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EIGENVALUE CONDITIONS FOR INDUCED SUBGRAPHS

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Abstract

Necessary conditions for an undirected graph G to contain a graph H as induced subgraph involving the smallest ordinary or the largest normalized Laplacian eigenvalue of G are presented.

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1. Introduction

We consider two fixed finite, undirected, and simple graphs: Let $G = (V, E)$ be a graph without isolated vertices, where $V = \{1, \ldots, n\}$ and E (with $|E| = m$) denote the vertex set and the edge set of G, respectively. Let $\delta \geq 1$ denote the minimum degree of G. Furthermore, let $d_H = \frac{2e}{h}$ be the average degree of a graph $H = (V(H), E(H))$, where $|V(H)| = h$ and $|E(H)| = e$.

The eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ of the adjacency matrix A of G are the *ordinary eigenvalues* (or shortly the eigenvalues) of G. Note that $-r \leq \lambda \leq \lambda_n = r$ for all eigenvalues λ of an r-regular graph G, and if G is connected, then $\lambda_1 = -\lambda_n$ if and only if G is bipartite [4, 7].

Let D be the *degree matrix* of G, that is an $(n \times n)$ diagonal matrix, where the degree d_i of vertex $i \in V$ is the i-th entry at the main diagonal. Moreover, let $0 = \eta_1 \leq \cdots \leq \eta_n$ be the eigenvalues of the *Laplacian* $L = D - A$ of G [1, 13]. If G is r-regular, then η is an eigenvalue of the Laplacian if and only if $r - \eta$ is an eigenvalue of A.

For G without isolated vertices, the *normalized Laplacian* is the $(n \times n)$ matrix $\mathcal{L} = (l_{ij})$ with $l_{ij} = 1$ if $i = j$, $l_{ij} = -\frac{1}{\sqrt{d_i d_j}}$ if $ij \in E$, and $l_{ij} = 0$ otherwise. The eigenvalues $0 = \sigma_1 \leq \cdots \leq \sigma_n$ of $\mathcal{L}$ are the *normalized Laplacian eigenvalues* of G [5, 6, 13]. It is known that $1 < \sigma_n \leq 2$ and that G is bipartite if and only if $\sigma_n = 2$ [10, 12, 13]. For an r-regular graph G, σ is a normalized Laplacian eigenvalue if and only if $r(1 - \sigma)$ is an eigenvalue of A.

For further notation and terminology we refer to [8].

In the present paper, we are interested in necessary conditions in terms of eigenvalues for the fact that G contains a copy of H as an induced subgraph. If all eigenvalues of G and all eigenvalues $\phi_1 \leq \cdots \leq \phi_h$ of the adjacency matrix A_H of H are taken into consideration, then Theorem 1 is a typical result of this kind.

Theorem 1 (Cauchy's Inequalities, Interlacing Theorem [4, 7])**.** *If H is an induced subgraph of G with eigenvalues $\phi_1 \leq \cdots \leq \phi_h$, then $\lambda_i \leq \phi_i \leq \lambda_{n-h+i}$ for $i = 1, \ldots, h$.*

In general, it is difficult to determine the spectra of large graphs G and H, however, the largest and the smallest eigenvalues of the matrices A, L, and $\mathcal{L}$ of a graph are well investigated ([1, 4, 5, 6]). Hence, we focus on simpler necessary conditions for H being an induced subgraph of G just involving smallest or largest eigenvalues. The inequalities (1) obtained from Theorem 1 are possible results of this type.

$$\lambda_1 \leq \phi_1 \quad \text{and} \quad \lambda_n \geq \phi_h. \tag{1}$$

If the largest Laplacian eigenvalue η_n of G and the degrees of the vertices of H in G are taken into account, then the assertion of Theorem 2 holds.

