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# Hidden Structures in Super Form Factors

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I, Rodolfo Panerai, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

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September 1, 2019

### Details of collaboration and publications

This thesis covers the original research that I carried out during my PhD program in collaboration with my supervisors (together with other authors) and is based on the three papers where the results were originally presented. These are

- Andreas Brandhuber, Edward Hughes, Rodolfo Panerai, Bill Spence, and Gabriele Travaglini. "The connected prescription for form factors in twistor space". In: *JHEP* 11 (2016), p. 143. arXiv: 1608.03277 [hep-th];
- Lorenzo Bianchi, Andreas Brandhuber, Rodolfo Panerai, and Gabriele Travaglini. "Form factor recursion relations at loop level". In: *JHEP* 02 (2019), p. 182. arXiv: 1812.09001 [hep-th];
- Lorenzo Bianchi, Andreas Brandhuber, Rodolfo Panerai, and Gabriele Travaglini. "Dual conformal invariance for form factors". In: *JHEP* 02 (2019), p. 134. arXiv: 1812.10468 [hep-th].

In addition to the above, during my PhD program I have also worked on topics which are fairly disconnected from the main line of research presented in this thesis. This has led to the following publications, whose results will not be covered here:

- Rodolfo Panerai. "Global equilibrium and local thermodynamics in stationary spacetimes". In: *Phys. Rev.* D93.10 (2016), p. 104021. arXiv: 1511.05963 [gr-qc];
- Joseph Hayling, Rodolfo Panerai, and Constantinos Papageorgakis. "Deconstructing Little Strings with $\mathcal{N} = 1$ Gauge Theories on Ellipsoids". In: *SciPost Phys.* 4.6 (2018), p. 042. arXiv: 1803.06177 [hep-th];
- Rodolfo Panerai, Matteo Poggi, and Domenico Seminara. "Supersymmetric Wilson loops in two dimensions and duality". In: *Phys. Rev.* D100.2 (2019), p. 025011. arXiv: 1812.01315 [hep-th].

# Abstract

Maximally supersymmetric Yang–Mills theory in four dimensions has remarkable features such as conformal symmetry at the quantum level, evidence of integrability and the existence of a well defined holographic dual. The associated perturbative S-matrix and the mysterious roots of its striking simplicity are part of an active area of research which has recently witnessed enormous progress in making many of its special features manifest.

These successes have led to the question of whether such hidden structures are necessarily confined to the realm of the S-matrix or whether they can also illuminate other aspects of the theory. The first step towards the study of more "off-shell quantities" is represented by supersymmetric form factors.

In the first part of the thesis, we propose formulas for any tree-level form factor of the stress-tensor multiplet, derived from twistor worldsheet models. These are the analogue of the ones introduced for amplitudes, both in the twistor and in the more recent ambitwistor formulation.

Another important line of research originates from the AdS/CFT correspondence. In this context, amplitudes are shown to be T-dual to polygonal lightlike Wilson loops. From the point of view of form factors, the dual holographic picture is that of a periodic lightlike Wilson line. The existence of such a picture constitutes a strong indication of invariance under dual conformal transformations.

In the second part of the thesis, we give a prescription for the definition of a canonical integrand for super form factors at one loop in terms of region variables in dual space. This allows us to derive recursion relations at loop level and to study the properties of the resulting expressions under the action of dual conformal generators. We show that the dual conformal anomaly for an arbitrary number of particles and generic helicities matches the expression known for the amplitude case.

# Contents

# 1. Introduction

Quantum field theories are fully determined by their correlators. In fact, the knowledge of all correlation functions of a given theory is sufficient to determine any property of the theory itself. In Minkowski space, most quantum field theories admit a regime in which they describe the dynamics of particles. These are the theories whose spectrum includes multiparticle asymptotic states. When taking a correlation function of renormalised fields and Fourier-transforming it to momentum space, one notices that, in tuning the momenta $p_i$, the correlator has a pole whenever these satisfy the *on-shell condition* $p_i^2 = m_i^2$, where $m_i$ is the physical mass of a particle. To study this singularity one can take the residue of the pole. In fact, one can repeat the process for every momentum in the correlator and study the quantity

$$A \sim \lim_{p_i \to m_i} (p_1^2 - m_1^2) \ldots (p_n^2 - m_n^2) \, \langle \varphi_1(p_1) \, \varphi_2(p_2) \ldots \varphi_n(p_n) \rangle \,, \tag{1.1}$$

where all momenta are brought on shell. According to the Lehmann–Symanzik–Zimmermann (LSZ) formula, the above is an element of the S-matrix of the theory, i.e. a transition amplitude between multiparticle asymptotic states,

$$A = {}_{\text{out}}\langle \mathbf{p}_1, \sigma_1; \, \ldots; \mathbf{p}_i, \sigma_i | \mathbf{p}_{i+1}, \sigma_{i+1}; \, \ldots; \mathbf{p}_n, \sigma_n \rangle_{\text{in}} \,. \tag{1.2}$$

The last fifteen years have witnessed enormous progress in the study of gauge-theory amplitudes with particular focus on $\mathcal{N} = 4$ super Yang–Mills (sYM), the four-dimensional gauge theory with maximal supersymmetry. The theory preserves its conformal symmetry at a quantum level, which means that all masses $m_i$ are vanishing. Despite the richness of its spectrum, a striking simplicity and new, unexpected mathematical structures have been unveiled in the planar limit of its S-matrix, with possibly the most prominent being an integrable algebra of Yangian type. More recently, amplitudes in $\mathcal{N} = 4$ sYM have been derived via new worldsheet formulas [7, 8] and in the context of positive geometry [9].

What is particularly fascinating about these findings is that all these emergent structures seem to be well concealed under the complexity of conventional computational techniques. In the typical approach to quantum field theory, where one starts from local fields and an action principle, many of these properties make their surprising appearance only at the latest steps of the calculations. In this picture, amplitudes are just sparse points in the vast kinematic space spanned by correlation functions.

Many of the advances in the amplitude program have been reached in conjunction with the discovery of novel computational techniques broadly referred to as *on-shell methods*. These stem from two major realisations. The first is that the many properties of the S-matrix can be made manifest by adopting a clever, unconstrained choice of variables, tailored to the on-shell dynamics. The second, perhaps more striking, is that in many cases on-shell quantities can be effectively computed solely in terms of other on-shell quantities.

These successes have led to the fascinating question of whether they could be a hint of a new approach to quantum field theory where the S-matrix is essential, rather than derived, and its remarkable properties are fundamental, rather than emergent. Can this imply a shift of paradigms away from locality? After all, the S-matrix, which is a central observable in quantum field theory, is extremely nonlocal in nature. In the scattering of massless particles, the kinematic data that selects an amplitude live at null infinity, at the boundary of Minkowski space, and amplitudes are obtained by summing over all processes that can take place in the bulk.

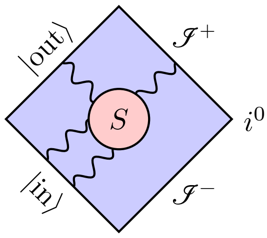



Figure 1.1: A Penrose diagram of Minkowski spacetime. The $|\text{in}\rangle$ and $|\text{out}\rangle$ asympotic states are defined on $\mathscr{I}^-$ and $\mathscr{I}^+$, respectively. In the definition of the S-matrix, interaction in the bulk are summed over.

More concretely, in light of these progresses one might wonder whether the on-shell methods developed in recent years must necessarily be confined to the realm of the S-matrix or whether they can also prove useful to illuminate other aspects of the theory. The first step towards the study of "more off-shell" quantities is represented by form factors, which differ from amplitudes by the insertion of some gauge-invariant composite operator $\mathcal{O}(x)$. Contrary to amplitudes, form factors are obtained by bringing all but one external momenta on shell,

$$F \sim \lim_{p_i^2 \to 0} p_1^2 \dots p_n^2 \, \langle \varphi_1(p_1)\, \varphi_2(p_2) \dots \varphi_n(p_n)\, \tilde{\mathcal{O}}(-q) \rangle \, . \tag{1.3}$$

This corresponds to the quantity

$$F = {}_{\text{out}}\langle \mathbf{p}_1, \sigma_1; \, \dots ; \mathbf{p}_n, \sigma_n | \mathcal{O}(0) | \Omega \rangle_{\text{in}} \, , \tag{1.4}$$

which appears, for example, in the context of effective field theories and captures a scattering process where particles interact with an external source. In quantum electrodynamics, the anomalous magnetic dipole moment of the electron is obtained with a form-factor computation.

The prediction agrees with the experimental results to about one part in a trillion [10], making it the most accurately verified prediction in the whole domain of science.

In this thesis we show how various on-shell methods can be applied to the study of form factors of operators from the stress-tensor multiplet of $\mathcal{N} = 4$ sYM. This line of research was initiated in [11, 12] where techniques such as *unitarity cuts* [13, 14], tree-level *recursion relations* [15, 16] and *MHV diagrams* [17, 18] were used directly to find new expressions for tree-level and one-loop form factors. These papers also provided first indications of extensions of the *amplitude/Wilson loop duality* [19–22] and formulations of form factors in *momentum twistor space* [23]. It also became clear soon after [24] that more advanced methods like *generalised unitarity* [25, 26] and the symbol of transcendental functions [27] could be employed effectively to obtain a plethora of novel results [28–32]. It also turned out that new geometric formulations like Grassmannians [33] and twistor strings could be extended, see [34] and [35–37], respectively. In Chapter 2 we give a general introduction to the amplitude program and review some of these results.

In Chapter 3 we present some results for tree-level form factors. A first crucial result of the amplitude program was the discovery of a representation for $\mathcal{N} = 4$ superamplitudes at tree level, found by Roiban, Spradlin and Volovich (RSV) in [38] as an integral over the moduli space of degree $k+1$ curves in super twistor space (for $\mathrm{N}^k$MHV amplitudes), following Witten's groundbreaking insights [39]. An important feature of the *RSV formula*, presented in Section 2.12, is that the integral is, in fact, localised on a discrete set of solutions of certain polynomial equations. However, despite being conceptually beautiful, the RSV representation proved hard to work with because of the difficulty in determining these solutions. In this respect, the recursion relation known as *BCFW recursion* [15, 16] emerged as a much more tractable and generalisable approach to compute amplitudes, also applicable in different theories, including gravity [40–42]. Important progress was made later in [43], which accomplished two goals: firstly, it showed that by rewriting the RSV formula using the link variables introduced in [44] – which have the neat property of linearising momentum conservation – one can overcome the roadblocks due to the complexity of the algebraic equations arising in [38]; and furthermore, it proved that a certain precisely formulated change of integration contour in the RSV formula, rewritten using *link variables*, expresses the amplitudes as a sum of residues that are identical to BCFW terms (with appropriate shifts). This is an intriguing result, as it relates two a priori very different formulations of gauge theory amplitudes. In Chapter 3 we extend the RSV formula to form factors. We propose a formula for tree-level super form factors from the $\mathcal{N} = 4$ stress-tensor multiplet, study some of its properties and show how it connects to other representations.

The next two chapters are devoted to form factors at loop level.

On the amplitude side, the successful extension of recursive techniques to integrands of planar loop amplitudes in $\mathcal{N} = 4$ SYM was accomplished in [45], following earlier work of [46].

A key insight of [45] is that at each loop order one can unambiguously define an object, the planar integrand, which can then be computed recursively. This relies on the fact that for a colour-ordered amplitude, one can re-write the momenta of the particles using *region momenta* as [20, 47]

$$p_i = x_i - x_{i+1} \, . \tag{1.5}$$

This change of variables automatically implements momentum conservation, and is a crucial ingredient in the duality between Wilson loops and scattering amplitudes in $\mathcal{N}=4$ SYM [19–21]. In this duality, the Wilson loop is stretched along a polygonal light-like contour which connects the points $x_i$. At strong coupling [19],this mapping can be interpreted as a T-duality transformation on the $\mathrm{AdS}_5$ coordinates.

In the weak coupling picture [20–22], the assignment of region momenta for the planar integrand shows the emergence of an anomalous hidden symmetry, known as *dual conformal invariance* (DCI) [48, 49].

Dual conformal symmetry is a highly non-trivial feature of scattering amplitudes in $\mathcal{N}=4$ super Yang-Mills (SYM) theory. Historically, it was first noticed that the integrals appearing in the perturbative expansion of the four-point amplitude enjoy conformal invariance when expressed in terms of dual variables [20, 47]. More precisely, they would be dual conformal invariant if they could be computed in four dimensions. The need for an infrared (IR) regulator breaks dual conformal invariance and generates an anomaly [48, 50], which is however under complete control [48] and at one loop induces relations among the supercoefficients of the box integrals entering the final result [51, 52]. Moreover, a one-loop unitarity-based derivation of this anomaly for arbitrary helicities and number of external legs was presented in [53]. In the Wilson loop picture, DCI is simply conformal invariance of the Wilson loop expectation value, whose anomaly is due to the presence of cusps along the contour.

It soon became also clear that tree-level scattering amplitudes are invariant under the full dual superconformal group [49] and the symmetry can be extended to an infinite dimensional Yangian algebra [54]. Since even at tree level the full amplitude is, strictly speaking, only covariant, not invariant, under dual conformal transformations, it is convenient to work with ratios of amplitudes. In practice, one usually divides the result by the tree-level MHV amplitude and the resulting ratio is then invariant up to anomalies due to IR divergences. A convenient way to show this invariance is to introduce momentum twistors [55]. These variables allow dual superconformal transformations to act linearly, and are helpful to systematically construct superconformal invariants [56]. More recently, dual conformal symmetry received renewed attention. On the one hand, the authors of [57] developed an IR regulator making dual conformal invariance of finite observables manifest at the integrand level, on the other hand a careful analysis has shown the emergence of hidden symmetries in the non-planar sector of amplitudes [58–61].

In Chapter 4 we show how one can define loop integrand functions for form factors and how these can be computed through loop recursion relations. In Chapter 5 we use the region variable assignment adopted in Chapter 4 to show that form factors enjoy dual conformal symmetry even at one loop and exhibit an anomaly which has the same form as the one found in the results for loop amplitudes.

# 2. Review

In this chapter we review material that is foundational for the original results that will be presented in Chapters 3, 4 and 5. At the same time, we set the notation and the conventions that will be used throughout the present thesis. Rather than giving a complete account of modern on-shell methods, the discussion will be tailored to the specific techniques that will be used and problems that will be analysed in later chapters. The topics are laid down following a logical progression rather than a historical account of the literature on the subject.

## 2.1 Spinors

With $\mathbb{R}^{p,q}$ we denote the semi-Riemannian manifold $\mathbb{R}^{p+q}$ with flat metric

$$\eta_{(p,q)} = \mathrm{diag}(\mathbb{1}_p, -\mathbb{1}_q). \tag{2.1}$$

The associated isometry group $\mathrm{T}_{p+q} \rtimes \mathrm{O}(p,q)$ is given by the semidirect product of the group of spacetime translations with the Lorentz group. The component of the Lorentz group connected to the identity, $\mathrm{SO}_0(p,q)$, has a double cover, the spin group $\mathrm{Spin}_0(p,q)$. This is defined through the central extension realised by the short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}_0(p,q) \longrightarrow \mathrm{SO}_0(p,q) \longrightarrow 1 \ . \tag{2.2}$$

In four dimensions, the choices $(4,0)$, $(1,3)$ and $(2,2)$ correspond, respectively, to Euclidean, Minkowski and Kleinian spacetimes; the other two possible choices are simply related to the former by a switch from a "west coast" to an "east coast" signature. We have

$$\mathrm{Spin}_0(4,0) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2) \ , \tag{2.3}$$

$$\mathrm{Spin}_0(1,3) \simeq \mathrm{SL}(2,\mathbb{C}) \ , \tag{2.4}$$

$$\mathrm{Spin}_0(2,2) \simeq \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \ . \tag{2.5}$$

Only in the Euclidean case the spin group is compact. Furthermore, for both Euclidean and Minkowski signature, the spin group is the universal cover of the Lorentz group. On the contrary, $\mathrm{Spin}(2,2)$ is not simply connected and has fundamental group $\pi_1(\mathrm{Spin}(2,2)) = \mathbb{Z} \times \mathbb{Z}$.

It is useful to consider the complexified Minkowski space $\mathbb{M}_{\mathrm{b}} \simeq \mathbb{C}^4$. In fact, the above spin groups can all be regarded as different real forms of the same Lie group, as their complexifications are all isomorphic to $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$. It is well known that for $\mathrm{SL}(2,\mathbb{C})$, the fundamental $(\frac{1}{2},0)$ and the conjugate $(0,\frac{1}{2})$ representations are inequivalent. Weyl spinors transforming with the two are indicated respectively with $\lambda_\alpha$ and $\tilde{\lambda}^{\dot{\alpha}}$, where $\alpha = 1,2$ and $\dot{\alpha} = \dot{1},\dot{2}$. For each of the two representations above, then, one can consider the associated representation defined through the map $A \mapsto (A^{\mathrm{t}})^{-1}$. These, however, are equivalent to the original ones. They correspond to spinors with "raised" and "lowered" indices, respectively, with

$$\lambda^\alpha = \varepsilon^{\alpha\beta}\lambda_\beta \ , \tag{2.6}$$

$$\tilde{\lambda}_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}^{\dot{\beta}} \tag{2.7}$$

where $\varepsilon$ is the antisymmetric tensor defined with

$$\varepsilon^{12} = \varepsilon^{\dot{1}\dot{2}} = \varepsilon_{21} = \varepsilon_{\dot{2}\dot{1}} = 1 \ . \tag{2.8}$$

The generators of the Lorentz group acting on dotted and undotted spinors read

$$(\sigma^{\mu\nu})_\alpha{}^\beta = \frac{\mathrm{i}}{4}(\sigma^\mu_{\alpha\dot{\gamma}}\,\bar{\sigma}^{\nu\,\dot{\gamma}\beta} - \sigma^\nu_{\alpha\dot{\gamma}}\,\bar{\sigma}^{\mu\,\dot{\gamma}\beta}) \ , \tag{2.9}$$

$$(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{\mathrm{i}}{4}(\bar{\sigma}^{\mu\,\dot{\alpha}\gamma}\,\sigma^\nu_{\gamma\dot{\beta}} - \bar{\sigma}^{\nu\,\dot{\alpha}\gamma}\,\sigma^\mu_{\gamma\dot{\beta}}) \ , \tag{2.10}$$

where

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} , \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{2.11}$$

and

$$\bar{\sigma}^{\mu\,\dot{\alpha}\alpha} = \varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\beta\dot{\beta}} \ . \tag{2.12}$$

Dotted and undotted spinors are also known as holomorphic and anti-holomorphic spinors. It is easy to see that one can construct Lorentz invariants with

$$\langle a\, b\rangle = \quad \varepsilon_{\alpha\beta}\lambda^\alpha_a\lambda^\beta_b = -\langle b, a\rangle \ , \tag{2.13}$$

$$[a\, b] = -\varepsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}^{\dot{\alpha}}_a\tilde{\lambda}^{\dot{\beta}}_b = -[b, a] \ . \tag{2.14}$$

The vector representation $(\frac{1}{2},\frac{1}{2})$ is the representation of the spin group which is isomorphic to the fundamental representation of the Lorentz group. This isomophism is realised by the linear map

$$f : \mathbb{C}^4 \to \mathrm{Mat}(2,\mathbb{C}) \ , \qquad f : p^\mu \mapsto p_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}}p^\mu \ . \tag{2.15}$$

With the above map, $p_\mu p^\mu = \det(p_{\alpha\dot\alpha})$. Any $p_{\alpha\dot\alpha} \in \mathrm{Mat}(2,\mathbb{C})$, being at most of rank two, can be written as

$$p_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha} + \mu_\alpha \tilde\mu_{\dot\alpha} \; . \tag{2.16}$$

However, if the associated vector is lightlike, then the matrix has rank one and can be decomposed as

$$p_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha} \tag{2.17}$$

for some $\lambda_\alpha$ and $\tilde\lambda_{\dot\alpha}$.

In complexified Minkowski space, the two spinors are independent. Different real spacetime signatures are obtained by imposing appropriate reality conditions on $\lambda_\alpha$ and $\tilde\lambda_{\dot\alpha}$.

