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# MASTERTHESIS

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## Defensive alliance polynomial

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# MASTERTHESIS

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# Abstract

In this master thesis, we define a new bivariate polynomial which we call the *defensive alliance polynomial* and denote it by $da(G; x, y)$. It is a generalization of the alliance polynomial and the strong alliance polynomial. We show the relation between $da(G; x, y)$ and the alliance, the strong alliance, the induced connected subgraph polynomials as well as the cut vertex sets polynomial. We investigate information encoded about $G$ in $da(G; x, y)$. We discuss the defensive alliance polynomial for the path graphs, the cycle graphs, the star graphs, the double star graphs, the complete graphs, the complete bipartite graphs, the regular graphs, the wheel graphs, the open wheel graphs, the friendship graphs, the triangular book graphs and the quadrilateral book graphs. Also, we prove that the above classes of graphs are characterized by its defensive alliance polynomial. We present the defensive alliance polynomial of the graph formed of attaching a vertex to a complete graph. We show two pairs of graphs which are not characterized by the alliance polynomial but characterized by the defensive alliance polynomial.

Also, we present three notes on results in the literature. The first one is improving a bound and the other two are counterexamples.

# Contents

# 1. Introduction

In this master thesis, we define a new bivariate polynomial which we call the *defensive alliance polynomial* and denote it by $da(G; x, y)$.

The defensive alliance polynomial is related to the concept of alliance. And for that, we introduce the concept of alliance in Chapter $2$. In Section 2.2, we present the alliance concept and related ideas then we show some bounds and results for some special classes of graphs. In Section 2.3, we present two graph polynomials related to the concept of alliance, the alliance polynomial and the strong alliance polynomial. In Section 2.4, we present three notes on results in the literature. The first one is improving a bound and the other two are counterexamples.

In Chapter $3$, we define $da(G; x, y)$ and show its relation with the alliance polynomial, the strong alliance polynomial, the induced connected subgraph polynomial and the cut vertex sets polynomial. In Section 3.3, we extract properties encoded about $G$ in $da(G; x, y)$, for example, the order, the size, the degree sequence, the connectivity, the number of components with maximum order, the maximum order of components and the number of cut vertices. In Section 3.4, we give the defensive alliance polynomial of some special classes of graphs and the characterization of these graphs by the defensive alliance polynomial. The special classes of the graphs are: the path graphs, the cycle graphs, the star graphs, the double star graphs, the complete graphs, the complete bipartite graphs, the regular graphs, the wheel graphs, the open wheel graphs, the friendship graphs, the triangular book graphs and the quadrilateral book graphs. To show the characterization of the complete bipartite graph by the defensive alliance polynomial, a relation is proved between the number of subsets of cardinality three which induce a connected subgraph and the number of $P_2$ and $C_3$ as subgraphs in $G$. In Section 3.5, we present the defensive alliance polynomial of the graph formed by attaching a vertex to a complete graph. In Section 3.6, we discuss two pairs of graphs which are characterized by the defensive alliance polynomial but cannot be characterized by the alliance polynomial. In Section 3.7, we list some questions about the defensive alliance polynomials for further research.

One of the ideas that we considered on this research is how to construct the graph from its defensive alliance polynomial especially for the class of trees. And for that, we studied the problem of the reconstruction conjecture. Unfortunately, no significant results were achieved by the time of writing this thesis. So, the knowledge obtained about the reconstruction conjecture was stated in the appendix A.

The examples for different concepts introduced in this thesis are presented separately in appendix $B$ for the reconstruction conjecture and appendix $C$ for the alliance.

In Appendix $D$, we present a Python program by which we can obtain the defensive alliance polynomial of graphs of small order.

## 1.1 Notion

We present the graph polynomials using the form:

$$\sum_{S\subseteq V(G)} [\,p_1(S)\,][\,p_2(S)\,]\cdots x^{f_x(S)}y^{f_y(S)}z^{f_z(S)}\cdots, \text{ where}$$

$$[\,p_i(S)\,] = \begin{cases} 1 & \text{if } G[S] \text{ has the property } p_i, \\ 0 & \text{otherwise.} \end{cases}$$

We say that a graph $G$ is characterized by a graph polynomial $f$ if for every graph $G$ such that $f(G) = f(H)$ we have that $G$ is isomorphic to $H$. The class of graphs $K$ is characterized by a graph polynomial $f$ if every graph $G \in K$ is characterized by $f$.

Also, when we say a set $S$ contributes a term $t$, we mean the set $S$ induces a connected subgraph $G[S]$ which yields the term $t$ in $da(G; x, y)$.

For any other notation or special graph class construction please consult Appendix E.

# 2. Alliances

## 2.1 Introduction

In Section 2.2, we present the alliance concept and related ideas. Then, we show some bounds and results for some special classes of graphs.

In Section 2.3, we present two graph polynomials related to the concept of alliance, the alliance polynomial and the strong alliance polynomial.

In Section 2.4, we present three notes on results in the literature. The first one is improving a bound and the other two are counterexamples.

## 2.2 Alliance concept

**Definition 2.1.** *Let $G$ be a simple graph and $S$ be a subset of $V(G)$. The degree of a vertex $u$ in $S$ denoted by $\delta_S(u)$ is $|\{\{u,v\} \in E(G) : v \in S\}|$.*

Let $G$ be a simple graph. An *alliance* is a non-empty subset of $V(G)$. Let $S$ be an alliance. $S$ is *defensive alliance* [KHH02] provided that

$$\delta_S(v) - \delta_{\bar{S}}(v) \geq -1 \ , \ \forall\ v \in S.$$

In other words, for every vertex from $S$, the number of neighbours in $G[S]$ is at least the number of its neighbours outside $G[S]$ minus one. For examples see Section C.0.1.

Further, $S$ is called *strong defensive alliance* provided that:

$$\delta_S(v) - \delta_{\bar{S}}(v) \geq 0 \ , \ \forall\ v \in S.$$

In other words, for every vertex from $S$, the number of neighbours in $G[S]$ is at least the number of its neighbours outside $G[S]$.

The idea of the defensive alliance arises from assuming that the vertices inside $S$ support each other as an alliance against attacks from outside the alliance.

In the case of the defensive alliance, every vertex could be supported by adjacent vertices inside the alliance plus itself against attacks from neighbors outside the alliance. This means that the number of attackers is less than or equal the number of defenders inside the alliance including the point itself. In the case of strong defensive alliance, the number of defenders including the attacked vertex is more than the number of attackers by at least one. For examples see Section C.0.2.

The concept can be generalized to the *defensive $k$-alliance* [RYS08]

$$\delta_S(v) - \delta_{\bar{S}} \geq k\ , \forall\ v \in S\ ,\ k \text{ is an integer in the range } -\Delta \leq k \leq \Delta.$$

Note that for $k = -1$ we get the defensive alliance and for $k = 0$ we get the strong defensive alliance. For examples see Section C.0.3.

The idea of defending could be replaced by other ideas, for example attacking. Let $S$ be a non-empty subset of $V(G)$ such that $G[S]$ is connected. $S$ is *offensive alliance* [KHH02] provided that

$$\delta_S(v) - \delta_{\bar{S}}(v) \geq -1,\ \forall\ v \in \partial(S).$$

In other words, for every vertex from $\partial(S)$, the number of neighbours in $G[S]$ is at least the number of its neighbours outside $G[S]$ plus one. For examples see Section C.0.4.

Further, $S$ is called *strong offensive alliance* provided that:

$$\delta_S(v) - \delta_{\bar{S}}(v) \geq 0,\ \forall\ v \in \partial(S).$$

In other words, for every vertex from $\partial(S)$, the number of neighbours in $G[S]$ is at least the number of its neighbours outside $G[S]$. For examples see Section C.0.5.

An alliance can be both defensive and offensive, in which case we call it a *dual alliance* or a *powerful alliance*, for examples see Section C.0.6. Also an alliance can be described as a *global alliance* [KHH02] if every vertex outside the alliance is adjacent to a vertex in the alliance, for examples see Section C.0.7. Lastly, an alliance (defensive or offensive) is called a *critical alliance* or a *minimal alliance* [KHH02] if it has no proper subset as an alliance of the same kind, for examples see Section C.0.8.

**Definition 2.2.** *[KHH02] Let $\mathbb{A}(G)$ be the set of all the critical defensive alliance in $G$. The* alliance number *denoted by $a(G)$ is defined by*

$$a(G) = min\{|S| : S\ \in\ \mathbb{A}(G)\}.$$

*The* upper alliance number *denoted by $A(G)$ is defined by*

$$A(G) = max\{|S| : S\ \in\ \mathbb{A}(G)\}.$$

**Definition 2.3.** *[KHH02] Let $\hat{\mathbb{A}}(G)$ be the set of all the strong critical defensive alliance in $G$. The* strong alliance number *denoted by $\hat{a}(G)$ is defined by*

$$\hat{a}(G) = min\{|S| : S\ \in\ \hat{\mathbb{A}}(G)\}.$$

*The* upper strong alliance number *denoted by $A(G)$ is defined by*

$$\hat{A}(G) = max\{|S| : S\ \in\ \hat{\mathbb{A}}(G)\}.$$

### Results [KHH02]

| Graph class | $a(G)$ | $A(G)$ | $\hat{a}(G)$ | $\hat{A}(G)$ |
|---|---|---|---|---|
| $P_n$ | $1$ | $2$ | $2$ | $2$ |
| $C_n$ | $2$ | $2$ | $2$ | $2$ |
| $S_n$ | $1$ | $1$ | $\lceil n/2 \rceil + 1$ | $\lceil n/2 \rceil + 1$ |
| $K_n$ | $\lceil n/2 \rceil$ | $\lceil n/2 \rceil$ | $\lfloor n/2 \rfloor + 1$ | $\lfloor n/2 \rfloor + 1$ |
| $W_n$ | $2$ | $\lfloor n/2 \rfloor$ | $\lceil n/2 \rceil + 1$ | $\lceil n/2 \rceil + 1$ |
| $K_{n,m}$ | $\lfloor n/2 \rfloor + \lfloor m/2 \rfloor$ | $\lceil n/2 \rceil + \lceil m/2 \rceil$ | | |

Sometimes, the strength of individual vertices differ from each other. For example, if a vertex is so powerful, then even if more than one vertex attack it, it can defend itself alone.

**Definition 2.4.** *[KHH02] Let $G$ be a graph with a mapping $w : V \mapsto Z$. An alliance $S$ in $G$ is called a* weighted alliance *provided that:*

$$\sum_{u \in N[v] \cap S} w(u) \geq \sum_{x \in N(v) \cap (S - v)} w(x), \forall v \in S.$$

For examples see Section C.0.9.

## 2.3 Alliance polynomials

The alliance polynomial [CGATu14] and the strong alliance polynomial [CHGRTu16] are examples of graph polynomials related to the concept of alliance. The alliance polynomial shows a strong power with respect to the ability of distinguishing between non-isomorphic graphs.

### 2.3.1 Alliance polynomial

**Definition 2.5.** *[CGATu14] Let $G$ be a graph with a mapping $k : \mathcal{P}(V) \mapsto Z$ with $k(S) = \min_{v \in S}\{\delta_S(v) - \delta_{\overline{S}}(v)\}$. The* alliance polynomial *denoted by $A(G;x)$ is defined as*

$$A(G;x) = \sum_{S \subseteq V(G)} [\text{ } S \textit{ is not empty } ][\text{ } G[S] \textit{ is connected } ] x^{k(S)+n}.$$

**Alliance polynomial of some common classes of graphs [CGATu14]**

$$
\begin{aligned}
A(P_n;x) =&(n-2)x^{n-2} + 2x^{n-1} + \frac{(n-1)(n-2)}{2}x^0 + x^1.\\
A(C_n;x) =&nx^{n-2} + (n)(n-2)x^n + x^{n+2}.\\
A(S_n;x) =&(n-2)x^{n-2} + 2x^{n-1} + \frac{(n-1)(n-2)}{2}x^0 + x^1.\\
A(K_n;x) =&\frac{(1+x^2)^n - 1}{x}.\\
A(K_{n,m};x) =&nx^n + mx^m + \sum_{i=1}^{n}\sum_{j=1}^{m}\binom{n}{i}\binom{m}{j}x^{\min\{2i-n,2j-m\}+n+m}.
\end{aligned}
$$

### 2.3.2 Strong alliance polynomial

**Definition 2.6.** *[CHGRTu16] The* strong alliance polynomial *denoted by* $a(G;x)$ *is defined by:* $a(G;x) = \sum_{i=a(G)}^{n} a_i x^i$, *where* $a_i$ *is the number of strong defensive alliances in* $G$ *with cardinality* $i$.

**Strong alliance polynomial of some common classes of graphs [CHGRTu16]**

$$
\begin{aligned}
a(P_n;x) =&\sum_{i=2}^{n}(n+1-i)x^i,\ for\ n \geq 2.\\
a(C_n;x) =&n\sum_{i=2}^{n-1}x^i + x^n,\ for\ n \geq 3.\\
a(K_n;x) =&\sum_{i=\lceil\frac{n+1}{2}\rceil}^{n}\binom{n}{i}x^i,\ for\ n \geq 1.\\
a(K_{n,m};x) =&\left(a(K_n;x) + \binom{n}{\frac{n}{2}}x^{\frac{n}{2}}\right)\left(a(K_m;x) + \binom{m}{\frac{m}{2}}x^{\frac{m}{2}}\right).\\
a(S_n;x) =&x\left(a\left(K_n;x\right) + \binom{n}{\frac{n}{2}}x^{\frac{n}{2}}\right),\ for\ n \geq 1.
\end{aligned}
$$

## 2.4 Notes

In this Section, we present three notes on the results in the literature. The first one is improving a bound and the other two are counterexamples.

### 2.4.1 Note 1

The authors of [KHH02], in Theorem $6$, stated that for the $m \times n$ grid graph $G_{m,n}$, $1 \leq a(G_{m,n}) \leq 4$. But the upper bound can be improved.

**Theorem 2.7.** *For the $m \times n$ grid graph $G_{m,n}$ , $1 \leq a(G_{m,n}) \leq 2$.*

*Proof.* If $n = 1$ or $m = 1$, then $a(G_{n,m}) = 1$, since we can always choose one of the endpoints. See Figure 2.1.