Theorem 2 (Bollobás, Nikiforov [3])**.** *If H is an induced subgraph of G, then $\left(\sum_{i \in V(H)} d_i - 2e\right) n \leq \eta_n h(n - h)$.*

In general, it is not easy to determine the value $\sum_{i \in V(H)} d_i$ exactly. If the degrees of G do not differ too much, then the inequality $\sum_{i \in V(H)} d_i \geq \delta h$ is reasonable and it follows

Corollary 3. *If H is an induced subgraph of G, then $\eta_n h \leq (d_H + \eta_n - \delta)n$.*

Note that Corollary 3 only makes sense if $\delta > d_H$. If G is r-regular, then $\delta = r$, $\eta_n = r - \lambda_1$, and $\sum_{i \in V(H)} d_i = rh$, hence, Theorem 2, Corollary 3, and the following Corollary 4, proved by Haemers already in [9], coincide in this case.

Corollary 4 (Haemers [9]). *If H is an induced subgraph of the r-regular graph G, then $(r - \lambda_1)h \le (d_H - \lambda_1)n$.*

The *identity matrix* is the $(n \times n)$ square matrix with ones on the main diagonal and zeros elsewhere. It is denoted simply by I if the size is immaterial or can be trivially determined by the context. In the sequel, $\underline{x}$ denotes a vector, where $\underline{1} = (1, 1, \ldots, 1)^T$ and $\underline{0} = (0, 0, \ldots, 0)^T$, and we write $\underline{x} \ge \underline{0}$ if $x_i \ge 0$ for each entry x_i of $\underline{x}$.

Our first result is Theorem 5 concerning the case that G is regular and involving the smallest eigenvalue λ_1 of G.

Theorem 5. *Let G be r-regular. If H is an induced subgraph of G, then $(A_H - \lambda_1 I)\underline{x} = \underline{1}$ is solvable, and, for any solution $\underline{x}$ of this equation,*

$$\frac{r - \lambda_1}{n} \le \min\left\{ \underline{z}^T (A_H - \lambda_1 I)\underline{z} \mid \underline{z} \in R^{|V(H)|},\ \underline{1}^T \underline{z} = 1 \right\} = \frac{1}{\underline{1}^T \underline{x}}.$$

Moreover, if $\lambda_1 < \phi_1$, then $A_H - \lambda_1 I$ is regular and $\underline{1}^T \underline{x}$ equals the sum of all entries of $(A_H - \lambda_1 I)^{-1}$.

If $\underline{z} = \left(\frac{1}{h}, \ldots, \frac{1}{h}\right)^T \in R^h$, then $\underline{1}^T \underline{z} = 1$ and $\underline{z}^T (A_H - \lambda_1 I)\underline{z} = \frac{2e - \lambda_1 h}{h^2}$. Thus, Theorem 5 is an extension of Corollary 4. If in Theorem 5, additionally, H is assumed to be ρ -regular, then $\underline{x} = \left(\frac{1}{\rho - \lambda_1}, \ldots, \frac{1}{\rho - \lambda_1}\right)^T$ is a solution of $(A_H - \lambda_1 I)\underline{x} = \underline{1}$, thus, $\frac{1}{\underline{1}^T \underline{x}} = \frac{(\rho - \lambda_1)}{h} = \frac{(d_H - \lambda_1)}{h}$, hence, Corollary 4 and Theorem 5 coincide in this case.

Now consider the following example, where the assertion of Theorem 5 is stronger than that one of Corollary 4 and inequalities (1) only lead to trivial statements. We ask for a necessary condition that the r-regular graph G contains $k \ge 1$ disjoint and independent copies of the path P_3 on 3 vertices, that is, H consists of k components each of them is isomorphic to P_3. The eigenvalues of P_3 are $-\sqrt{2}, 0, \sqrt{2}$ ([4]), hence, with Theorem 1 we may assume $\lambda_1 \le -\sqrt{2} < -\frac{4}{3}$. With $h = 3k$ and $d_H = \frac{4}{3}$, Corollary 4 leads to $k \le \frac{4 - 3\lambda_1}{9(r - \lambda_1)} n$.

