Fabrici, Igor; Harant, Jochen; Jendrol', Stanislav:

Paths of low weight in planar graphs

URN: urn:nbn:de:gbv:ilm1-2020200142

Original published in:	Discussiones mathematicae. Graph theory / Uniwersytet Zielonogórski, Wydział Matematyki, Informatyki i Ekonometrii. - Warsaw : De Gruyter Open. - 28 (2008), 1, p. 121-135.
Original published:	2008
ISSN:	2083-5892
DOI:	10.7151/dmgt.1396
[*Visited:*	2020-01-17]

Discussiones Mathematicae
Graph Theory 28 (2008) 121–135

PATHS OF LOW WEIGHT IN PLANAR GRAPHS

Igor Fabrici[1], Jochen Harant[2] and Stanislav Jendroľ[1]

[1]*Institute of Mathematics*
P.J. Šafárik University
Jesenná 5, SK–04154 Košice, Slovak Republic
e-mail: stanislav.jendrol@upjs.sk, igor.fabrici@upjs.sk

[2]*Institute of Mathematics*
Ilmenau Technical University
PF 10 05 65, D–98684 Ilmenau, Germany
e-mail: harant@mathematik.tu-ilmenau.de

Abstract

The existence of paths of low degree sum of their vertices in planar graphs is investigated. The main results of the paper are:

1. Every 3-connected simple planar graph G that contains a k-path, a path on k vertices, also contains a k-path P such that for its weight (the sum of degrees of its vertices) in G it holds

$$w_G(P) := \sum_{u \in V(P)} \deg_G(u) \leq \frac{3}{2}\, k^2 + \mathcal{O}(k)\,.$$

2. Every plane triangulation T that contains a k-path also contains a k-path P such that for its weight in T it holds

$$w_T(P) := \sum_{u \in V(P)} \deg_T(u) \leq k^2 + 13k\,.$$

3. Let G be a 3-connected simple planar graph of circumference $c(G)$. If $c(G) \geq \sigma |V(G)|$ for some constant $\sigma > 0$ then for any k, $1 \leq k \leq c(G)$, G contains a k-path P such that

$$w_G(P) = \sum_{u \in V(P)} \deg_G(u) \leq \Big(\frac{3}{\sigma} + 3\Big)k\,.$$

Keywords: planar graphs, polytopal graphs, paths, weight of an edge, weight of a path.

2000 Mathematics Subject Classification: 05C10, 05C38, 52B10.

1. Notation

We will adapt the convention that a graph is *planar* if it can be embedded in the plane (without edges crossing), and *plane* if it is already embedded in the plane. This paper will be concerned with simple plane graphs. The sets of vertices, edges and faces of such a graph G will be denoted by $V(G)$, $E(G)$ and $F(G)$, respectively, or by V, E and F if G is known from the context.

The *facial walk* of a face α of a connected plane graph G is the shortest closed walk induced by all edges incident with α. The *degree* of a face α is the length of its facial walk and is denoted by $\deg_G(\alpha)$ or $\deg(\alpha)$ if G is known from the context. The degree of a vertex x of a graph is the number of edges incident with x. Analogously the notation $\deg_G(x)$ or $\deg(x)$ is used for the degree of a vertex x. Let a k-*vertex* be a vertex of degree k. Let a k-*face* be defined similarly.

Let an (a, b)-*edge* be an edge f if an a-vertex and a b-vertex are endvertices of f.

A 3-face α is said to be the (a, b, c)-*triangle* or a *triangle of type* (a, b, c) if vertices incident with α have degrees a, b and c.

Let a k-*path* and a k-*cycle* be a path and a cycle on k vertices, respectively. Let a k-path be an $(a_1, a_2, \ldots, a_k)$-*path* if it passes through the vertices $u_1, u_2, \ldots, u_k$ in order with $a_i = \deg(u_i)$ for all $i = 1, 2, \ldots, k$.

Let an $(x; a, b, c)$-*star* be a star $K_{1,3}$ which is a subgraph of a graph G with a central x-vertex and an a-vertex, a b-vertex and a c-vertex as leaves.

Let $\mathcal{P}_\kappa(\delta, \rho)$ be the family of all κ-connected simple plane graphs with minimum vertex degree at least δ, $\delta \geq 3$, and minimum face degree at least ρ. It is easy to see that $\mathcal{P}_\kappa(\delta, \rho)$ is not empty only for $(\delta, \rho) \in \{(3,3), (3,4), (4,3), (3,5), (5,3)\}$. Let $\mathcal{P}(\delta, \rho) := \mathcal{P}_3(\delta, \rho)$ and $\mathcal{P}(\delta, \bar{3}) := \{G \in \mathcal{P}(\delta, 3)\colon G$ is a triangulation$\}$.