With the above identification, it is straightforward to check that one can write the scalar product of two lightlike vectors in terms of Lorentz invariants constructed with the associated spinors as

$$(p+q)^2 = 2p_\mu q^\mu = \langle p\, q\rangle [p\, q] \; . \tag{2.18}$$

One can regard $\lambda_\alpha$ and $\tilde\lambda_{\dot\alpha}$ as natural variables to represent lightlike momenta. In fact, there is no need to impose additional constraints on their components, as the $p_{\alpha\dot\alpha}$ built out of them is automatically localised on the lightcone. One might argue, however, that the price we pay is to introduce a redundancy, as can be obvious just by counting degrees of freedom. It is immediate to see that the decomposition in (2.17) is not unique, as one can always rescale the two spinors as

$$(\lambda_\alpha, \tilde\lambda_{\dot\alpha}) \mapsto (t\lambda_\alpha, t^{-1}\tilde\lambda_{\dot\alpha}) \; , \qquad t \in \mathbb{C} \setminus \{0\} \; . \tag{2.19}$$

The above transformation corresponds to the action of the so-called *little group*, or *stability group* of $p$. We will say more on this aspect in the next section. In real Minkowski spacetime, the two spinors are one the complex conjugate of the other and $t$ is a pure phase. Being two dimensional vectors, a generic spinor $\lambda_3^\alpha$ can always be decomposed as a linear combination of two linearly-independent spinors $\lambda_1^\alpha$ and $\lambda_2^\alpha$ as

$$\lambda_3^\alpha = \frac{\langle 3\, 2\rangle \lambda_1^\alpha - \langle 3\, 1\rangle \lambda_2^\alpha}{\langle 1\, 2\rangle} \; . \tag{2.20}$$

A direct consequence of the above formula is the *Schouten identity*

$$\langle 1\, 2\rangle\langle 3\, n\rangle + \langle 2\, 3\rangle\langle 1\, n\rangle + \langle 3\, 1\rangle\langle 2\, n\rangle = 0 \; . \tag{2.21}$$

Analogously, one finds

$$[1\, 2][3\, n] + [2\, 3][1\, n] + [3\, 1][2\, n] = 0 \; . \tag{2.22}$$

## 2.2 The action

In four-dimensional Minkowski spacetime, supersymmetry is generated by a set of $\mathcal{N}$ pairs of conjugate Weyl spinors $\mathbb{Q}_\alpha^A$ and $\bar{\mathbb{Q}}_A^{\dot\alpha}$, with $A = 1, ..., \mathcal{N}$. For renormalisable theories, one can only consider superalgebras with $\mathcal{N} \leq 4$, since for $\mathcal{N} > 4$, supersymmetric multiplets necessarily include fields with spin greater than one. The limit case, i.e. the theory with $\mathcal{N} = 4$, can be obtained as the dimensional reduction of the Yang–Mills theory describing the dynamics of a $\mathcal{N} = 1$ supersymmetric vector multiplet in ten dimensions.

Let us first look at the field content. All fields come from a single CPT-self-conjugate supermultiplet

$$(v_{\alpha\dot\alpha}, \psi_{A\,\alpha}, \bar\psi^{A\,\dot\alpha}, \varphi^{AB}) \,, \tag{2.23}$$

and take values in the Lie algebra of the gauge group $\mathbf{G}$. These are a gauge vector $v_{\alpha\dot\alpha}$, six real scalars $\varphi^{AB} = -\varphi^{BA}$, four Weyl fermions $\psi_{A\,\alpha}$ and four conjugate Weyl fermions $\bar\psi^{A\,\dot\alpha}$. Here $A, B, ... = 1, ..., 4$ are SU(4) R-symmetry indices. These that can be understood in terms of the breaking of Lorentz symmetry in the reduction from ten dimensions,

$$\begin{aligned} \mathrm{SO}_0(1,9) &\longmapsto \mathrm{SO}_0(1,3) \times \mathrm{SO}(6) \,, \\ \mathrm{Spin}_0(1,9) &\longmapsto \mathrm{Spin}_0(1,3) \times \mathrm{Spin}(6) \simeq \mathrm{Spin}_0(1,3) \times \mathrm{SU}(4) \,. \end{aligned} \tag{2.24}$$

So, while fermions transform in the fundamental and antifundamental representations of SU(4), scalars transform in the fundamental representation of SO(6). However, to make the R-symmetry structure manifest, the latter are conveniently arranged in the $\mathbf{6}$ of the double cover, SU(4). The following reality conditions hold:

$$(v_{\alpha\dot\alpha})^* = v_{\alpha\dot\alpha} \,, \tag{2.25}$$

$$(\varphi^{AB})^* = \varphi_{AB} = \tfrac{1}{2}\varepsilon_{ABCD}\varphi^{CD} \,, \tag{2.26}$$

$$(\psi_{A\,\alpha})^* = \bar\psi^A_{\dot\alpha} \,. \tag{2.27}$$

The action is completely fixed by its symmetries: it is, in fact, unique up to the choice of the gauge coupling $g_{\mathrm{YM}}$ and of the gauge group $\mathbf{G}$. Written in terms of spinor indices, it reads

$$\begin{aligned} S = \int \mathrm{d}^4x \; \mathrm{tr} \Big[ &-\frac{1}{8}\left(F_{\alpha\beta}F^{\alpha\beta} + \bar F_{\dot\alpha\dot\beta}\bar F^{\dot\alpha\dot\beta}\right) - \frac{1}{2}\mathrm{D}_{\alpha\dot\alpha}\varphi_{AB}\mathrm{D}^{\dot\alpha\alpha}\varphi^{AB} + \mathrm{i}\bar\psi^A_{\dot\alpha}\mathrm{D}^{\dot\alpha\alpha}\psi_{A\,\alpha} \\ &- \frac{\mathrm{i}}{2}g_{\mathrm{YM}}\psi^\alpha_A[\varphi^{AB}, \psi_{B\,\alpha}] - \frac{\mathrm{i}}{2}g_{\mathrm{YM}}\bar\psi^A_{\dot\alpha}[\varphi_{AB}, \bar\psi^{B\,\dot\alpha}] \\ &- \frac{1}{2}g^2_{\mathrm{YM}}[\varphi_{AB}, \varphi_{CD}][\varphi^{AB}, \varphi^{CD}] \Big] \,, \end{aligned} \tag{2.28}$$

where the self-dual and anti-self dual components of the field strength, namely $F_{\alpha\beta}$ and $\bar F_{\dot\alpha\dot\beta}$, are defined with

$$\mathrm{D}_{\alpha\dot\alpha} = \partial_{\alpha\dot\alpha} - \mathrm{i}g_{\mathrm{YM}}v_{\alpha\dot\alpha} \,, \tag{2.29}$$

$$[\mathrm{D}_{\alpha\dot\alpha}, \mathrm{D}_{\beta\dot\beta}] = -\mathrm{i} g_{\mathrm{YM}} (F_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta} + \varepsilon_{\alpha\beta}\bar F_{\dot\alpha\dot\beta}) \ . \tag{2.30}$$

The $\mathcal{N}=4$ super-Poincaré algebra $\mathfrak{iso}(1,3|4)$ reads

$$[\mathbf{M}_{\alpha\beta}, \mathbf{M}_{\gamma\delta}] = \tfrac{1}{2}\varepsilon_{\alpha\gamma}\mathbf{M}_{\beta\delta} + \tfrac{1}{2}\varepsilon_{\beta\gamma}\mathbf{M}_{\alpha\delta} + \tfrac{1}{2}\varepsilon_{\alpha\delta}\mathbf{M}_{\beta\gamma} + \tfrac{1}{2}\varepsilon_{\beta\delta}\mathbf{M}_{\alpha\gamma} \tag{2.31}$$

$$[\bar{\mathbf{M}}_{\dot\alpha\dot\beta}, \bar{\mathbf{M}}_{\dot\gamma\dot\delta}] = \tfrac{1}{2}\varepsilon_{\dot\alpha\dot\gamma}\bar{\mathbf{M}}_{\dot\beta\dot\delta} + \tfrac{1}{2}\varepsilon_{\dot\beta\dot\gamma}\bar{\mathbf{M}}_{\dot\alpha\dot\delta} + \tfrac{1}{2}\varepsilon_{\dot\alpha\dot\delta}\bar{\mathbf{M}}_{\dot\beta\dot\gamma} + \tfrac{1}{2}\varepsilon_{\dot\beta\dot\delta}\bar{\mathbf{M}}_{\dot\alpha\dot\gamma} \tag{2.32}$$

$$[\mathbf{M}_{\alpha\beta}, \mathbf{Q}^A_\gamma] = \tfrac{1}{2}\varepsilon_{\beta\gamma}\mathbf{Q}^A_\alpha + \tfrac{1}{2}\varepsilon_{\alpha\gamma}\mathbf{Q}^A_\beta \tag{2.33}$$

$$[\bar{\mathbf{M}}_{\dot\alpha\dot\beta}, \bar{\mathbf{Q}}_{A\,\dot\gamma}] = \tfrac{1}{2}\varepsilon_{\dot\alpha\dot\gamma}\bar{\mathbf{Q}}_{A\,\dot\beta} + \tfrac{1}{2}\varepsilon_{\dot\beta\dot\gamma}\bar{\mathbf{Q}}^{A}{}_{\dot\alpha} \tag{2.34}$$

$$[\mathbf{M}_{\alpha\beta}, \mathbf{P}_{\gamma\dot\gamma}] = \tfrac{1}{2}\varepsilon_{\beta\gamma}\mathbf{P}_{\alpha\dot\gamma} + \tfrac{1}{2}\varepsilon_{\alpha\gamma}\mathbf{P}_{\beta\dot\gamma} \tag{2.35}$$

$$[\bar{\mathbf{M}}_{\dot\alpha\dot\beta}, \mathbf{P}_{\gamma\dot\gamma}] = \tfrac{1}{2}\varepsilon_{\dot\alpha\dot\gamma}\mathbf{P}_{\gamma\dot\beta} + \tfrac{1}{2}\varepsilon_{\dot\beta\dot\gamma}\mathbf{P}_{\gamma\dot\alpha} \tag{2.36}$$

$$[\mathbf{R}^A{}_B, \mathbf{R}^C{}_D] = \delta_B{}^C\mathbf{R}^A{}_D - \delta_D{}^A\mathbf{R}^C{}_B \tag{2.37}$$

$$[\mathbf{R}^A{}_B, \mathbf{Q}^C_\alpha] = \delta_B{}^C\mathbf{Q}_{A\,\alpha} - \tfrac{1}{4}\delta_B{}^A\mathbf{Q}_{C\,\alpha} \tag{2.38}$$

$$[\mathbf{R}^A{}_B, \bar{\mathbf{Q}}_{C\,\dot\alpha}] = -(\delta_C{}^A\bar{\mathbf{Q}}_{B\,\dot\alpha} - \tfrac{1}{4}\delta_B{}^A\bar{\mathbf{Q}}_{C\,\dot\alpha}) \tag{2.39}$$

$$\{\mathbf{Q}^A_\alpha, \bar{\mathbf{Q}}_{B\,\dot\alpha}\} = \delta_B{}^A\mathbf{P}_{\alpha\dot\alpha} \ . \tag{2.40}$$

Unlike the cases with fewer supersymmetries, namely $\mathcal{N}=1$ and $\mathcal{N}=2$, where one can add a finite number of auxiliary fields to the vector multiplet so that the algebra closes off-shell, it is not known how to realise this for the theory with $\mathcal{N}=4$. An on shell representation is possible for a chiral half of the superalgebra. This will be discussed in Section 2.6.

The theory has a rich moduli space of vacua, parametrised by the expectation values of the six scalars. At the origin of the moduli space, where all scalars have vanishing expectation value, the theory is conformal. The scale invariance present at the classical level (the action does not contain any dimensionful parameter) is preserved at the quantum level and the $\beta$ function is identically vanishing for any value of the coupling $g_{\mathrm{YM}}$, which is exactly marginal.

For conformal field theories, the Poincaré algebra is extended to the conformal algebra $\mathfrak{so}(2,4)$ by the introduction of generators of dilatations and special conformal transformations

$$[\mathbf{D}, \mathbf{P}_{\alpha\dot\alpha}] = \mathbf{P}_{\alpha\dot\alpha} \tag{2.41}$$

$$[\mathbf{D}, \mathbf{K}_{\alpha\dot\alpha}] = -\mathbf{K}_{\alpha\dot\alpha} \tag{2.42}$$

$$[\mathbf{K}_{\alpha\dot\alpha}, \mathbf{P}_{\beta\dot\beta}] = \varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}\mathbf{D} + \varepsilon_{\dot\alpha\dot\beta}\mathbf{M}_{\alpha\beta} + \varepsilon_{\alpha\beta}\bar{\mathbf{M}}_{\dot\alpha\dot\beta} \tag{2.43}$$

$$[\mathbf{M}_{\alpha\beta}, \mathbf{K}_{\gamma\dot\gamma}] = \tfrac{1}{2}\varepsilon_{\beta\gamma}\mathbf{K}_{\alpha\dot\gamma} + \tfrac{1}{2}\varepsilon_{\alpha\gamma}\mathbf{K}_{\beta\dot\gamma} \tag{2.44}$$

$$[\bar{\mathbf{M}}_{\dot\alpha\dot\beta}, \mathbf{K}_{\gamma\dot\gamma}] = \tfrac{1}{2}\varepsilon_{\dot\alpha\dot\gamma}\mathbf{K}_{\gamma\dot\beta} + \tfrac{1}{2}\varepsilon_{\dot\beta\dot\gamma}\mathbf{K}_{\gamma\dot\alpha} \ . \tag{2.45}$$

When the same theory is both conformal and supersymmetric, the two algebras combine in a single superconformal algebra that closes with the addition of new fermionic conformal supercharges. In particular, for $\mathcal{N}=4$ sYM, one has the superconformal algebra $\mathfrak{su}(2,2|4)$

$$[\mathbf{D}, \mathbf{Q}^A{}_\alpha] = \tfrac{1}{2}\mathbf{Q}^A{}_\alpha \tag{2.46}$$

$$[\mathbf{D}, \bar{\mathbf{Q}}_{A\,\dot\alpha}] = \tfrac{1}{2}\bar{\mathbf{Q}}_{A\,\dot\alpha} \tag{2.47}$$

$$[\mathbf{D}, \mathbf{S}_{A\,\alpha}] = -\tfrac{1}{2}\mathbf{S}_{A\,\alpha} \tag{2.48}$$

$$[\mathbf{D}, \bar{\mathbf{S}}^{A}_{\dot\alpha}] = -\tfrac{1}{2}\bar{\mathbf{S}}^{A}_{\dot\alpha} \tag{2.49}$$

$$[\mathbf{K}_{\alpha\dot\alpha}, \mathbf{Q}^{A}_{\beta}] = \varepsilon_{\alpha\beta}\bar{\mathbf{S}}^{A}_{\dot\alpha} \tag{2.50}$$

$$[\mathbf{K}_{\alpha\dot\alpha}, \bar{\mathbf{Q}}_{A\,\dot\beta}] = \varepsilon_{\dot\alpha\dot\beta}\mathbf{S}_{A\,\alpha} \tag{2.51}$$

$$[\mathbf{P}_{\alpha\dot\alpha}, \mathbf{S}_{A\,\beta}] = \varepsilon_{\alpha\beta}\bar{\mathbf{Q}}_{A\,\dot\alpha} \tag{2.52}$$

$$[\mathbf{P}_{\alpha\dot\alpha}, \bar{\mathbf{S}}^{A}_{\dot\beta}] = \varepsilon_{\dot\alpha\dot\beta}\mathbf{Q}^{A}_{\alpha} \tag{2.53}$$

$$[\mathbf{M}_{\alpha\beta}, \mathbf{S}_{A\,\gamma}] = \varepsilon_{\beta\gamma}\mathbf{S}_{A\,\alpha} + \varepsilon_{\alpha\gamma}\mathbf{S}_{A\,\beta} \tag{2.54}$$

$$[\bar{\mathbf{M}}_{\dot\alpha\dot\beta}, \bar{\mathbf{S}}^{A}_{\dot\gamma}] = \varepsilon_{\dot\alpha\dot\gamma}\bar{\mathbf{S}}^{A}_{\dot\beta} + \varepsilon_{\dot\beta\dot\gamma}\bar{\mathbf{S}}_{A\,\dot\alpha} \tag{2.55}$$

$$[\mathbf{R}^{A}{}_{B}, \mathbf{S}_{C\,\alpha}] = -(\delta_{C}{}^{A}\mathbf{S}_{B\,\alpha} - \tfrac{1}{4}\delta_{B}{}^{A}\mathbf{S}_{C\,\alpha}) \tag{2.56}$$

$$[\mathbf{R}^{A}{}_{B}, \bar{\mathbf{S}}^{C}_{\dot\alpha}] = \delta^{C}{}_{B}\bar{\mathbf{S}}^{A\,\dot\alpha} - \tfrac{1}{4}\delta_{B}{}^{A}\bar{\mathbf{S}}^{C}_{\dot\alpha} \tag{2.57}$$

$$\{\mathbf{S}_{A\,\alpha}, \bar{\mathbf{S}}^{B}_{\dot\alpha}\} = \delta_{A}{}^{B}\mathbf{K}_{\alpha\dot\alpha} \tag{2.58}$$

$$\{\mathbf{Q}^{B}_{\beta}, \mathbf{S}_{A\,\alpha}\} = \delta_{A}{}^{B}\mathbf{M}_{\alpha\beta} - \varepsilon_{\alpha\beta}\mathbf{R}^{B}{}_{A} + \tfrac{1}{2}\varepsilon_{\alpha\beta}\delta_{A}{}^{B}(\mathbf{D}+\mathbf{Z}) \tag{2.59}$$

$$\{\bar{\mathbf{Q}}_{B\,\dot\beta}, \bar{\mathbf{S}}^{A}_{\dot\alpha}\} = \delta_{B}{}^{A}\bar{\mathbf{M}}_{\dot\alpha\dot\beta} + \varepsilon_{\dot\alpha\dot\beta}\mathbf{R}^{A}{}_{B} + \tfrac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\delta_{B}{}^{A}(\mathbf{D}-\mathbf{Z})\,. \tag{2.60}$$

Away from the origin of the moduli space, the nonvanishing expectation values of the scalars provide the length scales that break conformal invariance. In the process, some subgroup of **G** gets broken as well. In this thesis, however, we will only consider the vacuum annihilated by the full superconformal algebra $\mathfrak{su}(2,2|4)$.

Theories with conformal symmetry, like the one we are considering, are UV finite. This important property is a consequence of the fact that any observable at short distances can be obtained from a long-distance computation, through an appropriate rescaling. This fact, however, does not prevent generic composite operators, due to their inherent contact divergences, from developing anomalous dimensions as nontrivial functions of the gauge coupling $g_{\mathrm{YM}}$. Yet, the operators that we will consider in this thesis, will all be protected by supersymmetry and, as such, will all have vanishing anomalous dimension.

Moreover, despite the UV-finite character of the observables that will be studied in this thesis, namely amplitudes and form factors of BPS operators, their computation is still affected by divergences coming from the IR, classified as *soft* or *collinear*, that one needs to regulate. Intuitively, one can trace the origin of these divergences from the scattering of interacting massless particles. Starting with a given elastic process, one can consider the same process dressed with either soft emission, or hard emission in a collinear configuration. Since these "deformations" can be kept arbitrary small, they are effectively undetectable and as such, should be summed over in the computation of physical observables. These observables, like cross sections, turn out to be nonetheless finite. At a given order in perturbation theory, in fact, the phase-space integral of an amplitude involving extra undetectable particles precisely cancel the IR divergent part of higher-loop integrals. IR divergences can be regulated by evaluating

Feynman integrals in dimension $d = 4-2\epsilon$. The integrals are analytically continued from $\epsilon < 0$ and the result is written as a Laurent series in $\epsilon$. The IR divergent terms, i.e. the ones that appear with negative powers of $\epsilon$, must cancel for well-defined observables, allowing one to finally take the physical limit $\epsilon \to 0$.

## 2.3 Amplitudes and form factors

An amplitude is an S-matrix element between asymptotic multiparticle states describing particles involved in a given scattering process. As such, we expect it to depend only on the on-shell degrees of freedom that characterise the asymptotic states. In this section, we will briefly review how such states are defined and how the Poincaré group acts on them.

Let us first consider states describing a single stable particle. Such states should always exist in a relativistic quantum theory describing particles even in the presence of interactions because by definition a stable particle cannot disappear, nor transform into other particles. As in the original definition of Wigner [62], single-stable-particle states belong to irreducible representations of the Poincaré group, fixed by eigenvalues of the Casimir operators $P^2$ and $W^2$.

If we consider eigenstates of four-momentum, then, these have the form $|\mathbf{p}, \sigma_i\rangle$ where $\mathbf{p}$ labels the eigenvalue of $P_\mu$ (with fixed mass $m$) and $\sigma$ labels different states with the same $p$. For single-particle states, we require $\sigma$ to assume only discrete values.

A representation of the Poincaré group is realised, through the method of the induced representation, by choosing a reference momentum $k^\mu$ (with stability group or little group W) and, for every $p_\mu$, a particular Lorentz tranformation $p^\mu = (L_p)^\mu{}_\nu k^\nu$ which labels the cosets $\mathrm{SO}_0(1,3)/\mathrm{W}$ and defines

$$|\mathbf{p}, \sigma\rangle = U(L_p, 0)\,|\mathbf{k}, \sigma\rangle\ . \tag{2.61}$$

Then it is easy to construct the action of a Poincaré transformation over a given state as

$$U(\Lambda, a)\,|\mathbf{p}, \sigma\rangle = e^{-\mathrm{i}a\cdot p_\Lambda} \sum_{\sigma'} D_{\sigma\sigma'}(\mathrm{W}(\Lambda, p))\,|\Lambda\mathbf{p}, \sigma'\rangle\ ,, \tag{2.62}$$

where $D_{\sigma\sigma'}$ is a representation of the little group W. For massless particles, $\sigma$ labels the helicity $h$, eigenvalue of $W^0/m$. The action of the little group is diagonal,

$$U(\Lambda, a)\,|\mathbf{p}, h\rangle = e^{-\mathrm{i}a\cdot p_\Lambda}\, e^{\mathrm{i}\theta(\Lambda,p)h}\,|\Lambda\mathbf{p}, h\rangle\ , \tag{2.63}$$

which trivially implies that a massless particle has fixed helicity.