If we consider the system $(A_H - \lambda_1 I)\underline{x} = \underline{1}$, then, by Theorem 5, it is solvable and it follows $\underline{1}^T \underline{x} = k \underline{1}^T \underline{y}$, where $\underline{y}$ is a solution of $(A_{P_3} - \lambda_1 I)\underline{y} = \underline{1}$. It is easy to see that $\underline{1}^T \underline{y} = \frac{4 + 3\lambda_1}{2 - \lambda_1^2}$, thus, again by Theorem 5, $k \le \frac{2 - \lambda_1^2}{(4 + 3\lambda_1)(r - \lambda_1)} n$, which is stronger than $k \le \frac{4 - 3\lambda_1}{9(r - \lambda_1)} n$.

If, additionally, G is assumed to be bipartite, then $\lambda_1 = -r$ and $\lambda_n = r$. The inequalities (1) just imply $\sqrt{2} \le r$ in this case.

Next we consider again the case that G is not necessarily regular and try to establish a result similar to Theorem 5. Therefore, let $M(G,H)$ be the set of non-empty induced subgraphs H^* of H such that $B\underline{y} = \underline{1}$ has a solution $\underline{y} = (y_1, \ldots, y_t)^T$ with $y_s > 0$ for $s = 1, \ldots, t = |V(H^*)|$, where A_{H^*} denotes the adjacency matrix of H^* and $B = A_{H^*} + (\sigma_n - 1)\delta I$. In this case $\underline{y}$ is called a *positive solution* of $B\underline{y} = \underline{1}$. With $H^* = K_1$ and $y_1 = \frac{1}{(\sigma_n-1)\delta} > 0$ (note that $\sigma_n > 1$), it follows $K_1 \in M(G,H) \neq \emptyset$.

If $H^* \in M(G,H)$ and $\underline{y_1}$ and $\underline{y_2}$ are positive solutions of $B\underline{y} = \underline{1}$, then, since B is symmetric, $\underline{1}^T\underline{y_1} = \underline{y_2}^T B\underline{y_1} = \underline{y_2}^T\underline{1} = \underline{1}^T\underline{y_2}$, hence, the value $\underline{1}^T\underline{y}$ is independent on the choice of the positive solution $\underline{y}$. We define $g(G,H^*) = \underline{1}^T\underline{y}$, where $\underline{y}$ is an arbitrary positive solution of $B\underline{y} = \underline{1}$.

If the induced subgraph H^* of H is ρ-regular, then it is easy to see that $(A_{H^*} + (\sigma_n - 1)\delta I)\underline{y} = \underline{1}$ has a positive solution $\underline{y} = \left(\frac{1}{\rho+(\sigma_n-1)\delta}, \ldots, \frac{1}{\rho+(\sigma_n-1)\delta}\right)^T$, hence, $H^* \in M(G,H)$.

If H_1^* and H_2^* are independent induced subgraphs of H and $H_1^*, H_2^* \in M(G,H)$, then the disjoint union $H_1^* \cup H_2^*$ of H_1^* and H_2^* also belongs to $M(G,H)$ and $g(G, H_1^* \cup H_2^*) = g(G,H_1^*) + g(G,H_2^*)$.

Eventually, let $f(G,H) = \min_{H^* \in M(G,H)} \frac{1}{g(G,H^*)}$. Our second result is Theorem 6 involving the largest normalized Laplacian eigenvalue σ_n of G.

Theorem 6. *If H is an induced subgraph of G, then*

$$\frac{\sigma_n\delta^2}{2m} \leq \min\left\{\underline{z}^T(A_H + (\sigma_n - 1)\delta I)\underline{z} \mid \underline{z} \in R^{|V(H)|},\ \underline{1}^T\underline{z} = 1, \underline{z} \geq \underline{0}\right\} = f(G,H).$$

If G is r-regular, then the assertion of Theorem 6 is weaker than that one of Theorem 5 because $\lambda_1 = r(1 - \sigma_n)$, $\frac{2m}{\sigma_n\delta^2} = \frac{n}{r-\lambda_1}$, and $\min\left\{\underline{z}^T(A_H - \lambda_1 I)\underline{z} \mid \underline{z} \in R^{|V(H)|},\ \underline{1}^T\underline{z} = 1\right\} \leq \min\left\{\underline{z}^T(A_H - \lambda_1 I)\underline{z} \mid \underline{z} \in R^{|V(H)|},\ \underline{1}^T\underline{z} = 1, \underline{z} \geq \underline{0}\right\}$ in this case.