For a subgraph H of a planar graph G, the *weight* $w_G(H)$ of H is defined to be the sum of degrees of vertices of H in G; namely

$$w_G(H) = \sum_{u \in V(H)} \deg_G(u)\,.$$

2. Introduction and Results

It is well known that every planar graph contains a vertex of degree at most 5. In 1955 Kotzig [15, 16] proved that every 3-connected planar graph contains an edge of the weight at most 13 in general and most 11 in the absence of 3-vertices, respectively. These bounds are best possible.

In 1940 Lebesgue [17] proved that every graph G of minimum degree 5 contains a star $K_{1,3}$ such that its central vertex has, in G, degree at most 5 and every its leaf has degree at most 8. Recently van den Heuvel and McGuinness [11] proved that every planar graph of minimum degree at least 3 contains a $(3, a)$-edge for a $3 \leq a \leq 11$, or an $(a, 4, b)$-path for $3 \leq a \leq 7$ and $a \leq b \leq 11$, or a $(5; a, b, c)$-star for $3 \leq a \leq 6$, $a \leq b \leq 7$ and $b \leq c \leq 11$.

The following recent result, that strengthens all above mentioned ones, will be applied later in this paper.

Theorem 1 [9]**.** *Every planar graph G of minimum degree $\delta(G) \geq 3$ contains*

(i) *a $(3, a)$-edge for $3 \leq a \leq 10$, or*

(ii) *an $(a, 4, b)$-path for $a = 4$ and $4 \leq b \leq 10$,*
or $a = 5$ and $5 \leq b \leq 9$,
or $6 \leq a \leq 7$ and $6 \leq b \leq 8$, or

(iii) *a $(5; a, b, c)$-star for $4 \leq a \leq 5$, $5 \leq b \leq 6$ and $5 \leq c \leq 7$,*
or $a = b = c = 6$.

Moreover, for every $\mathcal{S} \in \{(3, 10)$-edge, $(4, 4, 9)$-path, $(5, 4, 8)$-path, $(6, 4, 8)$-path, $(7, 4, 7)$-path, $(5; 5, 6, 7)$-star, $(5; 6, 6, 6)$-star$\}$ there is a 3-connected plane graph H containing $\mathcal{S}$ and no other subgraph from the above list.

In generalizing Kotzig's theorem there are several other natural directions. Two possibilities are as follows.

Let $k \geq 1$ be an integer.

(A) Find the smallest integer $f = f(k, \delta, \rho)$ such that whenever a graph $G \in \mathcal{P}_3(\delta, \rho)$ contains at least k vertices, there is a connected subgraph H of G of order k whose weight

$$w_G(H) = \sum_{u \in V(H)} \deg_G(u) \leq f(k, \delta, \rho)\,.$$

(B) Find the smallest integer $w = w(k, \delta, \rho)$ such that whenever a graph $G \in \mathcal{P}_3(\delta, \rho)$ contains a k-path there is a k-path P_k in G with weight

$$w_G(P_k) = \sum_{u \in V(P_k)} \deg_G(u) \leq w(k, \delta, \rho)\,.$$

The possibility (A) was investigated by Enomoto and Ota [4]. They proved that for $k \geq 4$

$$8k - 5 \leq f(k, 3, 3) \leq 8k - 1$$

and conjectured the precise value of $f(k, 3, 3)$ to be $8k - 5$.

The problem (B) was formulated in [5]. The precise values of $w(k, \delta, 3)$ are known only for small k, e.g. $w(1, 3, 3) = 5$, $w(2, 3, 3) = f(2, 3, 3) = 13$, $w(2, 4, 3) = f(2, 4, 3) = 11$ (Kotzig [15]), $w(2, 5, 3) = f(2, 5, 3) = 11$ (Wernicke [21]), $w(3, 5, 3) = f(3, 5, 3) = 17$ (Franklin [7]), and $w(3, 3, 3) = f(3, 3, 3) = 21$ (Ando, Iwasaki and Kaneko [1]). For greater k only estimations are known, see e.g. surveys in [10, 13, 14]. In [5, 6] it was proved that

$$k \log_2 k \leq w(k, 3, 3) \leq 5k^2 \,.$$

Madaras [18] improved the upper bound showing that $w(k, 3, 3) \leq \frac{5}{2}k(k+1)$.

