{S}_2} e^{(S_n f_1)(\omega_1)} e^{(S_n f_2)(\omega_2)} \\ &= \sum_{\omega_1 \in \mathcal{S}_1} e^{(S_n f_1)(\omega_1)} \sum_{\omega_2 \in \mathcal{S}_2} e^{(S_n f_2)(\omega_2)}.\end{aligned}$$

Since $\mathcal{S}_1$ and $\mathcal{S}_2$ are arbitrary, we obtain

$$a_n(f_1 \times f_2, Q_1 \times Q_2) = a_n(f_1, Q_1) a_n(f_2, Q_2).$$

Therefore

$$\begin{aligned}P_{int}(f_1 \times f_2, Q_1 \times Q_2) &= \lim_{n \to \infty} \frac{1}{n} \log a_n(f_1 \times f_2, Q_1 \times Q_2) \\ &= \lim_{n \to \infty} \frac{1}{n} \log [a_n(f_1, Q_1) a_n(f_2, Q_2)] \\ &= P_{int}(f_1, Q_1) + P_{int}(f_2, Q_2).\end{aligned}$$

□

Next we show that the inner invariance pressure is invariant under appropriate conjugacies. Again, consider two control systems as in (8). A pair of maps (ρ, H) is called a conjugacy if $\rho : X_1 \to X_2$ and $H : U_1 \to U_2$ are homeomorphisms such that

$$\rho(F_1(x, u)) = F_2(\rho(x), H(u)) \text{ for all } x \in X_1, u \in U_1. \tag{9}$$

Note that this induces a map $h : \mathcal{U}_1 \to \mathcal{U}_2$ such that $h(\omega)_i = H(\omega_i)$ for all $i \in \mathbb{N}_0$ and the solutions satisfy

$$\rho(\varphi_1(k, x, \omega)) = \varphi_2(k, \rho(x), h(\omega)) \text{ for all } n \in \mathbb{N}_0. \tag{10}$$

Clearly, conjugacy is an equivalence relation.

Theorem 16 *Using the above notation, assume that* (ρ, H) *is a conjugacy between two systems of the form* (8)*, suppose that* $Q \subset X_1$ *is strongly invariant and let* $f \in C(U_2, \mathbb{R})$*. Then* $\rho(Q)$ *is strongly invariant in* X_2 *and the inner invariance pressure satisfies*

$$P_{int}(f \circ H, Q) = P_{int}(f, \rho(Q)).$$

Proof The set $\rho(Q)$ is compact by continuity of ρ. In order to see that it is strongly invariant, write $y = \rho(x) \in \rho(Q)$ with $x \in Q$. By strong invariance of Q there is $u \in U_1$ with $F_1(x, u) \in \text{int}Q$. Since ρ is an open map, the conjugacy condition implies.

$$F_2(y, H(u)) = F_2(\rho(x), H(u)) = \rho(F_1(x, u)) \in \rho(\text{int}Q) = \text{int}(\rho(Q)).$$

If $\mathcal{S}$ is a strongly (n, Q)-spanning set, then $h(\mathcal{S})$ is a strongly $(n, \rho(Q))$-spanning set: In fact, for $y = \rho(x) \in \rho(Q)$ there is $\omega \in \mathcal{S}$ with $\varphi_1(i, x, \omega) \in \text{int}(Q)$, $i = 1, \ldots, n$, therefore (10) implies

$$\varphi_2(i, y, h(\omega)) = \varphi_2(i, \rho(x), h(\omega)) = \rho(\varphi_1(i, x, \omega)) \in \rho(\text{int}(Q)) = \text{int}(\rho(Q)).$$

The same arguments show that for a strongly $(n, \rho(Q))$-spanning set $\widetilde{\mathcal{S}}$ the set $\mathcal{S} := h^{-1}(\widetilde{\mathcal{S}})$ is strongly (n, Q)-spanning. Note also that $(S_n f)(h(\omega)) = (S_n(f \circ H))(\omega)$. Hence

$$\sum_{h(\omega) \in h(\mathcal{S})} e^{(S_n f)(h(\omega))} = \sum_{\omega \in \mathcal{S}} e^{(S_n f)(h(\omega))} = \sum_{\omega \in \mathcal{S}} e^{(S_n (f \circ H))(\omega)}$$

and it follows that $a_n(f, \rho(Q)) = a_n(f \circ H, Q)$, and $P_{int}(f \circ H, Q) = P_{int}(f, \rho(Q))$, as claimed. □

Next we prove the power rule for inner invariance pressure. Consider a control system of the form (1) with compact strongly invariant set Q. Suppose we take $N \in \mathbb{N}$ steps at once. Then, naturally, the solution $\varphi(N, x, \omega)$ may be in $\text{int}Q$ while there may exist $i \in \{1, \ldots, N-1\}$ with $\varphi(i, x, \omega) \notin Q$. Hence, for a power rule in invariance problems of discrete-time systems one has to exclude this a-priori.