In general, it is not easy to calculate $\min\left\{\underline{z}^T(A_H + (\sigma_n - 1)\delta I)\underline{z} \mid \underline{1}^T\underline{z} = 1, \underline{z} \geq \underline{0}\right\}$, however, in special cases it can be done efficiently.

Therefore, we consider an example, where the graph G is non-regular (i.e., Corollary 4 and Theorem 5 are not applicable), $f(G,H)$ can be determined easily, and the necessary condition of Theorem 6 for the graph H to be an induced subgraph of G is stronger than that one of Theorem 2.

For positive integers p and q, where p is even, let $G = C_p \square P_3$ be the Cartesian product[1] of the cycle C_p and the path P_3 on 3 vertices (for $p = 20$, G is shown in the figure) and let H consist of q copies of $K_{1,4}$.

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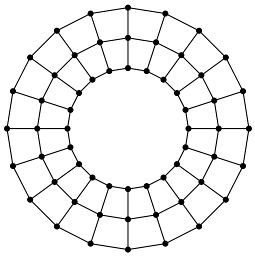
[1]Given graphs G_1 and G_2 with vertex set V_1 and V_2, respectively, their Cartesian product $G_1 \square G_2$ is the graph with vertex set $V_1 \times V_2$, where $(v_1, v_2)(w_1, w_2) \in E(G_1 \square G_2)$ when either $v_1 = w_1$ and $v_2w_2 \in E(G_2)$ or $v_2 = w_2$ and $v_1w_1 \in E(G_1)$.



We have $n = 3p$, $m = 5p$, $\delta = 3$, and, since G is bipartite, $\sigma_n = 2$. The Laplacian eigenvalues of C_p and of P_3 are $2 - 2\cos(\frac{2\pi j}{p})$ for $j = 0, \ldots, p-1$ and $0, 1, 3$, respectively ([4]). Moreover, if η' and η'' are Laplacian eigenvalues of G' and G'', respectively, then $\eta' + \eta''$ is a Laplacian eigenvalue of $G' \Box G''$ ([4]). Because p is even, it follows $\eta_n = 2 - 2\cos(\pi) + 3 = 7$.

It is easy to see that $\sum_{i \in V(H)} d_i - 2e = 10q$ and, using $h = 5q$, Theorem 2 implies $q \leq \frac{3}{7}p$ in this case.

If H^* is an induced subgraph of $K_{1,4}$, then $H^* = K_{1,s}$ or $H^* = \overline{K}_s$ (the edgeless graph on s vertices) for suitable $s \in \{1, 2, 3, 4\}$.

Let $H^* = K_{1,s}$ and consider the system $(A_{H^*} + (\sigma_n - 1)\delta I)\underline{y} = (A_{H^*} + 3I)\underline{y} = \underline{1}$. It is easy to see that $K_{1,4}, K_{1,3} \notin M(G, H)$, $K_{1,2}, K_{1,1} \in M(G, H)$, $g(G, K_{1,2}) = \frac{5}{7}$, and $g(G, K_{1,1}) = \frac{1}{2}$.

If $H^* = \overline{K}_s$, then $H^* \in M(G, H)$ and $(A_{H^*} + 3I)\underline{y} = \underline{1}$ lead to $g(G, H^*) = \frac{s}{3}$, hence, $f(G, H) = \frac{3}{4q}$. By Theorem 6, it follows $q \leq \frac{5}{12}p < \frac{3}{7}p$.