Applying Theorem 1 we are able to prove the first main result of this paper

Theorem 2. *Let k be an integer, $k \geq 4$. Then*

(i) *every plane triangulation T, that contains a k-path, also contains a k-path P such that $w_T(P) \leq k^2 + 13k$, and*

(ii) *every 3-connected planar graph, that contains a k-path, also contains a k-path P such that $w_G(P) \leq w(k, 3, 3) \leq \frac{3}{2}k^2 + \mathcal{O}(k)$.*

Note. As shown in [12], no analogue of Theorem 2 can be proved for the family $\mathcal{P}_2(3, 3)$. More precisely: For every pair of integers m, k, $m \geq k \geq 3$, there is a 2-connected planar graph G in which every k-path P_k has weight at least m, that is $w_G(P_k) \geq m$.

For $k = 2$ the situation is different. In 1972 Erdös conjectured that Kotzig’s theorem holds for all planar graphs G with minimum degree $\delta(G) \geq 3$. This conjecture was proved by Barnette, see [8].

The restriction to 4-connected planar graphs brings a different behaviour. In 2000 Mohar [19] proved that every 4-connected planar graph of order at least k contains a k-path P_k of weight

$$w_G(P_k) \leq 6k - 1\,;$$

the bound being tight. The difference is that every 4-connected plane graph contains a k-path whose weight is bounded from above by a function linear in k while on the other side there are 3-connected plane graphs in which all k-paths have weight bounded from below by a function which is not linear in k.

Developing the ideas of Mohar's proof [19] we show that a linear in k upper bound is also true for a wider family of plane graphs. Namely, the second main result of this paper is

Theorem 3. *For $G \in \mathcal{P}(\delta, \rho)$ let $c(G)$ be the length of a longest cycle of G. Let k be an integer, $3 \le k \le c(G)$. If $c(G) \ge \sigma|V(G)|$ for some positive number σ then G contains a k-path P_k such that*

$$w_G(P_k) < \Big(\Big(\frac{2\rho}{\rho-2} - \delta\Big)\frac{1}{\sigma} + \delta\Big)k\,.$$

In fact Theorem 3 is a corollary of the following more general result

Theorem 4. *Let G be a graph with $n = |V(G)|$ vertices, $e = |E(G)|$ edges, the length of a longest cycle $c = c(G)$, and minimum vertex degree $\delta = \delta(G)$. Let k be a positive integer $k \le c$. Then G contains a k-path P with*

$$w_G(P) = \sum_{u \in V(P)} \deg_G(u) \le \left(\frac{2e}{c} + \delta\left(1 - \frac{n}{c}\right)\right)k\,.$$

Immediately we have

Corollary 5. *Let G be a hamiltonian graph on n vertices, and let k be a positive integer, $k \le n$. Then G contains a k-path P such that*

$$w_G(P) \le \frac{2e}{n}\,k\,.$$

Every 4-connected planar graph G is known to be hamiltonian [20]. Hence $c(G) = |V(G)|$ and, because in this case $e \le 3n - 6$, we immediately obtain the above mentioned elegant Mohar's theorem [19].

Let S be a set of three vertices of a 3-connected planar graph G such that the graph $G - S$ obtained from G by removing S is disconnected (S is called a 3-*separator* in this case). It is known that $G - S$ consists of exactly two components $\mathcal{A}$ and $\mathcal{B}$. G is called to be *essentially* 4-*connected* if it is 3-connected and $|V(\mathcal{A})| = 1$ or $|V(\mathcal{B})| = 1$ for every 3-separator S of G.

Theorem 6. *Let G be an essentially 4-connected planar graph, and let k be an integer $1 \leq k \leq \frac{|V(G)|}{2}$. Then G contains a k-path P such that*

$$w_G(P) \leq 9k - 1\,.$$

Moreover, there exists an essentially 4-connected planar graph H in which every k-path P has weight

$$w_H(P) \geq \frac{15}{2}\,k - \frac{13}{2}\,.$$

The rest of the paper is organized as follows. In Sections 3 and 4 we prove Theorem 2. Theorems 3, 4 and 6 are proved in Section 5. In Section 6 we add some remarks concerning the results and open problems.

3. Proof of Theorem 2(i)

First we prove the following theorem

Theorem 7. *For a given positive integer $k \geq 4$, let G be a 3-connected plane graph in which every r-face, $r \geq 4$, contains at most two vertices of degree greater than k, and if it contains exactly two, then they are adjacent. Then G contains a k-path P of the weight*

$$w(P) \leq k^2 + 13k$$

that has at most four vertices of degree greater than k.