Multiparticle states in a noninteracting theory are created by taking tensor products of single-particle states. Therefore, the above representation for a massless particle can be immediately generalised to the Fock-space representation

$$U(\Lambda, a)\,|\mathbf{p}_1, h_1; ...; \mathbf{p}_n, h_n\rangle = \left(\prod_{j=1}^{n} e^{-\mathrm{i}a\cdot(\Lambda p_j)}\, e^{\mathrm{i}\theta(\Lambda,p_j)h_j}\right)|\Lambda\mathbf{p}_1, h_1; ...; \Lambda\mathbf{p}_n, h_n\rangle\ . \tag{2.64}$$

Here, we are keeping implicit non-spacetime labels that keep track of particle species and charges under internal symmetries of the theory. An analogous representation acts on dual state vectors. Asymptotic "in" and "out" states in an interacting theory are built from free multiparticle states by acting with appropriate Möller operators. One assumes to have a unitary representation of the Poincaré group that acts on them as in [**63**]. Therefore, by adopting the convention that every particle in the scattering process is "outgoing" (thus changing the sign of both momenta and helicities for "ingoing" particles) we derive the covariance property of an amplitude involving massless particles,

$$\mathscr{A}(\Lambda p_1, h_1; ...; \Lambda p_n, h_n) = \left(\prod_{j=1}^{n} e^{\mathrm{i}\theta(\Lambda, p_j)h_j}\right) \mathscr{A}(p_1, h_1; ...; p_n, h_n) \ . \tag{2.65}$$

At a first look, the above might be puzzling: although the asymptotic states are parametrized by a collection of real four-vectors (momenta) and elements of $\mathbb{Z}/2$ (helicities), there is no way to construct a function of these that could satisfy (2.65), unless we are dealing with the case of a purely scalar amplitude, where the little group representation $\theta(\Lambda, p)$ is trivial. Thinking in terms of Feynman rules immediately reveals the missing ingredient: polarizations. Here we use the term to indicate polarizations for particles of any spin. Polarizations carry both little-group indices and Lorentz indices.

However, for $h \geq 1$, polarizations are not uniquely-defined [**63**]. They are introduced to construct local fields which are subject to some gauge redundancy. In the case of a massless vector field, for example, polarization vectors are defined only up to a shift proportional to the four-momentum,

$$\varepsilon_\mu(\mathbf{p}, \lambda) \sim \varepsilon_\mu(\mathbf{p}, \lambda) + \alpha p_\mu \ . \tag{2.66}$$

But this complexity that emerges at the level of the Lagrangian description of the theory and of the associated Feynman rules is precisely what the amplitude program aims to avoid altogether by looking for a description of the S-matrix based only on on-shell data.

The solution is to replace the set of variables that we use to construct the amplitude with more natural ones. These are the spinors introduced in the previous section. In fact, one can univocally write any on-shell momentum $p_\mu$ in terms of a pair $\lambda_p^\alpha \tilde{\lambda}_p^{\dot\alpha} = p^{\dot\alpha\alpha}$ by first arbitrarily choosing the pair $\lambda_k^\alpha \tilde{\lambda}_k^{\dot\alpha}$ associated with the reference four-momentum $k^\mu$ and then transforming these with the appropriate spinor representations of the transformation $L_p$. Now, for a generic Lorentz transformation $\Lambda$,

$$\lambda_p^\alpha \mapsto e^{+\frac{\mathrm{i}}{2}\theta(\Lambda, p)}\, \lambda_{\Lambda p}^\alpha \ , \tag{2.67}$$

$$\tilde{\lambda}_p^{\dot\alpha} \mapsto e^{-\frac{\mathrm{i}}{2}\theta(\Lambda, p)}\, \tilde{\lambda}_{\Lambda p}^{\dot\alpha} \ . \tag{2.68}$$

This means that using these spinors one can build $\mathrm{SL}(2, \mathbb{C})$ invariants that transform properly as amplitudes under a Lorentz transformation of their defining momenta. The redundancy in the decomposition (2.17) associated with the little group action that we noted in the previous section becomes now a key element to construct amplitudes with the right covariance properties.

The helicity of a given particle dictates the overall power of the associated pair of spinors that appear in the expression for the amplitude. Holomorphic and anti-holomorphic spinors carry a little-group phase of helicity $-\frac{1}{2}$ and $+\frac{1}{2}$ respectively.

Finally, we note that if (2.65) has been obtained from (2.64) by setting $a = 0$, one can similarly set $\Lambda = \mathbb{1}$ and find

$$\mathscr{A}(p_1, h_1; ...; p_n, h_n) = e^{-\mathrm{i}a\cdot(p_1+...+p_n)} \mathscr{A}(p_1, h_1; ...; p_n, h_n) \ , \tag{2.69}$$

which implies that the amplitude has an overall delta of four-momentum conservation

$$\delta^{(4)}(p_1 + ... + p_n) \ . \tag{2.70}$$

Extending the above construction to form factors is quite straightforward. One needs to take into account the insertion of a local operator $\mathcal{O}(x)$. For simplicity, let us consider the case where $\mathcal{O}(x)$ has spin zero. From

$$\begin{aligned} \int \mathrm{d}x \, U(\Lambda, a) \, \mathcal{O}(x) \, U(\Lambda, a)^{-1} \, e^{-\mathrm{i}q\cdot x} &= \int \mathrm{d}x \, \mathcal{O}(\Lambda x + a) \, e^{-\mathrm{i}q\cdot x} \\ &= e^{\mathrm{i}(\Lambda q)\cdot a} \int \mathrm{d}x \, \mathcal{O}(x) \, e^{-\mathrm{i}(\Lambda q)\cdot x} \ , \end{aligned} \tag{2.71}$$

in analogy with (2.65) one has

$$\mathscr{F}(\Lambda p_1, h_1; ...; \Lambda p_n, h_n; \Lambda q) = \left( \prod_{j=1}^{n} e^{\mathrm{i}\theta(\Lambda, p_j) h_j} \right) \mathscr{F}(p_1, h_1; ...; p_n, h_n; q) \ . \tag{2.72}$$

Translation invariance, as in (2.69), gives

$$\mathscr{F}(p_1, h_1; ...; p_n, h_n; q) = e^{-\mathrm{i}a\cdot(p_1+...+p_n)} \, e^{\mathrm{i}a\cdot q} \, \mathscr{F}(p_1, h_1; ...; p_n, h_n; q) \ , \tag{2.73}$$

from which one can deduce the presence of an overall[1]

$$\delta^{(4)}(p_1 + ... + p_n - q) \ . \tag{2.74}$$

Before closing this section we want to comment on some issue which is specific to the S-matrix of $\mathcal{N} = 4$ sYM. In general, interacting conformal field theories do not admit a proper S-matrix. This fact can be attributed to the long-range character of the interactions that forbids the existence of multiparticle free asymptotic states. However, as discussed in Section 2.2, IR divergences forces us to perform computations at finite $\epsilon = 2 - d/2$, where conformal invariance is broken by the introduction of a mass scale $\mu$. Another, perhaps more physical, way to overcome this problem would be to perform computations for the theory on the Coulomb branch and then define the S-matrix at the origin of the moduli space simply by taking the zero-VEV limit [**64**].

[1] In our conventions, the off-shell momentum $q$ is incoming.

## 2.4 Symmetries

The action of spacetime symmetries on asymptotic states can be translated into an action on the amplitude $\mathscr{A}$, as a distribution in spinor variables. For example, the action of the generator of spacetime translations can be read off directly from (2.69), where we have the result of a finite transformation. The generator acts as a multiplicative operator

$$\mathsf{p}^{\dot{\alpha}\alpha} = \sum_{i=1}^{n} \lambda_i^{\alpha} \tilde{\lambda}_i^{\dot{\alpha}} \tag{2.75}$$

Generators of Lorentz transformations can again be deduced from the discussion of the previous section. In spinor-helicity notation they come in pairs of symmetric rank-two tensors

$$\mathsf{m}^{\alpha\beta} = \sum_{i=1}^{n} \lambda_i^{(\alpha} \frac{\partial}{\partial \lambda_{i\,\beta)}} \,, \tag{2.76}$$

$$\bar{\mathsf{m}}^{\dot{\alpha}\dot{\beta}} = \sum_{i=1}^{n} \tilde{\lambda}_i^{(\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_{i\,\dot{\beta})}} \,, \tag{2.77}$$

where the round brackets indicate a symmetrisation over the indices. Clearly, because of the symmetry of the theory under the Poincaré group, all its generators annihilate the amplitude $\mathscr{A}$, i.e.,

$$\mathsf{p}^{\dot{\alpha}\alpha}\, \mathscr{A}(\{\lambda_i, \tilde{\lambda}_i, h_i\}) = 0 \,, \tag{2.78}$$

$$\mathsf{m}^{\alpha\beta}\, \mathscr{A}(\{\lambda_i, \tilde{\lambda}_i, h_i\}) = 0 \,, \tag{2.79}$$

$$\bar{\mathsf{m}}^{\dot{\alpha}\dot{\beta}}\, \mathscr{A}(\{\lambda_i, \tilde{\lambda}_i, h_i\}) = 0 \,. \tag{2.80}$$

The first relation is a trivial consequence of momentum conservation. In fact, because of the presence of an overall delta function, the first equation will vanish in a distributional sense, i.e., $\mathsf{p}\,\delta(\mathsf{p}) = 0$. The second two are a direct consequence of the fact that the amplitude is written in terms of Lorentz-invariant contractions of spinors (angle and square brackets $\langle ij \rangle$ and $[ij]$) and these individually vanish under the action of $\mathsf{m}$ and $\bar{\mathsf{m}}$, as one could easily expect.

There is another operator, which does not correspond to a generator of any symmetry, but that can be naturally introduced as a consequence of the discussion of the previous section. This is the helicity operator

$$\mathsf{h}_i = \frac{1}{2}\left( -\lambda_i^{\alpha} \frac{\partial}{\partial \lambda_i^{\alpha}} + \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} \right) \,. \tag{2.81}$$

When acting on the amplitude this gives the helicity of the $i$-th particle, i.e.,

$$\mathsf{h}_j\, \mathscr{A}(\{\lambda_i, \tilde{\lambda}_i, h_i\}) = h_j\, \mathscr{A}(\{\lambda_i, \tilde{\lambda}_i, h_i\}) \,. \tag{2.82}$$

Conformal theories, like the theory we are focusing on, have additional generators which close the $\mathfrak{so}(2,4)$ conformal algebra. These are

$$\mathsf{d} = \sum_{i=1}^{n} \left( \frac{1}{2} \lambda_i^{\alpha} \frac{\partial}{\partial \lambda_i^{\alpha}} + \frac{1}{2} \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} + 1 \right) \,, \tag{2.83}$$

$$\mathsf{k}_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i^{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} \,. \tag{2.84}$$

The representation of the Poincaré algebra that acts on form factors is similar. The momentum generator reflects the presence of the off-shell momentum $q$ in the delta enforcing momentum conservation and reads

$$\mathsf{p}^{\dot{\alpha}\alpha} = -q^{\dot{\alpha}\alpha} + \sum_{i=1}^{n} \lambda_i^{\alpha} \tilde{\lambda}_i^{\dot{\alpha}} \,. \tag{2.85}$$

The other generators read

$$\mathsf{m}^{\alpha\beta} = q_{\dot{\alpha}}{}^{(\alpha} \frac{\partial}{\partial q_{\beta)\dot{\alpha}}} + \sum_{i=1}^{n} \lambda_i^{(\alpha} \frac{\partial}{\partial \lambda_{i\,\beta)}} \,, \tag{2.86}$$

$$\bar{\mathsf{m}}^{\dot{\alpha}\dot{\beta}} = q^{(\dot{\alpha}}{}_{\alpha} \frac{\partial}{\partial q_{\alpha\dot{\beta})}} + \sum_{i=1}^{n} \tilde{\lambda}_i^{(\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_{i\,\dot{\beta})}} \,, \tag{2.87}$$

$$\mathsf{k}_{\alpha\dot{\alpha}} = q^{\dot{\beta}\beta} \frac{\partial}{\partial q^{\dot{\alpha}\beta}} \frac{\partial}{\partial q^{\dot{\beta}\alpha}} + \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i^{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} \tag{2.88}$$

$$\mathsf{d} = q^{\dot{\alpha}\alpha} \frac{\partial}{\partial q^{\alpha\dot{\alpha}}} + \sum_{i=1}^{n} \left( \frac{1}{2} \lambda_i^{\alpha} \frac{\partial}{\partial \lambda_i^{\alpha}} + \frac{1}{2} \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} + 1 \right) \tag{2.89}$$

Alternatively, one can maintain the representation that acts on amplitudes and use momentum conservation to eliminate the explicit dependence on $q$ by trading it for the sum on all the on-shell momenta.

## 2.5 Gauge invariance

As anticipated, the discussion in Section 2.3 was particularly simple, as we only discussed the dependence of amplitudes and form factors on the kinematic data of the asymptotics states. However, our goal is to describe amplitdues and form factors as gauge-invariant observables in gauge theories, where particles in the spectrum are organised in representations of the gauge group $\mathbf{G}$.

Specifically, since we are dealing with the case of $\mathcal{N} = 4$ sYM, where all particles belong to the adjoint representation of $\mathbf{G}$, amplitudes will carry Lie-algebra indices, one for each particle involved in the scattering process. In the present thesis we will study the theory with $\mathbf{G} \simeq \mathrm{SU}(N_\mathrm{c})$, and, in particular, we will consider the limit where $N_\mathrm{c}$ is large. This means that we are effectively probing a specific regime of the theory, which comes from taking two limits. One is the limit of small coupling, that comes from considering amplitudes in their perturbative expansion in $g_{\mathrm{YM}}$. The other one is the *planar limit*, which consists in taking the leading order in the $1/N_\mathrm{c}$ expansion.

Feynman diagrams can be classified according to the powers in $g_{\mathrm{YM}}$ and $N_\mathrm{c}$ their expressions carry. These powers are, in turn, controlled by the topology of the diagram. Together with $n$, the number of loops $\ell$ in a diagram uniquely fixes the overall power in the gauge coupling, which is $g_{\rm YM}^{n-2+2\ell}$. Determining the power in $N_c$ is, instead, less straightforward, and involves studying the colour structure of the diagram. This can be done by introducing the *'t Hoft double-line notation* (or *fatgraphs*) [65]. Typically, one has to take into account the genus, i.e. the lowest genus of a two-dimensional orientable surface on which the diagram can be embedded without any crossing. However, we won't discuss this in full generality as for our purposes it will be sufficient to look at diagrams with $\ell = 0$ and $\ell = 1$, for which the genus is always zero.

Let us then see in detail what the color structure of amplitudes at tree and one-loop level is. As it turns out, amplitudes depend on color indices in a simple and controlled way, and one can use this fact to disentangle the color from the kinematical degrees of freedom. This technique goes under the name of *colour decomposition* [66]. We denote with $\mathrm{t}^a$ the generators of the Lie algebra $\mathfrak{su}(N_{\rm c})$, hence $a = 1, \dots, N_{\rm c}^2 - 1$. The Feynman rules associated with interaction vertices are written in terms of the structure constants of $\mathfrak{su}(N_{\rm c})$. These can be replaced by their definition in terms of generators, with

$$f^{abc} = -\frac{\mathrm{i}}{\sqrt{2}} \left[ \mathrm{tr}(\mathrm{t}^a \mathrm{t}^b \mathrm{t}^c) - \mathrm{tr}(\mathrm{t}^a \mathrm{t}^c \mathrm{t}^b) \right] . \tag{2.90}$$

Then one can combine together traces coming from different vertices in a same diagram with the Fierz identity

$$\mathrm{tr}(\mathrm{t}^a\, X)\ \mathrm{tr}(\mathrm{t}^a\, Y) = \mathrm{tr}(X\, Y) - \frac{1}{N_{\rm c}}\, \mathrm{tr}(X)\ \mathrm{tr}(Y) . \tag{2.91}$$

At tree level, this process can be used to bring all the generators associated with the external legs under a single trace. Indeed, the sum of all Feynman diagrams with $\ell = 0$ can be written as

$$\mathscr{A}_n^{(0)} = (2\pi)^4\, \delta^{(4)}(\mathsf{p})\, g_{\rm YM}^{n-2} \sum_{\sigma \in S_n/\mathbb{Z}_N} \mathrm{tr}(\mathrm{t}^{a_{\sigma(1)}} \mathrm{t}^{a_{\sigma(2)}} \dots \mathrm{t}^{a_{\sigma(n)}})\ A_n^{(0)}(\sigma(1), \dots, \sigma(n)) . \tag{2.92}$$

This is an explicit realisation of colour decomposition: the full amplitude is written as a sum of *partial amplitudes* $A_n^{(\ell)}$ multiplied by a particular trace structure which corresponds to a given cyclic ordering of the external legs. The partial amplitude receives contributions only from diagrams with that particular ordering.

At one loop, the structure is enriched by the presence of double-trace terms. From the sum of all diagrams with $\ell = 1$ one finds[2]

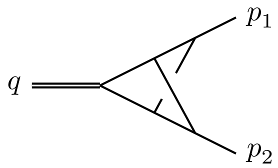
$$\begin{aligned} \mathscr{A}_n^{(1)} = (2\pi)^4\, \delta^{(4)}(\mathsf{p})\, g_{\rm YM}^{n} \Bigg( & N_{\rm c} \sum_{\sigma \in S_n/\mathbb{Z}_N} \mathrm{tr}(\mathrm{t}^{a_{\sigma(1)}} \dots \mathrm{t}^{a_{\sigma(n)}})\ A_n^{(1)}(\sigma(1), \dots, \sigma(n)) \\ & + \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor + 1} \sum_{\sigma \in S_n/\mathbb{Z}_N} \mathrm{tr}(\mathrm{t}^{a_{\sigma(1)}} \dots \mathrm{t}^{a_{\sigma(i-1)}})\, \mathrm{tr}(\mathrm{t}^{a_{\sigma(i)}} \dots \mathrm{t}^{a_{\sigma(n)}})\ A_{n;i}^{(1)}(\sigma(1), \dots, \sigma(n)]) \Bigg) . \end{aligned} \tag{2.93}$$

[2]With the introduction of regularization schemes the expression might be slightly changed, but the color structure will be preserved.

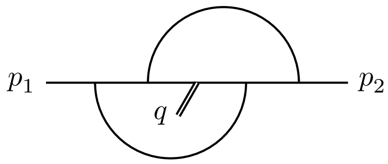
As shown above, the double-trace terms are subleading in the $1/N_{\rm c}$ expansion and, as such, can be discarded at large $N_{\rm c}$.

It follows from their very definition that partial amplitudes are both cyclic and gauge invariant.

Coming to form factors, most of what we said above still holds true. However, since we consider gauge-invariant operator insertions, the associated external off-shell leg does not carry free gauge indices. This means that partial form factors $F_n^{(\ell)}$ are fixed only by the ordering of the external on-shell legs and are defined as the sum of all diagrams obtained by inserting the single off-shell vertex in all possible ways. This fact leads to some subtlety in the identification of the terms that contribute at large $N_{\rm c}$. Consider, for example, a two-loops diagram with the following topology:




This diagram involves the computation of what, in literature, is referred to as a "nonplanar integral". However, since the off-shell leg is not involved in the ordering, one can redraw the diagram as




which shows that the diagram contributes to the leading order in the large-$N_{\rm c}$ exansion.

As a consequence of the fact that the inserted operator is a gauge singlet, tree and one-loop level form factors have the same color structure of (2.92) and (2.93), i.e.

$$\mathscr{F}_n^{(0)} = (2\pi)^4\, \delta^{(4)}(\mathsf{p})\, g_{\rm YM}^{n-2} \sum_{\sigma\in S_n/\mathbb{Z}_N} {\rm tr}({\rm t}^{a_{\sigma(1)}}{\rm t}^{a_{\sigma(2)}}...{\rm t}^{a_{\sigma(n)}})\; F_n^{(0)}(\sigma(1),...,\sigma(n)) \tag{2.94}$$

$$\mathscr{F}_n^{(1)}\big|_{\rm planar} = (2\pi)^4\, \delta^{(4)}(\mathsf{p})\, g_{\rm YM}^{n}\, N_{\rm c} \sum_{\sigma\in S_n/\mathbb{Z}_N} {\rm tr}({\rm t}^{a_{\sigma(1)}}\,...\,{\rm t}^{a_{\sigma(n)}})\; F_n^{(1)}(\sigma(1),...,\sigma(n))\;. \tag{2.95}$$

In fact, this structure is quite universal: for both amplitudes and form factors, at any loop order, the terms leading in the planar limit have the same single-trace structure and an overall power $N_{\rm c}^{\ell}$,[3]

$$\mathscr{X}_n^{(\ell)}\big|_{\rm planar} = (2\pi)^4\, \delta^{(4)}(\mathsf{p})\, g_{\rm YM}^{n-2+2\ell}\, N_{\rm c}^{\ell} \sum_{\sigma\in S_n/\mathbb{Z}_N} {\rm tr}({\rm t}^{a_{\sigma(1)}}\,...\,{\rm t}^{a_{\sigma(n)}})\; X_n^{(1)}(\sigma(1),...,\sigma(n))\;. \tag{2.96}$$

This has two immediate consequences. The first is that, when dealing with gauge theories in the planar limit, it is easier to work with partial amplitudes and form factors as these have

[3]Here and in the following, we will use the letter $X$ to denote a quantity that could be either an amplitude or a form factor

simpler expressions when compared to the "full" quantities, but contain the same information, as one can always reconstruct the latter from the former. The second is that at large $N_c$ one is naturally led to consider the expansion in the parameter

$$\lambda = g_{\mathrm{YM}}^2 N_{\mathrm{c}} \,, \tag{2.97}$$

known as the *'t Hooft coupling*.