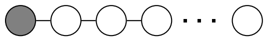


Figure 2.1: $G_{1,m}$ or $G_{n,1}$

Otherwise, we can form a minimal defensive alliance by choosing a vertex from the four corners and any vertex adjacent to it. See Figure 2.2. □

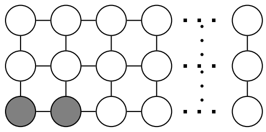


Figure 2.2: $G_{n,m}$

### 2.4.2 Note 2

The authors of [CGATu14], in the proof of Theorem $2.4$ stated that $X$ is a cut vertex set if and only if $V(G) \setminus X$ induces a disconnected graph. A counterexample graph $G$ shown in Figure 2.3, can be constructed.

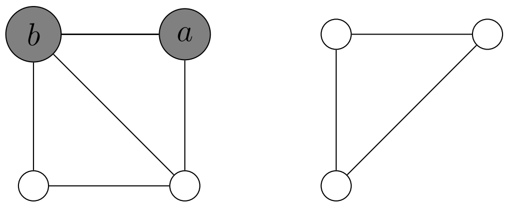



Figure 2.3: A disconnected graph $G$ with two components

Put $X = \{a, b\}$, the set $V(G) \setminus X$ induces a disconnected subgraph but $X$ is not a cut vertex set.

### 2.4.3 Note 3

The authors of [CGATu14], in Proposition $2.7$ stated that if $H$ is a proper subgraph of $G$, then all connected induced subgraphs of $H$ are a connected induced subgraphs of $G$ and at least one edge $e$ of $G$ is not contained in $H$. A counterexample graph $G$ shown in Figure 2.4, can be constructed.

Put $H = G[\ \{a, b, c, d\}\ ]$, then $H$ as a connected induced subgraph of $H$ is a connected induced subgraph of $G$ with zero edges $e$ of $G$ is not contained in $H$.

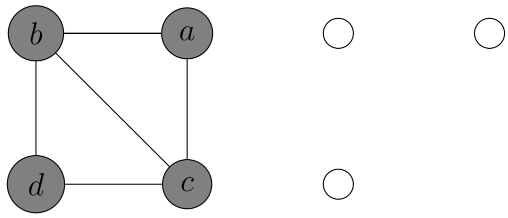



Figure 2.4: A disconnected graph $G$ with isolated vertices

# 3. The defensive alliance polynomial

## 3.1 Introduction

We introduce the *defensive alliance polynomial*, $da(G; x, y)$ which is a generalization of the alliance polynomial and the strong alliance polynomial.

In Section 3.2, we define $da(G; x, y)$ and show its relation with the alliance polynomial, the strong alliance polynomial, the induced connected subgraph polynomial [TAM11] and the cut vertex sets polynomial.

In Section 3.3, we extract properties encoded about $G$ in $da(G; x, y)$, for example, the order, the size, the degree sequence, the connectivity, the number of components with maximum order and the maximum order of components.

In Section 3.4, we give the defensive alliance polynomial of some special classes of graphs and characterization of these graphs by the defensive alliance polynomial. The special classes of the graphs are: the path graphs, the cycle graphs, the star graphs, the double star graphs, the complete graphs, the complete bipartite graphs, the regular graphs, the wheel graphs, the open wheel graphs, the friendship graphs, the triangular book graphs and the quadrilateral book graphs.

In Section 3.5, we present the defensive alliance polynomial of the graph formed by attaching a vertex to a complete graph.

In Section 3.6, we show two pairs of graphs which can be characterized by the defensive alliance polynomial but cannot be characterized by the alliance polynomial.

In Section 3.7, we present some questions which could be further researched.

## 3.2 Definition and relations with other graph polynomials

**Definition 3.1.** *The mappings $f_x$ and $f_y$ are defined as follows:*

$$f_x : \mathbb{P}(V(G)) \mapsto \mathbb{N} \textit{ with } f_x(S) = |S| \textit{ and}$$
$$f_y : \mathbb{P}(V(G)) \mapsto \mathbb{Z} \textit{ with } f_y(S) = \min_{u \in S}\{\delta_S(u) - \delta_{\bar{S}}(u) + n\}.$$

*The* defensive alliance polynomial *denoted by* da *is:*

$$da(G; x, y) = \sum_{S \subseteq V(G)} [S \textit{ is not empty}][\, G[S] \textit{ is connected}\,] x^{f_x(S)} y^{f_y(S)}.$$

### 3.2.1 Alliance polynomial

The *alliance polynomial* defined in [CGATu14] denoted by $A$ is:

$$A(G;y) = \sum_{S \subseteq V(G)} [S \text{ is not empty }][\ G[S] \text{ is connected }] y^{f_y(S)}.$$

**Proposition 3.2.**

$$A(G;y) \ = \ da(G;1,y).$$

*Proof.* The proof follows from comparing the definitions. □

### 3.2.2 Strong alliance polynomial

**Proposition 3.3.** *Let $S$ be a non-empty subset of $V(G)$ which induces a connected subgraph in $G$. $S$ is* strong defensive alliance *if $f_y(S) \geq n$.*

**Definition 3.4.** *The* strong alliance polynomial *defined in [CHGRTu16] denoted by $a$ is:*

$$a(G;x) = \sum_{S \subseteq V(G)} [S \textit{ is not empty }][\ G[S] \textit{ is connected }] \\ [S \textit{ is strong defensive alliance }] x^{f_x(S)}$$

**Proposition 3.5.**

$$a(G;x) = \sum_{k=0}^{n-1} [y^{n+k}] da(G;x,y).$$

*Proof.* The proof follows from comparing the definitions. □

### 3.2.3 Induced connected subgraph polynomial

**Definition 3.6.** *The* induced connected subgraph polynomial *defined in [TAM11] denoted by* q *is:*

$$q(G;x) = \sum_{S \subseteq V(G)} [S \textit{ is not empty }][\ G[S] \textit{ is connected }] x^{f_x(S)}.$$

**Proposition 3.7.**

$$q(G;x) = da(G;x,1).$$

*Proof.* The proof follows from comparing the definitions. □

### 3.2.4 Cut vertex sets polynomial

**Definition 3.8.** *Let $G$ be a simple connected graph. The* cut vertex sets polynomial *denoted by* v *is:*

$$v(G;x) = \sum_{S \subseteq V(G)} [S \textit{ is not empty}][\, S \textit{ is cut vertex set}]x^{f_x(S)}.$$

**Proposition 3.9.**

$$v(G;x) = x^n \left\{ \left(1 + \frac{1}{x}\right)^n - 1 - da\left(G; \frac{1}{x}, 1\right) \right\}.$$

*Proof.* Let $c_i$ be the number of the subsets of $V(G)$ with cardinality $i$ which induce a connected subgraph in $G$. Let $c_i^{'}$ be the number of the subsets of $V(G)$ with cardinality $i$ which induce a disconnected subgraph in $G$. We have

$$\binom{n}{i} = \begin{cases} 1 & \text{if } i = 0, \\ c_i + c_i^{'} & \text{if } i \geq 1. \end{cases}$$

Note that a set of size $i$ induces a disconnected subgraph in $G$ if and only if $V \setminus S$ is a cut vertex set. That is every set of cardinality $i$ counted by $c_i^{'}$ corresponds to a cut vertex set of cardinality $n - i$. Let $d_j$ be the number of cut vertex sets in $G$ of cardinality $j$, we have

$$\binom{n}{i} = \begin{cases} 1 & \text{if } i = 0, \\ c_i + d_{n-i} & \text{if } i \geq 1. \end{cases}$$

Multiplying both sides of the above equation by $z^i$ yields

$$d_{n-i}z^i = \binom{n}{i} z^i - c_i z^i.$$

By summing over $i$:

$$\sum_{i=1}^{n} d_{n-i} z^i = (1+z)^n - 1 - da(G; z, 1).$$

Now substitution of $z$ by $\frac{1}{x}$ and multiplication with $x^n$ provide the statement. □

## 3.3 Properties

### 3.3.1 The number of connected induced subgraphs

**Proposition 3.10.** *The number of connected induced subgraphs of order $k$ is* $[x^k]da(G;x,1)$.

*Proof.* A set $S$ where $S \subseteq V(G)$, contributes a term with $f_x(S) = k$ if and only if $S$ induces a connected subgraph in $G$ and $|S| = k$. By substituting $y = 1$ in $da(G;x,1)$, we sum the terms with the similar exponent of $x$. Hence, the coefficient of $x^k$ in $da(G;x,1)$ is the number of connected induced subgraphs of order $k$ in $G$. □

### 3.3.2 The order

**Proposition 3.11.** *The order of $G$ is $[x^1]da(G; x, 1)$.*

*Proof.* By putting $k = 1$ in Proposition 3.10 we get $[x^1]da(G; x, 1)$ as the number of connected subgraphs of order one, hence the order of $G$. $\square$

### 3.3.3 The size

**Proposition 3.12.** *The size of $G$ is $[x^2]da(G; x, 1)$.*

*Proof.* By putting $k = 2$ in Proposition 3.10 we get $[x^2]da(G; x, 1)$ as the number of connected subgraphs of order two, hence the size of $G$. $\square$

### 3.3.4 The connectivity

**Proposition 3.13.** *$G$ is connected if and only if $deg_x(da(G; x, y)) = n$.*

*Proof.* If $G$ is connected, then $V(G)$ contributes the term $x^n y^{f_y(V(G))}$. Since $G$ has only one subset of vertcies with cardinality $n$ this implies $deg_x(da(G; x, y)) = n$.

Now we prove the converse. If there exists a term in $da(G; x, y)$ where the exponent of $x$ equals $n$, then there exists a connected induced subgraph with order $n$. Since $V(G)$ is the only such subgraph, therefore $G$ is connected. $\square$

### 3.3.5 The degree sequence

**Proposition 3.14.** *Let $k$ be an integer in the range $0 \leq k \leq n - 1$. The number of vertices in $G$ with a degree $k$ is $[xy^{n-k}]da(G; x, y)$. Hence the degree sequence of $G$ can be obtained.*

*Proof.* Let $v$ be a vertex in $G$. The set $\{v\}$ induces a connected subgraph in $G$ which contributes the term $xy^{n-deg(v)}$ in $da(G; x, y)$. Hence $[xy^{n-deg(v)}]da(G; x, y)$ yields the number of all vertices with degree equal to $deg(v)$. $\square$

### 3.3.6 The maximum order of a component

**Proposition 3.15.** *Let $G$ be a simple graph. The maximum order of a component of $G$ is $deg(da(G; x, 1))$. Further, the number of components with maximum order $c$ is $[x^c]da(G; x, 1)$.*

*Proof.* From the definition of the defensive alliance polynomial, we can see that $deg(da(G;x,1))$ is the order of the maximum component of $G$. Let $c = deg(da(G;x,1))$ and $A = \{S : |S| = c$ and $S$ induces a component in $G\}$. Every set $S$ in $A$ contributes a term $x^c y^{f_y(S)}$ in $da(G;x,y)$. The number of these terms is $|A|$ which can be obtained from $[x^c]da(G;x,1)$. □

### 3.3.7 The number of cut vertices

**Proposition 3.16.** *Let $G$ be a simple connected graph. The number of cut vertices in $G$ is $n - [x^{n-1}]da(G;x,1)$.*

*Proof.* Let $v$ be a vertex in $V(G)$, every subset of $V(G)\setminus\{v\}$ contributes a connected subgraph in $G$ if and only if $v$ is not a cut vertex. Every such set $V(G)\setminus\{v\}$, contributes a term in $da(G;x,1)$ where the exponent of $x$ is $n-1$. The number of cut vertices is the order minus the sum of the above terms $= n - [x^{n-1}]da(G;x,1)$. □

### 3.3.8 The defensive alliance polynomial of pairwise disjoint graphs

**Proposition 3.17.** *Let $G_1$, $G_2$, $\cdots$, $G_k$ be pairwise disjoint graphs. Then*

$$da(\cup_{i=1}^{k} G_i;x,y) = \left(\sum_{i=1}^{k} \frac{da(G_i;x,y)}{y^{|G_i|}}\right) y^{\sum_{i=1}^{k}|G_i|}.$$

*Proof.* Let $i$ and $j$ be integers in the range $1, 2, \cdots, k$. Every connected subgraph in $G_i$ is disjoint from subgraphs in $G_j$ where $i \neq j$. But the exponent of $y$ in $da(G_i;x,y)$ is added to $|G_i|$, hence the sum of the orders of all the other graphs must be added. □

## 3.4 Defensive alliance polynomial of special classes of graphs and their characterization by it

### 3.4.1 The path graph

**Proposition 3.18.** *A simple graph $G$ is isomorphic to the path $P_n$ if and only if*

$$da(G;x,y) = 2xy^{n-1} + (n-2)xy^{n-2} + y^n \sum_{i=2}^{n-1}(n-i+1)x^i + x^n y^{n+1}, \text{ where } n \geq 2.$$

*Proof.* First, we show that a graph $G$ which is isomorphic to a path $P_n$, has the given defensive alliance polynomial. Let $G$ be of the form in Figure 3.1.

The non-empty subsets of $V(G)$ which induce connected subgraphs in $G$, can be partitioned into the following parts: The part $\{\{v_1\},\{v_n\}\}$ in which each set

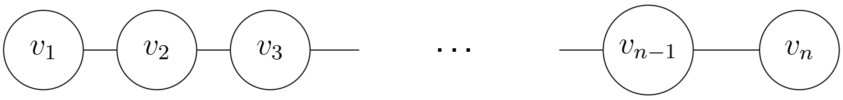


Figure 3.1: A path graph

contributes the term $xy^{n-1}$ and by summing, we get the term $2xy^{n-1}$. The part $\{\{v_2\}, \{v_3\}, \cdots, \{v_{n-1}\}\}$ in which each set contributes the term $xy^{n-2}$ and by summing, we get the term $(n-2)xy^{n-2}$. The part containing the sets of cardinality $i$ in the range of $2 \leq i \leq n-1$ in which each set contributes the term $x^i y^n$. By adding the terms we get

$$\begin{aligned}&(n-1)x^2y^n + (n-2)x^3y^n + \cdots + (n-(n-2))x^{n-1}y^n \\ &= y^n \sum_{i=2}^{n-1}(n-i+1)x^i.\end{aligned}$$

Finally, the part containing $V(G)$ in which $V(G)$ contributes the term $x^n y^{n+1}$.