Proof. We give a constructive proof of Theorem 7. For convenience a vertex x of G is called *major* if $\deg_G(x) > k$, and is called *minor* otherwise.

For a vertex x of G let $\mathcal{C}_x$ be a cycle induced in G by all edges of all faces incident with the vertex x but not having x as an endvertex (i.e., edges not incident with x). Clearly all neighbours of x are in $V(\mathcal{C}_x)$ and the length of $\mathcal{C}_x$ is at least $\deg_G(x)$.

Let $M = M(G)$ be a subgraph of G induced on major vertices of G. The graph G and the subgraph M have the properties mentioned in the lemmas bellow:

Lemma 1. *Let G contain a major vertex x with $\deg_M(x) = d$ and let $\deg_G(x) \geq kd + 1$. Then G contains a k-path P of which all vertices are minor and therefore $w(P) \leq k^2$.*

Proof. Since $\deg_G(x) \geq kd + 1$ then for the cycle $\mathcal{C}_x$ we have $|\mathcal{C}_x| > kd$, but on $\mathcal{C}_x$ there are exactly d major vertices. Hence at least one path between two consecutive major vertices of $\mathcal{C}_x$ contains at least k minor vertices. So we have a k-path P with $w(P) \leq k^2$. ■

Lemma 2. *If G contains a major vertex x with $\deg_M(x) \leq 2$, then G contains a k-path P all vertices, except possibly x, are minor and $w(P) \leq k^2 + k$.*

Proof. In the case $\deg_M(x) \leq 1$ the proof is clear. Let x be a 2-vertex in M then $\deg_G(x) \geq k+1$. If $\deg_G(x) \geq 2k+1$, then, by Lemma 1, G contains a required path. Let $\deg_G(x) \leq 2k$. On $\mathcal{C}_x$ there are two major vertices, say y and z, which divide $\mathcal{C}_x$ into two subpaths both consisting of minor vertices that contain two subpaths P_a and P_b with $a + b \geq k - 1$ starting in minor vertices u and v, respectively, which are neighbours of x in G. These two paths together with the edges ux and xv form an l-path, $l \geq k$. This path contains a k-path P as a subgraph all vertices of which, except possibly x, are minor. So for P we have

$$w_G(P) \leq k(k-1) + 2k = k^2 + k\,. \qquad ■$$

Due to Lemma 1 and Lemma 2 we may suppose that $\deg_M(x) = d \geq 3$ and $\deg_G(x) \leq kd$ for every major vertex x of G. Then we apply to M our Theorem 1. By it there is a major vertex u with $\deg_M(u) = a \leq 5$. Due to hypothesis of Theorem 7 all other vertices of $\mathcal{C}_u$ are minor. Among these major neighbours of u there are at most two vertices, say y and z, whose degrees are not known, and

(i) if $\deg_M(u) = 3$ then there is a major neighbour v with $\deg_M(v) = b \leq 10$, or

(ii) if $\deg_M(u) = 4$ then there are two more major neighbours of u, say v and w, such that $\deg_M(v) = b$ and $\deg_M(w) = c$ with $4 \leq b \leq 5$ and $4 \leq c \leq 10$ or $6 \leq b \leq 7$ and $6 \leq c \leq 8$, respectively, or

(iii) if $\deg_M(u) = 5$ then there are three more major neighbours v, w and x of degrees b, c, and d, respectively, where $b \leq 5, c \leq 6, d \leq 7$ or $b = c = d = 6$.

We may suppose that (1) and (2) below hold:

(1) *For the vertex* $t \in \{v, w, x\}$ *with* $\deg_M(t) = r$ $\deg_G(t) \le (r-1)k$.

***Proof of* (1).** Suppose $\deg_G(t) \ge (r-1)k+1$. Consider the cycle $\mathcal{C}_t$. Except of the vertex u of degree at most $5k$ there are other $r-1$ major vertices on $\mathcal{C}_t$. These major vertices split $\mathcal{C}_t$ into $r-1$ subpaths. At least one of these subpaths contains a k-path P on $k-1$ minor vertices and the k-th (possibly exceptional) vertex of this path is either a minor vertex or the vertex u which has degree at most $5k$. This means that the k-path P has at most one major vertex and we have $w_G(P) \le k(k-1) + 5k = k^2 + 4k$. ■