In the *'t Hooft limit*, the one we will consider, one takes $N_{\mathrm{c}} \to \infty$ while keeping $\lambda$ fixed.

## 2.6 Supersymmetry

We are left with the last set of indices that we still need to take into account. These are the indices associated with the R-symmetry group.

In supersymmetric theories, instead of thinking in terms of individual particles one is led to consider irreducible representations of the whole supersymmetry algebra, i.e. particle multiplets. $\mathcal{N} = 4$ sYM has a spectrum of 16 states (8 bosons and 8 fermions) that form a single on-shell CPT-self-conjugate supermultiplet. States can be classified by their helicity $h$ as

| | | |
|---|---|---|
| 1 gluon | $g$ | $(h = +1)$ |
| 4 gluinos | $\lambda_A$ | $(h = +\frac{1}{2})$ |
| 6 scalars | $s_{AB}$ | $(h = +0)$ |
| 4 gluinos | $\bar{\lambda}_{ABC}$ | $(h = -\frac{1}{2})$ |
| 1 gluon | $\bar{g}_{ABCD}$ | $(h = -1)$ |

Indices $A$, $B$, ... run from 1 to 4 and are SU(4) R-symmetry indices. Each one of the states above, in fact, transforms under the irreducible antisymmetric representation of SU(4) of dimension $\binom{4}{2-2h}$. The multiplicities trivially reflect the dimension of the associated representation. In constructing the representation it is useful to keep all indices lowered, although for the negative-helicity states one can switch to the conjugate representation with

$$\bar{\lambda}_{ABC} = \varepsilon_{ABCD}\, \lambda^D \,, \tag{2.98}$$

$$\bar{g}_{ABCD} = \varepsilon_{ABCD}\, \bar{g} \,. \tag{2.99}$$

In this spirit, one can consider not just simple amplitudes, but their supersymmetric extension, in which one scatters entire multiplets. This is achieved by introducing an *on-shell superspace*, first introduced by Nair [67]. This allows us to express any possible $n$-point amplitude in terms of a single partial *superamplitude* $\mathbb{A}_n(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})$, a polynomial in the Grassmann algebra generated by $4n$ Grassmann numbers $\{\eta_i^A\}$. Specifically, $\mathbb{A}_n(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})$ is defined as the sum of all possible $n$-point amplitudes (with fixed momenta $\{\lambda_i, \tilde{\lambda}_i\}$) where all SU(4) indices labelling the external states are saturated with the corresponding $\eta$'s.

One can think of superamplitudes as amplitudes defined on asymptotic superstates. A single-particle superstate reads

$$|\Phi(\mathbf{p})\rangle = |g(\mathbf{p})\rangle + \eta^A|\lambda_A(\mathbf{p})\rangle + \frac{1}{2!}\eta^B\eta^A|s_{AB}(\mathbf{p})\rangle + \frac{1}{3!}\eta^C\eta^B\eta^A|\lambda_{ABC}(\mathbf{p})\rangle + \frac{1}{4!}\eta^D\eta^C\eta^B\eta^A|g_{ABCD}(\mathbf{p})\rangle \ . \tag{2.100}$$

On $\mathbb{A}_n(\lambda_i, \tilde{\lambda}_i, \eta_i)$, the supersymmetry generators act as

$$\mathsf{q}^{A\,\alpha} = \sum_{i=1}^n \mathsf{q}_i^{A\,\alpha} = \sum_{i=1}^n \lambda_i^\alpha\,\eta_i^A \ , \tag{2.101}$$

$$\bar{\mathsf{q}}_A{}^{\dot\alpha} = \sum_{i=1}^n \tilde{\lambda}_i^{\dot\alpha}\,\frac{\partial}{\partial\eta_i^A} \ , \tag{2.102}$$

Notice how this superspace representation is chiral, since the conjugate $\bar{\eta}_A = (\eta^A)^*$ does not appear.

Superamplitudes are annihilated by the generators in (2.101) and (2.102) as a result of supersymmetric Ward identities. The vanishing under the action of $\mathsf{q}^{A\,a}$, being the latter a multiplicative operator, means that amplitudes necessarily contain an overall Grassmann $\delta^{(8)}(\mathsf{q}^{A\,\alpha})$. More detail on Grassmann delta functions can be found in Appendix B.1. Moreover, by imposing the vanishing under the action of the R-charge generator

$$\mathsf{r}^A{}_B = \sum_{i=1}^n \left(\eta_i^A\,\frac{\partial}{\partial\eta_i^B} - \frac{1}{4}\delta_B^A\,\eta_i^C\,\frac{\partial}{\partial\eta_i^C}\right) \ , \tag{2.103}$$

one finds that, for $n > 3$,

$$\mathbb{A}_n = A_{n,0} + A_{n,1} + \ldots + A_{n,n-4} \ , \tag{2.104}$$

where $A_{n,k}$ is a homogeneous polynomial in the $\eta$'s of degree $4(k+2)$. The coefficients are partial amplitudes that are said to be N$^k$MHV or, equivalently, of MHV degree $k$.

We complete the representation of the full superconformal algebra by giving the explicit form of the generators

$$\mathsf{s}_{A\,\alpha} = \sum_{i=1}^n \frac{\partial}{\partial\lambda_i^\alpha}\frac{\partial}{\partial\eta_i^A} \ , \tag{2.105}$$

$$\bar{\mathsf{s}}_{\dot\alpha}^A = \sum_{i=1}^n \eta_i^A\,\frac{\partial}{\partial\tilde{\lambda}_i^{\dot\alpha}} \ , \tag{2.106}$$

$$\mathsf{z} = 1 + \frac{1}{2}\sum_{i=1}^n \left(\lambda_i^\alpha\,\frac{\partial}{\partial\lambda_i^\alpha} - \tilde{\lambda}_i^{\dot\alpha}\,\frac{\partial}{\partial\tilde{\lambda}_i^{\dot\alpha}} - \eta_i^A\,\frac{\partial}{\partial\eta_i^A}\right) \ . \tag{2.107}$$

We now move to form factors. In the same spirit we want to consider, not just the insertion of a particular composite operator, but rather of an entire supersymmetric multiplet. Multiplets of local operators are parametrised by the more conventional superspace

$$(x^{\dot\alpha\alpha}, \theta_{A\,\alpha}, \bar{\theta}^{A\,\dot\alpha}) \ . \tag{2.108}$$

However, as mentioned in Section 2.2, all known representations of the supersymmetry algebra on the $\mathcal{N} = 4$ vector multiplet close only on-shell. If interested in an off-shell representation, one is forced to consider the multiplet generated by a chiral half of the supersymmetry algebra. This is known as the *chiral* (or *self-dual*) multiplet, and consists in the fields

$$(F_{\alpha\beta}, \psi^A_\alpha, \varphi^{AB}) \tag{2.109}$$

on which the superalgebra generated solely by the $\mathbb{Q}^\alpha_A$ acts with

$$\mathbb{Q}^\alpha_A \varphi^{BC} = 2\mathrm{i}\sqrt{2}\delta^{[B}_A \psi^{C]\alpha} \;, \tag{2.110}$$

$$\mathbb{Q}^\alpha_A \psi^B_\beta = \delta^B_A F^\alpha_\beta + \mathrm{i}g_{\mathrm{YM}}\delta^\alpha_\beta[\varphi^{BC}, \varphi_{CA}] \;, \tag{2.111}$$

$$\mathbb{Q}^\alpha_A F_{\beta\gamma} = 2\sqrt{2}g_{\mathrm{YM}}\delta^\alpha_{(\beta}[\varphi_{AB}, \lambda^B_{\gamma)}] \;. \tag{2.112}$$

The components of the chiral vector multiplet can be arranged in an $\mathcal{N} = 4$ half-superfield

$$W_{AB}(x,\theta) = \varphi_{AB}(x) + 2\mathrm{i}\sqrt{2}\theta^{\alpha[A}\psi^{B]}_\alpha(x) + \mathrm{i}\sqrt{2}\theta^{[A}_\alpha\theta^{B]}_\beta F^{\alpha\beta}(x) + O(g_{\mathrm{YM}}) \;. \tag{2.113}$$

This superfield is subject to a constraint which has a natural interpretation in a particular parametrisation of the superspace, called *harmonic superspace* [**68**]. We consider the projectors $u^A_{+a}$ and $u^A_{-a'}$ parametrising the coset

$$\frac{\mathrm{SU}(4)}{\mathrm{SU}(2)\times\mathrm{SU}(2)'\times\mathrm{U}(1)} \simeq \mathbf{Gr}(4,2) \;, \tag{2.114}$$

where $a$ and $a'$ are, respectively, SU(2) and SU(2)$'$ indices, and $\pm$ is the U(1) charge. With these, we define

$$\theta_{+a\,\alpha} = \theta_{A\,\alpha}u^A_{+a} \;, \tag{2.115}$$

$$\theta_{-a'\,\alpha} = \theta_{A\,\alpha}u^A_{-a'} \;, \tag{2.116}$$

and the component $W_{++}$, where

$$u^A_{+a}u^B_{+b}W_{AB} = \varepsilon_{ab}W_{++} \;. \tag{2.117}$$

In terms of these new variables, the constraint reads

$$\frac{\partial}{\partial\theta_{-a'\,\alpha}}W_{++} = 0 \;, \tag{2.118}$$

so that

$$W_{++}(x,\theta^+,u) = \varphi_{++} + \mathrm{i}\sqrt{2}\theta^{+a}_\alpha\varepsilon_{ab}\varepsilon^{\alpha\beta}\psi^{+b}_\beta - \mathrm{i}\frac{\sqrt{2}}{2}\theta^{+a}_\alpha\varepsilon_{ab}\theta^{+b}_\beta F^{\alpha\beta} + O(g_{\mathrm{YM}}) \;, \tag{2.119}$$

where the projected fields are defined with

$$\varphi_{++}(x,u) = -\frac{1}{2}u^A_{+a}\varepsilon^{ab}u^B_{+b}\varphi_{AB}(x) \;, \tag{2.120}$$

$$\psi_{+a\,\alpha}(x,u) = u^A_{+a}\psi^\alpha_A(x) \; . \tag{2.121}$$

Accordingly, the chiral supercharges $\mathbb{Q}^A_\alpha$ are projected as

$$\mathbb{Q}^{+a}_\alpha = \bar{u}^{+a}_A \mathbb{Q}^A_\alpha \; , \tag{2.122}$$
$$\mathbb{Q}^{-a'}_\alpha = \bar{u}^{-a'}_A \mathbb{Q}^A_\alpha \; . \tag{2.123}$$

with the conjugate matrices $\bar{u}^{+a}_A$ and $\bar{u}^{-a'}_A$ that satisfy the orthonormality

$$\bar{u}^{+a}_A u^A_{+b} = \delta^a_b \; , \qquad \bar{u}^{-a'}_A u^A_{-b'} = \delta^{a'}_{b'} \; , \tag{2.124}$$
$$\bar{u}^{+a}_A u^A_{-b'} = 0 \; , \qquad \bar{u}^{-a'}_A u^A_{+b'} = 0 \; , \tag{2.125}$$

and completeness relations

$$\bar{u}^{+a}_A u^B_{+a} + \bar{u}^{-a'}_A u^B_{-a'} = \delta^B_A \; . \tag{2.126}$$

We now define the chiral stress-tensor supermultiplet[4] [69, 70]

$$\begin{aligned}\mathcal{T}(x,\theta^+,u) &= \mathrm{tr}(W^{++}W^{++})(x,\theta^+,u) \\ &= \mathrm{tr}(\varphi_{++}\varphi_{++})(x) + \ldots + \frac{1}{3}(\theta_+)^4\mathscr{L}(x) \; ,\end{aligned} \tag{2.127}$$

and the super-form factor

$$\mathbb{F}_n(q,\gamma^+,u) = \int \mathrm{d}^4x\,\mathrm{d}^4\theta^+\; e^{-\mathrm{i}q\cdot x - \mathrm{i}\theta_{+a}\cdot\gamma^{+a}}\langle\Phi(\mathbf{p}_1)\ldots\Phi(\mathbf{p}_n)|\mathcal{T}(x,\theta^+,u)|0\rangle \; . \tag{2.128}$$

From now on we will drop the explicit dependence on $u$, when not necessary.

We can write down the action of the chiral supersymmetry generators on the form factor, acting both on the on-shell states and on the operator insertion, as an extension of (2.101),

$$\begin{aligned}\mathsf{q}^{+a\,\alpha} &= \sum_{i=1}^n \lambda^\alpha_i\,\eta^{+a}_i \; , \\ \mathsf{q}^{-a'\,\alpha} &= \sum_{i=1}^n \lambda^\alpha_i\,\eta^{-a'}_i \; ,\end{aligned} \tag{2.129}$$

where

$$\begin{aligned}\eta^{+a}_i &= \bar{u}^{+a}_A\eta^A_i \; , \\ \eta^{-a'}_i &= \bar{u}^{-a'}_A\eta^A_i \; .\end{aligned} \tag{2.130}$$

The super form factor (2.128) is annihilated by the generators in (2.129). This means that $\mathbb{F}_n$ carries an overall

$$\delta^{(4)}(\lambda_1\eta^+_1 + \ldots + \lambda_n\eta^+_n - \gamma^+)\,\delta^{(4)}(\lambda_1\eta^-_1 + \ldots + \lambda_n\eta^-_n) \tag{2.131}$$

[4] With $\mathscr{L}$ we denote the on-shell Lagrangian, i.e. the function obtained by imposing the e.o.m.'s on the $\mathcal{N}=4$ Lagrangian introduced in Section 2.2.

For simplicity we will denote the above expression with

$$\delta^{(8)}(\mathsf{q}^{A\,\alpha}) = \delta^{(8)}(\lambda_1\eta_1 + \ldots + \lambda_n\eta_n - \gamma) \tag{2.132}$$

but we will always set $\gamma^- = 0$.

Expanding $\mathbb{F}_n$ in homogeneous Grassmann polynomials $F_{n,k}$ of degree $4(k+2)$ one finds, for $n > 2$,

$$\mathbb{F}_n = F_{n,0} + F_{n,1} + \ldots + F_{n,n-2} \,. \tag{2.133}$$

## 2.7 Tree recursion relations

Tree-level amplitudes are rational functions. This is obvious from thinking in terms of Feynman rules. For local theories, interaction vertices take the form of either constants or polynomials in the momenta, therefore the only poles that a tree-level amplitude can have are those associated with internal-line propagators. The discussion in this section will be tailored to the case of superamplitudes in $\mathcal{N} = 4$ sYM.

Poles at finite momenta in a given superamplitude $\mathbb{A}_n^{(0)}$ are in one-to-one correspondence with factorization channels, proper subsets of the set of all external momenta in the amplitude. Again, this becomes obvious from the diagrammatic expansion. Moreover, since we are dealing with partial amplitudes, factorisation channels can involve only adjacent momenta. For each factorisation channel $\{p_i, p_{i+1}, \ldots, p_j\}$, the amplitude has a pole of the form

$$\sim -\frac{\mathrm{i}}{(p_i + p_{i+1} + \ldots + p_j)^2} \,. \tag{2.134}$$

Most notably, the residue of such a pole is given by the product of two subamplitudes: this is ultimately a consequence of unitarity. At the diagrammatic level, this follows from the fact that all the diagrams with the propagator (2.134), i.e. the ones contributing to the residue, can be formed by combining the diagrams that appear in the expression of the two subamplitudes. This crucial observation has lead Britto, Cachazo, Feng and Witten [**15**, **16**] to a recursion relation that allows to construct tree-level amplitudes for a certain class of theories by essentially gluing together amplitudes with fewer legs.

The idea is to introduce a deformation of the amplitude through a single complex variable $z$. There are in principle many ways in which one could do that, provided that one does not break three fundamental constraints, which are the on-shellness of individual external momenta and the vanishing of both their total sum $\mathsf{p}$ and of the total supermomentum $\mathsf{q}$. The starting point of the BCFW recursion relation is to pick a pair of adjacent[5] external legs, say 1 and $n$, and to define the complex function $\mathbb{A}_n^{(0)}(z)$ as the original amplitude with the shifts

$$\lambda_1 \mapsto \hat{\lambda}_1 = \lambda_1 - z\lambda_n \,,$$

[5]the requirement that the shifted legs should be adjacent is not necessary but turns out to be a particularly convenient choice

$$\begin{aligned} \tilde{\lambda}_n &\mapsto \hat{\tilde{\lambda}}_n = \tilde{\lambda}_n + z\tilde{\lambda}_1 \ , \\ \eta_n &\mapsto \hat{\eta}_n = \eta_n + z\eta_1 \ , \end{aligned} \tag{2.135}$$

collectively denoted as a $\langle n\, 1]$ shift.

This deformation satisfies the three constraints mentioned above and thus ensures that $\mathbb{A}_n^{(0)}(z)$ is a proper amplitude for every value of $z$, although defined in complexified Minkowski space. The original amplitude can be recovered by setting $z = 0$. Now let us consider

$$\omega = z^{-1}\mathbb{A}_n^{(0)}(z)\, \mathrm{d}z \ , \tag{2.136}$$

a one form on $\mathbb{CP}^1$ that takes values in the Grassmann algebra generated by the $\eta$'s.

From the considerations above one can easily conclude that $\omega$ is a meromorphic differential. Indeed, its poles are associated with the factorization channels in which $p_1$ and $p_n$ cannot appear together. Actually, there is the notable addition of a new pole, which is located at $z = 0$, and, possibly, of another pole at $z = \infty$.

Let us consider first the pole at zero. The residue,

$$\operatorname*{Res}_{z=0} \omega = \mathbb{A}_n^{(0)}(0) \tag{2.137}$$

gives precisely the original amplitude that we want to compute. Now, being $\mathbb{CP}^1$ compact, the sum of the residues of $\omega$ must vanish. This means that

$$\mathbb{A}_n^{(0)}(0) = -\sum_{\text{poles } z_j} \operatorname*{Res}_{z=z_j} \omega \tag{2.138}$$

were the sum is over nonvanishing $z_j$, poles of $\omega$. For each pole at finite $z$ we consider the associated factorization channel $\{p_1, ..., p_j\}$ and the sum

$$P_j(z) = \hat{p}_1 + ... + p_j = -p_{j+1} - ... - \hat{p}_n \ . \tag{2.139}$$

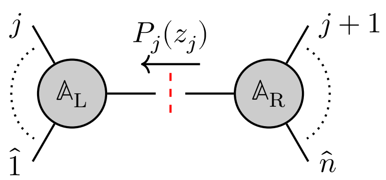
of the (shifted) momenta in the channel. It is easy to check that $P_j^2(z)$ can be written, in terms of its zero

$$z_j = \frac{P_j^2(0)}{\langle n|P_j(0)|1]} \ , \tag{2.140}$$

as

$$P_j^2(z) = \frac{z_j - z}{z_j} P_j^2(0) \ . \tag{2.141}$$

Graphically, the amplitude splits as



where $\mathbb{A}_{\mathrm{L}}^{(0)}$ and $\mathbb{A}_{\mathrm{R}}^{(0)}$ are the "left" and "right" subamplitudes that factorise in the limit. When summing over all poles one obtains

$$-\operatorname*{Res}_{z=z_j} \omega = -\mathrm{i} \int \mathrm{d}\eta_P^4 \; \mathbb{A}_{\mathrm{L},j}^{(0)} \, P_j^{-2}(0) \, \mathbb{A}_{\mathrm{R},j}^{(0)} \,, \tag{2.142}$$

where the integral is effectively a sum over all the possible states flowing through the internal line, a direct consequence of unitarity.

What about the pole at infinity? It will be present if and only if $\lim_{z\to\infty} \mathbb{A}(z)$ is nonvanishing. The limit corresponds to a process in which a hard particle scatters through a soft background. In $\mathcal{N}=4$ sYM, supersymmetry guarantees the absence of such a pole.

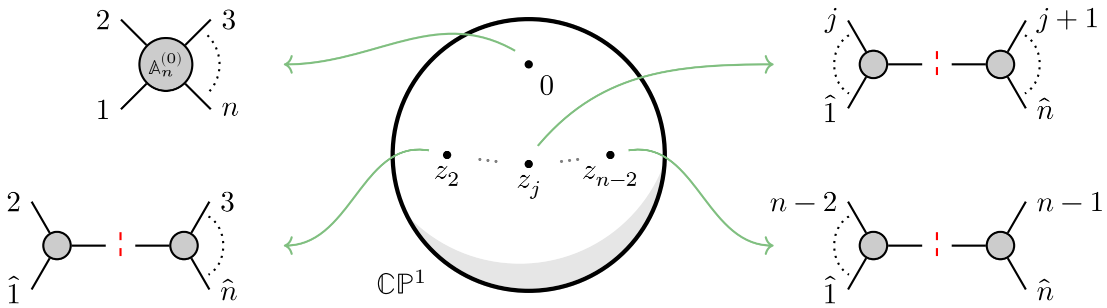



Figure 2.1: In $\omega$, each factorisation channel for the partial amplitude corresponds to a particular pole in $z$. The full amplitudes is given by the residue in $z=0$.