If H^* with $|V(H^*)| \geq 1$ is an arbitrary induced subgraph of H and $\underline{z} = (z_1, \ldots, z_h)^T$ with $z_i = \frac{1}{|V(H^*)|}$ if $i \in V(H^*)$ and $z_i = 0$ otherwise, then $\underline{1}^T \underline{z} = 1$ and $\underline{z}^T(A_H + (\sigma_n - 1)\delta I)\underline{z} = \frac{d_{H^*} + (\sigma_n - 1)\delta}{|V(H^*)|}$, where d_{H^*} denotes the average degree of H^*. Thus, Corollary 7 is a consequence of Theorem 6.

Corollary 7. *If H is an induced subgraph of G, then $\frac{\sigma_n \delta^2}{2m} \leq \frac{d_{H^*} + (\sigma_n - 1)\delta}{|V(H^*)|}$, where H^* is an arbitrary induced subgraph of H with $|V(H^*)| \geq 1$.*

Obviously, Corollary 7 is an extension of Corollary 4 if G is regular. We conclude with an example, where Corollary 3 is weaker than Corollary 7 for not necessarily regular G. Therefore, let $V(H)$ be an independent set of G, i.e. $d_H = 0$. By Corollary 3 and Corollary 7, it follows that $h \leq \frac{\eta_n - \delta}{\eta_n} n$ and $h \leq \frac{2(\sigma_n - 1)}{\sigma_n \delta} m$ if G contains h independent vertices, respectively. In [11], it is shown that there are infinitely many graphs G such that $\frac{2(\sigma_n - 1)}{\sigma_n \delta} m < \frac{\eta_n - \delta}{\eta_n} n$.

2. Proofs

In [11], the following Lemma 8 is proved. For completeness we give a proof here.

Lemma 8. *If $x_1, \dots, x_n$ are real numbers, then*

$$\sigma_n \left(\sum\nolimits_{i=1}^n d_i x_i\right)^2 - 2(\sigma_n - 1)m \sum\nolimits_{i=1}^n d_i x_i^2 \le 4m \sum\nolimits_{ij \in E} x_i x_j. \tag{2}$$

Proof. It is easy to see that σ is an eigenvalue of $\mathcal{L}$ if and only if $\mu = 1 - \sigma$ fulfills $det(A - \mu D) = 0$, see [10, 12, 14]. Let $\mu_i = 1 - \sigma_{n-i+1}$ for $i = 1, \dots, n$.
Note that D is positive definite since $\delta \ge 1$. Define $\underline{x}^T D \underline{y}$ as the inner product for vectors $\underline{x}, \underline{y} \in R^n$ and let $\underline{x}$ and $\underline{y}$ be called *D-orthogonal* if $\underline{x}^T D \underline{y} = 0$. If $\underline{x}^T D \underline{x} = 1$ then $\underline{x}$ is called *D-normal.* A set of D-normal vectors being pairwise D-orthogonal is a *D-orthonormal* set.

We consider the generalized eigenvalue problem $A\underline{x} = \mu D \underline{x}$ for $\mu \in R$ and $\underline{x} \in R^n$ with $\underline{x} \ne \underline{0}$. If the pair $(\mu, \underline{x})$ is a solution of this equation, then $\underline{x}$ is a *D-eigenvector of G* and μ is the corresponding *D-eigenvalue of G*.

We use the well known fact (e.g. see [14]) that there is a D-orthonormal basis of R^n consisting of D-eigenvectors of G. Next we will show the following assertion. If $\{\underline{u_1}, \dots, \underline{u_n}\}$ is a D-orthonormal basis of R^n such that $\underline{u_i}$ is a D-eigenvector with corresponding D-eigenvalue μ_i for $i = 1, \dots, n$, then, for any vector $\underline{x} \in R^n$,

$$(\mu_2 - \mu_1)(\underline{x}^T D \underline{u_2})^2 + \dots + (\mu_n - \mu_1)(\underline{x}^T D \underline{u_n})^2 + \mu_1 \underline{x}^T D \underline{x} = \underline{x}^T A \underline{x}. \tag{3}$$

To see this, let $\underline{x}$ be given. There are real numbers $a_1, \dots, a_n$ such that $\underline{x} = a_1 \underline{u_1} + \dots + a_n \underline{u_n}$.