(2) *For the vertex* u $\deg_G(u) \le 2k$ *holds.*

***Proof of* (2).** Suppose the contrary. Let the cycle $\mathcal{C}_u$ have at least $2k+1$ vertices. There are at most five major vertices among them (all are neighbours of u), namely the vertices y and z, and at most three from the set $\{v, w, x\}$ as described above. The vertices y and z divide the cycle $\mathcal{C}_u$ into two subpaths P_p and P_q with $p + q \ge 2k-1$. Hence $\max\{p, q\} \ge k$ and on $\mathcal{C}_u$ there is a k-path P all vertices of which except of at most three (from the set $\{v, w, x\}$) are minor. This path has in the case (i) the weight $w_G(P) \le k(k-1) + (b-1)k \le k^2 - k + 10k - k = k^2 + 8k$ because P contains at most one major neighbour of degree $\le 9k$. In the case (ii) (the case (iii)) the path P has weight

$$w_G(P) \le k(k-2) + (b-1)k + (c-1)k \le k^2 + 11k$$

$$(w_G(P) \le k(k-3) + (b-1)k + (c-1)k + (d-1)k \le k^2 + 12k)$$

and contains at most two major vertices (at most three major vertices, respectively). ■

Because u is a major vertex, $|\mathcal{C}_u| \ge k+1$ and the vertices y and z divide the cycle $\mathcal{C}_u$ into two subpaths that contain two subpaths P_p and P_q with $p + q \ge k-1$ starting in vertices u^* and v^* which are neighbours of u in G. These two subpaths together with the vertex u and edges u^*u and uv^* form an l-path, $l \ge k$, which contains as a subgraph a k-path P passing through at least $(k-4)$ minor vertices and at most four major vertices all from the set $\{u, v, w, x\}$.

Applying Properties (1) and (2), and distinguishing three cases (i), (ii) and (iii) according to the degree of u in M we obtain, similarly as in the proof of (2), the following upper bound on $w_G(P)$, the weight of P,

$$w_G(P) \leq k^2 + 13k\,.$$

From Theorem 7 we immediately have the upper bound for Theorem 2(i).■

4. Proof of Theorem 2(ii)

To prove Theorem 2(ii) consider first the following construction and then modify the proof of Theorem 2(i). Suppose $H \in \mathcal{P}(3,3)$.

Let $G_0 = H, G_1, \ldots, G_p = G$ be a sequence of plane graphs defined as follows: If G_i, $i = 0, 1, \ldots, p-1$, is a 3-connected plane graph having an r-face α, $r \geq 4$, incident with two non-adjacent major vertices u and v we insert a diagonal $d = uv$ into α joining the vertices u and v. The result is a 3-connected plane graph $G_{i+1} = G_i + d$. If G_i does not contain any face α having the above-mentioned property we put $i = p$ and $G = G_p$.

It is easy to see that the graph $G_p = G$ satisfies the hypothesis of Theorem 7. If G contains a major vertex x of degree $\deg_M(x) \leq 2$ then, analogously as in the proof of Lemma 2, one can prove that G contains a k-path P of the weight $w(P) \leq k^2 + k$ with at most one major vertex. Clearly this path is also present in the graph H. If for each major vertex x in G $\deg_M(x) \geq 3$ then, by Theorem 1, there is a major vertex u with $3 \leq \deg_M(u) \leq 5$ such that on $\mathcal{C}_u$ there are at most two major vertices y and z of unknown degree and at most three major vertices with known degree bounds. Because of our above construction there is, on $\mathcal{C}_u - \{y, z\}$, at most one edge or one pair of adjacent edges, incident neither with y nor z that is not present in H. These edges together with the vertices y and z divide the cycle $\mathcal{C}_u$ into at most three subpaths consisting of at least $k-2$ vertices with at least $k-4$ minor ones among them. All these subpaths are clearly present in H. Two longest ones of them joined through the vertex u form in H a path Q on at least $\frac{2}{3}(k-2)+1 = \frac{2k-1}{3}$ vertices. On the path Q there are the vertex u, at most two vertices from the set $\{v, w, x\}$ and all remaining vertices are minor. Hence, by Theorem 1, for the weight of Q we have $w(Q) \leq (\lceil\frac{2k-1}{3}\rceil - 3)k + 5k + 7k + 8k = \lceil\frac{2k-1}{3}\rceil k + 17k$.