As in (2.138), the amplitude is obtained by summing over all factorisation channels where exactly one shifted leg appears, as depicted in Figure 2.1. This gives

$$\mathbb{A}_n^{(0)} = -\mathrm{i} \sum_{j=2}^{n-2} \int \mathrm{d}\eta_P^4 \; \mathbb{A}_{\mathrm{L},j}^{(0)} \, P_j^{-2}(0) \, \mathbb{A}_{\mathrm{R},j}^{(0)} \tag{2.143}$$

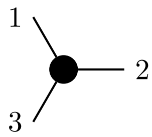
In practice, in using the above, one computes one component at the time in the superamplitude, incrementing at each step the MHV degree $k$. The process can be applied recursively until one ends up with the basic amplitudes which constitute the true building blocks that can be used to bootstrap all tree-level amplitudes. In our case, these are three point amplitudes. Now, it is well known that one cannot find three real lightlike four-momenta $p_1$, $p_2$ and $p_3$ such that $p_1+p_2+p_3=0$. A three-point amplitude for a massless theory, in fact, exists only for complex momenta. No actual physical process is associated with such an amplitude.

Three-point kinematics can be realized either by setting

$$\tilde{\lambda}_1 \propto \tilde{\lambda}_2 \propto \tilde{\lambda}_3 \,, \tag{2.144}$$

which is the case that corresponds to a so-called $\mathrm{MHV}_3$ amplitude

$A_{3,0}^{(0)} =$

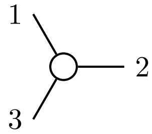



$$= \mathrm{i} \, \frac{\delta^{(8)}(\lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3)}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle} \,, \tag{2.145}$$

or by setting

$$\lambda_1 \propto \lambda_2 \propto \lambda_3 \,, \tag{2.146}$$

which gives an $\overline{\mathrm{MHV}}_3$ amplitude

$$A^{(0)}_{3,-1} = \quad$$



$$= -\mathrm{i}\, \frac{\delta^{(4)}([2\,3]\eta_1 + [3\,1]\eta_2 + [1\,2]\eta_3)}{[1\,2][2\,3][3\,1]} \,. \tag{2.147}$$

Much of this construction can be repeated for form factors. Poles in form factors at tree level are still associated with factorisations, where now the residues correspond to the gluing of an amplitude with a form factor. However, since momentum conservation now involves the off-shell leg as well, the two sums of momenta in (2.139) cannot go on-shell simultaneously. For finite $q$, a single pole $z_j$ splits in two different poles, $z_{j,\mathrm{L}}$ and $z_{j,\mathrm{R}}$ that correspond to the two possible ways of inserting the off-shell leg: on the left side and on the right side. This is depicted in Figure 2.2

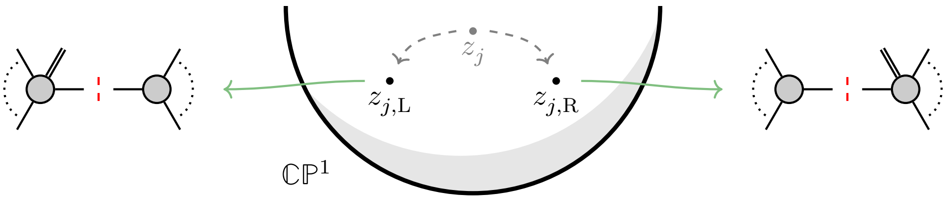
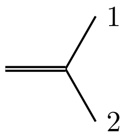



Figure 2.2: For a given $n$, form factors have twice as many factorisation channels compared to amplitudes since the off-shell leg can be inserted on either side of the factorisation.

The final formula, then, reads

$$\mathbb{F}^{(0)}_n = -\mathrm{i} \sum_{j=3}^{n} \int \mathrm{d}\eta^4_P \, \left( \mathbb{F}^{(0)}_{\mathrm{L},j}\, P^{-2}_{j,\mathrm{L}}(0)\, \mathbb{A}^{(0)}_{\mathrm{R},j} + \mathbb{A}^{(0)}_{\mathrm{L},j}\, P^{-2}_{j,\mathrm{R}}(0)\, \mathbb{F}^{(0)}_{\mathrm{R},j} \right) \,. \tag{2.148}$$

To bootstrap the computation of form factors one needs, in addition to the tree-point amplitudes (2.147) and (2.145), the minimal form factor

$$F^{(0)}_{2,0} = \quad$$



$$= \frac{\delta^{(8)}(\lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3 - \gamma)}{\langle 1\,2\rangle\langle 2\,1\rangle} \,. \tag{2.149}$$

As an immediate application of the recursion, one can prove by induction that the MHV amplitudes and form factors are given, respectively, by

$$A^{(0)}_{n,0} = \mathrm{i}\, \frac{\delta^{(8)}(\mathsf{q})}{\langle 1\,2\rangle\langle 2\,3\rangle \cdots \langle n\,1\rangle} \,, \tag{2.150}$$

for $n \geq 4$, and

$$F^{(0)}_{n,0} = \frac{\delta^{(8)}(\mathsf{q})}{\langle 1\,2\rangle\langle 2\,3\rangle \cdots \langle n\,1\rangle} \,, \tag{2.151}$$

for $n \geq 3$. See Appendix C for details.

## 2.8 Unitarity

One-loop integrals are generically expressed in terms of logarithms and dilogarithms which can have branch cut singularities as complex functions of kinematic invariants. At one loop, any amplitude or form factor can be written in terms of a well-defined integral basis. By emplyoing integral-reduction techniques one can, in fact, bring an arbitary one-loop quantity to the ansatz

$$\mathbb{X}^{(1)} = \sum_i C_{4,i} I_{4,i} + \sum_j C_{3,j} I_{3,j} + \sum_k C_{2,k} I_{2,k} + R \, , \tag{2.152}$$

where

$$I_n = \frac{1}{2\pi^{2-\epsilon} r_\Gamma} \int \mathrm{d}^{4-2\epsilon}\ell \, \frac{1}{\ell^2(\ell-K_1)^2 ... (\ell-K_{n-1})^2} \tag{2.153}$$

with

$$r_\Gamma = \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \tag{2.154}$$

are scalar integrals. The sums run over the possible ways of assigning the external momenta to the corners of the integral and the coefficients $C$ are rational functions of the external kinematics. Tadpoles are absent in theories of massless particles. Finally, $R$ denote rational terms. These arise from the regularisation procedure and are absent for supersymmetric Yang–Mills theories, so we will not consider them in our analysis. Furthermore, we will also discard terms with bubble integrals $I_2$ since these are UV divergent, as it is obvious from simple power counting, and therefore cannot be present in the expression of protected quantities.

The ansatz for any one-loop form factor is then

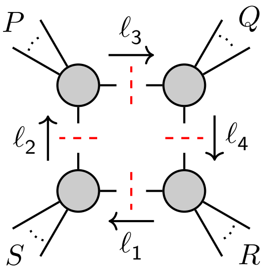
$$\mathbb{F}^{(1)} = \sum_i C_{4,i} I_{4,i} + \sum_j C_{3,j} I_{3,j} \, . \tag{2.155}$$

For amplitudes things are even simpler, as one can also exclude triangle integrals $I_3$ [**71**], so that the ansatz reads

$$\mathbb{A}^{(1)} = \sum_i C_{4,i} I_{4,i} \, . \tag{2.156}$$

The problem of computing quantities at one-loop has now been translated to the problem of finding the correct expression of all the coefficients in (2.156) and (2.155). One way to determine these coefficients is through the analysis of the discontinuities of the one-loop result. In this approach, one compares the discontinuities of the ansatz, as functions of the $C$'s, with the results obtained through unitarity cuts. This is a well-known generalisation of the optical theorem that relates the discontinuity in a particular kinematic channel with the gluing of the two tree-level amplitudes generated by cutting the channel [**72**, **73**].

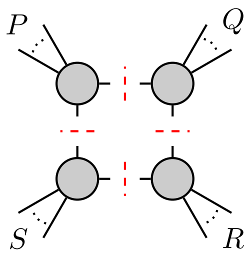
A more recent technique, that goes under the name of *generalised unitarity* [**13**, **14**, **26**], takes a more radical approach: it is possible to directly determine a coefficient by cutting all the legs of the associated integral. Let us first see how this works for amplitudes, where only box integrals are present. Every cut on a loop leg constrains the loop momentum by reducing one of its degrees of freedom. In four dimensions, the four deltas coming from the cuts fully localise the loop integral on two distinct complex solutions. Each solution contributes with



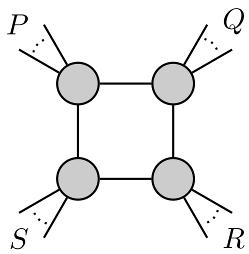

$$= \int \left( \prod_{j=1}^{4} \mathrm{d}^4 \eta_{\ell_a} \right) \mathbb{A}^{(0)}_{\mathrm{DL}}(-\ell_1, \ell_2)\, \mathbb{A}^{(0)}_{\mathrm{UL}}(-\ell_2, \ell_3)\, \mathbb{A}^{(0)}_{\mathrm{UR}}(-\ell_3, \ell_4)\, \mathbb{A}^{(0)}_{\mathrm{DR}}(-\ell_4, \ell_1)\ , \tag{2.157}$$

where essentially four tree-level amplitudes are glued together.

Interestingly, box coefficients can be seen as a generalisation of the results coming from BCFW recursion relations at tree level. Specifically, any term in the BCFW recursion can be written as a particular box coefficient, modulo an overall kinematic prefactor. To see how this works, we introduce a new kind of diagram, where the cut on the four legs is not drawn. This new diagram is defined with




$$= \mathrm{i}\,\Delta\, A^{(0)}_{n,0}$$

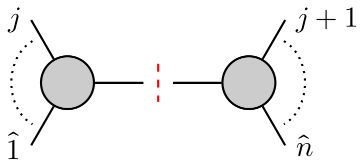



$$, \tag{2.158}$$

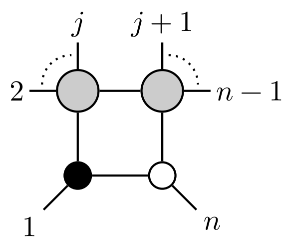
where

$$\Delta = \sqrt{[(P+Q)^2(P+S)^2 - P^2R^2 + Q^2S^2]^2 - 4(P+Q)^2(P+S)^2Q^2S^2}\ . \tag{2.159}$$

In the literature, these diagrams are known as *on-shell diagrams*[6] and provide a unified tool to study amplitudes at tree and loop level. In Appendix C is shown how a generic BCFW term can be written as a so-called *BCFW bridge*,




$$= A^{(0)}_{n,0}$$

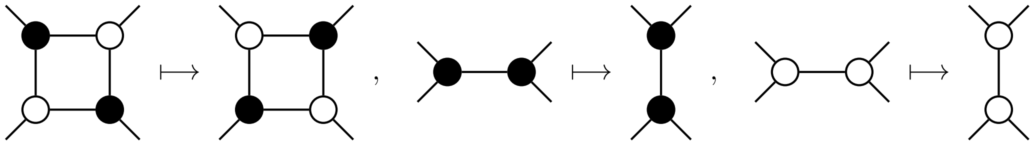



$$, \tag{2.160}$$

[6] We adopt an unconventional normalisation, where the expression of the diagram does not contain the tree-level MHV factor $A^{(0)}_{n,0}$.

where the black and the white corners are precisely those defined in (2.145) and (2.147). In fact, with the BCFW bridge one can recursively replace each grey blob with its expression in terms of amplitudes with smaller $n$ or $k$. This process can be repeated until one is left with three-point vertices only. We will see how this works concretely later in this section.

Before getting to an actual computation, we need to mention that the description provided by on-shell diagrams is redundant. In fact, starting from a given diagram, by repeatedly acting on it with the fundamental moves

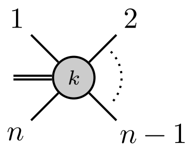


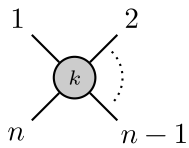
$$(2.161)$$

one can generate different diagrams that are all associated with the same expression. This will turn out to be useful in the next section.

Coming to form factors, their box coefficients can be determined with quadruple cuts in full analogy with the case of amplitudes. In particular, (2.157) and (2.158) still holds provided that one replaces $\mathbb{A}$'s with $\mathbb{F}$'s where appropriate.

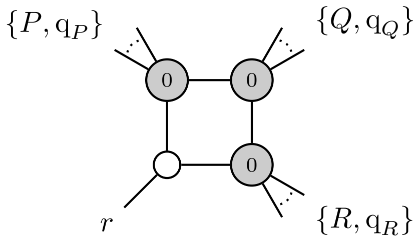
We will present some results that will be useful later. First, we introduce the following graphical convention: in an on-shell diagram, a number inside a grey blob identifies the associated MHV level. In other words,

$F^{(0)}_{n,k} =$ 

, $A^{(0)}_{n,k} =$ 

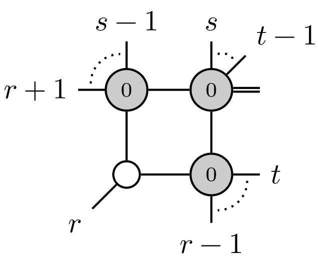
. (2.162)

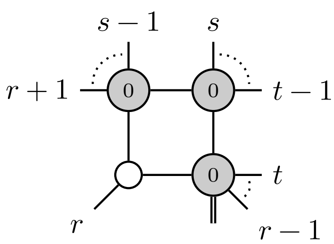
In Appendix C, the explicit computation of a generic on-shell diagram of the form




$$(2.163)$$

for both amplitudes and form factors is presented. These quantities are also known as $R$-invariants; the origin of this term will be explained in the next section. The results for amplitude $R$-invariants can be simply obtained by taking the $q \to 0$ limit of the analogous expressions obtained for form factors, where two inequivalent types of $R$-invariants can be identified [74],

$R'_{rst} =$ 

, $R''_{rst} =$ 

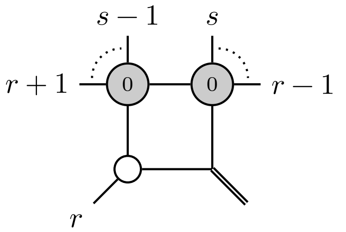
. (2.164)

As shown in the Appendix, if $s \neq t$ [74, 75],

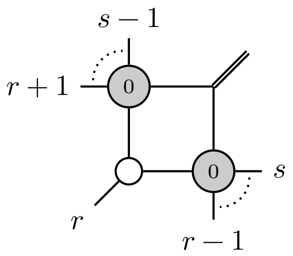
$$R^{\bullet}_{rst} = \frac{\langle s-1\, s\rangle\langle t-1\, t\rangle\, \delta^{(4)}(\langle \mathrm{q}_r + \mathrm{q}_P|QR|r\rangle - \langle \mathrm{q}_R|QP|r\rangle)}{Q^2\langle r|RQ|s-1\rangle\langle r|RQ|s\rangle\langle r|PQ|t-1\rangle\langle r|PQ|t\rangle}\ , \tag{2.165}$$

where the $\bullet$ indicates that this formula applies to both types of $R$-invariants. We denote the total outgoing momentum and supermomentum in the upper-left, upper-right and lower-right corners respectively as $\{P, \mathrm{q}_P\}$, $\{Q, \mathrm{q}_Q\}$ and $\{R, \mathrm{q}_R\}$ as shown in (2.163).

In particular, no modification is needed for the corner case

$R''_{rsr} =$  , (2.166)


which does not have a counterpart in the context of amplitudes. However, the previous formula does not apply to the specific case $s = t$:

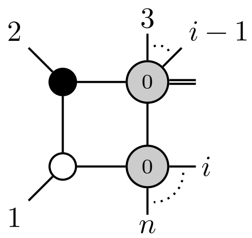
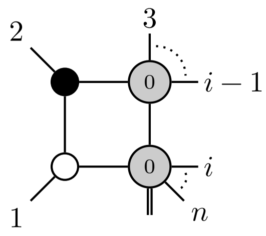
$R'_{rss} =$  , (2.167)

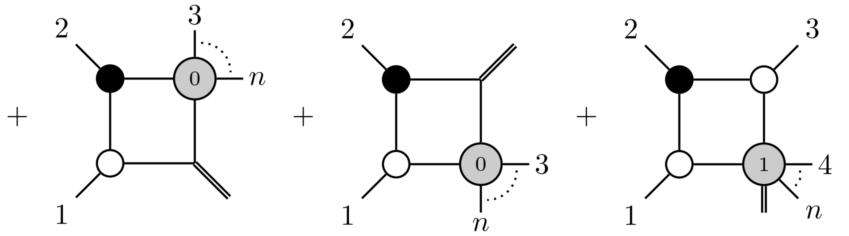

for which the correct result turns out to be given by

$$R'_{rss} = -\frac{\langle s-1\, s\rangle\, \delta^{(4)}(\langle \mathrm{q}_r + \mathrm{q}_P|QR|r\rangle - \langle \mathrm{q}_R|QP|r\rangle)}{Q^4\langle r|RQ|s-1\rangle\langle r|PQ|s\rangle\langle r|PQ|r\rangle}\ . \tag{2.168}$$

We close this section by deriving the result for the complete tree-level, $n$-point NMHV form factor. In [75] it was shown that the tree-level NMHV form factor can be written as a combination of $R$-invariants.

We use a BCFW shift of the $\langle 1\, 2]$ kind. For an $n$-point form factors the recursion, written in terms of BCFW bridges, gives

$F^{(0)}_{n,1} = \sum_{i=4}^{n}$  $+ \sum_{i=5}^{n}$ 



(2.169)

The last diagram can be written in terms of $R$-invariants by recursively inserting the NMHV $(n-1)$-point form factor in the lower-right corner.

To understand how many $R$-invariants contribute to that diagram, one can use the following argument. An $n$-point NMHV form factor is expressed in terms of $2n-5$ diagrams containing products of MHV amplitudes and form factors and one diagram containing the combination of a NMHV $(n-1)$-point form factor and a $\overline{\text{MHV}}$ three-point amplitude. If one denotes with $a_n$ the number of $R$-invariants associated to the $n$-point NMHV form factor, one can replace the NMHV $(n-1)$-point form factor with its $a_{n-1}$ $R$-invariants. This gives a recursive relation,

$$a_n = a_{n-1} + 2n - 5 \,, \tag{2.170}$$

which is solved by

$$a_n = (n-2)^2 \,. \tag{2.171}$$

Consequently, the diagram involving a NMHV form factor times a $\overline{\text{MHV}}$ three-point amplitude should decompose into $(n-3)^2$ box coefficients. The precise combination for a $[1\,2\rangle$ shift is

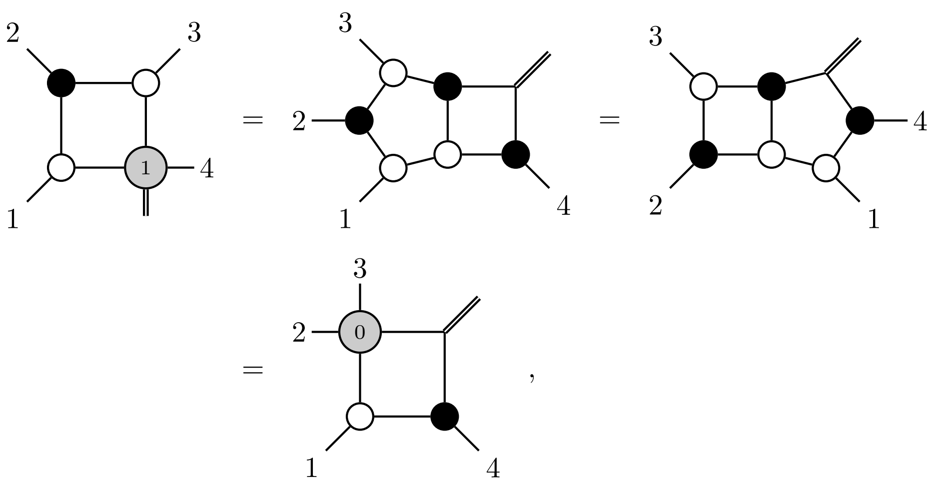
$$F^{(0)}_{n,1} = F^{(0)}_{n,0}\left(\sum_{j=3}^{n}\sum_{i=3}^{j} R'_{1ij} + \sum_{j=5}^{n+1}\sum_{i=3}^{j-2} R''_{1ij}\right) \,, \tag{2.172}$$

where we make the identification $n+1\sim 1$. The number of $R$-invariants in this expression is

$$\#R' + \#R'' = \frac{(n-2)(n-1)}{2} + \frac{(n-2)(n-3)}{2} = (n-2)^2 \,. \tag{2.173}$$

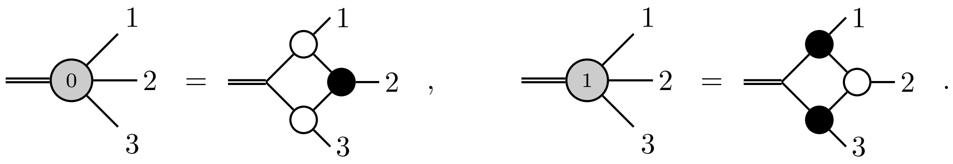
Finally we want to illustrate how the NMHV$\times\overline{\text{MHV}}$ diagram can be written in terms of the $R$-invariants introduced earlier by using the diagrammatic rules (2.161).