Then $\underline{x}^T A \underline{x} = \mu_1 a_1^2 + \dots + \mu_n a_n^2$, $\underline{x}^T D \underline{x} = a_1^2 + \dots + a_n^2$, and $\underline{x}^T D \underline{u_i} = a_i$ for $i = 1, \dots, n$. The desired equality (3) is equivalent to $(\mu_2 - \mu_1)a_2^2 + \dots + (\mu_n - \mu_1)a_n^2 + \mu_1(a_1^2 + \dots + a_n^2) = \mu_1 a_1^2 + \ldots + \mu_n a_n^2$.

As a consequence,

$$(\mu_n - \mu_1)(\underline{x}^T D \underline{u_n})^2 + \mu_1 \underline{x}^T D \underline{x} \le \underline{x}^T A \underline{x}. \tag{4}$$

The vector $\frac{1}{\sqrt{2m}}\underline{1}$ is a D-normal D-eigenvector of G with corresponding D-eigenvalue $\mu_n = 1$, thus, inequality (4) and $\sigma_n = 1 - \mu_1$ imply the lemma. ■

***Proof of Theorem* 5.** Inequality (2) and $\lambda_1 = r(1 - \sigma_n)$, if G is r-regular, imply the fact that if G is r-regular and $x_1, \dots, x_n$ are real numbers, then

$$(r - \lambda_1) \left(\sum\nolimits_{i=1}^n x_i\right)^2 + \lambda_1 n \sum\nolimits_{i=1}^n x_i^2 \le 2n \sum\nolimits_{ij \in E} x_i x_j. \tag{5}$$

Let U be an induced subgraph of G isomorphic to H and $\phi : V(H) \to V(U)$ be a graph isomorphism from H to U.

For real numbers $z_1, \dots, z_h$ with $\sum_{q=1}^h z_q = 1$, let $x_1, \dots, x_n$ be defined as follows: If $i \in V(U)$, then there is a suitable $q \in \{1, \dots, h\}$ such that $i = \phi(v_q)$. Set $x_i = z_q$ in this case. If $i \in V \setminus V(U)$, then let $x_i = 0$.

With $\underline{z} = (z_1, \ldots, z_h)^T$, we obtain $\sum_{i \in V} x_i = \sum_{q=1}^h z_q = 1$, $\sum_{i \in V} x_i^2 = \sum_{q=1}^h z_q^2$, and $2\sum_{ij \in E} x_i x_j = 2\sum_{v_q v_{q'} \in E(H)} z_q z_{q'} = \underline{z}^T A_H \underline{z}$.

Inequality (5) implies $(r - \lambda_1) + \lambda_1(\sum_{q=1}^h z_q^2)n \leq \underline{z}^T A_H \underline{z} n$, hence, with $B = (A_H - \lambda_1 I)$, $1 \leq \frac{n}{(r-\lambda_1)} \min \underline{z}^T B \underline{z} = \frac{n}{(r-\lambda_1)} MIN$, where the minimum is taken over all vectors $\underline{z} = (z_1, \ldots, z_h)^T$ with $\sum_{q=1}^h z_q = 1$.

Note that this minimum exists, because $\lambda_1 \leq \phi_1$ follows from Theorem 1, hence, all eigenvalues $\phi_1 - \lambda_1, \phi_2 - \lambda_1, \ldots, \phi_h - \lambda_1$ of B are non-negative. It follows that B is positive semidefinite.

To investigate this value MIN, we consider the Lagrange function $L(\underline{z}, \kappa) = \underline{z}^T B \underline{z} - 2\kappa(\sum_{q=1}^h z_q - 1)$ with Lagrange multiplier 2κ and the necessary optimality conditions $L_{z_q} = 0$ for $q = 1, \ldots, h$ (for more details an Lagrange Theory see [2]).