Hence we have

Theorem 8. *Every graph $G \in \mathcal{P}(3,3)$ contains a $\lceil\frac{2k-1}{3}\rceil$-path Q of the weight $w(Q) \leq \lceil\frac{2k-1}{3}\rceil k + 17k$ with at most four major vertices.*

From this theorem we immediately have our Theorem 2(ii). ■

5. Light Paths with Linear Weights

In this section we are going to prove Theorems 3, 4 and 6. We start with

***Proof of Theorem* 4.** Let $\mathcal{C}$ be a longest cycle of G. For the number e of edges of G,

$$2e = \sum_{x\in V(G)} \deg_G(x) \; = \sum_{x\in V(\mathcal{C})} \deg_G(x) \; + \sum_{x\in V(G)\setminus V(\mathcal{C})} \deg_G(x)\,.$$

Thus,

$$\sum_{x\in V(\mathcal{C})} \deg_G(x) = 2e - \sum_{x\in V(G)\setminus V(\mathcal{C})} \deg_G(x) \le 2e - \delta\cdot|V(G)\setminus V(\mathcal{C})| = 2e - \delta(n-c).$$

For $x \in V(\mathcal{C})$, let $P(x)$ denote the path on $\mathcal{C}$ starting in x and following a fixed orientation of $\mathcal{C}$ such that $|V(P(x))| = k$ (because $k \le c$ this is possible). Then every vertex of $\mathcal{C}$ is contained in exactly k of these c paths. Hence,

$$\sum_{x\in V(\mathcal{C})} \Big(\sum_{y\in V(P(x))} \deg_G(y) \Big) = k \sum_{x\in V(\mathcal{C})} \deg_G(x) \le (2e + \delta(c-n))k\,.$$

Among these c paths there is one, say P, with

$$\sum_{x\in V(P)} \deg_G(x) \le \left(\frac{2e}{c} + \delta\Big(1 - \frac{n}{c}\Big) \right) k\,,$$

and Theorem is proved. ■

***Proof of Theorem* 3.** Theorem 3 is a simple consequence of Theorem 4. For a connected plane graph G with e edges, f faces and minimum face degree ρ we have

$$2e = \sum_{\alpha\in F(G)} \deg_G(\alpha) \ge \rho f\,.$$

This immediately yields $f \le \frac{2e}{\rho}$. Using this inequality and Euler's polyhedral formula we obtain $e \le \frac{\rho(n-2)}{\rho-2}$. Applying this fact together with the inequality $c \geq \sigma n$ on parameters c and n in Theorem 4 we obtain

$$w_G(P) \leq \Big(\frac{2e}{c} + \delta\Big(1 - \frac{n}{c}\Big)\Big)k \leq \Big(\frac{2\rho(n-2)}{c(\rho-2)} + \delta - \frac{\delta n}{c}\Big)k$$

$$= \Big(\Big(\frac{2\rho}{\rho-2} - \delta\Big)\frac{n}{c} + \delta - \frac{4\rho}{c(\rho-2)}\Big)k < \Big(\Big(\frac{2\rho}{\rho-2} - \delta\Big)\frac{1}{\sigma} + \delta\Big)k$$

which is the statement of Theorem 3. Note that $(\frac{2\rho}{\rho-2} - \delta)$ is positive for all five admissible pairs (δ, ρ). ∎

***Proof of Theorem* 6.** Theorem 4 and next theorem give the upper bound in Theorem 6.

Theorem 9. *For every* 3*-connected essentially* 4*-connected plane graph* G *on* n *vertices there is*

$$n \leq 2c - 4$$

where $c = c(G)$ *is the length of a longest cycle of* G*.*

Proof. Consider G to be embedded into the plane π. For a cycle $\mathcal{C}$ of G, the bounded and the unbounded region of $\pi \setminus \mathcal{C}$ are denoted by $\mathrm{int}(\mathcal{C})$ and $\mathrm{out}(\mathcal{C})$, respectively. A cycle $\mathcal{C}$ of G is called to be int-*feasible* if, for every $x \in V(G) \cap \mathrm{int}(\mathcal{C})$, $\mathrm{deg(x)} = 3$, $\mathrm{N(x)} \subseteq \mathrm{V}(\mathcal{C})$, and any two $y, z \in N(x)$ are not adjacent on $\mathcal{C}$. A cycle to be out-*feasible* is defined similarly. Recall that $N(x)$ denotes the neighbourhood of x.

Lemma 3. *Given an* int*-feasible cycle* $\mathcal{C}$ *of* G *on at least* 4 *vertices,* $|V(G) \cap \mathrm{int}(\mathcal{C})| \leq \frac{|\mathrm{V}(\mathcal{C})|}{2} - 2$.

Proof. By induction on $c = |V(\mathcal{C})|$. If $c = 4$ then $\mathcal{C}$ is int-feasible only if $|V(G) \cap \mathrm{int}(\mathcal{C})| = 0$.