If we take, for example, $F^{(0)}_{4,1}$, with the above one finds



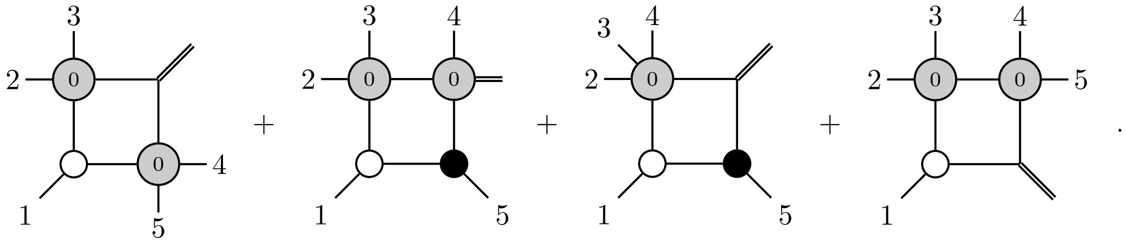
(2.174)

where we used [34]



$$(2.175)$$

The above allows us to identify the last term in the recursion as an $R$-invariant and explicitly check (5.14). Similarly, for the $n = 5$ case the last term in (2.169) can be recast as



$$(2.176)$$

## 2.9 Hidden Symmetries

As discussed in the previous sections, the kinematics of any amplitude can be encoded in the set of variables $\{\lambda_i, \tilde{\lambda}_i, \eta_i\}$. However, the converse is not necessarily true, i.e. an arbitrary choice of such variables in general will not correspond to a well defined superamplitude, which requires the additional constraints

$$\mathsf{p} = \sum_{i=1}^{n} \lambda_i \tilde{\lambda}_i = 0 \,, \tag{2.177}$$

$$\mathsf{q} = \sum_{i=1}^{n} \lambda_i \eta_i = 0 \,. \tag{2.178}$$

Interestingly, an arbitrary choice of $\{\lambda_i, \tilde{\lambda}_i, \eta_i\}$, instead, always corresponds to a well-defined form factor, where the off-shell momentum $q$ and supermomentum $\gamma$ are implicitly defined by

$$\sum_{i=1}^{n} \lambda_i \tilde{\lambda}_i = q \,, \tag{2.179}$$

$$\sum_{i=1}^{n} \lambda_i \eta_i = \gamma \,. \tag{2.180}$$

In the case of amplitudes, one can switch to a set of variables in which momentum conservation is automatically realised. These are *dual supercoordinates* [22] $(x_i, \theta_i) \in \mathbb{C}^{4|8}$ defined as

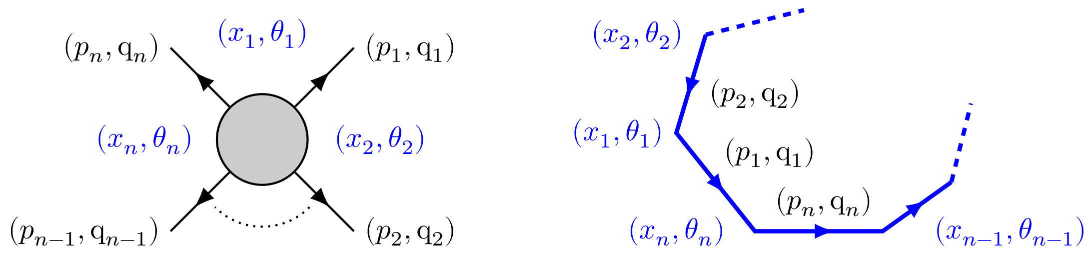
$$x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}} = p_i^{\alpha\dot{\alpha}} = \lambda_i^{\alpha} \tilde{\lambda}_i^{\dot{\alpha}} \,, \tag{2.181}$$

$$\theta_i^{\alpha A} - \theta_{i+1}^{\alpha A} = \mathrm{q}_i^{\alpha A} = \lambda_i^{\alpha} \eta_i^{A} \,. \tag{2.182}$$

Momentum and supermomentum conservation are imposed with the identification

$$(x_{n+1}, \theta_{n+1}) \sim (x_1, \theta_1) \,, \tag{2.183}$$

so that the coordinates form a closed polygon with light-like edges, as shown in Figure 2.3.




(a) (b)

Figure 2.3: In (a) region variables are assigned to a partial amplitude. In (b) is shown how these coordinates form a light-like polygon in dual space.

One can change variables and rewrite amplitudes in terms of a set of variables $\{x_i, \lambda_i, \theta_i\}$.

Clearly, dual coordinates, known also as *region variables*, are all defined up to a constant shift. Amplitudes will depend on such coordinates only through their difference and as such, will be annihilated by the generator of translations and supertranslations in dual space

$$\mathsf{P}^{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i^{\dot{\alpha}\alpha}} \,, \tag{2.184}$$

$$\mathsf{Q}_{\alpha A} = \sum_{i=1}^{n} \frac{\partial}{\partial \theta_i^{\alpha\, A}} \,. \tag{2.185}$$

The superalgebra can be closed with the additional generators

$$\bar{\mathsf{Q}}_{\dot{\alpha}}^{A} = \sum_{i=1}^{n} \theta_i^{\alpha A} \frac{\partial}{\partial x_i^{\dot{\alpha}\alpha}} + \eta_i^A \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} \,, \tag{2.186}$$

$$\mathsf{M}_{\alpha\beta} = \sum_{i=1}^{n} x^{\dot{\alpha}}{}_{i(\alpha} \frac{\partial}{\partial x_i^{\dot{\alpha}\beta)}} + \theta_{i(\alpha}^{A} \frac{\partial}{\partial \theta_i^{\beta)\, A}} + \lambda_{i(\alpha} \frac{\partial}{\partial \lambda_i^{\beta)}} \,, \tag{2.187}$$

$$\bar{\mathsf{M}}_{\dot{\alpha}\dot{\beta}} = \sum_{i=1}^{n} x_{i(\dot{\alpha}}{}^{\alpha} \frac{\partial}{\partial x_i^{\dot{\beta})\alpha}} + \tilde{\lambda}_{i(\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\beta})}} \,, \tag{2.188}$$

$$\mathsf{R}^{A}{}_{B} = \sum_{i=1}^{n} \theta_i^{\alpha A} \frac{\partial}{\partial \theta_i^{\alpha\, B}} + \eta_i^A \frac{\partial}{\partial \eta_i^B} - \frac{1}{4}\delta_B^A \theta_i^{\alpha C} \frac{\partial}{\partial \theta_i^{\alpha\, C}} - \frac{1}{4}\delta_B^A \eta_i^C \frac{\partial}{\partial \eta_i^C} \,, \tag{2.189}$$

which form an $\mathcal{N} = 4$ superalgebra represented on dual superspace. Here, we are keeping explicit the dependence on $\tilde{\lambda}$'s and $\eta$'s.

Now, one could extend the above to form a whole $\mathcal{N} = 4$ superconformal algebra acting on superspace. This is achieved with generators

$$\mathsf{Z} = \sum_{i=1}^{n} -\frac{1}{2}\lambda_i^{\alpha} \frac{\partial}{\partial \lambda_i^{\alpha}} + \frac{1}{2}\tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} + \frac{1}{2}\eta_i^C \frac{\partial}{\partial \eta_i^C} \,, \tag{2.190}$$

$$\mathsf{D} = \sum_{i=1}^{n} -x_i^{\alpha\dot\alpha}\frac{\partial}{\partial x_i^{\dot\alpha\alpha}} - \frac{1}{2}\theta_i^{\alpha A}\frac{\partial}{\partial\theta_i^{\alpha\,A}} - \frac{1}{2}\lambda_i^{\alpha}\frac{\partial}{\partial\lambda_i^{\alpha}} - \frac{1}{2}\tilde\lambda_i^{\dot\alpha}\frac{\partial}{\partial\tilde\lambda_i^{\dot\alpha}}\,, \tag{2.191}$$

$$\mathsf{S}_\alpha^A = \sum_{i=1}^{n} -\theta^B_{i\,\alpha}\theta_i^{\beta A}\frac{\partial}{\partial\theta_i^{\beta\,B}} + x_{i\,\alpha}{}^{\dot\beta}\theta_i^{\beta A}\frac{\partial}{\partial x_i^{\dot\beta\beta}} + \lambda_{i\,\alpha}\theta_i^{\gamma A}\frac{\partial}{\partial\lambda_i^{\gamma}} + x_{i+1\,\alpha}{}^{\dot\beta}\eta_i^A\frac{\partial}{\partial\tilde\lambda_i^{\dot\beta}} - \theta^B_{i+1\,\alpha}\eta_i^A\frac{\partial}{\partial\eta_i^B}\,, \tag{2.192}$$

$$\bar{\mathsf{S}}_{\dot\alpha A} = \sum_{i=1}^{n} x_{i\,\dot\alpha}{}^{\beta}\frac{\partial}{\partial\theta_i^{\beta\,A}} + \tilde\lambda_{i\,\dot\alpha}\frac{\partial}{\partial\eta_i^A}\,, \tag{2.193}$$

$$\mathsf{K}_{\alpha\dot\alpha} = \sum_{i=1}^{n} x_{i\,\alpha}{}^{\dot\beta}x_{i\,\dot\alpha}{}^{\beta}\frac{\partial}{\partial x_i^{\dot\beta\beta}} + x_{i\,\dot\alpha}{}^{\beta}\theta^B_{i\,\alpha}\frac{\partial}{\partial\theta_i^{\beta\,B}} + x_{i\,\dot\alpha}{}^{\beta}\lambda_{i\,\alpha}\frac{\partial}{\partial\lambda_i^{\beta}} + x_{i+1\,\alpha}{}^{\dot\beta}\tilde\lambda_{i\,\dot\alpha}\frac{\partial}{\partial\tilde\lambda_i^{\dot\beta}} + \tilde\lambda_{i\,\dot\alpha}\theta^B_{i+1\,\alpha}\frac{\partial}{\partial\eta_i^B}\,. \tag{2.194}$$

However, there is no obvious reason why superamplitudes should be annihilated by these dual superconformal generators. And in fact, they are not. But the quantity $\tilde{\mathbb{A}}_n^{(\ell)}$ defined with

$$\mathbb{A}_n^{(\ell)} = A_{n,0}^{(0)}\,\tilde{\mathbb{A}}_n^{(\ell)}\,. \tag{2.195}$$

is, remarkably, invariant under dual superconformal transformations for $\ell = 0$. The full amplitude is, instead, covariant [76], with

$$\mathsf{K}_{\alpha\dot\alpha}\mathbb{A}_n^{(0)} = -\sum_{j=1}^{n} x_{j\,\alpha\dot\alpha}\,\mathbb{A}_n^{(0)}\,, \tag{2.196}$$

$$\mathsf{S}^{\alpha\,A}\mathbb{A}_n^{(0)} = -\sum_{j=1}^{n}\theta_j^{\alpha\,A}\,\mathbb{A}_n^{(0)}\,. \tag{2.197}$$

At $\ell \neq 0$ the invariance is spoiled by perturbative anomalies, but these appear in a restricted and controlled fashion. At one loop, one can show that the anomaly of the dual special conformal generator $\mathsf{K}$ takes the form

$$\mathsf{K}_{\alpha\dot\alpha}\tilde{\mathbb{A}}_n^{(0)} = \frac{2}{\epsilon}\frac{r_\Gamma}{(4\pi)^{2-\epsilon}}\,\tilde{\mathbb{A}^{(0)}}_n\sum_{j=1}^{n} x_{j\,\alpha\dot\alpha}(-s_{j-1\,j})^{-\epsilon}\,. \tag{2.198}$$

This surprising behaviour of amplitudes under the action of dual superconformal generators can be understood through the string theory description of scattering amplitudes at strong coupling [19]. A certain T-duality takes the planar theory to itself, loosely speaking interchanging the dual coordinates and space-time coordinates. A scattering process has as its dual the expectation value of a polygonal Wilson loop whose edges are precisely the lightlike momenta of the particles involved in the scattering. Notice that the duality makes sense at the level of partial amplitudes, where particles are ordered and, as such, provide a well-defined ordering for the edges that form the Wilson-loop contour.

## 2.10 Twistors

The representation of the conformal algebra $\mathfrak{so}(2,4)$ acting on amplitudes presented in Section 2.4 is peculiar. As Witten realised in [39], it can be brought to a more natural form by

Fourier-tranforming the $\tilde{\lambda}$'s[7] so that

$$\tilde{\lambda}^{\dot{\alpha}} \mapsto \mathrm{i}\frac{\partial}{\partial \mu_{\dot{\alpha}}}\,, \qquad\qquad -\mathrm{i}\frac{\partial}{\partial \tilde{\lambda}_{\dot{\alpha}}} \mapsto \mu^{\dot{\alpha}}\,. \tag{2.199}$$

In terms of these new variables, the generators

$$\mathsf{p}^{\dot{\alpha}\alpha} = \mathrm{i}\sum_{i=1}^{n} \lambda_i^{\alpha}\frac{\partial}{\partial \mu_{\dot{\alpha}}}\,, \tag{2.200}$$

$$\mathsf{m}^{\alpha\beta} = \sum_{i=1}^{n} \lambda_i^{(\alpha}\frac{\partial}{\partial \lambda_i^{\beta)}}\,, \tag{2.201}$$

$$\bar{\mathsf{m}}^{\dot{\alpha}\dot{\beta}} = \sum_{i=1}^{n} \mu_i^{(\dot{\alpha}}\frac{\partial}{\partial \mu_{\dot{\beta})}}\,, \tag{2.202}$$

$$\mathsf{d} = \sum_{i=1}^{n}\left(\frac{1}{2}\lambda_i^{\alpha}\frac{\partial}{\partial \lambda_i^{\alpha}} - \frac{1}{2}\mu_i^{\dot{\alpha}}\frac{\partial}{\partial \mu_i^{\dot{\alpha}}}\right)\,, \tag{2.203}$$

$$\mathsf{k}_{\alpha\dot{\alpha}} = \mathrm{i}\sum_{i=1}^{n} \mu_{\dot{\alpha}}\frac{\partial}{\partial \lambda_i^{\alpha}}\,. \tag{2.204}$$

are all first-order differential operators.

This representation can be made even simpler by combining together the variables $\lambda_\alpha$ and $\mu^{\dot{\sigma}}$ in a single object

$$Z^{\hat{\mathsf{A}}} = (\lambda_\alpha, \mu^{\dot{\alpha}}) \in \mathbb{T}_{\mathrm{b}} \simeq \mathbb{C}^4\,, \tag{2.205}$$

on which the action of the complexified conformal algebra is that of $\mathfrak{sl}(4,\mathbb{C})$, which is isomorphic to the complexification of $\mathfrak{so}(2,4)$. If one chooses a particular reality condition on the spinors, corresponding to a certain signature for the metric, one ends up with some real form of $\mathfrak{sl}(4,\mathbb{C})$. The simplest choice corresponds to the Kleinian signature, for which $\mathbb{T}_{\mathrm{b}} \simeq \mathbb{R}^4$ and the conformal group is $\mathrm{SL}(4,\mathbb{R})$.

One can extend $\mathbb{T}_{\mathrm{b}}$ by introducing four Grassmann coordinates $\chi_A$ which are obtained transforming the $\eta$'s with

$$\eta^A \mapsto i\frac{\partial}{\partial \chi_A}\,, \qquad\qquad -\mathrm{i}\frac{\partial}{\partial \eta^A} \mapsto \chi_A\,. \tag{2.206}$$

and define

$$\mathcal{Z}^{\mathsf{A}} = (\lambda_\alpha, \mu^{\dot{\alpha}}, \chi_A) \in \mathbb{T} \simeq \mathbb{C}^{4|4}\,, \tag{2.207}$$

on which the superconformal algebra acts with $\mathfrak{sl}(4|4,\mathbb{C})$.

*Twistor space* is defined as the projectivisation $\mathbb{PT} \simeq \mathbb{CP}^{3|4}$. Its relation with space-time (often referred to as the twistor correspondence) is non-local and is captured by the so-called incidence relation

$$\mu^{\dot{\alpha}} = -\mathrm{i}x^{\dot{\alpha}\alpha}\lambda_\alpha\,, \tag{2.208}$$

[7] The choice is clearly arbitrary: one could choose to transform the $\lambda$'s instead.

$$\chi_A = \theta_A^\alpha \lambda_\alpha \ , \tag{2.209}$$

which connects the twistor coordinates with the spacetime coordinates $(x^{\dot{\alpha}\alpha}, \theta_A^\alpha)$. This can be illustrated in terms of the double fibration

$$\begin{tikzcd} & \mathbb{PS} \arrow[dl] \arrow[dr] & \\ \mathbb{PT} & & \mathbb{M} \end{tikzcd} \qquad \begin{tikzcd} & (\lambda_\alpha, x^{\dot{\alpha}\alpha}, \theta_A^\alpha) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ (\lambda_\alpha, -\mathrm{i}x^{\dot{\alpha}\alpha}\lambda_\alpha, \theta_A^\alpha\lambda_\alpha) & & (x^{\dot{\alpha}\alpha}, \theta_A^\alpha) \end{tikzcd}$$

where $\mathbb{PS} \simeq \mathbb{CP}^1 \times \mathbb{M}$.

Let us forget for a second about the supersymmetric extension and focus on the geometric picture that comes from the bosonic part of the twistor correspondence. Through the incidence relation, a point in complexified Minkowski space $\mathbb{M}_\mathrm{b}$ corresponds to a linearly embedded $\mathbb{CP}^1 \subset \mathbb{PT}_\mathrm{b}$, where $\lambda_\alpha$ are homogeneous coordinates on $\mathbb{CP}^1$ and $\mu^{\dot{\alpha}}$ parametrise the embedding. Conversely, let us consider the intersection point of two such $\mathbb{CP}^1 \subset \mathbb{PT}_\mathrm{b}$, associated with points $x, y \in \mathbb{M}_\mathrm{b}$. The incidence relation tells us that $x$ and $y$ are lightlike separated, since $(x-y)^{\dot{\alpha}\alpha}\lambda_\alpha = 0$ which implies that $(x-y)^{\dot{\alpha}\alpha} = \lambda^\alpha\tilde{\lambda}^{\dot{\alpha}}$ for some $\tilde{\lambda}^{\dot{\alpha}}$. This means that a point in twistor space $\mathbb{PT}_\mathrm{b}$ spans a so-called $\alpha$-plane in $\mathbb{M}_\mathrm{b}$, whose tangent vectors are lightlike and of the form $\lambda^\alpha\tilde{\lambda}^{\dot{\alpha}}$.

A point $x \in \mathbb{M}_\mathrm{b}$ can be identified by any two points $Z_1, Z_2 \in \mathbb{PT}_\mathrm{b}$ which belong to the $\mathbb{CP}^1 \subset \mathbb{PT}_\mathrm{b}$ associated with $x$. This is realised by the *bi-twistor*

$$X^{\hat{\mathsf{A}}\hat{\mathsf{B}}} = Z_1^{[\hat{\mathsf{A}}} Z_2^{\hat{\mathsf{B}}]} = \langle 1\,2 \rangle \begin{pmatrix} \varepsilon_{\alpha\beta} & -\mathrm{i}x_\alpha{}^{\dot{\beta}} \\ \mathrm{i}x^{\dot{\alpha}}{}_\beta & -\frac{1}{2}x^2\varepsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \ . \tag{2.210}$$

To be able to compute distances, one must break conformal invariance by introducing the so-call *infinity* bi-twistor, i.e. a bi-twistor which corresponds to a point at infinity:

$$I^{\hat{\mathsf{A}}\hat{\mathsf{B}}} = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \ , \qquad\qquad I_{\hat{\mathsf{A}}\hat{\mathsf{B}}} = \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix} \ . \tag{2.211}$$

With this, the metric reads

$$g(X,Y) = \frac{X^{\hat{\mathsf{A}}\hat{\mathsf{B}}} Y_{\hat{\mathsf{A}}\hat{\mathsf{B}}}}{I_{\hat{\mathsf{C}}\hat{\mathsf{D}}} X^{\hat{\mathsf{C}}\hat{\mathsf{D}}} I_{\hat{\mathsf{E}}\hat{\mathsf{F}}} Y^{\hat{\mathsf{E}}\hat{\mathsf{F}}}} \ . \tag{2.212}$$

Twistors are the natural variables to describe the kinematics in dual space. The coordinates of $\mathbb{PS}$ should remind the reader about the variables introduced in the previous section.[8]

Indeed, one can describe the dual-space kinematics in terms of supertwistors $\mathcal{Z}_1, ..., \mathcal{Z}_n$ that parametrise the lightlike edges of the polygons [**55**] and intersect in the vertices $(x_i, \theta^A)$, with

$$x_j^{\dot{\alpha}\alpha} = \mathrm{i}\frac{\lambda_{j-1}^\alpha \mu_j^{\dot{\sigma}} - \lambda_j^\alpha \mu_{j-1}^{\dot{\sigma}}}{\langle j-1\,j \rangle} \ , \tag{2.213}$$

[8]Notice how, in dual space, the R-symmetry index for the $\theta$'s is raised rather than lowered. This reflects in the fact that the dual conformal algebra introduced in the previous section differs from the conventional ones introduced in Section 2.2 and 2.6 for the position of the R-symmetry indices.

$$\theta_j^{\alpha\,A} = \mathrm{i}\frac{\lambda_{j-1}^{\alpha}\chi_j^A - \lambda_j^{\alpha}\chi_{j-1}^A}{\langle j-1\,j\rangle}\,. \tag{2.214}$$

There are two main advantages in this choice of variables. The first is that, written in terms of twistors, the kinematics of a scattering process is fully unconstrained, as twistors necessarily correspond to lightlike directions and momentum conservation is ensured by the dual-space description based on a closed polygon. The second is that the action of dual conformal invariance on twistors is extremely simple and one can easily introduce invariant quantities by appropriately saturating the twistor indices.