We obtain that the equations $B\underline{z} = \kappa \underline{1}$ and $\underline{1}^T \underline{z} = 1$ are simultaneously solvable.

Next we will show that κ is unique. If $B\underline{z_1} = \kappa_1 \underline{1}$, $\underline{1}^T \underline{z_1} = 1$, $B\underline{z_2} = \kappa_2 \underline{1}$, and $\underline{1}^T \underline{z_2} = 1$, then $\kappa_1 = \kappa_1 \underline{1}^T \underline{z_2} = \underline{z_1}^T B \underline{z_2} = \kappa_2 \underline{z_1}^T \underline{1} = \kappa_2$.

With $1 \leq \frac{n}{(r-\lambda_1)} MIN$, it follows $MIN = \underline{z}^T B \underline{z} = \kappa > 0$.

If $\underline{x} = \frac{1}{\kappa} \underline{z}$, then $B\underline{x} = \underline{1}$ and $\underline{1}^T \underline{x} = \frac{1}{\kappa}$.

If $\lambda_1 < \phi_1$, then B is regular and $1 = \underline{1}^T \underline{z} = \kappa \underline{1}^T B^{-1} \underline{1}$, hence, $\underline{1}^T \underline{x} = \underline{1}^T B^{-1} \underline{1}$. ∎

***Proof of Theorem* 6.** The proof of Theorem 6 is similar to that one of Theorem 5.

Let $x_i \geq 0$ for $i = 1, \ldots, n$ and, since $\sigma_n > 1$, inequality (2) implies

$$\sigma_n (\sum_{i=1}^n d_i x_i)^2 - \frac{2(\sigma_n - 1)m}{\delta} \sum_{i=1}^n (d_i x_i)^2 \leq \frac{4m}{\delta^2} \sum_{ij \in E} (d_i x_i)(d_j x_j).$$

Substituting $w_i = d_i x_i$ for $i = 1, \ldots, n$, it follows

$$\sigma_n \delta^2 - 2(\sigma_n - 1)m\delta \sum_{i=1}^n w_i^2 \leq 4m \sum_{ij \in E} w_i w_j, \tag{6}$$

for arbitrary $w_i \geq 0$ for $i = 1, \ldots, n$ with $\sum_{i=1}^n w_i = 1$.

Again, let U be an induced subgraph of G isomorphic to H and $\phi : V(H) \to V(U)$ be a graph isomorphism from H to U, and, for real numbers $z_1, \ldots, z_h \geq 0$ with $\sum_{q=1}^h z_q = 1$, let $w_1, \ldots, w_n$ be defined as follows: If $i \in V(U)$, then there is a suitable $q \in \{1, \ldots, h\}$ such that $i = \phi(v_q)$. Set $w_i = z_q$ in this case. If $i \in V \setminus V(U)$, then let $w_i = 0$.

Inequality (6) implies $\frac{\sigma_n \delta^2}{2m} \leq \min(\underline{z}^T A_H \underline{z} + (\sigma_n - 1)\delta \underline{z}^T \underline{z}) = MIN$, where the minimum is taken over $\mathcal{S}_h = \{\underline{z} = (z_1, \ldots, z_h)^T \mid z_q \geq 0 \text{ for } q = 1, \ldots, h,\ \sum_{q=1}^h z_q = 1\}$. Note that this minimum exists because $\underline{z}^T A_H \underline{z} + (\sigma_n - 1)\delta \underline{z}^T \underline{z}$ is a continuous function and $\mathcal{S}_h$ is a compact set.

Let $\underline{z} = (z_1, \ldots, z_h)^T \in \mathcal{S}_h$ with $\underline{z}^T A_H \underline{z} + (\sigma_n - 1)\delta \underline{z}^T \underline{z} = MIN$. Furthermore, let H' be the induced subgraph of H with vertex set $V(H') = \{q \in V(H) \mid z_q > 0\} \neq \emptyset$.