Starting from control system (1) define the following control system. Given $N \in \mathbb{N}$, the control range is $U^N = U \times \cdots \times U$ and the set of corresponding controls is denoted by $\mathcal{U}^N$. Then a bijective relation between the controls in $\mathcal{U}$ and in $\mathcal{U}^N$ is given by

$$i : \mathcal{U} \to \mathcal{U}^N : \omega = (\omega_k) \mapsto (\omega_k^N) := (\omega(Nk), \ldots, \omega(Nk + N - 1)).$$

The solutions will be given by $\varphi^N(0, x, \omega) = x$ and for $k \geq 1$

$$\varphi^N(k, x, i(\omega)) = \varphi(nN, x, \omega).$$

Then, these are the solutions of a control system of the form

$$x_{k+1} = F^{(N)}(x_k, v_k),\ \ v_k \in U^N, \tag{11}$$

and the solutions can be written as

$$\varphi^N(k, x, \omega) = \varphi_{N, \theta^{N(k-1)}(\omega)} \circ \cdots \circ \varphi_{N,\omega}(x).$$

As argued above, in the definition of the strong invariance pressure of system (11) we only consider solutions which remain in Q for all times between the steps of length N.

Proposition 17 *In the above setting we denote by* $P_{inv}^N(f, Q)$ *the inner invariance pressure of* (11)*. Then for every* $f \in C(U, \mathbb{R})$

$$P_{int}^N(g, Q) = N \cdot P_{int}(f, Q),$$

where $g \in C(U^N, \mathbb{R})$ *is given by* $g(\omega_0, \ldots, \omega_{N-1}) := \sum_{i=0}^{N-1} f(\omega_i)$*.*

Proof If $\mathcal{S} \subset \mathcal{U}$ is a strongly (nN, Q)-spanning set for (1), then $\mathcal{S}^N := \{i(\omega);\ \omega \in \mathcal{S}\}$ is a strongly (n, Q)-spanning set for (11). Analogously, if $\mathcal{S}^N$ is a strongly (n, Q)-spanning set for (11), then $i^{-1}(\mathcal{S}^N)$ is a strongly (nN, Q)-spanning set for (1). Therefore

$$\sum_{\omega \in \mathcal{S}^N} e^{(S_n g)(\omega)} = \sum_{\omega \in i^{-1}(\mathcal{S}^N)} e^{(S_{nN} f)(\omega)}.$$

We denote

$$a_n^N(f, Q) := \inf_{\mathcal{S}^N} \left\{ \sum_{\omega \in \mathcal{S}^N} e^{(S_n f)(\omega)} \right\},$$

where the infimum is taken over all the strongly (n, Q)-spanning sets $\mathcal{S}^N$ for (11). Then $a_n^N(g, Q) = a_{nN}(f, Q)$ and so

$$P_{int}^N(g, Q) = \lim_{n\to\infty} \frac{1}{n} \log a_n^N(g, Q) = N \lim_{n\to\infty} \frac{1}{nN} \log a_{nN}(f, Q) = N \cdot P_{int}(f, Q).$$

□

The following simple example illustrates inner invariance pressure. A more elaborate case will be discussed in the next section in the framework of outer invariance pressure for continuous-time systems.

Example 18 Consider a scalar linear system of the form

$$x_{k+1} = \alpha x_k + u_k, u_k \in U := [-1, 1],$$

with $\alpha > 1$ and let $Q := \left[-\frac{1}{\alpha-1} + \varepsilon, \frac{1}{\alpha-1} - \varepsilon\right]$, where $\varepsilon > 0$ is small. Let $f \in C(U, \mathbb{R})$ be given by $f(u) = |u|$, $u \in [-1, 1]$. We claim that

$$P_{int}(f, Q) = h_{inv,int}(Q) = \log \alpha.$$

Concerning the formula for the inner invariance entropy of Q one knows that

$$h_{inv,int}(Q) \le h_{inv,out}(Q) = \log \alpha.$$

The converse inequality follows by the same volume argument as in Colonius, Kawan and Nair [2, Example 3.2].

In order to show $P_{int}(f, Q) \ge \log \alpha$, consider for $n \in \mathbb{N}$ a finite strongly (n, Q)-spanning set $\mathcal{S}$. For $\omega \in \mathcal{S}$ define

$$Q_\omega := \{x \in Q; \varphi(j, x, \omega) \in \operatorname{int} Q \text{ for } j = 1, \dots, n\}.$$

Then $Q = \bigcup_{\omega\in\mathcal{S}} Q_\omega$ and hence the Lebesgue measure λ satisfies $\lambda(Q) \le \sum_{\omega\in\mathcal{S}} \lambda(Q_\omega)$. Furthermore, for $x \in Q_\omega$ we have

$$\varphi(n, x, \omega) = \alpha^n x + \sum_{i=0}^{n-1} \alpha^{n-1-i} u_i \in Q,$$

which implies that $\lambda(Q) \ge \alpha^n \lambda(Q_\omega)$. Thus

$$\lambda(Q) \le \sum_{\omega\in\mathcal{S}} \lambda(Q_\omega) \le \#\mathcal{S} \cdot \max_{\omega\in\mathcal{S}} \lambda(Q_\omega) \le \#\mathcal{S} \cdot \alpha^{-n} \lambda(Q)$$

and hence $\#\mathcal{S} \geq \alpha^n$. Since $f(u) \geq 0$, it follows that

$$a_n(f, Q) = \inf \left\{ \sum_{\omega \in \mathcal{S}} e^{(S_n f)(\omega)};\ \mathcal{S} \text{ strongly } (n, Q)\text{-spanning} \right\} \geq \alpha^n$$

implying

$$P_{int}(f, Q) = \lim_{n \to \infty} \frac{1}{n} \log a_n(f, Q) \geq \log \alpha.$$

In order to prove $P_{int}(f, Q) \leq \log \alpha$, we use that the inner invariance entropy is given by $h_{inv,int}(Q) = \log \alpha$. If a solution with $x_0 \in Q$ and control values $u_i \in U$ satisfies for $k \geq 1$

$$\varphi(k, x, \omega) = \alpha^k x + \sum_{i=0}^{k-1} \alpha^{k-1-i} u_i \in \operatorname{int} Q,$$

then it follows for every $\delta \in (0, 1)$ that $\delta u_i \in \delta U = [-\delta, \delta] \subset [-1, 1] = U$ for all i and

$$\delta\varphi(k, x_0, \omega) = \alpha^k \delta x_0 + \sum_{i=0}^{k-1} \alpha^{k-1-i} \delta u_i \in \operatorname{int}(\delta Q) \subset \operatorname{int}(Q).$$

Hence the solution with initial point $\delta x_0 \in \delta Q$ and control values $\delta u_i \in \delta U$ remains in $\operatorname{int}(\delta Q)$. Observe that $f(\delta u_i) = |\delta u_i| \leq \delta$.