Now we prove the converse. Let $n$ be an integer where $n \geq 2$, and $H$ is a graph with the defensive alliance polynomial,

$$da(H; x, y) = 2xy^{n-1} + (n-2)xy^{n-2} + y^n \sum_{i=2}^{n-1}(n-i+1)x^i + x^n y^{n+1}.$$

By Proposition 3.11, the order of $H$ equals $n$. By Proposition 3.12, the size of $H$ equals $n-1$. By Proposition 3.13, $H$ is connected. Hence, $H$ is a tree. By Proposition 3.14, the degree sequence of $H$ is $(2, 2, \cdots, 2, 1, 1)$. Consequently, $H$ is isomorphic to the path graph $P_n$. □

### 3.4.2 The cycle graph

**Proposition 3.19.** *A simple graph $G$ is isomorphic to the cycle $C_n$ if and only if*

$$da(G; x, y) = nxy^{n-2} + ny^n \sum_{i=2}^{n-1} x^i + x^n y^{n+2}, \text{ where } n \geq 3.$$

*Proof.* First, we show that a graph $G$ which is isomorphic to a cycle $C_n$, has the given defensive alliance polynomial.

The non-empty subsets of $V(G)$ which induce connected subgraphs in $G$, can be partitioned into the following parts: The part containing the sets of cardinality one in which each set contributes the term $xy^{n-2}$ and by summing, we get the term $nxy^{n-2}$. The part containing the sets of cardinality $i$ in the range of $2 \leq i \leq n-1$ in which each set contributes the term $x^i y^n$. By adding the terms we get

$$\begin{aligned}&nx^2y^n + nx^3y^n + \cdots + nx^{n-1}y^n \\ &= ny^n \sum_{i=2}^{n-1} x^i,\end{aligned}$$

Finally, the part containing $V(G)$ in which $V(G)$ contributes the term $x^n y^{n+2}$.

Now we prove the converse. Let $n$ be an integer where $n \geq 3$, and $H$ is a graph with the defensive alliance polynomial

$$da(H; x, y) = nxy^{n-2} + ny^n \sum_{i=2}^{n-1} x^i + x^n y^{n+2}.$$

By Proposition 3.11, the order of $H$ equals $n$. By Proposition 3.13, $H$ is connected. By Proposition 3.14, the degree sequence of $H$ is $(2, 2, \cdots, 2)$.

Consequently, $H$ is isomorphic to the cycle graph $C_n$. □

### 3.4.3 The star graph

**Definition 3.20.** *Let $n$ be a positive integer. The* star graph *denoted by $S_n$ is defined by the graph join $nK_1 + K_1$. Further the vertex with the maximum degree is called the* center*.*

**Proposition 3.21.** *A simple graph $G$ is isomorphic to the star $S_n$ if and only if*

$$da(G; x, y) = xy + nxy^{n-1} + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{i} x^{i+1} y^{2i} + \sum_{i=\lceil \frac{n+1}{2} \rceil}^{n} \binom{n}{i} x^{i+1} y^{n+1}, \textit{ where } n \geq 1.$$

*Proof.* First, we show that a graph $G$ which is isomorphic to a star $S_n$, has the given defensive alliance polynomial. Let $G$ be of the form in Figure 3.2.

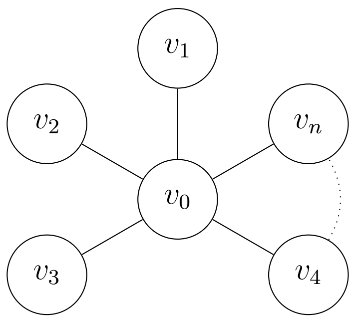



Figure 3.2: A star graph

The non-empty subsets of $V(G)$ which induce connected subgraphs in $G$, can be partitioned into the following parts: The part $\{\{v_0\}\}$ in which $\{v_0\}$ contributes the term $xy$. The part $\{\{v_1\}, \{v_2\}, \cdots, \{v_n\}\}$ in which each set contributes the term $xy^{n-1}$ and by summing, we get the term $nxy^{n-1}$. The part containing the sets of cardinality $i$ in the range of $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$ in which each set contributes the term $x^{i+1}y^{2i+1}$ and by summing, we get $\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{i} x^{i+1} y^{2i+1}$. The part containing the sets of cardinality $i$ in the range of $\lceil \frac{n+1}{2} \rceil \leq i \leq n$ in which each set contributes the term $x^{i+1}y^{n+1}$ and by summing, we get $\sum_{\lceil \frac{n+1}{2} \rceil}^{i=n} \binom{n}{i} x^{i+1} y^{n+1}$.

Now we prove the converse. Let $n$ be an integer where $n \geq 1$, and $H$ is a graph with the defensive alliance polynomial

$$da(H;x,y) = xy + nxy^{n-1} + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{i} x^{i+1}y^{2i} + \sum_{i=\lceil \frac{n+1}{2} \rceil}^{n} \binom{n}{i} x^{i+1}y^{n+1}.$$

By Proposition 3.11, the order of $H$ equals $n+1$. By Proposition 3.12, the size of $H$ equals $n$. By Proposition 3.13, $H$ is connected. Hence, $H$ is a tree. By Proposition 3.14, the degree sequence of $H$ is $(n, 1, 1 \cdots, 1)$. Consequently, $H$ is isomorphic to the star graph $S_n$. $\square$

### 3.4.4 The double star graph

**Definition 3.22.** *Let $r$ and $t$ be positive integers. The* star graph *denoted by $S_{r,t}$ is defined by the graph union $S_r \cup S_t$ and connecting the two centers of the two stars.*

**Proposition 3.23.** *A simple graph $G$ is isomorphic to the double star $S_{r,t}$ if and only if*

$$[x]da(G;x,y) = (r+t)y^{r+t+1} + y^{r+1} + y^{t+1} \text{ and}$$
$$[x^{r+t+2}]da(G;x,y) = y^{r+t+3}, \text{ where } r \text{ and } t \text{ are positive integers.}$$

*Proof.* First, we show that a graph $G$ which is isomorphic to a double star $S_{r,t}$, has the above properties in the proposition. Let $G$ be of the form in Figure 3.3.

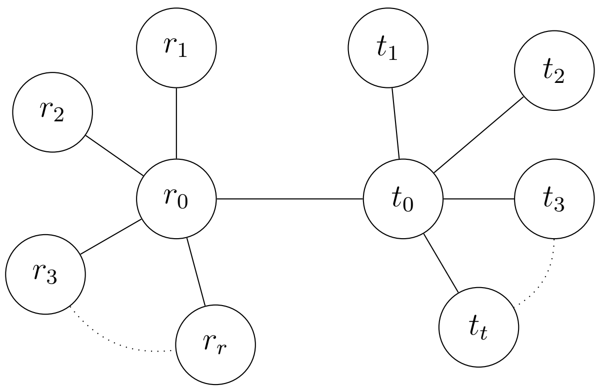



Figure 3.3: A double star graph

The subsets of $V(G)$ with cardinality one which induce connected subgraphs in $G$, can be partitioned into the following parts: The part $\{\{r_0\}\}$ in which $\{r_0\}$ contributes the term $xy^{t+1}$. The part $\{\{t_0\}\}$ in which $\{t_0\}$ contributes the term $xy^{r+1}$. The part $\{\{r_1\}, \{r_2\}, \cdots, \{r_r\}, \{t_1\}, \{t_2\}, \cdots, \{t_t\}\}$ in which each set contributes the term $xy^{r+t+1}$ and by summing, we get the term $(r+t)xy^{r+t+1}$.

The set $V(G)$ contributes the term $x^{r+t+2}y^{r+t+3}$.

Now we prove the converse. Let $r$ and $t$ be integers and $H$ is a graph with

$$\begin{aligned}[x]da(G;x,y) =&(r+t)y^{r+t+1} + y^{r+1} + y^{t+1} \text{ and}\\ [x^{r+t+2}]da(G;x,y) =&y^{r+t+3}.\end{aligned}$$

By Proposition 3.11, the order of $H$ equals $r+t+2$. By Proposition 3.13, $H$ is connected. By Proposition 3.14, the degree sequence of $H$ is $(r+1, s+1, 1, 1, \cdots, 1)$. Let the vertex with degree $r+1$ be $r_0$ and the vertex with degree $t+1$ be $t_0$. Connect $r_0$ with $r+1$ vertices. If all those vertices connected to $r_0$ are with degree one, then the graph will be disconnected which is contradiction. Then $r_0$ is connected to $t_0$. By connecting the rest of the vertcies to $t_0$, $H$ is reconstructed. Consequently, $H$ is isomorphic to the double star graph $S_{r,t}$. □

### 3.4.5 The complete graph

**Proposition 3.24.** *A simple graph $G$ is isomorphic to the complete graph $K_n$ if and only if*

$$da(G;x,y) = \frac{(1+xy^2)^n - 1}{y}, \text{ where } n \geq 1 \text{ and } y \neq 0.$$

*Proof.* First, we show that a graph $G$ which is isomorphic to a complete graph $K_n$, has the given defensive alliance polynomial.

The non-empty subsets of $V(G)$ which induce connected subgraphs in $G$, can be partitioned into one part: The part containing the sets of cardinality $i$ in the range of $1 \leq i \leq n$ in which each set contributes the term $x^i y^{2i-1}$ and by summing, we get:

$$\begin{aligned}&\binom{n}{1}x^1y^1 + \binom{n}{2}x^2y^3 + \cdots + \binom{n}{n}x^ny^{2n-1}\\ &= \sum_{i=1}^{n}\binom{n}{i}x^iy^{2i-1}\\ &= \frac{1}{y}(\sum_{i=0}^{n}\binom{n}{i}(xy^2)^i - 1)\\ &= \frac{(1+xy^2)^n - 1}{y}.\end{aligned}$$

Now we prove the converse. Let $n$ be an integer where $n \geq 1$ and $H$ is a graph with the defensive alliance polynomial, $da(H;x,y) = \frac{(1+xy^2)^n-1}{y}$. By Proposition 3.11, the order of $H$ equals $n$. By Proposition 3.14, the degree sequence of $H$ is $(n-1, n-1, \cdots, n-1)$. Consequently, $H$ is isomorphic to the complete graph $K_n$. □

### 3.4.6 The complete bipartite graph

**Lemma 3.25.** *Let $G$ be a simple graph. Let $k_3$ be the number of the subsets which induce connected subgraphs in $G$ with order three. Let the number of connected subgraphs in $G$ with order three and size two be $S_{3,2}$ and with order three and size three be $S_{3,3}$, then*

$$k_3 = S_{3,2} - 2S_{3,3}.$$

*Proof.* Any induced connected subgraph in $G$ with order three will be isomorphic either to a cycle or a path of order three. If the induced connected subgraph in $G$ with order three is a cycle then it will count three subgraphs which are isomorphic to a path of order three. $\square$

**Lemma 3.26.** *Let $G$ be a $\Delta$-regular simple graph. then*

$$S_{3,2} = n\binom{\Delta}{2}.$$

*Proof.* The number of connected subgraphs in $G$ with order three and size two containing a specific vertex $v$ as the common vertex between the two edges is formed by choosing any two vertices from the neighbors is $= \binom{\Delta}{2}$. By multiplying with the number of all vertices $n$, the result follows. $\square$

**Lemma 3.27.** *Let $G$ be a $\Delta$-regular connected simple graph with order $2\Delta$. $G$ is isomorphic to $K_{\Delta,\Delta}$ if and only if $k_3 = n\binom{\Delta}{2}$.*

*Proof.* First, we show that if a graph $G$ is isomorphic to $K_{\Delta,\Delta}$ then $k_3 = n\binom{\Delta}{2}$.

$G$ is isomorphic to $K_{\Delta,\Delta}$ then $G$ has no cycles of order three. By Lemma 3.25 and Lemma 3.26, the result follows.

Now we prove the converse. By Lemma 3.25, $k_3 = S_{3,2}$ and $S_{3,3} = 0$. $G$ is free of cycles of order three. Any vertex $v$ is adjacent to $\Delta$ pairwise nonadjacent vertices which have a degree $\Delta$ and need to be adjacent to $\Delta - 1$ other vertices which are not adjacent to $v$. By constructing the graph, we obtain that $G$ is isomorphic to $K_{\Delta,\Delta}$. $\square$

**Proposition 3.28.** *A simple graph $G$ is isomorphic to the complete bipartite graph $K_{n,m}$ if and only if*

$$da(G;x,y) = nxy^n + mxy^m + y^{n+m}\sum_{i=1}^{n}\sum_{j=1}^{m}\binom{n}{i}\binom{m}{j}x^{i+j}y^{\min\{2i-n,2j-m\}},$$

*where $n, m$ are positive integers.*

*Proof.* First, we show that a simple graph $G$ which is isomorphic to the complete bipartite graph $K_{n,m}$, has the given defensive alliance polynomial. Let $K_{n,m}$ be of the form $G(U \cup W, E)$ where $|U| = n$, $|W| = m$ and $U, W$ are the parts of $K_{n,m}$.

The non-empty subsets of $V(G)$ which induce connected subgraphs in $G$, can be partitioned into the following parts: The part containing the sets of cardinality one from $U$ in which each set contributes the term $xy^n$ and by summing, we get the term $nxy^n$. The part containing the sets of cardinality one from $W$ in which each set contributes the term $mxy^m$ and by summing, we get the term $mxy^m$. The part containing the sets of cardinality more than one in which we choose subset of cardinality $i$ from $U$ and another subset of cardinality $j$ from $W$ which contributes the term $y^{n+m}\left(x^{i+j}y^{\min\{2i-m,2j-n\}}\right)$ and by summing, we get the term $y^{n+m}\sum_{i=1}^{n}\sum_{j=1}^{m}\binom{n}{i}\binom{m}{j}x^{i+j}y^{\min\{2j-m,2i-n\}}$.

Now we prove the converse. Let $n, m$ be positive integers, and $H$ is a graph with

$$da(H;x,y) = nxy^n + mxy^m + y^{n+m}\sum_{i=1}^{n}\sum_{j=1}^{m}\binom{n}{i}\binom{m}{j}x^{i+j}y^{\min\{2i-n,2j-m\}}.$$

By Proposition 3.11, the order of $H$ equals $n+m$. By Proposition 3.12, the size of $H$ equals $nm$. By Proposition 3.13, $H$ is connected. By Proposition 3.14, the degree sequence of $H$ is $(n, n, \cdots, n, m, m, \cdots, m)$. Partition $V(H)$ into two sets $W, U$ where $W$ contains all vertices with degree $n$ and $U$ contain all vertices of degree $m$.

- Case $1$: $n \neq m$, assume $n > m$. Note that $[x^2y^{m+2}]da(G;x,y) = 0$, since this happens only if there is no edge between two vertices with degree $n$. By counting the edges and joining the vertices from $W$ to $U$, $H$ is isomorphic to $K_{n,m}$

- Case $2$: $n = m$ then $H$ is regular. Note that:

$$\begin{aligned} k_3 =& [x^3]da(G;x,1) \\ =& 2n\binom{n}{2}. \end{aligned}$$

  Consequently, by Lemma 3.27, $H$ is isomorphic to the complete bipartite graph $K_{n,n}$.