Let $c > 4$, $d = |V(G) \cap \mathrm{int}(\mathcal{C})| > 0$, and ϕ be an orientation of $\mathcal{C}$. Consider a fixed $x \in V(G) \cap \mathrm{int}(\mathcal{C})$ and let x_1, x_2, x_3 be the neighbours of x on $\mathcal{C}$ met in this sequence following ϕ. For $i = 1, 2, 3$, let $\mathcal{C}_i$ be the cycle obtained by the union of the path on $\mathcal{C}$ from x_i to x_{i+1} following ϕ and the two edges xx_i and $xx_{i+1} (x_4 = x_1), c_i = |V(\mathcal{C}_i)|$, and $d_i = |V(G) \cap \mathrm{int}(\mathcal{C}_\mathrm{i})|$. Obviously, $\mathcal{C}_i$ is int-feasible and $c_i \geq 4$, $i = 1, 2, 3$.

We have $c_1+c_2+c_3 = c+6, d_1+d_2+d_3 = d-1$, and, by induction hypothesis, $d_i \le \frac{c_i}{2} - 2 (i = 1,2,3)$. This implies $d \le \frac{c}{2} - 2$. ■

A consequence of a result of Tutte [20] is the following

Lemma 4. *G contains a cycle $\mathcal{T}$ (a so-called Tutte-cycle) such that* $\deg(x) = 3$ *and $N(x) \subseteq V(\mathcal{T})$ for every $x \in V(G) \setminus V(\mathcal{T})$.*

If a Tutte-cycle $\mathcal{T}$ of G is not int-feasible (assume the edge yz belongs to $\mathcal{T}$ for certain $y, z \in N(x), x \in V(G) \cap \text{int}(\mathcal{T})$) then the cycle obtained from $\mathcal{T}$ by removing the edge yz and adding the edges xy and yz is also a (longer!) Tutte-cycle of G. Hence, we may assume that there is a Tutte-cycle $\mathcal{C}$ being both int-feasible and out-feasible. Since $c = |V(\mathcal{C})| \ge 6$ if $s = |V(G) \setminus V(\mathcal{C})| > 0$ and $n = c \ge 4$ if $s = 0$, we may apply Lemma 3. Because $\mathcal{C}$ is also out-feasible, by symmetry, we have $|V(G) \cap \text{out}(\mathcal{C})| \le \frac{|V(\mathcal{C})|}{2} - 2$. Hence, $s \le c - 4$, $n \le 2c - 4$. ■

To prove the second part of Theorem 6, take a 4-connected plane triangulation H containing only 5- and 6-vertices such that the distance between arbitrary two 5-vertices is at least k. Such triangulations are well known, see e.g. [3]. They are the duals to the famous fullerene graphs. Let $K(H)$ be the graph obtained from H by inserting a new vertex into each face α and joining it to all vertices incident with α. Then $K(H)$ contains 12 vertices of degree 10, $|V(H)|-12$ vertices of degree 12, the remaining vertices of $K(H)$ are independent 3-vertices, and every 3-separator forms the neighbourhood of a 3-vertex. Since every k-path P contains at most one vertex of degree 10, the proof is complete. ■

6. Remarks

6.1. Comparison of the upper bounds

Theorem 4 provides the upper bound $(\frac{3n}{c} + 3 - \frac{12}{c})k$ on $w_G(P)$ for a k-path P of graphs G from the family $\mathcal{P}(3,3)$ if G has circumference $c = c(G)$ and $k \le c$. Here $n = |V(G)|$.

Theorem 2, on the other side, gives for $G \in \mathcal{P}(3, \bar{3})$, with G containing a k-path, the upper bound $k(k+13)$ on $w_G(P)$.

It is easy to see that for plane triangulations the first mentioned upper bound is better than the second one if and only if

$$\frac{3n-12}{c} - 10 \le k \le c \text{ and } c \ge -5 + \sqrt{3n+13}\,.$$

Note that the best known lower bound for circumference of a 3-connected planar graph G is $\Omega(n^{\log_3 2})$ recently proved by Chen and Yu [2].

7. Matchings and Kotzig's Type Theorems

The idea of the proof of Theorem 4 can be used e.g. in proving the following Kotzig's type theorem.