Take $0 < \delta < \frac{1}{\alpha-1} - \varepsilon$. Then for $x_0 \in Q = \left[-\frac{1}{\alpha-1} + \varepsilon, \frac{1}{\alpha-1} - \varepsilon\right]$ there are $n_0 \in \mathbb{N}$ and $\omega = (u_i)$ with $u_i \in U = [-1, 1]$ such that

$$\varphi(n_0, x_0, \omega) \in (-\delta, \delta) \text{ and } \varphi(k, x_0, \omega) \in \operatorname{int} Q \text{ for all } k = 1, \ldots, n_0 - 1. \tag{12}$$

This is seen as follows. If $x_0 \in \left[0, \frac{1}{\alpha-1} - \varepsilon\right]$, we can make a step to the left of x_0 of length l where $l \in (0, (\alpha - 1)\varepsilon]$ is arbitrary. In fact, using the control value $u_0 = -1 \in [-1, 1]$ one obtains for $x_1 = \alpha x_0 + u_0$ that

$$x_1 - x_0 = \alpha x_0 - x_0 - 1 \leq (\alpha - 1)\left(\frac{1}{\alpha - 1} - \varepsilon\right) - 1 = -(\alpha - 1)\varepsilon < 0.$$

Similarly, for $u_0 = -(\alpha - 1)x_0 \in [-1, 1]$, one computes $x_1 = x_0$ and hence, by continuity, one can make steps of length l to the left.

Analogously for $x_0 \in \left[\frac{1}{1-\alpha} + \varepsilon, 0\right]$ one can make steps of length $l \in (0, (\alpha - 1)\varepsilon]$ to the right. Going several steps, if necessary, one can reach the interval $(-\delta, \delta)$ from each point of Q. These arguments also show that we can stay in the interval $(-\delta, \delta)$ if we start in it. Together we have shown that there is a time $n_0 \in \mathbb{N}$ such that for every $x_0 \in Q$ there is a control ω with (12).

By continuity, there are finitely many controls $\omega_1, \ldots, \omega_N$ such that for every $x_0 \in Q$ there is ω_i with $\varphi(n_0, x_0, \omega_i) \in (-\delta, \delta)$ and $\varphi(k, x_0, \omega_i) \in \operatorname{int} Q$ for $k = 1, \ldots, n_0$.

Now choose a finite (n, Q)-spanning set $\mathcal{S}$ with minimal cardinality $\#\mathcal{S} = r_{inv,int}(n, Q)$. This yields the set $\mathcal{S}_\delta := \{\delta\omega; \omega \in \mathcal{S}\}$ of controls with values in $[-\delta, \delta]$ which keep every element in δQ. Concatenations of the controls in $\mathcal{S}_\delta$ with the controls $\omega_1, \ldots, \omega_N$ yields an $(n_0 + n, Q)$-spanning set $\mathcal{S}'$ with cardinality $\#\mathcal{S}' \leq N \cdot \#\mathcal{S}$. For $k \in \{n_0, \ldots, n_0 + n - 1\}$, the controls in $\mathcal{S}'$ have values in $[-\delta, \delta]$, hence $f(u) = |u| \leq \delta$ here.

We compute for $\omega' = (u'_i) \in \mathcal{S}'$

$$(S_{n_0+n} f)(\omega') = \sum_{i=0}^{n_0+n-1} f(u'_i) = \sum_{i=0}^{n_0-1} f(u'_i) + \sum_{i=n_0}^{n_0+n-1} f(u'_i)$$
$$\leq n_0 \max_{u\in[-1,1]} |u| + n \max_{u\in[-\delta,\delta]} |u| = n_0 + n\delta.$$

This yields

$$a_{n+n_0}(f, Q) \leq \sum_{\omega'\in\mathcal{S}'} e^{(S_{n+n_0} f)(\omega)} \leq \#\mathcal{S}' \cdot e^{n_0+n\delta} \leq N \cdot \#\mathcal{S} \cdot e^{n_0+n\delta}$$
$$= N \cdot r_{inv,int}(n, Q) \cdot e^{n_0+n\delta},$$

and hence

$$P_{int}(f, Q) = \limsup_{n\to\infty} \frac{1}{n+n_0} \log a_{n+n_0}(f, Q)$$
$$\leq \limsup_{n\to\infty} \left[\frac{1}{n+n_0} \log N + \frac{n}{n+n_0} \frac{1}{n} \log r_{inv,int}(n, Q) + \frac{n_0+n\delta}{n+n_0} \right]$$
$$\leq \lim_{n\to\infty} \frac{1}{n} \log r_{inv,int}(n, Q) + \limsup_{n\to\infty} \frac{n_0+n\delta}{n+n_0}.$$

Since $\frac{n_0+n\delta}{n+n_0} \leq 2\delta$ for n large enough, it follows that $P_{int}(f, Q) \leq h_{inv,int}(Q) + 2\delta$ which implies $P_{int}(f, Q) \leq h_{inv,int}(Q)$ using that $\delta > 0$ is arbitrary.