□

### 3.4.7 The regular graph

**Proposition 3.29.** *A simple graph $G$ is isomorphic to a $\Delta$-regular graph if and only if $[x]da(G;x,y) = ny^{n-\Delta}$.*

*Proof.* First, we show that a graph $G$ which is isomorphic to a $\Delta$-regular graph has $[x]da(G;x,y) = ny^{n-\Delta}$. Every subset of $V(G)$ which induces a connected subgraph in $G$, contributes a term $xy^{n-\Delta}$ and by summing, we get the term $nxy^{n-\Delta}$.

Now we prove the converse. Let $H$ be a graph with $[x]da(G;x,y) = ny^{n-\Delta}$. By Proposition 3.11, the order of $H$ equals $n$. By Proposition 3.14, the degree sequence of $H$ is $(\Delta, \Delta, \cdots, \Delta)$. Consequently, $H$ is isomorphic to a $\Delta$-regular graph. □

**Lemma 3.30.** *Let $G$ be a $\Delta$-regular graph. A subset of $V(G)$ of cardinality $k$ induces a component in $G$ if and only if it contributes in $da(G;x,y)$ a term $x^ky^{\Delta+n}$.*

*Proof.* Every component of order $k$ in a $\Delta$-regular graph, contributes a term with $x^k y^{\Delta+n}$.

To prove the converse, let $S$ be a subset of $V(G)$ of cardinality $k$ which contributes in $da(G;x,y)$ a term $x^k y^{\Delta+n}$. For sake of contradiction, assume that $S$ is not a component. Hence, there is a vertex in $S$ which is connected to other vertices outside $S$. Let the maximum number of vertices connected to a vertex in $S$ from outside of $S$ to be $t$. Hence $S$ contributes in $da(G;x,y)$ a term $x^k y^{n+(\Delta-t)-t} = x^k y^{n+\Delta-2t}$, contradiction since $t \neq 0$. Consequently, $t = 0$ and $S$ contributes a component in $G$. $\square$

**Lemma 3.31.** *For a $\Delta$-regular graph $G$, the number of components with cardinality $k$ is $= [x^k y^{\Delta+n}]da(G;x,y)$.*

*Proof.* From Lemma 3.30, every subset of $V(G)$ with cardinality $k$, induces a component in $G$ if and only if this subset contributes in $da(G;x,y)$ a term $x^k y^{\Delta+n}$. By summing the terms, the result follows. $\square$

**Corollary 3.32.** *Let $G$ be a connected $\Delta$-regular graph. $[x^n]da(G;x,y) = y^{\Delta+n}$.*

*Proof.* From Lemma 3.31, the result follows. $\square$

### 3.4.8 The wheel graph

**Definition 3.33.** *Let $n$ be a positive integer larger than three. The* wheel graph *denoted by $W_n$ is defined by the graph join $C_n + K_1$.*

**Proposition 3.34.** *A simple graph $G$ is isomorphic to the wheel $W_n$ if and only if*

$$\begin{aligned}
[x]da(G;x,y) =& ny^{n-2} + y \text{ and} \\
[x^n]da(G;x,y) =& (n+1)y^{n+2} \text{ and} \\
[x^{n+1}]da(G;x,y) =& y^{n+4}, \text{ where } n \geq 3.
\end{aligned}$$

*Proof.* First, we show that a graph $G$ which is isomorphic to a wheel $W_n$ has the above properties in the proposition. Let $G$ be of the form in Figure 3.4.

The subsets of $V(G)$ with cardinality one which induce connected subgraphs in $G$, can be partitioned into the following parts: The part $\{\{v_0\}\}$ in which $\{v_0\}$ contributes the term $xy$. The part $\{\{v_1\},\{v_2\},\cdots,\{v_n\}\}$ in which every set contributes the term $xy^{n-2}$ and by summing, we get the term $nxy^{n-2}$.

The set $V(G)$ contributes the term $x^{n+1}y^{n+4}$. And if we delete any vertex from $V(G)$, we get a set which contributes the term $x^n y^{n+2}$ and by summing, we get $(n+1)x^n y^{n+2}$.

Now we prove the converse. Let $n$ be an integer, $n \geq 3$, and $H$ is a graph with

$$\begin{aligned}
[x]da(H;x,y) =& ny^{n-2} + y \text{ and} \\
[x^n]da(H;x,y) =& (n+1)y^{n+2} \text{ and} \\
[x^{n+1}]da(H;x,y) =& y^{n+4}.
\end{aligned}$$

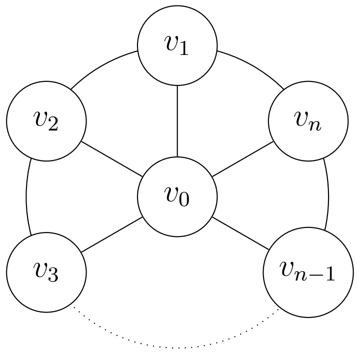



Figure 3.4: A wheel graph

By Proposition 3.11, the order of $H$ equals $n+1$. By Proposition 3.13, $H$ is connected. By Proposition 3.14, the degree sequence of $H$ is $(n, 3, 3, \cdots, 3)$. By Proposition 3.16, the number of cut vertices is zero. Hence all the subgraphs $G \setminus \{v\}$ where $v \in V(G)$, are all connected. Let $v_0$ be the vertex with degree $n$. The specific graph $G \setminus \{v_0\}$ is connected and with degree sequence $(2, 2, \cdots, 2)$ which is isomorphic to the cycle graph $C_n$. By connecting the vertex $v_0$ to every vertex in $C_n$, $H$ is constructed which is isomorphic to the wheel graph $W_n$. □

### 3.4.9 The open wheel graph

**Definition 3.35.** *Let $n$ be a positive integer larger than two. The* open wheel graph *denoted by $W_n^{'}$ is defined by the graph join $P_n + K_1$. This graph is sometimes also known as* Fan.

**Proposition 3.36.** *A simple graph $G$ is isomorphic to the open wheel $W_n^{'}$ if and only if*

$$\begin{aligned}
[x]da(G; x, y) =& 2y^{n-1} + (n-2)y^{n-2} + xy \text{ and} \\
[x^n]da(G; x, y) =& 3y^{n+1} + (n-2)y^{n+2} \text{ and} \\
[x^{n+1}]da(G; x, y) =& y^{n+3}, \text{ where } n \geq 4.
\end{aligned}$$

*Proof.* First, we show that a graph $G$ which is isomorphic to an open wheel $W_n^{'}$, has the above properties in the proposition. Let $G$ be of the form in Figure 3.5.

The subsets of $V(G)$ with cardinality one which induce connected subgraphs in $G$, can be partitioned into the following parts: The part $\{\{v_0\}\}$ in which $\{v_0\}$ contributes the term $xy$. The part $\{\{v_2\}, \{v_3\}, \cdots, \{v_{n-1}\}\}$ in which every set contributes the term $xy^{n-2}$ and by summing, we get the term $(n-2)xy^{n-2}$. The part $\{\{v_1\}\}, \{v_n\}\}$ in which each set contributes the term $xy^{n-1}$ and by summing, we get $2xy^{n-1}$.

The set $V(G)$ contributes the term $x^{n+1}y^{n+3}$.

Each of the subsets $V(G) \setminus \{v_2\}, V(G) \setminus \{v_{n-1}\}$ and $V(G) \setminus \{v_0\}$ contributes the term $x^n y^{n+1}$ and by summing, we get the term $3x^n y^{n+1}$. Each subset of cardinality $n$ but not the previous, contributes the term $x^n y^{n+2}$ and by summing, we get $(n-2)x^n y^{n+2}$.

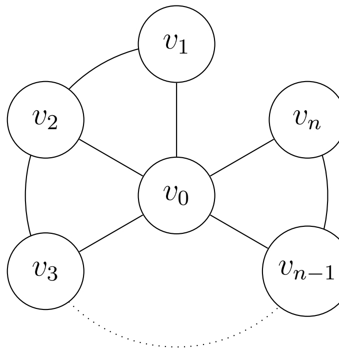



Figure 3.5: An open wheel graph

Now we prove the converse. Let $n$ be an integer, $n \geq 4$, and $H$ is a graph with

$$\begin{aligned}[x]da(H;x,y) =&2y^{n-1} + (n-2)y^{n-2} + xy \text{ and}\\ [x^n]da(H;x,y) =&3y^{n+1} + (n-2)y^{n+2} \text{ and}\\ [x^{n+1}]da(H;x,y) =&y^{n+3}.\end{aligned}$$

By Proposition 3.11, the order of $H$ equals $n+1$. By Proposition 3.13, $H$ is connected. By Proposition 3.14, the degree sequence of $H$ is $(n, 3, 3, \cdots, 3, 2, 2)$. By Proposition 3.16, the number of cut vertices is zero. Hence all the graphs $G \setminus \{v\}$ where $v \in V(G)$, are all connected. Let the vertex with degree $n$ be $v_0$. The specific subgraph $G \setminus \{v_0\}$ is connected and with degree sequence $(2, 2, \cdots, 2, 1, 1)$ which is isomorphic to the path graph $P_n$. By connecting the vertex $v_0$ to every vertex in $P_n$, $H$ is constructed which is isomorphic to the open wheel graph $W_n^{'}$. $\square$

### 3.4.10 The friendship graph

**Definition 3.37.** *Let $n$ be a positive integer. The* friendship graph *denoted by $F_n$ is defined by the graph join $nK_2 + K_1$. This graph is also known as* Windmill graph.

**Proposition 3.38.** *A simple graph $G$ is isomorphic to the friendship $F_n$ if and only if*

$$[x]da(G;x,y) = 2ny^{2n-1} + y, \text{ where } n \text{ is a positive integer.}$$

*Proof.* First, we show that a graph $G$ which is isomorphic to a friendship graph $F_n$, has the above properties in the proposition. Let $G$ be of the form in Figure 3.6.

The subsets of $V(G)$ with cardinality one which induce connected subgraphs in $G$, can be partitioned into the following parts: The part $\{\{v_0\}\}$ in which $\{v_0\}$ contributes the term $xy$. The part $\{\{v_1\}, \{v_2\}, \cdots, \{v_{2n}\}\}$ in which every set contributes the term $xy^{2n-1}$ and by summing we get the term $2nxy^{2n-1}$.

Now we prove the converse. Let $n$ be a positive integer, and $H$ is a graph with

$$[x]da(H;x,y) = 2ny^{2n-1} + y.$$

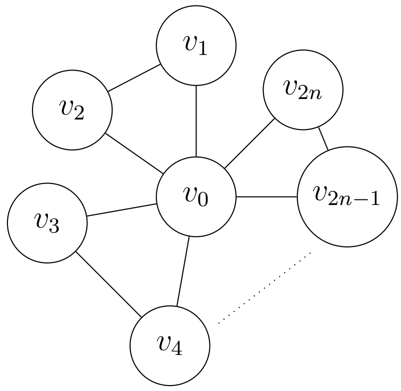



Figure 3.6: A friendship graph

By Proposition 3.11, the order of $H$ equals $2n+1$. By Proposition 3.14, the degree sequence of $H$ is $(2n, 2, 2, \cdots, 2)$. We construct the graph by first connecting by an edge the vertex with degree $2n$ to every other vertex. Second every other vertex choose any arbitrary vertex not the one with degree $2n$ and connect it with an edge to complete its degree. Hence the constructed graph $H$ is isomorphic to the friendship graph $F_n$. □

### 3.4.11 The triangular book graph

**Definition 3.39.** *Let $n$ be a positive integer. The* triangular book graph *denoted by $B_n$ is defined by the graph join $nK_1 + K_2$.*

**Proposition 3.40.** *A simple graph $G$ is isomorphic to the triangular book graph $B_n$ if and only if*

$$[x]da(G; x, y) = 2y + ny^n, \text{ where } n \text{ is a positive integer.}$$

*Proof.* First, we show that a simple graph $G$ which is isomorphic to a triangular book graph $B_n$, has the above properties in the proposition. Let $G$ be of the form in Figure 3.7.

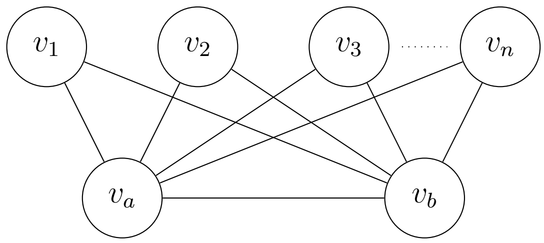



Figure 3.7: A triangular book graph

The subsets of $V(G)$ with cardinality one which induce connected subgraphs in $G$, can be partitioned into the following parts: The part $\{\{v_1\}, \{v_2\}, \cdots, \{v_n\}\}$ in which every set contributes the term $xy^n$ and by summing, we get the term $nxy^n$.

The part $\{\{v_a\}, \{v_b\}\}$ in which every set contributes the term $xy$ and by summing, we get $2xy$.

Now we prove the converse. Let $n$ be a positive integer, and $H$ is a graph with

$$[x]da(H; x, y) = 2y + ny^n.$$

By Proposition 3.11, the order of $H$ equals $n+2$. By Proposition 3.14, the degree sequence of $H$ is $(n+1, n+1, 2, 2, \cdots, 2)$. By connecting the two vertices with degree $n+1$ to every other vertex, $H$ is constructed which is isomorphic to the triangular book graph $B_n$. □

### 3.4.12 The quadrilateral book graph

**Definition 3.41.** *Let $n$ be a positive integer. The* quadrilateral book graph *denoted by $B_{n,2}$ is defined by the graph join $nK_2 + k_2$.*

**Proposition 3.42.** *A simple graph $G$ is isomorphic to the quadrilateral book graph $B_{n,2}$ if and only if*

$$[x]da(G; x, y) = 2y^{n+1} + 2ny^{2n} \text{ and}$$
$$[x^2]da(G; x, y) = ny^{2n+2} + (2n+1)y^{n+3} \text{ and}$$
$$[x^{2n+1}]da(G; x, y) = (2n+2)y^{2n+2} \text{ and}$$
$$[x^{2n+2}]da(G; x, y) = y^{2n+4}, \text{ where } n \text{ is a positive integer.}$$

*Proof.* First, we show that a simple graph $G$ which is isomorphic to a quadrilateral book graph $B_{n,2}$, has the above properties in the proposition. Let $G$ be of the form in Figure 3.8.