If $t = |V(H')| = 1$, then $H' = K_1 \in M(G, H)$ with $V(H') = \{q\}$, $z_q = 1$, and $MIN = (\sigma_n - 1)\delta > 0$. Hence, $\underline{y} = (\frac{1}{(\sigma_n - 1)\delta})$ is a positive solution of $(A_{H'} + (\sigma_n - 1)\delta I)\underline{y} = \underline{1}$ and it follows $g(G, H') = \underline{1}^T \underline{y} = \frac{1}{(\sigma_n - 1)\delta} = \frac{1}{MIN}$ and $1 \leq \frac{2m}{\sigma_n \delta^2 g(G, H')}$.

If $t \geq 2$, then $0 < z_q < 1$ for all $q \in V(H')$. Thus, $MIN = \min(\underline{u}^T A_{H'} \underline{u} + (\sigma_n - 1)\delta \underline{u}^T \underline{u})$, where the minimum is taken over the relative interior $rint(\mathcal{S}_t) = \{\underline{u} = (u_1, \ldots, u_t)^T \mid u_s > 0 \text{ for } s = 1, \ldots, t,\ \sum_{s=1}^t u_s = 1\}$ of $\mathcal{S}_t$, consequently, this minimum is a local minimum at the hyperplane $\mathcal{H}_t = \{\underline{u} = (u_1, \ldots, u_t)^T \mid \sum_{s=1}^t u_s = 1\}$.

To investigate this value MIN, we consider the Lagrange function $L(\underline{u}, \kappa) = \underline{u}^T A_{H'} \underline{u} + (\sigma_n - 1)\delta \underline{u}^T \underline{u} - 2\kappa(\sum_{s=1}^t u_s - 1)$ with Lagrange multiplier 2κ and the necessary optimality conditions $L_{u_s} = 0$ for $s = 1, \ldots, t$.

With $B = A_{H'} + (\sigma_n - 1)\delta I$, we obtain that the system $B\underline{u} = \kappa \underline{1}$, $\underline{1}^T \underline{u} = 1$ has a positive solution $\underline{u}$.

Next we will show that κ is unique. If $B\underline{u_1} = \kappa_1 \underline{1}$, $\underline{1}^T \underline{u_1} = 1$, $B\underline{u_2} = \kappa_2 \underline{1}$, and $\underline{1}^T \underline{u_2} = 1$, then $\kappa_1 = \kappa_1 \underline{1}^T \underline{u_2} = \underline{u_1}^T B \underline{u_2} = \kappa_2 \underline{u_1}^T \underline{1} = \kappa_2$.

With $1 \leq \frac{2m}{\sigma_n \delta^2} MIN$, it follows $MIN = \underline{u}^T B \underline{u} = \kappa > 0$.

If $\underline{y} = \frac{1}{\kappa} \underline{u}$, then $B\underline{y} = \underline{1}$ has a positive solution $\underline{y}$, consequently, $H' \in M(G, H)$. Moreover, $g(G, H') = \underline{1}^T \underline{y} = \frac{1}{\kappa} = \frac{1}{MIN}$ and we obtain $1 \leq \frac{2m}{\sigma_n \delta^2 g(G, H')}$.

To see that $f(G, H) = \frac{1}{g(G, H')}$, assume there is $H'' \in M(G, H)$ with $g(G, H'') > g(G, H')$. Then there exists $\underline{u} \in rint(\mathcal{S}_t)$ with $t = |V(H'')|$ such that $\underline{u}^T A_{H''} \underline{u} + (\sigma_n - 1)\delta \underline{u}^T \underline{u} < MIN$.

Let $x_i = u_i$ if $i \in V(H'')$ and $x_i = 0$ for $i \in V(H) \setminus V(H'')$.

It follows $\underline{x} = (x_1, \ldots, x_h)^T \in \mathcal{S}_{|V(H)|}$ and $\underline{x}^T A_H \underline{x} + (\sigma_n - 1)\delta \underline{x}^T \underline{x} < MIN$, contradicting the definition of MIN. ■

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