As announced in Remark 2, we conclude this section with some comments on other versions of invariance pressure that can be constructed in analogy to versions of invariance entropy, cf. Kawan [7].

Call a pair (K, Q) of nonempty subsets of X admissible for control system (1), if K is compact and for each $x \in K$ there is $\omega \in \mathcal{U}$ such that $\varphi(k, x, \omega) \in Q$ for all $k \in \mathbb{N}_0$. Then for $n \in \mathbb{N}$ a subset $\mathcal{S} \subset \mathcal{U}$ is called (n, K, Q)-spanning if for all $x \in K$ there is $\omega \in \mathcal{S}$ with $\varphi(k, x, \omega) \in Q$ for $k = 0, 1, \ldots, n$. For $f \in C(U, \mathbb{R})$ define

$$a_n(f, K, Q) := \inf \left\{ \sum_{\omega\in\mathcal{S}} e^{(S_n f)(\omega)};\ \mathcal{S}\ (n, K, Q)\text{-spanning} \right\}.$$

Then one can define the invariance pressure as

$$P(f, K, Q) := \limsup_{n\to\infty} \frac{1}{n} \log a_n(f, K, Q).$$

Another version of invariance pressure can be defined as follows. For $\varepsilon > 0$, the ε-neighborhood $N_\varepsilon(Q)$ of $Q \subset X$ is the set $N_\varepsilon(Q) := \{y \in X;$ there is $x \in Q$ with $d(x, y) < \varepsilon\}$. Given a closed set $Q \subset X$, $\varepsilon > 0$ and $n \in \mathbb{N}$, a set $\mathcal{S} \subset \mathcal{U}$ is called $(n, Q, N_\varepsilon(Q))$-spanning, if for all $x \in Q$ there is $\omega \in \mathcal{S}$ with $\varphi(k, x, \omega) \in N_\varepsilon(Q)$ for all $k = 1, \ldots, n$. For $f \in C(U, \mathbb{R})$ define

$$a_n(\varepsilon, f, Q) := \inf \left\{ \sum_{\omega\in\mathcal{S}} e^{(S_n f)(\omega)};\ \mathcal{S}\ (n, Q, N_\varepsilon(Q))\text{-spanning} \right\},$$

and

$$P(\varepsilon, f, Q) := \limsup_{n\to\infty} \frac{1}{n} \log a_n(\varepsilon, f, Q).$$

Then we define the outer invariance pressure as

$$P_{out}(f, Q) := \lim_{\varepsilon \to 0} P(\varepsilon, f, Q).$$

Clearly, $P_{out}(f, Q) = \sup_{\varepsilon > 0} P(\varepsilon, f, Q) \leq P_{int}(f, Q)$.

4 Invariance Pressure of Continuous-Time Systems

In this section we discuss invariance pressure for control systems given by ordinary differential equation and show that it can be characterized using discretized time. Then we will derive a formula for the outer invariance pressure of linear control systems.

Throughout we assume that X is a d-dimensional smooth manifold, $U \subset \mathbb{R}^m$ is Borel measurable and $\mathcal{U} = \{\omega : \mathbb{R} \to U;$ Lebesgue integrable$\}$. Consider the continuous-time control system

$$\dot{x}(t) = F(x(t), \omega(t)), \tag{13}$$

where $F : X \times U \to TX$ is continuous, TX is the tangent bundle and for each $u \in \mathbb{R}^m$ the map $F_u := F(\cdot, u) : X \to TX$ is a vector field. We assume that $Q \subset X$ is compact and that for all $x \in Q$ and $\omega \in \mathcal{U}$ a unique solution $\varphi(t, x, \omega) \in Q, t \geq 0$, exists. Furthermore, we assume that Q is controlled invariant, i.e., for every $x \in Q$ there exists $\omega \in \mathcal{U}$ such that $\varphi(t, x, \omega) \in Q$ for all $t \geq 0$.

In analogy to the discrete-time case, we call a subset $\mathcal{S} \subset \mathcal{U}$ a (τ, Q)-spanning set, if $\tau > 0$ and for every $x \in Q$ there exists $\omega \in \mathcal{S}$ such that $\varphi(t, x, \omega) \in Q$ for all $t \in [0, \tau]$.

For $\tau \geq 0$ and $f \in C(U, \mathbb{R})$ define $(S_\tau f)(\omega) = \int_0^\tau f(\omega(t))dt$ and

$$a_\tau(f, Q) := \inf \left\{ \sum_{\omega \in \mathcal{S}} e^{(S_\tau f)(\omega)};\ \mathcal{S}\ (\tau, Q)\text{-spanning} \right\}.$$

The central definition is the following.

Definition 19 The invariance pressure in Q of $f \in C(U, \mathbb{R})$ for the control system (13) is

$$P_{inv}(f, Q) = \limsup_{\tau \to \infty} \frac{1}{\tau} \log a_\tau(f, Q)$$

and the invariance pressure of (13) is the map $P_{inv}(\cdot, Q) : C(U, \mathbb{R}) \to \mathbb{R}$.

The next theorem shows that for the invariance pressure the time may be discretized.