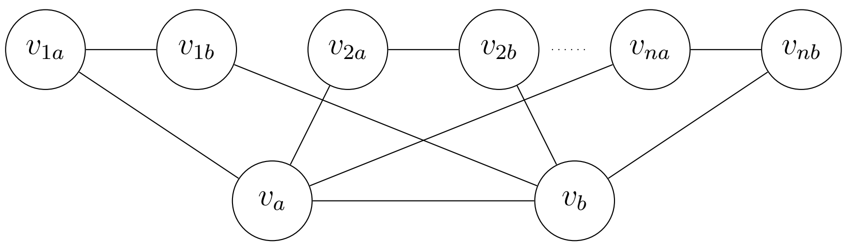



Figure 3.8: A quadrilateral book graph

The subsets of $V(G)$ with cardinality one which induce connected subgraphs in $G$, can be partitioned into the following parts: The part $\{\{v_{1a}\}, \{v_{1b}\}, \{v_{2a}\}, \{v_{2b}\}, \cdots, \{v_{na}\}, \{v_{nb}\}\}$ in which every set contributes the term $xy^{2n}$ and by summing, we get the term $2nxy^{2n}$. The part $\{\{v_a\}, \{v_b\}\}$ in which every set contributes the term $xy^{n+1}$ and by summing, we get $2xy^{n+1}$.

The subsets of $V(G)$ with cardinality two which induce connected subgraphs in $G$, can be partitioned into the following parts: The part $\{\{v_{1a}, v_{1b}\}, \{v_{2a}, v_{2b}\}, \cdots, \{v_{na}, v_{nb}\}\}$ in which every set contributes the term $x^2y^{2n+2}$ and by summing, we get the term $nx^2y^{2n+2}$. The part $\{\{v_a, v_b\}, \{v_a, v_{1a}\}, \{v_a, v_{2a}\}, \cdots, \{v_a, v_{na}\}, \{v_b, v_{1b}\}, \{v_b, v_{2b}\}, \cdots, \{v_b, v_{nb}\}\}$ in which every set contributes the term $x^2y^{n+3}$ and by summing, we get $(2n+1)x^2y^{n+3}$,

The set $V(G)$ contributes the term $x^{2n+2}y^{2n+4}$. And if we delete any vertex from $V(G)$, we get a set which contributes the term $x^{2n+1}y^{2n+2}$ and by summing, we get $(2n+2)x^{2n+1}y^{2n+2}$.

Now we prove the converse. Let $n$ be a positive integer, and $H$ is a graph with

$$\begin{aligned}
[x]da(H;x,y) =& 2y^{n+1} + 2ny^{2n} \text{ and} \\
[x^2]da(H;x,y) =& ny^{2n+2} + (2n+1)y^{n+3} \text{ and} \\
[x^{2n+1}]da(H;x,y) =& (2n+2)y^{2n+2} \text{ and} \\
[x^{2n+2}]da(H;x,y) =& y^{2n+4} \ .
\end{aligned}$$

By Proposition 3.11, the order of $H$ equals $2n+2$. By Proposition 3.12, the size of $H$ equals $3n+1$. By Proposition 3.13, $H$ is connected. By Proposition 3.14, the degree sequence of $H$ is $(n+1, n+1, 2, 2, \cdots, 2)$. By Proposition 3.16, the number of cut vertices is zero. Let the two vertices with degree $n+1$ be $v_a$ and $v_b$ respectively. A subset of cardinality two which induces a connected subgraph in $G$, and contains two vertices of degree two is the only subset of cardinality two which contributes a term $[x^2y^{2n+2}]$. Then, the number of edges connecting two vertices of degree two is $[x^2y^{2n+2}]da(G;x,y)$ and equals $n$. The number of the rest edges is $2n+1$. But the number of edges which are incident to vertices of degree two are necessary only $2n$. Hence, the last edge is necessarily between the two vertices of degree $n+1$. At this point we have a graph like the one in Figure 3.9 where the number in the vertices is

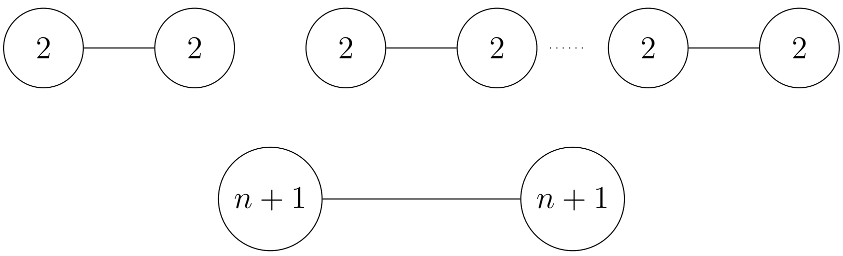



Figure 3.9: A quadrilateral book graph

its degree. The two vertices of degree $n+1$ need to be connected to $n$ vertices of degree two. But a vertex with degree $n+1$ will never be connected to two adjacent vertices of degree two, since this will make this vertex of degree $n+1$ a cut vertex which contradicts the statement that $H$ has no cut vertices. This means that every vertex of degree $n+1$ will be connected to only non-adjacent vertices of degree two, which yields the quadrilateral book graph $H$. $\square$

## 3.5 Attaching a vertex to a complete graph

**Proposition 3.43.** *Let $v_0$ be a vertex and $n$ a positive integer. Let $H$ be a simple graph formed from $K_n \cup \{v_0\}$ by joining some vertices to $v_0$. Let $V(H) \setminus \{v_0\} = R \cup S$*

*where $R = \{r_1, r_2, \cdots, r_r\}$, $r = |R|$ where $R$ is the set of vertices in $H$ which are adjacent to $v_0$ and $S = \{s_1, s_2, \cdots, s_s\}$, $s = |S|$ where $S$ is the set of vertices in $H$ which are not adjacent to $v_0$. Let $G$ be a simple graph. $G$ is isomorphic to $H$ if and only if*

$$\begin{aligned} da(G; x, y) =& (1 + xy^2)da(K_r; x, y) + y\, da(K_s; x, y) + y\, da(K_r; x, y)da(K_s; x, y) \\ &+ xy^{n+1-r} + xy\, da(K_r; x, y) \sum_{j=1}^{s} \binom{s}{j} x^j y^{\min\{2j, s+1\}} . \end{aligned}$$

*Proof.* The subsets of $V(G)$ with cardinality one which induce connected subgraphs in $G$, can be partitioned into the following parts: The part $\{\{v_0\}\}$ in which $\{v_0\}$ contributes the term $xy^{n+1-r}$.
The part containing the sets of cardinality $i$ in the range of $1 \leq i \leq r$ formed only from vertices in $R$ in which each set contributes the term $x^i y^{2i-1}$ and by summing, we get:

$$\begin{aligned} &\binom{r}{1} x^1 y^1 + \binom{r}{2} x^2 y^3 + \cdots + \binom{r}{r} x^r y^{2r-1} \\ &= \sum_{i=1}^{r} \binom{r}{i} x^i y^{2i-1} \\ &= \frac{1}{y} \left( \sum_{i=0}^{r} \binom{r}{i} \left(xy^2\right)^i - 1 \right) \\ &= \frac{\left(1 + xy^2\right)^r - 1}{y} \\ &= da(K_r; x, y). \end{aligned}$$

The part containing the sets of cardinality $i$ in the range of $1 \leq i \leq s$ arises only from the vertices in $S$ in which each set contributes the term $x^i y^{2i}$ and by summing, we get:

$$\begin{aligned} &\binom{s}{1} x^1 y^2 + \binom{s}{2} x^2 y^4 + \cdots + \binom{s}{s} x^s y^{2s} \\ &= \sum_{i=1}^{s} \binom{s}{i} x^i y^{2i} \\ &= y\, \frac{1}{y} \left( \sum_{i=0}^{s} \binom{s}{i} \left(xy^2\right)^i - 1 \right) \\ &= y\, \frac{\left(1 + xy^2\right)^s - 1}{y} \\ &= y\, da(K_s; x, y). \end{aligned}$$

The part containing the sets of cardinality $i$ in the range of $2 \leq i \leq r + 1$ results from $\{v_0\}$ and the vertices in $R$ in which each set contributes the term $x^{i+1} y^{2i+1}$ and by summing, we get:

$$\begin{aligned}
&\binom{r}{1}x^2y^3+\binom{r}{2}x^3y^5+\cdots+\binom{r}{r}x^{r+1}y^{2r+1}\\
&=\sum_{i=1}^{r}\binom{r}{i}x^{i+1}y^{2i+1}\\
&=xy^2\frac{1}{y}\left(\sum_{i=0}^{r}\binom{r}{i}\left(xy^2\right)^i-1\right)\\
&=xy^2\frac{\left(1+xy^2\right)^r-1}{y}\\
&=xy^2da(K_r;x,y).
\end{aligned}$$

The part containing the sets formed from subsets of $R$ of cardinality $i$ in the range of $1\le i\le r$ and subsets of $S$ of cardinality $j$ in the range of $1\le j\le s$ in which each set contributes the term $x^{i+j}y^{(r+s+1)+(i+j-1)-(r+s+1-i-j)}$ and by summing, we get:

$$\begin{aligned}
&y\binom{r}{1}x^1y^1\binom{s}{1}x^1y^1+y\binom{r}{1}x^1y^1\binom{s}{2}x^2y^3+\cdots+y\binom{r}{1}x^1y^1\binom{s}{3}x^3y^5\\
&+y\binom{r}{2}x^2y^3\binom{s}{1}x^1y^1+y\binom{r}{2}x^2y^3\binom{s}{2}x^2y^3+\cdots+y\binom{r}{2}x^2y^3\binom{s}{3}x^3y^5\\
&\vdots\\
&+y\binom{r}{r}x^ry^{2r-1}\binom{s}{1}x^1y^1+y\binom{r}{r}x^ry^{2r-1}\binom{s}{2}x^2y^3+\cdots\\
&+y\binom{r}{r}x^ry^{2r-1}\binom{s}{s}x^sy^{2s-1}\\
&=y\sum_{i=1}^{r}\binom{r}{i}x^iy^{2i-1}\sum_{j=1}^{s}\binom{s}{j}x^jy^{2j-1}\\
&=y\ \frac{1}{y}\left(\sum_{i=0}^{r}\binom{r}{i}\left(xy^2\right)^i-1\right)\ \frac{1}{y}\left(\sum_{j=0}^{s}\binom{s}{j}\left(xy^2\right)^j-1\right)\\
&=y\ \frac{\left(1+xy^2\right)^r-1}{y}\ \frac{\left(1+xy^2\right)^s-1}{y}\\
&=y\ da(K_r;x,y)\ da(K_s;x,y).
\end{aligned}$$

The part containing the sets formed from $v_0$ and subsets of $R$ of cardinality $i$ in the range of $1\le i\le r$ and subsets of $S$ of cardinality $j$ in the range of $1\le j\le s$ in which each set contributes the term $x^{i+j+1}y^{2i+min\{2j,s+1\}}$ and by summing, we get:

$$
\begin{aligned}
& xy\binom{r}{1}x^1y^1\binom{s}{1}x^1y^{\min\{2,s+1\}} + xy\binom{r}{1}x^1y^1\binom{s}{2}x^2y^{\min\{4,s+1\}} + \cdots \\
& + xy\binom{r}{1}x^1y^1\binom{s}{s}x^sy^{\min\{2s,s+1\}} \\
& + xy\binom{r}{2}x^2y^3\binom{s}{1}x^1y^{\min\{2,s+1\}} + xy\binom{r}{2}x^2y^3\binom{s}{2}x^2y^{\min\{4,s+1\}} + \cdots \\
& + xy\binom{r}{2}x^2y^3\binom{s}{s}x^sy^{\min\{2s,s+1\}} \\
& \vdots \\
& + xy\binom{r}{r}x^ry^{2r-1}\binom{s}{1}x^1y^{\min\{2,s+1\}} + xy\binom{r}{r}x^ry^{2r-1}\binom{s}{2}x^2y^{\min\{4,s+1\}} + \cdots \\
& + xy\binom{r}{r}x^ry^{2r-1}\binom{s}{s}x^sy^{\min\{2s,s+1\}} \\
& = xy\sum_{i=1}^{r}\binom{r}{i}x^iy^{2i-1}\sum_{j=1}^{s}\binom{s}{j}x^jy^{\min\{2j,s+1\}} \\
& = xy\,\frac{1}{y}\left(\sum_{i=0}^{r}\binom{r}{i}\left(xy^2\right)^i - 1\right)\sum_{j=1}^{s}\binom{s}{j}x^jy^{\min\{2j,s+1\}} \\
& = xy\,\frac{\left(1+xy^2\right)^r - 1}{y}\sum_{j=1}^{s}\binom{s}{j}x^jy^{\min\{2j,s+1\}} \\
& = xy\ da(K_r;x,y)\sum_{j=1}^{s}\binom{s}{j}x^jy^{\min\{2j,s+1\}}.
\end{aligned}
$$

Now we prove the converse. Let $r$ and $s$ be integers and $H$ is a graph with the defensive alliance polynomial,

$$
\begin{aligned}
da(H;x,y) = (1+xy^2)da(K_r;x,y) + y\ da(K_s;x,y) + y\ da(K_r;x,y)da(K_s;x,y) \\
+xy^{n+1-r} + xy\ da(K_r;x,y)\sum_{j=1}^{s}\binom{s}{j}x^jy^{\min\{2j,s+1\}}.
\end{aligned}
$$

By Proposition 3.11, the order of $H$ equals $r+s+1$. Let $r+s=n$. By Proposition 3.14, the degree sequence of $H$ consist of $n$ $r$ times then $(n-1)$ $s$ times then r one time: $(n,n,\cdots,n,n-1,n-1\cdots,n-1,r)$. By constructing first all the vertices with degree $n$. Note that no term left of the form $ax^2y^{(r+s+1)+1-(r-1)}$, hence no vertex with degree $n-1$ is connected to the vertex of degree $r$. Hence we choose arbitrary $s$ vertices and connect them to each other. $\square$

## 3.6 the distinctive power of the defensive alliance polynomial

The authors in [CGATu14], showed how the alliance polynomial can characterize some classes of graphs which were not characterized by other well-known graph polynomials like the tutte polynomial, the domination polynomial, the independence polynomial, the matching polynomial, the bivariate polynomial, and the subgraph component polynomial.

As a generalization for the alliance polynomial, the defensive alliance polynomial has at least the same power. In this section, we present two pairs of graphs that cannot be characterized by the alliance polynomial but can be characterized by the defensive alliance polynomial.