Regarding this last point, we notice that for any four twistors $Z_a, Z_b, Z_c, Z_d$, the four-bracket defined as

$$\langle a,b,c,d\rangle = \varepsilon_{\hat{A}\hat{B}\hat{C}\hat{D}} Z_a^{\hat{A}} Z_b^{\hat{B}} Z_c^{\hat{C}} Z_d^{\hat{D}} \tag{2.215}$$

is manifestly invariant under the action of $\mathrm{SL}(4,\mathbb{C})$. In terms of dual-space kinematics one finds

$$\langle i,j-1,j,k\rangle = \langle i|x_{ij}x_{jk}|k\rangle\,\langle j-1\,j\rangle\;, \tag{2.216}$$

and since

$$x_i^{\dot{\alpha}\alpha}\lambda_{i\,\alpha} = x_{i+1}^{\dot{\alpha}\alpha}\lambda_{i\,\alpha}\;, \tag{2.217}$$

one can replace $x_i$ in (2.216) by $x_{i+1}$, and $x_k$ by $x_{k+1}$.

A quantity which is invariant under the full superconformal group $\mathrm{SL}(4|4,\mathbb{C})$ is given by the five bracket

$$[a,b,c,d,e] = \frac{\delta^{(4)}(\langle a,b,c,d\rangle\chi_e + \text{cyclic})}{\langle a,b,c,d\rangle\langle b,c,d,e\rangle\langle c,d,e,a\rangle\langle d,e,a,b\rangle\langle e,a,b,c\rangle} \tag{2.218}$$

defined for an arbitrary set of five superwistors $\mathcal{Z}_a, \dots, \mathcal{Z}_e$.

After some manipulations one can show that the amplitude $R$-invariants introduced in Section 2.8 are just a specific instance of this general invariant [**55**]:

$$R_{rst} = [s-1,s,t-1,t,r]\;. \tag{2.219}$$

So far we have only discussed how twistors and momentum twistors can be used in describing amplitudes. Their application to form factors is discussed in Chapter 3 and Chapter 5.

## 2.11 Loop recursion relations

As previously mentioned, the presence of IR divergences forces us to regularise the terms coming from loop diagrams. As a consequence of this procedure, loop integrals display fewer symmetries than tree-level amplitudes. For this reason, some of the techniques that have been developed to study results at loop level focus, not on loop amplitudes themselves, but rather on their loop integrands, i.e. the sum of all Feynman diagrams contributing to a given partial amplitude at a certain loop order, where the loop integration is left unperformed. Since no integration is involved, we define loop integrands with the same Feynman rules that are used at tree level, where no modification is introduced to account for a particular regularisation scheme. In our case, for example, loop integrands are defined as fully four-dimensional quantities.

Before moving forward, however, there is a problem we have to address: if it is clear how to write down an integrand in terms of Feynman rules, it is also clear that this function is not uniquely defined. In fact, the integrand carries a dependence on loop momenta, that are determined only up to shifts[9]. In general, it is not even clear how to relate loop momenta between different diagrams. For planar graphs, however, the problem can by easily addressed. One can uniquely fix the integrand function by using region variables. This is achieved by assigning a new coordinate $x_{\ell_j}$ to the $j$'th loop as in Figure 2.4.

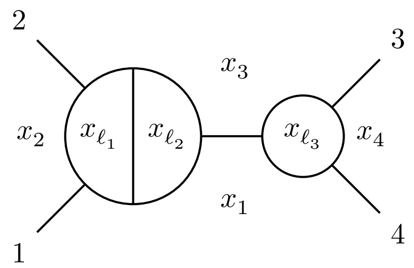



Figure 2.4: Assignment of region variables for planar loop diagrams. Each loop gets assigned a coordinate $x_{\ell_j}$. Loop momenta are defined by taking differences, as with external momenta.

The loop integrand for a given partial amplitude $A^{\ell}_{n,k}$ will be denoted with $\mathcal{A}^{(\ell)}_{n,k}$. For simplicity, when dealing with one-loop integrands, we will denote the loop coordinate with $x_0$.

The same ideas that lead to the formulation of tree-level recursion relations for amplitudes can be extended to produce results for loop integrands [**45**]. What makes this possible is the fact that loop integrands have much simpler analytic properties than their integrated counterpart. The BCFW shift in (2.135), when translated in terms of dual coordinates, takes the simple form

$$x_1 \mapsto \hat{x}_1 = x_1 - z\lambda_n\tilde{\lambda}_1 \tag{2.220}$$

As in the case of tree-level amplitudes, poles correspond to internal lines going on shell (also referred to as cut legs). At loop level, one has to distinguish between two cases. If the cut leg splits the amplitude in two disconnected parts $\mathcal{A}^{(m)}_{\mathrm{L}}$ and $\mathcal{A}^{(\ell-m)}_{\mathrm{R}}$, then one has a configuration which is, essentially, the same as in the tree-level case, with the notable difference that now one has to sum over $m$, i.e. over all the possible ways in which the loops are distributed among

[9]More generally, integrands are defined up to terms that integrate to zero. However, the integrand is more constrained when is defined in terms of Feynman diagrams.

the two subamplitudes. If, instead, the cut leg belongs to a loop, then the associated pole corresponds to an amplitude $\mathcal{A}_{n+2}^{(\ell-1)}$ in a forward-limit configuration: the two new on-shell legs coming from cutting a loop propagator have opposite momenta. It has to be noted, however, that the new kind of cut contributions can be problematic when arising from bubbles and tadpoles. Fortunately, supersymmetry takes care of this problem by ensuring that such terms will all cancel when performing state sums. This is discussed in detail in [**46**].

Let us see how to take into account poles associated with forward limits. We have $\ell$ such poles, all of the form depicted in Figure 2.5

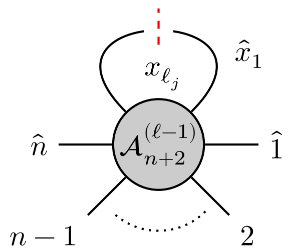



Figure 2.5: Cutting a loop leg produces a single amplitude in a forward configuration.

We need to compute the residue of $\mathcal{A}^\ell(z)/z\,\mathrm{d}z$ associated with such a pole. In terms of dual coordinates the propagator reads

$$-\frac{\mathrm{i}}{(x_{\ell_j}-\hat{x}_1(z))^2} = -\frac{\mathrm{i}}{(x_{\ell_j}-x_1)^2 + z\langle n|x_0 - x_1|1]} \,. \tag{2.221}$$

which gives the location of the pole at

$$z_{\ell_j} = -\frac{(x_{\ell_j}-x_1)^2}{\langle n|x_{\ell_j}-x_1|1]} \,, \tag{2.222}$$

and a residue

$$-\operatorname*{Res}_{z_{\ell_j}}\left(-\frac{\mathrm{i}}{z(x_{\ell_j}-\hat{x}_1(z))^2}\right) = -\frac{\mathrm{i}}{(x_{\ell_j}-x_1)^2} \,, \tag{2.223}$$

which is nothing but the unshifted propagator, as in the original tree-level recursion. Therefore

$$-\operatorname{Res}\left(\frac{\mathcal{A}^{(\ell)}(z)}{z}\,\mathrm{d}z\right) = -\mathrm{i}\int \mathrm{d}^4\eta\,\frac{\mathcal{A}_{\text{forw.}}^{(\ell-1)}(z_{\ell_j})}{(x_{\ell_j}-x_1)^2} \,, \tag{2.224}$$

where the Grassmann integration implements the state sum over the cut loop leg.

A summary of all the poles is given in Figure 2.6. As for the recursion at tree level, the pole at infinity is not present.

It is natural to try to extend this construction for form factors. However, contrary to what happens at tree level, this turns out to be a nontrivial task. In fact, the very notion of an integrand function for form factors is an elusive concept. In Chapter 4 we give a prescription on how to define such a function and how to compute it via a BCFW-like recursion relation.

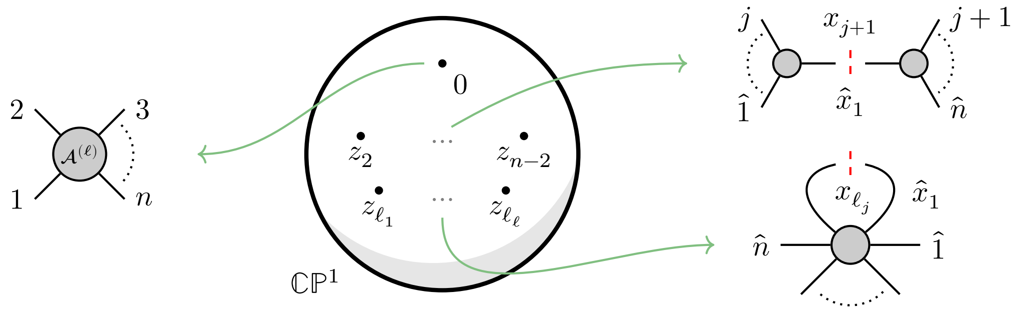



Figure 2.6: Poles associated with the BCFW recursion relation at $\ell$-loops.

## 2.12 Worldsheets

One of the fascinating aspects of the recent progresses in the amplitude program is the discovery of formulas for amplitudes at tree (and more recently loop) level that are suggestive of a stringy origin. Twistor string theory, as introduced by Witten in the seminal paper [39], is a twisted B-model coupled to D1-instantons supported on holomorphic curves in twistor space. Over time, other string-theoretical derivations of the same result have been presented [77, 78]. However, for all the above models the string spectra include both $\mathcal{N}=4$ sYM and $\mathcal{N}=4$ conformal supergravity. The conformal supergravity spoils gauge theory amplitudes at loop level and there is no obvious mechanism for decoupling.

The formula reads[10]

$$A^{(0)}_{n,k} = \int \mathrm{d}\mathcal{M}_{n,k} \prod_{a=1}^{n} \frac{\delta^{(4|4)}(\mathcal{Z}_a - \xi_a \mathcal{P}(\sigma_a))}{\xi_a(\sigma_a - \sigma_{a+1})} \,, \tag{2.225}$$

where

$$\mathcal{P}(\sigma, \{\mathcal{A}\}) = \sum_{d=0}^{k+1} \mathcal{A}_d \, \sigma^d \,, \tag{2.226}$$

is a degree-$(k+1)$ polynomial in $\sigma$ with values is $\mathbb{PT}$. It parametrises the embedding of the worldsheet in the target space. The supertwistors $\mathcal{A}_d$ act as moduli of the curve. The integration measure reads

$$\mathrm{d}\mathcal{M}_{n,k} = \frac{1}{\mathrm{vol}\,\mathrm{GL}(2)} \, \mathrm{d}^{4k+8|4k+8}\mathcal{A} \; \mathrm{d}^n\sigma \; \mathrm{d}^n\xi \,, \tag{2.227}$$

where the $\mathrm{GL}(2,\mathbb{C})$ that we quotient by corresponds to Möbius transformations on the worldsheet and an overall rescaling of the supertwistors $\mathcal{Z}_a$.

Despite its integral form, the formula is really a sum over a discrete set of solutions determined by the constraints imposed by the deltas. We first need to clarify how the integration over the deltas should be performed, since the integral is defined over a complex domain. While

[10] Here, and also in Chapter 3 we use a slight abuse of notation. The expression in (2.225), in fact, differs from a conventional partial amplitude for the presence of an overall delta of momentum conservation.

more detail can be found in Appendix B.2, here we simply point out that the presence of the deltas has the net effect of reducing the integral to a sum, as

$$\int \mathrm{d}z_1 \dots \mathrm{d}z_m \; g(z)\, \delta(f_1(z)) \cdots \delta(f_m(z)) = \sum_{z_0 \in f^{-1}(0)} \operatorname{Res} \omega\big|_{z_0} \,, \tag{2.228}$$

where

$$\omega = \frac{g(z)\, \mathrm{d}z_1 \wedge \dots \wedge \mathrm{d}z_m}{f_1(z) \dots f_m(z)} \,. \tag{2.229}$$

However, for most of the manipulations that we will perform in Chapter 3 it will be useful to approach the problem differently. By imposing a Kleinian signature on the spacetime one has $\mathbb{PT} \simeq \mathbb{RP}^{3|4}$ and the integral becomes real. At this point, the deltas are the conventional real distributions and have a simple representation in terms of their Fourier transform.

The number of solutions over which the integral localises, for some $n$ and $k$, is given by

$$\left\langle \begin{matrix} n-3 \\ k \end{matrix} \right\rangle \,, \tag{2.230}$$

where the symbol denotes the Eulerian number defined as

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = \sum_{j=0}^{k+1} (-1)^j \begin{pmatrix} n+1 \\ j \end{pmatrix} (k-j+1)^n \,. \tag{2.231}$$

After some manipulation the formula in (2.225), which is known as *connected prescription formula* or *Roiban–Spradlin–Volovich (RSV) formula*, can be recast as

$$\begin{aligned} A_{n,k} = \int \frac{1}{\operatorname{vol} \mathrm{GL}(2)} \prod_{a=1}^{n} \frac{\mathrm{d}^2 \sigma_a}{(a\, a+1)} \\ \times \prod_{J=1}^{k+2} \delta^{(2|4)}(\tilde{\lambda}_J - \tilde{\lambda}(\sigma_J), \eta_J - \eta(\sigma_J)) \prod_{i=k+3}^{n} \delta^{(2)}(\lambda_i - \lambda(\sigma_i)) \,, \end{aligned} \tag{2.232}$$

where the $(\sigma_i^1, \sigma_i^2)$ are now homogenous coordinates on $\mathbb{CP}^1$ and round brackets are defined, as for angle brackets, with

$$(i\, j) = \sigma_i^{\alpha} \sigma_j^{\beta} \varepsilon_{\alpha\beta} \,. \tag{2.233}$$

This formula can be obtained directly from a string theory defined in four-dimensional ambitwistor space (for more detail about the construction, the reader should refer to [79]). In the next chapter, we will show how the two formulas are related. The derivation will be tailored to the extension of these formulas to form factors. Notice how in (2.232) the particles have been arbitrarily split into two sets.

Both in the original twistor and the more recent ambitwistor string construction, the correct expressions for gauge amplitudes are obtained by taking only the single-trace terms coming from the correlators of the appropriate vertex operators. Multi-trace contributions are discarded, as these correspond to gluon scattering processes with exchange of internal conformal supergravity states. As mentioned above, the contributions associated with conformal supergravity are responsible for the failure to naïvely extend the above construction to loop level.

In the next chapter we will discuss how (2.225) and (2.232) can be modified to compute tree-level form factors of the $\mathcal{N} = 4$ stress-tensor supermultiplet.

# 3. Worldsheet formulas

The main goal of this chapter is to extend the RSV formula, described in Section 2.12, to super form factors of the stress-tensor multiplet. As we shall see, this task turns out to be surprisingly simple, suggesting also potential new directions to explore for different operator insertions.

An additional, more recent, motivation for this line of enquiry stems from the CHY scattering equations [7, 80], which describe scattering amplitudes at tree level in a variety of theories with and without supersymmetry, and in different numbers of dimensions. Specialising to four dimensions, a new remarkable closed formula for the S-matrix of Yang-Mills theories with different amounts of supersymmetry was derived in [79] starting from ambitwistor strings. Taking gluon scattering as an example, these four-dimensional scattering equations treat positive and negative helicity gluons in a different, complementary way, similarly to the link representations of [43, 44]. It is then natural to ask how different representations of the same S-matrix of gauge theory can be related.

This question was answered in [81], which wrote down a map between the polynomial and rational form of the scattering equations, appearing in the RSV formula and in [79], respectively. A first observation we will make is that the connection is (and, in fact, was) immediate once one makes use of the link representation of the RSV formula discussed in [43]. We will then move on to discuss how to extend the RSV formula to form factors. Our starting point will be an interesting formula written down in [82] which conjectures an extension of the four-dimensional scattering equations for Yang-Mills theory to form factors of the local operator $\mathrm{tr}(F_{\alpha\beta}F^{\alpha\beta})$. In that case, very few modifications to the formula for amplitudes are needed – specifically, two auxiliary gluons of positive helicity $x$ and $y$ are added.[1] Importantly, the amplitudes generated by this formula depend only on $p_x + p_y$ (or $p_x + p_y$ and the supermomentum $\mathrm{q}_x + \mathrm{q}_y$ in the supersymmetric version we introduce in Section 3.1) rather than on the two momenta separately. We note another important feature of this formula: it contains certain Parke-Taylor like denominators of the form $(a\,b)$, with $\sigma_{a,b}$ parameterising punctures on the Riemann sphere,[2] which only involve adjacent *physical* particles, i.e. they do not include $x$ and $y$.

After establishing in Section 3.1 a quick path to relate the RSV formula of [38] and the

[1]The choice of positive helicity is such that all-plus and single-minus gluon form factors of $\mathrm{tr}(F_{\alpha\beta}F^{\alpha\beta})$ are now non-vanishing.

[2]The precise meaning of this notation will be explained in the next section.

four-dimensional scattering equations of [79], our next goal is to write down a formula (3.4), analogous to the RSV result, describing supersymmetric form factors of the chiral stress tensor multiplet operator in twistor space.

In Section 3.2 we show how the proposed formula encodes the correct dependence on the Grassmann variable by explicitly producing the correct supermomentum conservation delta function, and by deriving the expression for the form factor that saturates the upper bound for $k$, at any $n$. We will then show how this proposal is equivalent to a simple supersymmetric extension of the scattering equations formula for form factors presented in [82]. In Section 3.3 we show that our formula can naturally be expressed in terms of link variables, in the same vein as the RSV formula. This link-variable formulation turns out to be very advantageous from the point of view of simplifying calculations, as we demonstrate in several examples in Section 3.4. Importantly, we confirm a feature of the link variable representation of the RSV formula found in [43], namely that a simple deformation of the integration contour in the link variable space and the global residue theorem lead to an alternative representation of the amplitudes which coincides with the BCFW recursion relations for form factors [11]. Finally, we conclude in Section 3.5 with comments on a possible derivation of the proposed formulas directly from a string-theory perspective.

## 3.1 The connected-prescription formula

As in Section 2.12, we choose a set of supertwistor variables $\mathcal{Z}_a$, $a = 1, ..., n$ describing the $n$ particles, with $\mathcal{Z} = (\lambda_\alpha, \mu^{\dot\alpha}, \eta^A)$. Then, as in [34, 82], we describe the form factor insertion through two extra particles $x$ and $y$. The momentum and supermomentum carried by the form factor will then be

$$-q = \lambda_x\tilde\lambda_x + \lambda_y\tilde\lambda_y \,, \tag{3.1}$$

$$-\gamma = \lambda_x\eta_x + \lambda_y\eta_y \,. \tag{3.2}$$

In twistor space, this amounts to introducing two extra supertwistors $\mathcal{Z}_x$ and $\mathcal{Z}_y$.

In analogy with the amplitude formulation, we introduce a degree-$(k+1)$ curve from $\mathbb{CP}^1$ to $\mathbb{CP}^{3|4}$, where $k$ is the MHV degree of the superamplitude. This polynomial has the form

$$\mathcal{P}(\sigma, \{\mathcal{A}\}) := \sum_{d=0}^{k+1} \mathcal{A}_d\, \sigma^d \,, \tag{3.3}$$

where the supertwistors $\mathcal{A}_d$ are the supermoduli of the curve.

We propose that all form factors of the chiral stress-tensor supermultiplet in $\mathcal{N} = 4$ sYM are described in twistor space by the following simple generalisation of the RSV formula:

$$F^{(0)}_{n,k} = \langle \mathcal{Z}_x \mathcal{I} \mathcal{Z}_y\rangle^2 \int \mathrm{d}\mathcal{M}_{n+2,k}\, \frac{\prod_{a=x,y} \delta^{(4|4)}(\mathcal{Z}_a - \xi_a \mathcal{P}(\sigma_a, \{\mathcal{A}\}))}{\xi_x\, \xi_y\, (\sigma_x - \sigma_y)^2} \prod_{a=1}^{n} \frac{\delta^{(4|4)}(\mathcal{Z}_a - \xi_a \mathcal{P}(\sigma_a, \{\mathcal{A}\}))}{\xi_a\, (\sigma_a - \sigma_{a+1})} \,, \tag{3.4}$$

where $\sigma_{n+1} = \sigma_1$ and $\mathcal{I}$ is the infinity twistor, so that $\langle \mathcal{Z}_x \mathcal{I} \mathcal{Z}_y \rangle = \langle x\, y \rangle$.