Theorem 20 *If U is compact, then the invariance pressure of system* (13) *satisfies for every* $\tau > 0$

$$P_{inv}(f, Q) = \limsup_{n \to \infty} \frac{1}{n\tau} \log a_{n\tau}(f, Q) \text{ for all } f \in C(U, \mathbb{R}). \tag{14}$$

Proof For every $f \in C(U, \mathbb{R})$, the inequality

$$P_{inv}(f, Q) \geq \limsup_{n \to \infty} \frac{1}{n\tau} \log a_{n\tau}(f, Q) \tag{15}$$

is obvious. For the converse note that the function $g(u) := f(u) - \inf f$ is nonnegative (if $f \geq 0$, it is not necessary to consider the function g). Let $(\tau_k)_{k \geq 1}, \tau_k \in (0, \infty)$ and $\tau_k \to \infty$.

Then for every $k \geq 1$ there exists $n_k \geq 1$ such that $n_k\tau \leq \tau_k < (n_k + 1)\tau$ and $n_k \to \infty$ for $k \to \infty$. Since $g \geq 0$ it follows that

$$a_{\tau_k}(g, Q) \leq a_{(n_k+1)\tau}(g, Q)$$

and consequently

$$\frac{1}{\tau_k} \log a_{\tau_k}(g, Q) \leq \frac{1}{n_k\tau} \log a_{(n_k+1)\tau}(g, Q).$$

This yields

$$\limsup_{k\to\infty} \frac{1}{\tau_k} \log a_{\tau_k}(g, Q) \leq \limsup_{k\to\infty} \frac{1}{n_k\tau} \log a_{(n_k+1)\tau}(g, Q).$$

Since $\frac{1}{n_k\tau} = \frac{n_k+1}{n_k} \frac{1}{(n_k+1)\tau}$ and $\frac{n_k+1}{n_k} \to 1$ for $k \to \infty$, we obtain

$$\begin{aligned} \limsup_{k\to\infty} \frac{1}{\tau_k} \log a_{\tau_k}(g, Q) &\leq \limsup_{k\to\infty} \frac{1}{(n_k + 1)\tau} \log a_{(n_k+1)\tau}(g, Q) \\ &\leq \limsup_{n\to\infty} \frac{1}{n\tau} \log a_{n\tau}(g, Q). \end{aligned}$$

Together with (15) applied to $f - \inf f$, this shows that

$$P_{inv}(f - \inf f, Q) = \limsup_{n\to\infty} \frac{1}{n\tau} \log a_{n\tau}(f - \inf f, Q),$$

and as in Proposition 13 (ii) we have

$$\begin{aligned} P_{inv}(f, Q) = P_{inv}(f - \inf f, Q) + \inf f &= P_{inv}(g, Q) + \inf f \\ &= \limsup_{n\to\infty} \frac{1}{n\tau} \log a_{n\tau}(f - \inf f, Q) + \inf f \\ &= \limsup_{n\to\infty} \frac{1}{n\tau} \log e^{-n \inf f} a_{n\tau}(f, Q) + \inf f \\ &= \limsup_{n\to\infty} \frac{1}{n\tau} \log a_{n\tau}(f, Q). \end{aligned}$$

□

The above result can be rephrased in the following form. Define the invariance pressure at time 1 of system (13) by

$$P^1_{inv}(f, Q) = \limsup_{n\to\infty} \frac{1}{n} \log a_n(f, Q),\ f \in C(U, \mathbb{R}),$$

where

$$a_n(f, Q) := \inf \left\{ \sum_{\omega\in\mathcal{S}} e^{(S_n f)(\omega)};\ \mathcal{S}\ (n, Q)\text{-spanning} \right\}.$$

Corollary 21 *If U is compact, then the invariance pressure of system (13) satisfies*

$$P_{inv}(f, Q) = P^1_{inv}(f, Q) \text{ for all } f \in C(U, \mathbb{R}).$$

Remark 22 Compactness of U has been used in the proof of Theorem 20 only in order to guarantee that $\inf f > -\infty$ for every $f \in C(U, \mathbb{R})$. Thus the property in (14) holds for arbitrary U if the considered functions f are bounded below.

Next we determine the outer invariance pressure for a class of problems with linear control systems. For a control system of the form (13) the outer invariance entropy is defined as follows (cf. Kawan [7, p. 44]). The ε-neighborhood of $Q \subset X$ be denoted by $N_\varepsilon(Q) := \{y \in X;$ there is $x \in Q$ with $d(x, y) < \varepsilon\}$.

Given a closed set $Q \subset X$, $\varepsilon > 0$ and $\tau > 0$, a set $\mathcal{S} \subset \mathcal{U}$ is called $(\tau, Q, N_\varepsilon(Q))$-spanning, if for all $x \in Q$ there is $\omega \in \mathcal{S}$ with $\varphi(t, x, \omega) \in N_\varepsilon(Q)$ for all $t \in [0, \tau]$. Denote by $r_{inv}(\tau, \varepsilon, Q)$ the minimal number of elements that a $(\tau, Q, N_\varepsilon(Q))$-spanning set can have and

$$h_{inv}(\varepsilon, Q) := \limsup_{\tau\to\infty} \frac{1}{\tau} \log r_{inv}(\tau, \varepsilon, Q). \tag{16}$$

Definition 23 The outer invariance entropy of a closed subset $Q \subset X$ is defined by

$$h_{inv,out}(Q) := \lim_{\varepsilon\to 0} h_{inv}(\varepsilon, Q) \le \infty.$$

It is obvious that $h_{inv,out}(Q) = \sup_{\varepsilon>0} h_{inv}(\varepsilon, Q) \le h_{inv}(Q)$.