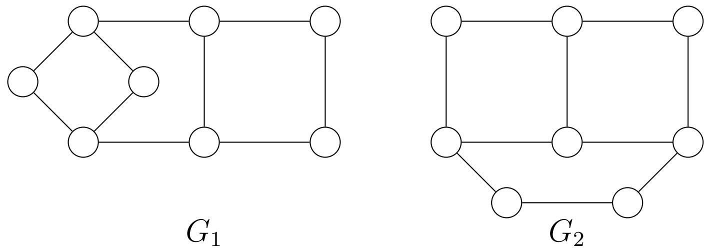



Figure 3.10: First pair of graphs

The alliance polynomial of the two graphs in the Figure 3.10 is:

$$A(G_1;x) = A(G_2;x) = x^{10} + 7x^9 + 37x^8 + 63x^7 + 4x^6 + 4x^5.$$

By applying the program in Section D.1 we obtain the defensive alliance polynomial of $G_1$:

$$\begin{aligned} da(G_1;x,y) =& x^8y^{10} + 2x^7y^9 + 6x^7y^8 + x^6y^9 + 14x^6y^8 + 7x^6y^7 \\ &+ 2x^5y^9 + 10x^5y^8 + 16x^5y^7 + 2x^4y^9 + 4x^4y^8 + 17x^4y^7 \\ &+ 2x^3y^8 + 14x^3y^7 + x^2y^8 + 9x^2y^7 + 4xy^6 + 4xy^5. \end{aligned}$$

By applying the program in Section D.1 we obtain the defensive alliance polynomial of $G_2$:

$$\begin{aligned} da(G_2;x,y) =& x^8y^{10} + 3x^7y^9 + 5x^7y^8 + x^6y^9 + 15x^6y^8 + 7x^6y^7 \\ &+ x^5y^9 + 11x^5y^8 + 15x^5y^7 + 2x^4y^9 + 2x^4y^8 + 19x^4y^7 \\ &+ 3x^3y^8 + 13x^3y^7 + x^2y^8 + 9x^2y^7 + 4xy^6 + 4xy^5. \end{aligned}$$

Another pair of graphs:
The alliance polynomial of the two graphs in the Figure 3.11 is:

$$A(G_3;x) = A(G_4;x) = 8x^9 + 26x^8 + 20x^7 + 11x^6 + 2x^5 + x^4.$$

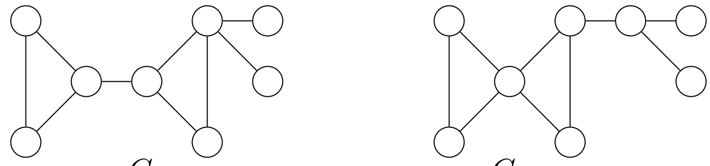



Figure 3.11: Second pair of graphs

By applying the program in Section D.1 we obtain the defensive alliance polynomial of $G_3$ is:

$$\begin{aligned} da(G_3;x,y) =& x^8y^9 + 3x^7y^9 + 2x^7y^8 + 9x^6y^8 + x^6y^7 + x^5y^9 \\ &+ 7x^5y^8 + 3x^5y^7 + x^5y^6 + 2x^4y^9 + 3x^4y^8 + 5x^4y^7 \\ &+ 2x^4y^6 + x^3y^9 + 4x^3y^8 + 5x^3y^7 + x^3y^6 + x^2y^8 \\ &+ 4x^2y^7 + 4x^2y^6 + 2xy^7 + 3xy^6 + 2xy^5 + xy^4. \end{aligned}$$

By applying the program in Section D.1 we obtain the defensive alliance polynomial of $G_4$ is:

$$\begin{aligned} da(G_4;x,y) =& x^8y^9 + 3x^7y^9 + 2x^7y^8 + 2x^6y^9 + 7x^6y^8 + x^6y^7 \\ &+ x^5y^9 + 7x^5y^8 + 3x^5y^7 + x^5y^6 + 5x^4y^8 + 5x^4y^7 \\ &+ 2x^4y^6 + x^3y^9 + 4x^3y^8 + 5x^3y^7 + x^3y^6 + x^2y^8 \\ &+ 4x^2y^7 + 4x^2y^6 + 2xy^7 + 3xy^6 + 2xy^5 + xy^4. \end{aligned}$$

## 3.7 Open questions

In this section we list some open questions which we think could be interesting to do further research about defensive alliance polynomial.

- Can we find a pair of non isomorphic graphs which are $da-$equivalent (have the same defensive alliance polynomial)?
- Can we implement an algorithm which construct a graph (at least one graph) from its defensive alliance polynomial?
- For a graph $G$, can we find a relation between $da(G;x,y)$ and $da(G \setminus \{v\};x,y)$ where $v$ is a vertex in the graph $G$?
- For a graph $G$, can we find a relation between $da(G;x,y)$ and $da(G \setminus \{e\};x,y)$ where $e$ is an edge in the graph $G$?
- For a graph $G$, can we find a relation between $da(G;x,y)$ and $da(G + \{v\};x,y)$ where $v$ is any vertex not in $G$?

- For two graphs $G$ and $H$, can we find a relation between $da(G;x,y)$, $da(H;x,y)$ and $da(G+H;x,y)$?
- For a graph $G$, can we find a relation between $da(G;x,y)$ and $da(K_{|V(G)|};x,y)$?
- For a graph $G$, can we find a relation between $da(G;x,y)$ and $da(E_{|V(G)|};x,y)$?
- For a graph $G$, can we find a relation between $da(G;x,y)$ and $da(\bar{G};x,y)$?
- Can we characterize uniquely more special classes of graphs by their defensive alliance polynomial?
- Can we find relations with other graph polynomials?
- Can we compare the distinctive power of the defensive alliance polynomial with other graph polynomials?

# A. The reconstruction conjecture

## A.1 Introduction

In Section A.1.1, we present the reconstruction of a graph from a special multi-set of its subgraphs. We define the notion of a deck and what information we can obtain about a graph from its deck. Then, we explain Kelly's lemma and state the reconstruction conjecture.

In Section A.3, we present the reconstruction of some special classes of graphs: the regular graphs, the tree graphs, the path graphs, the cycle graphs, the star graphs, the double star graphs, the complete graphs, the complete bipartite graphs, the wheel graphs, the open wheel graphs, the friendship graphs, the triangular book graphs, the quadrilateral book graph and the disconnected graphs. Lastly, we discuss the complement of a graph and its relation to the reconstructability of a graph.

In Section A.4, we list some variants to the idea of reconstructing a graph from its deck.

Note that through this appendix, we mean by a graph, unlabeled graph.

### A.1.1 Definitions and properties

**Definition A.1.** *Let $G$ be a simple graph. The minimum order of a component in $G$ is $c_{min}(G)$. The maximum order of a component in $G$ is $c_{max}(G)$.*

**Definition A.2.** *Let $G(V, E)$ be a simple graph and $v$ a vertex in $V(G)$. The graph $G - v$ denoted by $G_v$ is a* vertex-deleted subgraph *of $G$ obtained by removing $v$ from $V(G)$ and all the incident edges to $v$. For examples see Section B.0.1.*

**Definition A.3.** *Let $G(V, E)$ be a simple graph. The* deck *denoted by $D(G)$ is the multi-set $\{G_v : v \in V(G)\}$.*

*$D'$ is the multi-set $\{G_v : v \in V(G), k(G_v) = k(G)\}$.*

*$D''$ is the multi-set $\{G_v : v \in V(G), k(G_v) = k(G), c_{min}(G_v) = c_{min}(G) - 1\}$.*

**Definition A.4.** *Let $G(V, E)$ be a simple graph. A graph $H$ is said to be a* construction *of $G$ if $V(G) = V(H)$ and for every $v$ in $V(G)$, $G_v$ is isomorphic to $H_v$. Further $G$ is called* reconstructable *if every construction of $G$ is isomorphic to $G$.*

*Note: Not every graph is reconstructable, for example $K_2$ or $2K_1$.*

**Definition A.5.** *Let $\tau$ be a class of graphs. The class $\tau$ is called* recognizable *if for every graph $G$ in $\tau$, every construction of $G$ is isomorphic to a graph in $\tau$.*

**Definition A.6.** *A function $s$ defined on a class $\tau$ of graphs, is reconstructable, if for each graph $G$ in $\tau$, $s$ takes the same value for all constructions of $G$.*

### A.1.2 Kelly's lemma

**Lemma A.7** (Kelly's lemma)**.** *[BH74] Let $G$ and $H$ be two simple graphs where $|V(H)| < |V(G)|$. Let $s(H, G)$ be the number of subgraphs in $G$ isomorphic to $H$.*

*$s(H, G)$ is reconstructable.*

*Proof.* Each subgraph in $G$ isomorphic to $H$ occurs in exactly $|V(G)| - |V(H)|$ of the subgraphs $G_v$. Therefore

$$s(H, G) = \frac{\sum_{v \in V(G)} s(H, G_v)}{|V(G)| - |V(H)|}.$$

□

Section B.0.3 shows an example.

**The order [Har77]** By applying Kelly's lemma to a graph $G$ for a subgraph $K_1$ we get the number of vertices of $G$ which is the order of $G$. For examples see Section B.0.3.

**The size [Har77]** By applying Kelly's lemma to a graph $G$ for a subgraph $K_2$ we get the number of edges of $G$ which is the size of $G$. For examples see Section B.0.3.

**The degree sequence [BH74]** By obtaining the order and the size, we could also obtain the degree sequence. Note that in every $G_v$, the edges incident from $v$ is removed, and the number of such edges is the degree of $v$ in $G$. So for every $v$ in $V(G)$, deg($v$) = $|E(G)| - |E(G_v)|$.

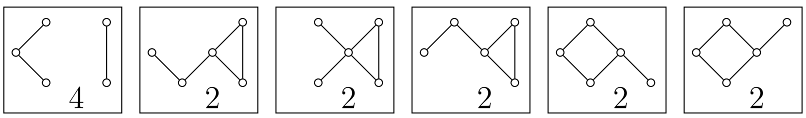



Figure A.1: Degree sequence of $G$ is $(4, 2, 2, 2, 2, 2)$.

#### The number of components

**Lemma A.8.** *Let $G$ be a simple graph. The number of components of $G$ is reconstructable.*

*Proof.* When a vertex $v$ in $G$ is deleted, either the number of components of $G$ will increase if $v$ is a cut vertex, or it will stay the same if $v$ is not cut vertex. Hence, the minimum number of components in any $G_v$ is the number of components of $G$. □

#### The minimum order of a component

**Lemma A.9.** *Let $G$ be a simple graph. The minimum order of a component in $G$ is reconstructable.*

*Proof.* Consider the $D^{'}(G)$, the minimum order of a component in $G$ equals one plus the minimum order of any component in all $G_v$ in $D^{'}(G)$ since it is obtained from the component of minimum order in $G$ by deleting a non cut vertex. $\square$

#### The maximum order of a component

**Lemma A.10.** *Let $G$ be a simple graph. The maximum order of a component in $G$ is reconstructable.*

*Proof.* Consider the $D^{''}(G)$, the maximum order of a component in $G$ equals the maximum order of any component in all $G_v$ in $D^{''}(G)$ since $D^{''}(G)$ is obtained from deleting a non cut vertex from a component in $G$ with minimum order. $\square$

## A.2 The reconstruction conjecture RC [BH74]

All finite simple undirected graphs with at least three vertices are reconstructable [BH74].

## A.3 The recognition and the reconstruction of special graph classes

### A.3.1 Graphs with a node with degree $n-1$

**Lemma A.11.** *Let $G$ be a graph with a vertex with degree $n-1$ where $n \geq 3$. $G$ is reconstructable.*

*Proof.* Choose from the deck of $G$, the specific subgraph $G \setminus \{v\}$, where $v$ is a vertex in $G$ with degree $n-1$. Add $v$ and connect it to every other vertex to reconstruct $G$. $\square$

### A.3.2 The regular graphs

**Lemma A.12.** *[Har77] Regular graphs with order at least three are recognizable.*

*Proof.* Let $G$ be a graph and let $\tau$ be the class of regular graphs. By obtaining the degree sequence from the deck of $G$, we recognize that $G$ is isomorphic to a graph in $\tau$. $\square$

**Lemma A.13.** *[Har77] Regular graphs with order at least three are reconstructable.*

*Proof.* Let $G$ be a $\Delta-$regular graph . Let $G_v$ be a specific vertex-deleted subgraph of $G$. Add a vertex to $G_v$ and from this vertex only one unique choice is to join it with all the vertices in $G_v$ with degree $\Delta - 1$. □

For examples see Section B.0.4.

### A.3.3 The tree graphs

**Lemma A.14.** *[Har77] Tree graphs with order at least three are reconstructable.*

### A.3.4 The path graphs

**Corollary A.15.** *The path $P_n$ graph is reconstructable.*

*Proof.* By Lemma A.14. □

### A.3.5 The cycle graphs

**Corollary A.16.** *The cycle $C_n$ graph is reconstructable.*

*Proof.* By Lemma A.13. □

### A.3.6 The star graphs

**Corollary A.17.** *The star $S_n$ graph is reconstructable.*

*Proof.* By Lemma A.14. □

### A.3.7 The double star graphs

**Corollary A.18.** *The double star $S_{r,t}$ graph is reconstructable.*

*Proof.* By Lemma A.14. □

### A.3.8 The complete graph graphs

**Corollary A.19.** *The complete graph $K_n$ is reconstructable.*

*Proof.* By Lemma A.13. □

### A.3.9 The complete bipartite graph graphs

**Lemma A.20.** *The class of complete bipartite graphs is reconstructable.*

*Proof.* Let $K_{n,m}$ be on the form $G(U \cup W, E)$ where $|U| = n$, $|W| = m$ and $U, W$ are the partition sets of $K_{n,m}$. There are three cases according to $n$ and $m$.

Case 1: $n = m$ then the graph is regular which is reconstructable class.

Case 2: $n = m + 1$. Obtain the degree sequence which will be on the form $\{n, n, \cdots, n, n-1, n-1, \cdots, n-1\}$. Choose specific $G_v$ in which the degree sequence

is of the form $\{n, n, \cdots, n, n-2, n-2, \cdots, n-2\}$. Add a vertex to $G_v$ and connect it to the vertices which needs to complete its degree.
Or Choose specific $G_v$ in which the degree sequence is of the form $\{n-1, n-1, \cdots, n-1, n-1, n-1, \cdots, n-1\}$. Add a vertex to $G_v$ and connect it to any $n$ vertices.