Theorem 10. *For a graph G let n, e, δ and m be the number of its vertices, edges, the minimum degree, and the edge independence number, respectively. Let M be a matching of G of the cardinality m. Then G contains an edge $h \in M$ of weight*

$$w_G(h) \le \frac{2e - \delta(n-2m)}{m}\,.$$

Proof. Let $M = \{h_1, h_2, \ldots, h_m\}$ be a maximum matching of G. Put $V_1 = V(M)$ and $V_2 = V \setminus V_1$. Then

$$\sum_{h\in M} w_G(h) = \sum_{u\in V_1} \deg_G(u) = 2e - \sum_{v\in V_2} \deg_G(v) \le 2e - \delta(n-2m)\,,$$

which immediately yields the required inequality. ■

For a graph G having perfect matching there is $m = \frac{n}{2}$, so we get

Corollary 11. *If a graph G has a perfect matching M then G contains an edge $h \in M$ of weight*

$$w_G(h) \le \frac{4e}{n}\,.$$

The number e edges of planar graph G is bounded by $e \le 3n-6$ or $e \le 2n-4$ in general and in the absence of 3-faces, respectively. Using these inequalities in Corollary 11 we obtain

Corollary 12. *If a planar graph G has a perfect matching M then it contains an edge h of weight*

$$w_G(h) \le 11 \text{ and } w_G(h) \le 7$$

in general and in the absence of 3-faces, respectively.

Acknowledgement

This work was supported by Science and Technology Assistance Agency under the contract No. APVT-20-004104. Support of Slovak VEGA Grant 1/3004/06 is acknowledged as well. The authors wish to thank the referees for their thoughtful suggestions.

References

[1] K. Ando, S. Iwasaki and A. Kaneko, *Every* 3*-connected planar graph has a connected subgraph with small degree sum*, Annual Meeting of Mathematical Society of Japan, 1993, Japanese.

[2] G. Chen and X. Yu, *Long cycles in* 3*-connected graphs*, J. Combin. Theory (B) **86** (2002) 80–99.

[3] E. Etourneau, *Existence and connectivity of planar having* 12 *vertices of degree* 5 *and,* $n-12$ *vertices of degree* 6, Colloq. Math. Soc. János Bolyai **10** (1975) 645–655.

[4] H. Enomoto and K. Ota, *Connected subgraphs with small degree sum in* 3*-connected planar graphs*, J. Graph Theory **30** (1999) 191–203.

[5] I. Fabrici and S. Jendrol', *Subgraphs with restricted degrees of their vertices in planar* 3*-connected graphs*, Graphs Combin. **13** (1997) 245–250.

[6] I. Fabrici and S. Jendrol', *Subgraphs with restricted degrees of their vertices in planar graphs*. Discrete Math. **191** (1998) 83–90.

[7] P. Franklin, *The four color problem*, Amer. J. Math. **44** (1922) 225–236.

[8] B. Grünbaum, *New views on some old questions of combinatorial geometry*, Int. Teorie Combinatorie, Rome **1** (1976) 451–468.

[9] J. Harant and S. Jendrol', *On the existence of specific stars in planar graph*, Graphs and Combinatorics **23** (2007) 529–543.

[10] J. Harant, S. Jendrol' and M. Tkáč, *On* 3*-connected plane graphs without triangular faces*, J. Combin. Theory (B) **77** (1999) 150–161.

[11] J. van den Heuvel and S. McGuinness, *Coloring the square of a planar graph*, J. Graph Theory **42** (2003) 110–124.

[12] S. Jendrol', *Paths with restricted degrees of their vertices in planar graphs*, Czechoslovak Math. J. **49** (1999) 481–490.

[13] S. Jendrol', T. Madaras, R. Soták and Z. Tuza, *On light cycles in plane triangulations*, Discrete Math. **197/198** (1999) 453–467.

[14] S. Jendrol' and H.-J. Voss, *Light subgraphs of graphs embedded in the plane and in the projective plane — a survey*, P.J. Šafárik University Košice, IM Preprint series (A) No. 1 (2004).

[15] A. Kotzig, *Contribution to the theory of Eulerian polyhedra*, Mat. Čas. SAV (Math. Slovaca) **5** (1955) 101–113.

[16] A. Kotzig, *Extremal polyhedral graphs*, Ann. New York Acad. Sci. **319** (1979) 565–570.

[17] H. Lebesgue, *Quelques conséquences simples de la formule d'Euler*, J. Math. Pures Appl. **19** (1940) 27–43.

[18] T. Madaras, *Note on weights of paths in polyhedral graphs*, Discrete Math. **203** (1999) 267–269.

[19] B. Mohar, *Light paths in* 4*-connected graphs in the plane and other surfaces*, J. Graph Theory **34** (2000) 170–179.

[20] W.T. Tutte, *A theorem on planar graphs*, Trans. Amer. Math. Soc. **82** (1956) 99–116.

[21] P. Wernicke, *Über den kartographischen Vierfarbensatz*, Math. Ann. **58** (1904) 413–426.

Received 22 November 2006
Revised 4 May 2007
Accepted 4 May 2007