We consider linear control systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \ u(t) \in U, \tag{17}$$

where $A \in \mathbb{R}^{d\times d}$, $B \in \mathbb{R}^{d\times m}$ and $\varnothing \ne \mathrm{int} U$ with $U \subset \mathbb{R}^m$.

The following result is a consequence of Kawan [7, Theorem 3.1].

Theorem 24 *Suppose that $Q \subset \mathbb{R}^d$ is a compact controlled invariant set for system* (17) *with* $\mathrm{int} Q \ne \varnothing$. *Then*

$$h_{inv,out}(Q) = \sum_{i=1}^{d} \max(0, \mathrm{Re}\, \mu_i),$$

where summation is over all eigenvalues μ_i of A.

Remark 25 The existence of a compact controlled invariant set Q with nonempty interior can be guaranteed if the matrix pair (A, B) is controllable (i.e., rank $[B, AB, \dots, A^{d-1}B] = d$) and the matrix A is hyperbolic (i.e., it has no eigenvalues on the imaginary axis).

Theorem 24 will be used to prove a result on outer invariance pressure which we define in the following way. For the general system (13), $f \in C(U, \mathbb{R})$ and $\varepsilon > 0$ let

$$a_\tau(\varepsilon, f, Q) := \inf\left\{\sum_{\omega\in\mathcal{S}} e^{(S_\tau f)(\omega)};\ \mathcal{S}\ (\tau, Q, N_\varepsilon(Q))\text{-spanning}\right\},$$

$$P_{inv}(\varepsilon, f, Q) := \limsup_{\tau\to\infty} \frac{1}{\tau} \log a_\tau(\varepsilon, f, Q).$$

Definition 26 For $f \in C(U, \mathbb{R})$ the outer invariance pressure in Q is defined by $P_{out}(f, Q) = \lim_{\varepsilon\to 0} P_{inv}(\varepsilon, f, Q)$ and the outer invariance pressure of the control system (13) is the map $P_{out}(\cdot, Q) : C(U, \mathbb{R}) \to \mathbb{R}$.

We get the following formula for the outer invariance pressure of linear systems.

Theorem 27 *Consider the linear control system* (17) *with compact convex control range U. Let $Q \subset \mathbb{R}^d$ be compact and let $f \in C(U, \mathbb{R})$ be a map such that there are $u_0 \in U$ and $x_0 \in \mathrm{int} Q$ with $f(u_0) = \min_{u\in U} f(u)$ and $Ax_0 + Bu_0 = 0$ (i.e., x_0 is an equilibrium for*

u_0), and assume that there is $T_0 > 0$ such that for every $x \in Q$ there are $T \in (0, T_0]$ and $\omega \in \mathcal{U}$ with

$$\varphi(T, x, \omega) = x_0 \text{ and } \varphi(t, x, \omega) \in Q \text{ for all } t \in (0, T]. \tag{18}$$

Then the outer invariance pressure is

$$P_{out}(f, Q) = f(u_0) + h_{inv,out}(Q) = f(u_0) + \sum_{i=1}^{d} \max(0, \operatorname{Re} \mu_i), \tag{19}$$

where summation is over all eigenvalues μ_i of A.

Proof Note that our assumption on Q implies that Q is controlled invariant. Then the second equality in (19) is an immediate consequence of Theorem 24. We will prove the first equality in (19) in three steps.

Step 1 First we will simplify the assertion. Define $g(v) := f(u + u_0)$ on $V := U - u_0$. Then $g(0) = f(u_0) \leq f(u) = g(u - u_0)$ for all $u \in U$, hence $g(0) = \min_{v \in V} g(v)$. Consider the control system

$$\dot{y}(t) = Ay(t) + Bv(t), v(t) \in V. \tag{20}$$

A trajectory $\varphi(\cdot, x, \omega)$ of (17) determines a trajectory $\psi(\cdot, x - x_0, \omega - u_0)$ of (20) (here u_0 is identified with the corresponding constant control function) and conversely, since

$$\begin{aligned}
\psi(t, x - x_0, \omega - u_0) &= e^{At}(x - x_0) + \int_0^t e^{A(t-s)} B(\omega(s) - u_0) ds \\
&= e^{At} x + \int_0^t e^{A(t-s)} B\omega(s) ds - \left[e^{At} x_0 + \int_0^t e^{A(t-s)} B u_0 ds \right] \\
&= \varphi(t, x, \omega) - x_0.
\end{aligned}$$

Thus $\varphi(t, x, \omega) \in N_\varepsilon(Q)$ implies that $\psi(t, x - x_0, \omega - u_0) \in N_\varepsilon(Q) - x_0 = N_\varepsilon(Q - x_0)$. The controllability condition for (17) implies that for every $x - x_0 \in Q - x_0$ there is $\omega \in \mathcal{U}$ with

$$\psi(T, x - x_0, \omega - u_0) = 0 \text{ and } \psi(t, x - x_0, \omega - u_0) \in Q - x_0 \text{ for all } t \in [0, T].$$

Furthermore, $0 \in \operatorname{int}(Q - x_0)$ since $x_0 \in \operatorname{int} Q$. It follows that the $(\tau, Q, \operatorname{int} Q)$-spanning sets $\mathcal{S}$ of system (17) give rise to $(\tau, Q - x_0, \operatorname{int}(Q - x_0))$-spanning sets $\mathcal{S} - u_0$ of system (20) and conversely. Then it follows that the outer invariance pressure $P_{out}(f, Q)$ of system (17) coincides with the outer invariance pressure $P_{out}(g, Q - x_0)$ of system (20).