Case 3: $n \geqslant m+2$. Obtain the degree sequence which will be on the form $\{n, n, \cdots, n, m-a, m-a, \cdots, m-a\}$ where $a \geqslant 2$. Choose any $G_v$ , add a vertex and connect it to all the vertices which needs to complete its degree. □

### A.3.10 The wheel graphs

**Corollary A.21.** *The wheel $W_n$ graph is reconstructable.*

*Proof.* By Lemma A.11. □

### A.3.11 The open wheel graphs

**Corollary A.22.** *The open wheel $W_n'$ graph is reconstructable.*

*Proof.* By Lemma A.11. □

### A.3.12 The friendship graphs

**Corollary A.23.** *The friendship $F_n$ graph is reconstructable.*

*Proof.* By Lemma A.11. □

### A.3.13 The triangular book graph graphs

**Corollary A.24.** *The triangular book $B_n$ graph is reconstructable.*

*Proof.* By Lemma A.11. □

### A.3.14 The quadrilateral book graph graphs

**Corollary A.25.** *The quadrilateral book $B_{n,2}$ graph is reconstructable.*

*Proof.* Obtain the degree sequence. Choose specific $G_v$ in the deck with one vertex of maximum degree deleted, then connect it to all the other vertices which needs to complete its degree. □

### A.3.15 Disconnected graphs

**Lemma A.26.** *[CKS73] A graph $G$ is connected if and only if at least two of its vertex-deleted subgraphs in its deck are connected.*

*Proof.* Consider the longest path in $G$. The endpoints of this path can not be cut vertices. Both of them are only connected to vertices in the path since if they are connected to vertices not on path, the path could be extended which is contradiction for longest path.

To prove the converse, assume we have two specific connected vertex-deleted subgraphs, $G_u$ and $G_v$. Consider $G_u$, either it was obtained from a connected graph or obtained from a graph composed of an isolated vertex and $G_u$. But if we consider $G_v$, it is connected with no isolated vertex, then $G$ must be connected with no isolated vertices. □

**Corollary A.27.** *[CKS73] A graph $G$ is disconnected if and only if at most one of its vertex-deleted subgraphs in its deck is connected.*

*Proof.* The contrapositive of the previous Lemma. □

**Corollary A.28.** *[CKS73] Disconnected graphs with order at least three are recognizable.*

**Lemma A.29.** *[CKS73] Disconnected graphs with order at least three are reconstructable.*

*Proof.* If there were any isolated vertices then construct the graph by choosing a $G_v$ which needs $0$ edges to be added and just add an isolated vertex to reconstruct $G$. Assuming no isolated vertices exist, and according to order of components in $G$ we have different cases:

| Number of components with order: | $c_{min}(G)$ | $c_{min}(G)+1$ | $(c_{min}(G)+2)\geqslant$ |
|---|---|---|---|
| Case 1 | $\geqslant 1$ | $\geqslant 0$ | $\geqslant 1$ |
| Case 2 | $1$ | $k-1$ | $0$ |
| Case 3 | $\geqslant 2$ | $\geqslant 0$ | $0$ |

Case $1$:

| Number of components with order: | $c_{min}(G)$ | $c_{min}(G)+1$ | $(c_{min}(G)+2)\geqslant$ |
|---|---|---|---|
| Case 1 | $\geqslant 1$ | $\geqslant 0$ | $\geqslant 1$ |

Consider a graph $G_u$ from $D^{''}(G)$. Note that the component of order $c_{min}(G)-1$ in $G_u$ is obtained from removing a non cut vertex from a component of order $c_{min}(G)$ in $G$. Also, note that all the components in $G_u$ of order at least $(c_{min}(G)+1)$ are also components in $G$.

Let the number of components of order $(c_{min}(G))$ in $G_u$ be $a$. Consider a graph $G_w$ in $D^{'}(G)\setminus D^{''}(G)$ and the number of components of order $(c_{min}(G)) = a+1$. Note in $G_w$ there is no component of order $c_{min}(G)-1$, thus $G_w$ has a component obtained from $G$ by removing a non cut vertex from a component with order at least $c_{min}(G)+2$ but it cannot be from a component of order $c_{min}(G)+1$ since the number of components of order $c_{min}(G)$ equals $a+1$. Note in $G_w$ the components of order $(c_{min}(G))$ are the components of order $c_{min}(G)$ in $G$. So from $G_w$ we obtain the components in $G$ with order $(m+1)$ and from $G_u$ we obtain all other components, Thus $G$ is reconstructed.

Case $2$:

| Number of components with order: | $c_{min}(G)$ | $c_{min}(G)+1$ | $(c_{min}(G)+2)$ |
|---|---|---|---|
| Case 2 | $1$ | $k-1$ | $0$ |

From a graph in $D^{''}(G)$ we can obtain all graphs of order $c_{min}(G)+1$ in $G$. We need to obtain the components of order $c_{min}(G)$ in $G$. Note all graphs in $D^{'}(G) \setminus D^{''}(G)$ contain a pair of components of order $c_{min}(G)$. According to these pairs we have three cases:

Case $2.1$: All pairs are isomorphic to a graph $H$. Then $H$ is in $G$.

Case $2.2$: All pairs are composed of two non isomorphic graphs. One of them is in $G$ and the other is obtained by removing a non cut-vertex from any component in $G$ with order $c_{min}(G)+1$. By removing a non cut-vertex from any component of order $c_{min}(G)+1$, we can differentiate between the pair and obtain the one in $G$.

Case $3.3$: The pairs are sometimes different. The component in $G$ with order $c_{min}(G)$ will always be there in every pair.

Case $3$:

| Number of components with order: | $c_{min}(G)$ | $c_{min}(G)+1$ | $(c_{min}(G)+2)$ |
|---|---|---|---|
| Case 3 | $k$ | $0$ | $0$ |

Note the components in every graph in $D^{''})$ of order $c_{min}(G)$ are also components in $G$. Note for every component $H$ in a $G_v$ in $D^{''}$, the number of components isomorphic to $H$ in $G$ is the maximum number of component isomorphic to $H$ in any graph in $D^{''}(G)$, unless all the components of order $c_{min}(G)$ in $G$ are isomorphic, then we increment the number by one. □

### A.3.16 Complement

**Lemma A.30.** *[Har77] A graph $G$ is reconstructable if and only if $\bar{G}$ is reconstructable.*

*Proof.* There is a bijection $f$ between $G$ and $\bar{G}$ which maps every $v$ in $V(G)$ to a $u$ in $V(\bar{G})$ and if $\{v_1, v_2\} \in E(G)$ then $\{f(v_1), f(v_2)\} \notin E(\bar{G})$ and if $\{v_1, v_2\} \notin E(G)$ then $\{f(v_1), f(v_2)\} \in E(\bar{G})$. Then for every vertex $v$ in $V(G)$ the bijection holds between $G_v$ and $\bar{G}_{f(v)}$. Thus if $G$ is constructible then $\bar{G}$ is constructible.

The converse is proved by similar way. □

This Lemma helps us to determine if a graph is constructible from its complement. For example, if the complement of the graph is disconnected then the graph is reconstructable since the class of disconnected graphs are reconstructable.

## A.4 Other variants of decks

- The deck is a set not a multi-set.
- Edge deleted deck, such that edges is deleted from the graph $G$ not vertices.
- Special kind of deck, for example in case of trees: maximal subtrees.

# B. The reconstruction conjecture examples

### B.0.1 Vertex-deleted subgraph

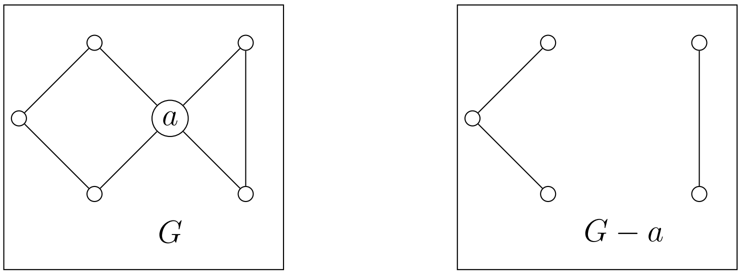



Figure B.1: A vertex-deleted subgraph.

### B.0.2 Deck

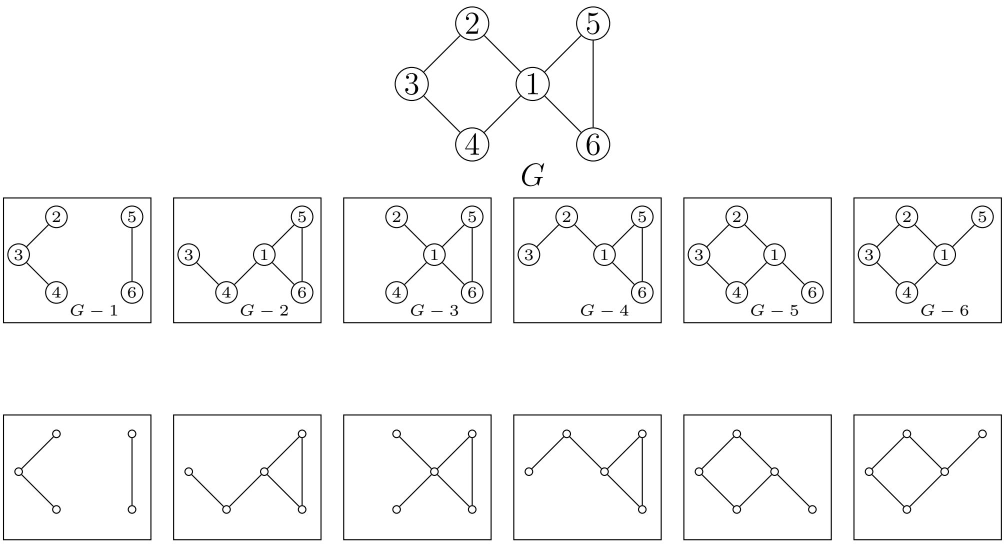



Figure B.1: A deck.

### B.0.3 Kelly's lemma example

By applying Kelly's lemma on a graph $G$ for a subgraph $C_3$ and $C_4$ we get in $G$ the number of cycles of length three and four respectively.

The Figure B.2 shows how to obtain the number of subgraphs of $G$ which is isomorphic to $C_3$, from the deck of $G$.

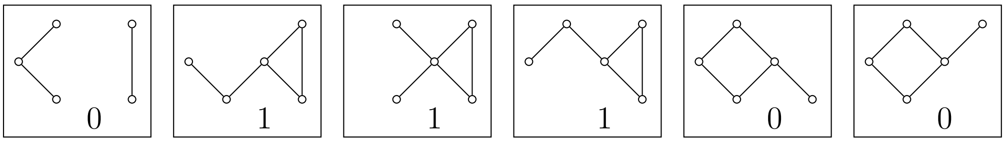



Figure B.2: $s(C_3, G) = \frac{0+1+1+1+0+0}{6-3} = 1.$

The Figure B.3 shows how to obtain the number of subgraphs of $G$ which is isomorphic to $C_4$, from the deck of $G$.

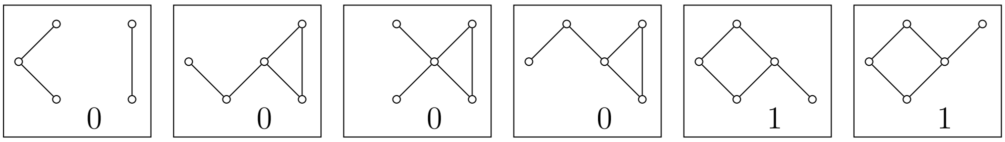



Figure B.3: $s(C_4, G) = \frac{0+0+0+0+1+1}{6-4} = 1.$

By applying Kelly's lemma on a graph $G$ for a subgraph $K_2$ we get in $G$ the number of edges which is known to be the size of $G$. By using the subgraph $K_1$ we get the number of vertices in $G$ which is known as the order of $G$.

The Figure B.4 shows how to obtain the number of subgraphs of $G$ which is isomorphic to $K_2$, from the deck of $G$.

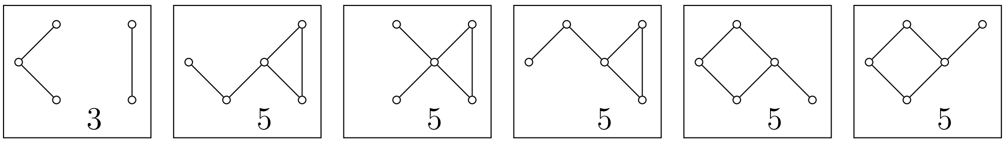



Figure B.4: $s(K_2, G) = \frac{3+5+5+5+5+5}{6-2} = 7.$

The Figure B.5 shows how to obtain the number of subgraphs of $G$ which is isomorphic to $K_1$, from the deck of $G$.

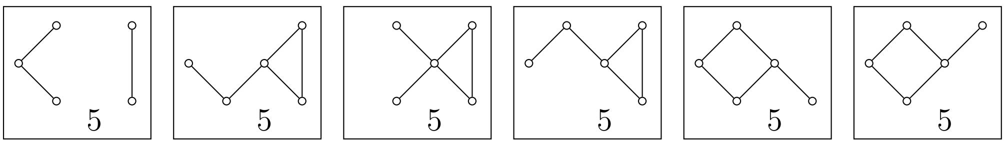



Figure B.5: $s(K_1, G) = \frac{5+5+5+5+5+5}{6-1} = 6.$

### B.0.4 Regular graphs example

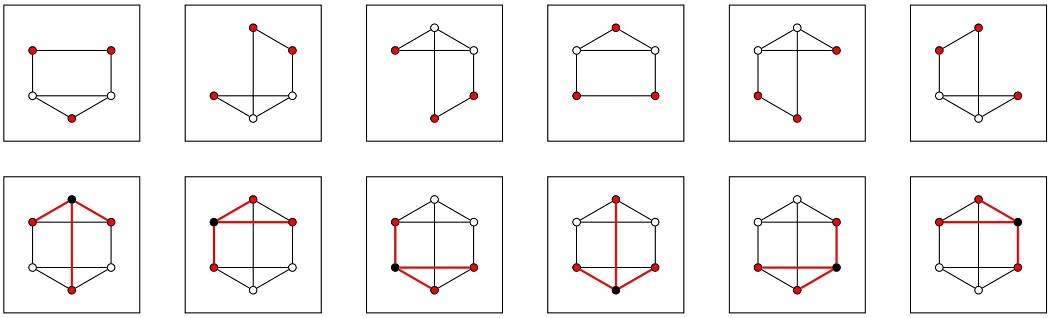


Figure B.6: Regular graph.

# C. The alliance examples

### C.0.1 Defensive alliance

Only one vertex without any allies will lose whenever attacked by more than one neighbour.

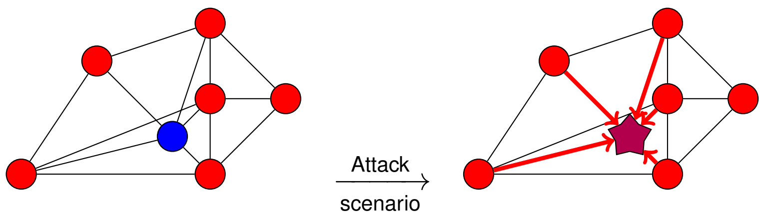



Figure C.1: No allies to defend.

With an alliance of two, still every attack with more than two will lead to lose on any of the alliance vertices.

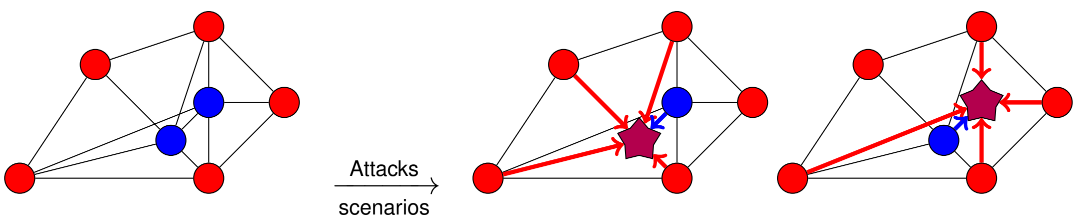



Figure C.2: Number of defending allies is not enough.

With an alliance of three, no possible attack will lead to lose of any vertex in the alliance. It is well defended.

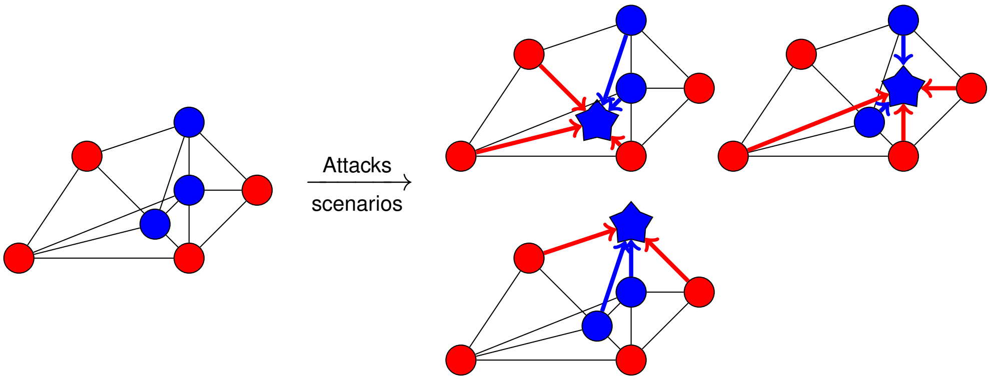
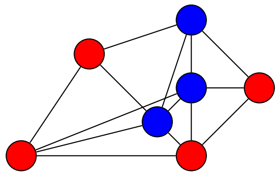


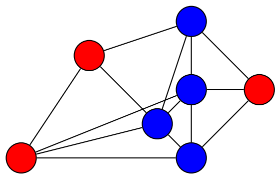

Figure C.3: Number of defending allies is enough.

### C.0.2 Defensive alliance and strong defensive alliance

The set of the three blue vertices is *defensive alliance* but not *strong defensive alliance*.

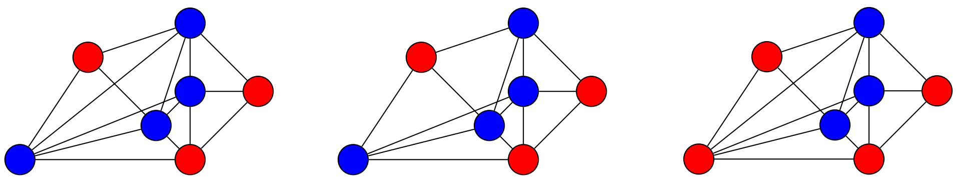


The set of the four blue vertices is *defensive alliance* and *strong defensive alliance*.



### C.0.3 K-defensive alliance

Consider the blue set in the following figures:



$1$-defensive alliance $0$-defensive alliance $-1$-defensive alliance

Figure C.4: $K-$defensive alliance.

### C.0.4 Offensive alliance

Only one vertex without any allies will lose whenever it attack vertices which is supported by neighbours.

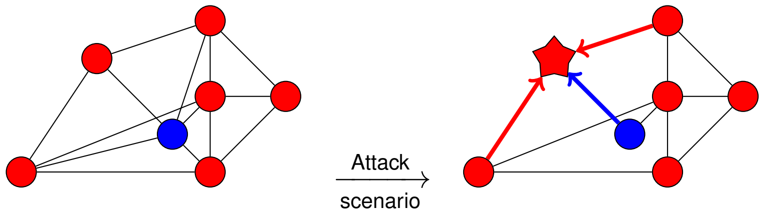



Figure C.5: No allies to attack with.

With an alliance of two, still every attack on a vertex supported by just one neighbour will fail.

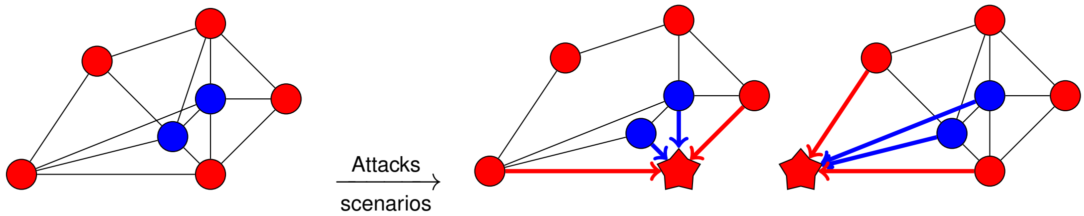



Figure C.6: Number of attacking allies is not enough.

With an alliance of four, every possible attack will lead to nearly a win despite of support.

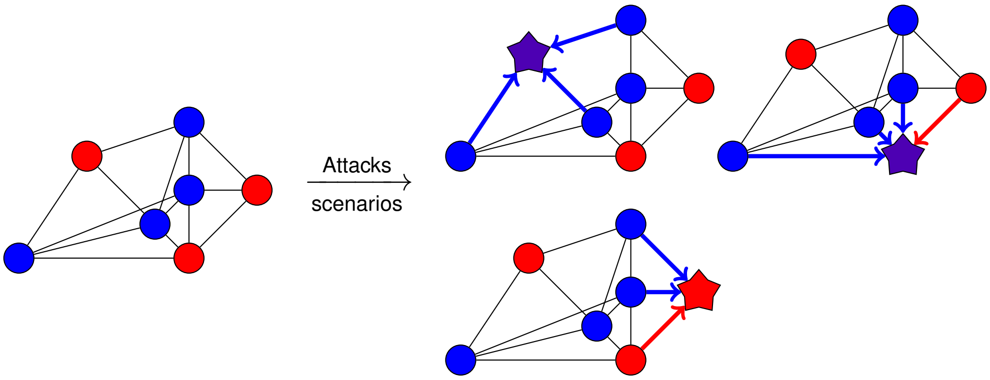
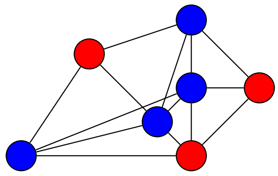


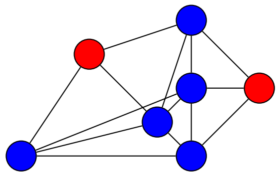

Figure C.7: Number of attacking allies is enough.

### C.0.5 Offensive alliance and strong offensive alliance

The set of the four blue vertices is *offensive alliance* but not *strong offensive alliance*.

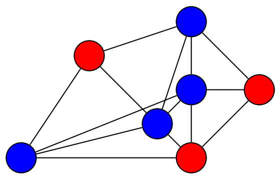


The set of the five blue vertices is *offensive alliance* and *strong offensive alliance*.



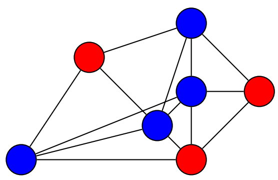
### C.0.6 Dual alliance

The set of the four blue vertices is *dual alliance* or *powerful alliance*. Both *offensive alliance* and *defensive alliance*.



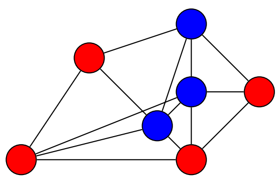
### C.0.7 Global alliance

The set of the four blue vertices is *global offensive alliance*.



### C.0.8 Critical alliance

The set of the three blue vertices is *critical(minimal) defensive alliance*.



### C.0.9 Weighted alliance

Consider the blue set of vertices in the following figure:

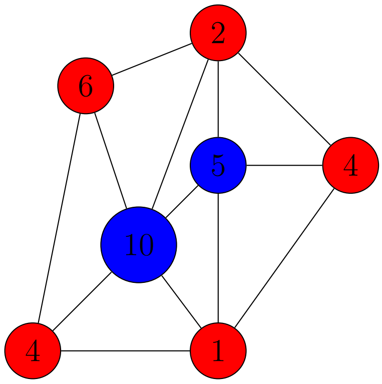



Figure C.8: Weighted alliance.

# D. The python code to obtain the defensive alliance polynomial

A python program to obtain the $da(G; x, y)$ for a graph $G$ with a small order.

## D.1 The python code to obtain the defensive alliance polynomial

```
import networkx as nx
import matplotlib.pyplot as plt
from sympy import *
import warnings
warnings.filterwarnings("ignore")
import pandas

#
#       power set of a set
#
def get_subsets(fullset):
        listrep = list(fullset)
        subsets = []
        for i in range(2**len(listrep)):
                subset = []
                for k in range(len(listrep)):
                        if i & 1<<k:
                                subset.append(listrep[k])
                subsets.append(subset)
return subsets

#
# Create a graph*As an example: complete Bipartite
#
completeBipartite = nx.complete_bipartite_graph( 5,3);
graph = completeBipartite
```

```
#
#       Function to compute the minimuin degree of a subgraph,
#               2deg-main deg
#
def ranking(subgraph , maingraph ):
        vertexList = subgraph.nodes();
        miniuim = len( vertexList );
        for node in vertexList:
                deg=2*subgraph.degree(node)-maingraph.degree(node);
                if deg < miniuim:
                        miniuim = deg;
return miniuim;

#
#       Start
#
subsets = get_subsets(graph.nodes())
order = len( graph.nodes() )

#
#       Create a polynomial
#
x = Symbol('x')
y = Symbol('y')
result = x**1;
for subgraphVertexSet in subsets:
        #       Remove null graph
        if len(subgraphVertexSet) ==0 :
                continue;
        #       Induce the subgraph
        H = graph.subgraph( subgraphVertexSet )
        #       Make sure of connectivity
        if nx.is_connected(H) != True:
                continue;
        result +=  x**len(H.nodes()) *y**( ranking(H,graph) +order )
result -= x**1
print result
#
#       Draw the graph
#
nx.draw_networkx(graph)
plt.show()
```

# E. Notation

| | |
|---|---|
| $G$ | A pair $(V, E)$ where $V$ is a set of vertices and $E$ is a set of two-elements subsets of $V$ |
| $V(G)$ | The set $V$ in the graph $G$ |
| $E(G)$ | The set $E$ in the graph $G$ |
| $n$ | The order of the graph $G$ defined by $\|V(G)\|$ |
| $m$ | The size of the graph $G$ defined by $\|E(G)\|$ |
| $\Delta(G)$ | The maximum degree of a vertex in the graph $G$ |
| $\delta_S(u)$ | $\|\{\{u,v\} \in E(G) : v \in S\}\|$, where $S \subseteq V(G)$ |
| $\bar{S}$ | $V(G) \setminus S$, where $S \subseteq V(G)$ |
| $H \subseteq G$ | $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ |
| $G[S]$ | The subgraph induced by $S$ in the graph $G$, where $S \subseteq V(G)$ |
| $H \cup G$ | The graph $(V(H) \cup V(G)\ ,\ E(H) \cup E(G))$ |
| $H + G$ | The graph obtained from $H \cup G$ and joining every vertex in $G$ with every vertex in $H$ |
| $E_n$ | The *edgeless graph* with order $n$ and no edges |
| $K_n$ | The *complete graph* with order $n$ and every two vertices in $G$ are adjacent |
| $P_n$ | The *path graph* such that a label for the vertices exist as $v_1, v_2, \ldots, v_n$ then its edge are $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$ |
| $C_n$ | The *cycle graph* such that a label for the vertices exist as $v_1, v_2, \ldots, v_n$ then its edge are $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1$ |
| $S_n$ | The *star graph* defined by the graph join $nK_1 + K_1$ |
| $W_n$ | The *wheel graph* defined by the graph join $C_n + K_1$ |
| $W_n'$ | The *open wheel graph* defined by the graph join $P_n + K_1$. This graph is sometimes also known as *Fan* |
| $F_n$ | The *friendship graph* defined by the graph join $nK_2 + K_1$. This graph is also known as *Windmill graph* |
| $B_n$ | The *triangular book graph* defined by the graph join $nK_1 + K_2$ |
| $B_{n,2}$ | The *quadrilateral book graph* defined by the graph join $nK_2 + k_2$ |
| *Tree* | A connected graph with no cycles |
| *Component* | A maximal connected subgraph of a graph $G$ |
| *Cut vertex* | A vertex in $G$ whose removal results in a graph of more components than $G$ |

| | |
|---|---|
| $N(v)$ | The *open neighborhood* of $v$ in $G$ defined by $\{u : u \in V(G), \{u,v\} \in E(G)\}$ |
| $N[v]$ | The *closed neighborhood* of $v$ in $G$ defined by $N(v) \cup \{v\}$ |
| $\partial(S)$ | The *closed neighborhood* of the set of vertices $S$ in $G$ defined by $\bigcup\limits_{v \in S} (N[v] \setminus S)$ |
| $[x^k]f(G;x,y)$ | The *polynomial* of the terms $x^k$ in the graph polynomial $f(G;x,y)$ |
| $[x^k y^l]f(G;x,y)$ | The *coefficient* of the term $x^k y^l$ in the graph polynomial $f(G;x,y)$ |
| $k(G)$ | The number of components in the graph $G$ |
| $c_{min}(G)$ | The minimum order of a component in the graph $G$ |
| $c_{max}(G)$ | The maximum order of a component in the graph $G$ |

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# Declaration of Authenticity

I, Hany Ibrahim, certify that this thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any university; and that to the best of my knowledge and belief it does not contain any material previously published or written by another person where due reference is not made in the text.

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