These considerations imply that without loss of generality, we can assume that $0 \in U$ and that $Q \subset \mathbb{R}^d$ is a compact set with $0 \in \operatorname{int} Q$ such that for every $x \in Q$ there are $T > 0$ and $\omega \in \mathcal{U}$ with

$$\varphi(T, x, \omega) = 0 \text{ and } \varphi(t, x, \omega) \in Q \text{ for all } t \in (0, T],$$

and that $f \in C(U, \mathbb{R})$ with $f(0) = \min_{u \in U} f(u)$ (we just write U instead of $U - u_0$, Q instead of $Q - x_0$ and f instead of g).

Using the same arguments as in the proof of Proposition 13(ii), we find that

$$P_{out}(f, Q) = P_{out}(f - f(0), Q) + f(0).$$

Hence we can further assume without loss of generality that $0 = f(0) = \min_{u \in U} f(u)$. Then the claim takes the form $P_{out}(f, Q) = h_{inv,out}(Q)$.

Step 2 Next we show $P_{out}(f, Q) \geq h_{inv,out}(Q)$. Clearly, it is sufficient to show for all $\varepsilon > 0$ that $P_{inv}(\varepsilon, f, Q) \geq h_{inv}(\varepsilon, Q)$. Using (18) together with the fact that 0 is an equilibrium, one finds that for every $\tau \geq T_0$ and every $x \in Q$ that there is a control ω_x with $\varphi(\tau, x, \omega_x) = 0$ and $\varphi(t, x, \omega_x) \in Q$ for all $t \in [0, \tau]$. By uniform continuity in $t \in [0, \tau]$ there is a neighborhood of x such that for every y in this neighborhood one has

$$\varphi(t, y, \omega_x) \in N_\varepsilon(Q) \text{ for all } t \in [0, \tau].$$

Then compactness of Q implies that there is a finite $(\tau, Q, N_\varepsilon(Q))$-spanning set.

Let $\delta > 0$. Then for all τ large enough one finds a finite $(\tau, Q, N_\varepsilon(Q))$-spanning set $\mathcal{S}$ with

$$\begin{aligned} P(\varepsilon, f, Q) &= \limsup_{\tau' \to \infty} \frac{1}{\tau'} a_{\tau'}(\varepsilon, f, Q) \geq \frac{1}{\tau} \log a_\tau(\varepsilon, f, Q) - \delta \\ &\geq \frac{1}{\tau} \log \sum_{\omega \in \mathcal{S}} e^{(S_\tau f)(\omega)} - 2\delta. \end{aligned}$$

Since $\mathcal{S}$ is $(\tau, Q, N_\varepsilon(Q))$-spanning, it follows that $\#\mathcal{S} \geq r_{inv}(\varepsilon, \tau, Q)$ and, by assumption, we also know that $f(u) \geq f(0) = 0$ for all $u \in U$. This implies for all τ large enough that

$$P(\varepsilon, f, Q)) \geq \frac{1}{\tau} \#\mathcal{S} - 2\delta \geq \frac{1}{\tau} r_{inv}(\tau, \varepsilon, Q) - 2\delta.$$

For $\tau \to \infty$ it follows that

$$P(\varepsilon, f, Q) \geq \limsup_{\tau \to \infty} \frac{1}{\tau} r_{inv}(\tau, \varepsilon, Q) - 2\delta.$$

Since $\delta > 0$ is arbitrary, it follows that this inequality also holds for $\delta = 0$. For $\varepsilon \to 0$, this yields, using also Theorem 24,

$$P_{out}(f, Q) = \lim_{\varepsilon \to 0} P(\varepsilon, f, Q) \geq \lim_{\varepsilon \to 0} \limsup_{\tau \to \infty} \frac{1}{\tau} r_{inv}(\tau, \varepsilon, Q) = h_{inv,out}(Q).$$

Step 3 Finally we show $P_{out}(f, Q) \leq h_{inv,out}(Q)$. Fix $\varepsilon > 0$. The assertion will follow if we can show that for every $\delta > 0$

$$P(\varepsilon, f, Q) \leq h_{inv}(\varepsilon, Q) + \delta.$$

The strategy will be similar as in Example 18: Every point in Q is steered into a small neighborhood of $0 \in \mathbb{R}^d$ and kept there by a spanning set constructed using linearity of the system equation.

Take $\delta > 0$. Since $0 \in \text{int}Q$ there is $\alpha \in (0, 1)$ such that the α-ball $N_\alpha(0)$ around 0 with radius α is contained in $\text{int}Q$. We may choose $\alpha > 0$ small enough such that $|u| < \alpha$ implies $f(u) \leq \delta$. The variation-of-constants formula shows that for $\beta > 0$ every trajectory $\varphi(t, x_0, u), t \geq 0$, of system (17) satisfies

$$\beta\varphi(t, x_0, u) = e^{At}\beta x_0 + \int_0^t e^{A(t-s)} B\beta u(s) ds = \varphi(t, \beta x_0, \beta u), t \geq 0.$$

Take $\beta < \alpha$ small enough such that $\beta Q \subset N_\alpha(0)$ in $\mathbb{R}^d$ and $\beta U \subset N_\alpha(0)$ in $\mathbb{R}^m$. The controls βu take values in βU which is a subset of U by convexity of U. Note also that $N_\alpha(0) \subset Q$ implies $N_{\alpha\beta}(0) \subset \beta Q$.

As in Step 2, there is for every $x \in Q$ a control $\omega_x \in \mathcal{U}$ with

$$\varphi(T_0, x, \omega_x) = 0 \text{ and } \varphi(t, x, \omega_x) \in Q \text{ for all } t \in (0, T_0].$$

By uniform continuity on $[0, T_0]$ one finds for all y in a neighborhood of x that

$$\|\varphi(T_0, y, \omega_x)\| < \alpha\beta \text{ and } \varphi(t, y, \omega_x) \in N_\varepsilon(Q) \text{ for all } t \in [0, T_0].$$

Then compactness of Q implies that there are finitely many controls $\omega_1, \dots, \omega_N$ such that for every $x \in Q$ there is ω_i with

$$\|\varphi(T_0, x, \omega_i)\| < \alpha\beta \text{ and } \varphi(t, y, \omega_i) \in N_\varepsilon(Q) \text{ for all } t \in [0, T_0]. \tag{21}$$

Thus we have found finitely many controls steering every point in Q into $N_{\alpha\beta}(0) \subset \beta Q \subset N_\alpha(0) \subset \operatorname{int} Q$. Next we construct controls keeping every point in the ball $N_{\alpha\beta}(0)$ in the ε-neighborhood of $N_\varepsilon(Q)$ (on arbitrarily large time intervals).

Fix $\tau > 0$ and let $\mathcal{S} = \{\omega'_1, \dots, \omega'_M\}$ be a $(\tau, Q, N_\varepsilon(Q))$-spanning set with $\#\mathcal{S} = r_{inv}(\tau, \varepsilon, Q)$. Then it follows that $\mathcal{S}_\beta := \{\beta\omega'_1, \dots, \beta\omega'_M\}$ is $(\tau, \beta Q, N_\varepsilon(Q))$-spanning. The controls βu take values in $\beta U \subset N_\alpha(0) \cap U$. Obviously, $\#\mathcal{S}_\beta = M = \#\mathcal{S} = r_{inv}(\tau, \varepsilon, Q)$.

The concatenations of the controls $\omega_1, \dots, \omega_N$ with the controls in $\mathcal{S}_\beta$ are given for $i = 1, \dots, N$ and $j = 1, \dots, M$ by

$$\omega_{ij}(t) := \begin{cases} \omega_i(t) & \text{for } t \in [0, T_0] \\ \beta\omega'_j(t - T_0) & \text{for } \quad t > T_0 \end{cases}.$$

Now consider $\tau' := \tau + T_0$. Then the set

$$\mathcal{S}' = \{\omega_{ij};\ i \in \{1, \dots, N\} \text{ and } j \in \{1, \dots, M\}\}$$

is $(\tau', Q, N_\varepsilon(Q))$-spanning. This follows, since by (21) one has that all points $\varphi(T_0, x, \omega_i) \in N_{\alpha\beta}(0) \subset \beta Q$. On the interval $[T_0, \tau']$ each control only takes values in $\beta U \subset N_\alpha(0)$, hence $f(u) \le \delta$ here. We have $\#\mathcal{S}' = N \cdot M = N \cdot r_{inv}(\tau, \varepsilon, Q)$ and compute for $\omega_{ij} \in \mathcal{S}'$

$$\begin{aligned}(S_{\tau'} f)(\omega_{ij}) &= \int_0^{\tau'} f(\omega_{ij}(\sigma))d\sigma = \int_0^{T_0} f(\omega_{ij}(\sigma))d\sigma + \int_{T_0}^{\tau'} f(\omega_{ij}(\sigma))d\sigma \\ &\le T_0 \max_{u\in U} f(u) + (\tau' - T_0) \max_{|u|\le\alpha} f(u) \le T_0 \max_{u\in U} f(u) + \tau\delta.\end{aligned}$$

This yields

$$\begin{aligned}\log a_{\tau'}(\varepsilon, f, Q) &\le \log \sum_{\omega_{ij}\in\mathcal{S}'} e^{(S_{\tau'} f)(\omega_{ij})} \le \log \sum_{\omega_{ij}\in\mathcal{S}'} e^{T_0 \max_{u\in U} f(u) + \tau\delta} \\ &\le \log \#\mathcal{S}' + T_0 \max_{u\in U} f(u) + \tau\delta \\ &\le \log N + T_0 \max_{u\in U} f(u) + \tau\delta + \log r_{inv}(\tau, \varepsilon, Q).\end{aligned}$$

Note that

$$\lim_{\tau'\to\infty} \frac{\tau}{\tau'} \frac{1}{\tau} \log r_{inv}(\tau, \varepsilon, Q) = h_{inv}(\varepsilon, Q).$$

Let $\tau_k \to \infty$ such that for $\tau'_k = \tau_k + T_0$

$$P(\varepsilon, f, Q) = \lim_{k\to\infty} \frac{1}{\tau'_k} \log a_{\tau'_k}(\varepsilon, f, Q).$$

For k large enough

$$\frac{1}{\tau'_k} \left[\log N + T_0 \max_{u\in U} f(u) + \tau_k\delta\right] \le 2\delta,$$

hence it follows that

$$P(\varepsilon, f, Q) = \lim_{k\to\infty} \frac{1}{\tau'_k} \log a_{\tau'_k}(\varepsilon, f, Q) \le h_{inv}(\varepsilon, Q) + 2\delta.$$

Since $\delta > 0$ is arbitrary, this implies $P(\varepsilon, f, Q) \le h(\varepsilon, Q)$ and the proof is complete. $\square$

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