Note that we do not involve the coordinates for particles $x$ and $y$ in the string of Parke-Taylor type denominators, similarly to [82] (but at variance with e.g. (3.27) of [34], which includes terms of the type $(n\, x)(x\, y)(y\, 1)$ in the denominator).

We now show how from (3.4) we can deduce the scattering equation representation of [82] for form factors (or, more precisely, its generalisation describing supersymmetric form factors of the stress tensor multiplet operator). The proof parallels that of [43].

To begin with, we divide the particles into two sets containing $k+2$ and $n-k$ particles, which we label with indices $J$ and $i$, respectively, with the auxiliary particles belonging to the second set. We will denote by $\mathsf{m}$ the first set of $k+2$ particles, and by $\mathsf{p}$ that of the remaining $n-k-2$ (physical) particles, and also define $\bar{\mathsf{p}} = \mathsf{p} \cup \{x, y\}$. A particularly convenient choice when working with, say, component gluon amplitudes is then to assign gluons of negative (positive) helicity to the first (second) group, with the fictitious particles $x$ and $y$ being included in the second set. This parallels the assignments made in [82] for the non-supersymmetric scattering equations for form factors, where these two particles are treated as gluons of positive helicity.

Next, one Fourier transforms all the twistor variables of the $i$-particles to dual twistor variables: $\mathcal{Z}_i \to \mathcal{W}_i$. In terms of these new variables, we have

$$F^{(0)}_{n,k} = \int \mathrm{d}\mathcal{M}_{n+2,k}\, \frac{\prod_{J\in\mathsf{m}} \delta^{(4|4)}(\mathcal{Z}_J - \xi_J \mathcal{P}(\sigma_J, \{\mathcal{A}\}))}{\xi_x\, \xi_y\, (\sigma_x - \sigma_y)^2\, \prod_{a=1}^{n} \xi_a\, (\sigma_a - \sigma_{a+1})} \times \left\langle \frac{\partial}{\partial \mathcal{W}_x}\, \mathcal{I}\, \frac{\partial}{\partial \mathcal{W}_y} \right\rangle^2 \prod_{i\in\bar{\mathsf{p}}} \exp\left(i\, \xi_i\, \mathcal{W}_i \cdot \mathcal{P}(\sigma_i, \{\mathcal{A}\})\right) . \tag{3.5}$$

This procedure has the advantage that there are now as many $\delta$-functions as moduli, and the integration over the $\mathcal{A}$'s can be performed explicitly, with the net effect of localising the polynomial $\mathcal{P}(\sigma, \{\mathcal{A}\})$ onto

$$\mathcal{P}(\sigma) = \sum_{J\in\mathsf{m}} \frac{\mathcal{Z}_J}{\xi_J} \prod_{K\neq J} \frac{\sigma_K - \sigma}{\sigma_K - \sigma_J} . \tag{3.6}$$

One is then left with

$$F^{(0)}_{n,k} = \int \frac{1}{\mathrm{vol\, GL}(2)}\, \frac{\mathrm{d}\sigma_x\, \mathrm{d}\xi_x\, \mathrm{d}\sigma_y\, \mathrm{d}\xi_y}{\xi_x\, \xi_y\, (\sigma_x - \sigma_y)^2} \prod_{a=1}^{n} \frac{\mathrm{d}\sigma_a\, \mathrm{d}\xi_a}{\xi_a\, (\sigma_a - \sigma_{a+1})} \times \left\langle \frac{\partial}{\partial \mathcal{W}_x}\, \mathcal{I}\, \frac{\partial}{\partial \mathcal{W}_y} \right\rangle^2 \exp\left( i \sum_{i\in\bar{\mathsf{p}}, J\in\mathsf{m}} \mathcal{W}_i \cdot \mathcal{Z}_J\, \frac{\xi_i}{\xi_J} \prod_{K\neq J} \frac{\sigma_K - \sigma_i}{\sigma_K - \sigma_J} \right) . \tag{3.7}$$

We can simplify the integrals with the change of variables $(\xi_i, \xi_J) \mapsto (t_i, t_J)$ with

$$t_i = \xi_i \prod_K (\sigma_K - \sigma_i)\, , \qquad t_J^{-1} = \xi_J \prod_{K\neq J} (\sigma_K - \sigma_J)\, , \tag{3.8}$$

and spinor coordinates $\sigma_\alpha = t^{-1}(1, \sigma)$, so that

$$\prod_{a=1}^{n} \frac{\mathrm{d}t_a\, \mathrm{d}\sigma_a}{t_a(\sigma_a - \sigma_{a+1})} = \prod_{a=1}^{n} \frac{\mathrm{d}^2\sigma_a}{(a\, a+1)}\, , \tag{3.9}$$

with $(a\,b) = \epsilon_{\alpha\beta}\sigma_a^\alpha\sigma_b^\beta$. The formula, then, reduces to

$$F_{n,k}^{(0)} = \int \frac{1}{\mathrm{vol}\,\mathrm{GL}(2)} \frac{\mathrm{d}^2\sigma_x\,\mathrm{d}^2\sigma_y}{(x\,y)^2} \prod_{a=1}^{n} \frac{\mathrm{d}^2\sigma_a}{(a\,a+1)} \left\langle \frac{\partial}{\partial \mathcal{W}_x}\, \mathcal{I}\, \frac{\partial}{\partial \mathcal{W}_y} \right\rangle^2 \exp\left( i \sum_{i\in\bar{\mathsf{p}}, J\in\mathsf{m}} \frac{\mathcal{W}_i \cdot \mathcal{Z}_J}{(i\,J)} \right). \tag{3.10}$$

It is now easy to go back to spinor variables by performing a Fourier transform. The result is[3]

$$\begin{aligned} F_{n,k}^{(0)} = \langle x\,y\rangle^2 \int \frac{1}{\mathrm{vol}\,\mathrm{GL}(2)} \frac{\mathrm{d}^2\sigma_x\,\mathrm{d}^2\sigma_y}{(x\,y)^2} \prod_{a=1}^{n} \frac{\mathrm{d}^2\sigma_a}{(a\,a+1)} \\ \times \prod_{i\in\bar{\mathsf{p}}} \delta^{(2)}(\lambda_i - \lambda(\sigma_i)) \prod_{J\in\mathsf{m}} \delta^{(2|4)}(\tilde{\lambda}_J - \tilde{\lambda}(\sigma_J), \eta_J - \eta(\sigma_J))\,, \end{aligned} \tag{3.11}$$

where we have defined the functions

$$\lambda(\sigma) = \sum_{J\in\mathsf{m}} \frac{\lambda_J}{(\sigma\,\sigma_J)}\,, \qquad \tilde{\lambda}(\sigma) = \sum_{i\in\bar{\mathsf{p}}} \frac{\tilde{\lambda}_i}{(\sigma_i\,\sigma)}\,, \qquad \eta(\sigma) = \sum_{i\in\bar{\mathsf{p}}} \frac{\eta_i}{(\sigma_i\,\sigma)}\,. \tag{3.12}$$

(3.11) is nothing but the supersymmetric form of the scattering equation for form factors presented in [**82**]. By performing in reverse the same steps of this proof, one can of course derive the connected prescription for form factors (3.4) from the scattering equations.

## 3.2 A test

We now want to show that the formula encodes the correct dependence on $\gamma^+$, namely that supermomentum conservation indeed takes the form of (2.131).

The Grassmann deltas in (3.11) can be contracted as in (2.130) to obtain

$$\prod_{J\in\mathsf{m}} \delta^{(2)}\left( \eta_J^+ - \sum_{i\in\bar{\mathsf{p}}} \frac{\eta_i^+}{(J\,i)} \right) \delta^{(2)}\left( \eta_J^- - \sum_{i\in\bar{\mathsf{p}}} \frac{\eta_i^-}{(J\,i)} \right)\,. \tag{3.13}$$

The first factors, on the support of the holomorphic deltas give

$$\delta^{(4)}\left( \sum_{J\in\mathsf{m}} \eta_J^+\lambda_J - \sum_{i\in\bar{\mathsf{p}}}\sum_{J\in\mathsf{m}} \frac{\eta_i^+\lambda_J}{(J\,i)} \right) = \delta^{(4)}\left( \sum_{a=1}^{n} \eta_a^+\lambda_a + \eta_x^+\lambda_x + \eta_y^+\lambda_y \right)\,. \tag{3.14}$$

which is precisely what we expect, given (3.2). Similarly, one can extract the correct delta of supermomentum conservation for the components labelled with minus, keeping in mind that one needs to set $\gamma^- = 0$.

As a test of the proposed formula, we compute the form factor associated with the upper bound for $k$, namely, $k = n-2$, also known as N$^{\max}$MHV. In this case, $\bar{\mathsf{p}} = \{x, y\}$ and $\mathsf{m} = \{1, \ldots, n\}$. Solving the rational scattering equations is trivial here, since they are linear in the $(i\,J)^{-1}$ variables. The delta functions are

$$\prod_{J=1}^{n} \delta^{(2|4)}\left( \tilde{\lambda}_J - \frac{\tilde{\lambda}_x}{(J\,x)} - \frac{\tilde{\lambda}_y}{(J\,y)},\, \eta_J - \frac{\eta_x}{(J\,x)} - \frac{\eta_y}{(J\,y)} \right) \tag{3.15}$$

[3] We recall that $\mathcal{Z} = (\lambda, \mu, \eta)$ and $\mathcal{W} = (\tilde{\mu}, \tilde{\lambda}, \tilde{\eta})$, with $\mathcal{Z}\cdot\mathcal{W} = \lambda^\alpha\tilde{\mu}_\alpha + \mu_{\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}} + \tilde{\eta}_A\eta^A$.

$$\times\, \delta^{(2)}\left(\lambda_x - \sum_{J=1}^{n} \frac{\lambda_J}{(x\,J)}\right)\, \delta^{(2)}\left(\lambda_y - \sum_{J=1}^{n} \frac{\lambda_J}{(y\,J)}\right)\,. \tag{3.16}$$

The delta functions in the second line of the above bring momentum conservation, i.e.

$$\delta^{(2)}\left(\lambda_x - \sum_{J=1}^{n} \frac{\lambda_J}{(x\,J)}\right)\, \delta^{(2)}\left(\lambda_y - \sum_{J=1}^{n} \frac{\lambda_J}{(y\,J)}\right) = [x\,y]^2\, \delta^4(p_1 + \ldots + p_n + p_x + p_y)\,. \tag{3.17}$$

The GL(2) redundancy in the integration can be fixed by imposing

$$\sigma_x = (1, 0)\,, \qquad\qquad \sigma_y = (0, 1)\,. \tag{3.18}$$

With this choice we can perform the integration directly in terms of the round brackets. In fact, by using the Schouten identity we can replace the terms in the denominator with contractions of the form

$$(a\,y)(x\,a+1) - (x\,a)(a+1\,y) = (a\,a+1)\,, \tag{3.19}$$

which, because of (3.18), produce individual spinor components.

The antiholomorphic delta functions enforce

$$(J\,x) = \frac{[x\,y]}{[J\,y]}\,, \qquad\qquad (J\,y) = \frac{[y\,x]}{[J\,x]}\,, \tag{3.20}$$

and bring the Jacobian factor

$$\prod_{J=1}^{n} \frac{(J\,x)^2\,(J\,y)^2}{[x\,y]}\,. \tag{3.21}$$

Putting everything together, one arrives at the expression

$$F^{(0)}_{n,-1} = \frac{q^4}{[1\,2]\ldots[n\,1]} \prod_{J=1}^{n} \delta^{(4)}\left(\eta_J + \frac{[J\,y]}{[y\,x]}\eta_x + \frac{[J\,x]}{[x\,y]}\eta_y\right) \delta^{(4)}(p_1 + \ldots + p_n + p_x + p_y)\,, \tag{3.22}$$

which agrees with the result found in [**11**].

## 3.3 The link representation

In [**43**], Spradlin and Volovich presented an interesting formula for the $S$-matrix of $\mathcal{N} = 4$ SYM using the link variables introduced in [**44**], and we now give the corresponding formula for the form factors of the stress tensor multiplet operator.

The link representation is obtained by introducing auxiliary variables

$$c_{iJ} = \frac{1}{(i\,J)}\,, \tag{3.23}$$

where we note that the first and second index run over the sets $\bar{\mathsf{p}}$ and $\mathsf{m}$, respectively. This identification is achieved by introducing

$$1 = \int \mathrm{d}c_{iJ}\, \delta\left(c_{iJ} - 1/(iJ)\right)\,. \tag{3.24}$$

Doing so, we can recast (3.11) as

$$F^{(0)}_{n,k} = \langle x\,y\rangle^2 \int \prod_{i\in\bar{\mathsf{p}},J\in\mathsf{m}} \mathrm{d}c_{iJ}\; U(c_{iJ}) \prod_{i\in\bar{\mathsf{p}}} \delta^{(2)}(\lambda_i - c_{iJ}\lambda_J) \prod_{J\in\mathsf{m}} \delta^{(2|4)}(\tilde\lambda_J + c_{iJ}\tilde\lambda_i, \eta_J + c_{iJ}\eta_i)\,, \tag{3.25}$$

where we are implicitly summing over repeated indices in the deltas, and

$$U(c_{iJ}) = \int \frac{1}{\mathrm{vol}\,\mathrm{GL}(2)} \frac{\mathrm{d}^2\sigma_x\,\mathrm{d}^2\sigma_y}{(x\,y)^2} \prod_{a=1}^{n} \frac{\mathrm{d}^2\sigma_a}{(a\,a+1)} \prod_{i\in\bar{\mathsf{p}},J\in\mathsf{m}} \delta\Big(c_{iJ} - \frac{1}{(i\,J)}\Big)\,. \tag{3.26}$$

There are several reasons why it is interesting to study the link representation form (3.25). Firstly, it has the advantage of linearising momentum conservation in terms of the $c_{iJ}$ variables. Secondly, the quantity $U(c_{iJ})$ defined in (3.26) appears to be much more easily computed for any $k$, which is a considerable advantage in comparison to the scattering equations. Finally, it was shown in [**43**] that by using the (global) residue theorem, one can arrive at an alternative representation of the amplitudes which precisely matches BCFW diagrams, thus establishing a direct connection between the twistor-string representation of amplitudes and on-shell recursion relations. We will see that the same is also true for our representation of form factors, as we will explain in Section 3.4.

In performing explicit calculations, a natural way to fix the GL(2) gauge freedom is to fix the four variables corresponding to the two auxiliary legs $x$ and $y$,

$$\sigma_x = (1,0)\,, \qquad \sigma_y = (0,1)\,, \qquad (x\,y) = 1\,. \tag{3.27}$$

We can then change variables from the spinors $\sigma_a^\alpha$ to the brackets $(x\,a)$ and $(y\,a)$ so that

$$U(c_{iJ}) = \int \prod_{a=1}^{n} \frac{\mathrm{d}(x\,a)\;\mathrm{d}(y\,a)}{(a\,a+1)} \prod_{i\in\bar{\mathsf{p}},J\in\mathsf{m}} \delta\Big(c_{iJ} - \frac{1}{(i\,J)}\Big)\,. \tag{3.28}$$

All the other brackets can be obtained from those used as integration variables using the Schouten identity,

$$(x\,y)(a\,b) = (x\,a)(y\,b) - (y\,a)(x\,b)\,. \tag{3.29}$$

In (3.28) we have $2n$ integration variables and $(k+2)(n-k)$ delta functions, which means that $U(c_{iJ})$ contains $k(n-k-2)$ delta functions after integration. In (3.25), four of the Grassmann-even delta functions enforce momentum conservation, leaving $2n$ delta functions and $(k+2)(n-k)$ variables $c_{iJ}$ to integrate over. This leaves $k(n-k-2)$ integration variables, which we denote by $\tau_k$. Thus (3.25) can be written as

$$F^{(0)}_{n,k} = J\,\langle x\,y\rangle^2\, \delta^{(4)}\Big(q - \sum_{a=1}^{n} p_a\Big) \int \mathrm{d}^{k(n-k-2)}\tau\; U(c_{iJ}) \prod_{J\in\mathsf{m}} \delta^{(4)}(\eta_J + c_{iJ}\eta_i)\,, \tag{3.30}$$

for some $c_{iJ}(\tau)$ linear in $\tau$, and an appropriate Jacobian $J$.

### 3.4 Examples

In this section we work out explicitly the form of $U(c_{iJ})$ defined in (3.26) for various form factors. We will always use the gauge fixing (3.27) so that $U(c_{iJ})$ is computed using (3.28).

#### *3.4.1 The MHV form factor*

Let us start by considering an MHV form factor. In this case the two sets are

$$\begin{aligned} \bar{\mathsf{p}} &= \{1, \dots, \widehat{J_1}, \dots, \widehat{J_2}, \dots, n, x, y\} \, , \\ \mathsf{m} &= \{J_1, J_2\} \, , \end{aligned} \tag{3.31}$$

where hatted entries are to be omitted from the set. This corresponds to the case where, for the component operator $\mathscr{L}$ and a purely gluonic on-shell state, there are exactly two gluons with negative and $n-2$ with positive helicity.

The $U$ function in (3.28) is given by

$$U^{\text{MHV}} = \frac{1}{(c_{xy;J_1J_2})^2 \, c_{J_1-1\,J_2} \, c_{J_1+1\,J_2} \, c_{J_1\,J_2-1} \, c_{J_1\,J_2+1}} \prod_{a \neq J_1-1, J_1, J_2-1, J_2} \frac{1}{c_{a\,a+1;J_1J_2}} \, , \tag{3.32}$$

where $c_{ab;JK} = c_{aJ}c_{bK} - c_{aK}c_{bJ}$. Performing the integration over the link variables is again straightforward, and one arrives at the MHV super form factor of the chiral part of the stress tensor multiplet

$$F^{(0)}_{n,0} = \frac{1}{\langle 1\,2\rangle \cdots \langle n\,1\rangle} \, \delta^{(8)}\Big( \sum_{a=1}^{n} \lambda_a \eta_a + \lambda_x \eta_x + \lambda_y \eta_y \Big) \, , \tag{3.33}$$

which agrees with the known result [**12**] if we identify $-\gamma_+ = \lambda_x\eta_x + \lambda_y\eta_y$.

#### *3.4.2 An NMHV form factor*

We now want to show how to perform a computation of a specific component. We choose, for instance, the case in which the external on-shell states are gluons with helicities $(---+)$. We then set $\mathsf{m} = \{1, 2, 3\}$ and $\bar{\mathsf{p}} = \{4, x, y\}$. The function (3.28) reads

$$U^{---+} = \int \prod_{a=1}^{4} \frac{\mathrm{d}(x\,a)\,\mathrm{d}(y\,a)}{(a\,a+1)} \prod_{J=1}^{3} \delta\Big(c_{xJ} - \frac{1}{(x\,J)}\Big)\,\delta\Big(c_{yJ} - \frac{1}{(y\,J)}\Big)\,\delta\Big(c_{4J} - \frac{1}{(4\,J)}\Big) \, . \tag{3.34}$$

With nine delta functions and eight integrations, there is one delta function remaining after all integrations are carried out. The integrations over $(x\,J)$ and $(y\,J)$ are straightforward, and one can then choose to solve the two delta functions involving $(4\,1)$ and $(4\,2)$, producing a Jacobian, and insert this solution into the remaining delta function for $(4\,3)$. Collecting all terms from this process, one finds that that

$$U^{---+} = \frac{c_{x2}\,c_{y2}}{c_{42}\,c_{xy;21}\,c_{xy;23}} \, \delta(S_{123;4xy}) \, , \tag{3.35}$$

where, following the notation introduced in [43], we define

$$S_{ijk;lmn} := c_{mi}\, c_{mj}\, c_{lk}\, c_{nk}\, c_{ln;ij} - c_{ni}\, c_{nj}\, c_{lk}\, c_{mk}\, c_{lm;ij} - c_{li}\, c_{lj}\, c_{mk}\, c_{nk}\, c_{mn;ij} \,. \tag{3.36}$$

We comment that in this case

$$S_{lmn,LMN} = \det([1/c_{iJ}]) \prod_{\substack{i=l,m,n \\ J=L,M,N}} c_{iJ} \tag{3.37}$$

where $[1/c_{iJ}]$ is the matrix with elements $1/c_{iJ}$. As in (3.30), the form factor can be obtained by integrating out the remaining delta function. However, there is a more efficient way to derive the final result which avoids solving the constraint of $\delta(S_{123;4xy})$ altogether.

In general, the complex delta function has the property

$$\int \mathrm{d}z_1 \ldots \mathrm{d}z_m \; g(z)\; \delta(f_1(z)) \cdots \delta(f_m(z)) = \sum_{z_0 \in f^{-1}(0)} \mathrm{Res}\; \omega\big|_{z_0} \,, \tag{3.38}$$

where

$$\omega := \frac{g(z)\, \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_m}{f_1(z) \ldots f_m(z)} \,. \tag{3.39}$$

In our case, this means that the integral in (3.30) can be written as a sum of residues of

$$\omega_U = \frac{c_{x2}\, c_{y2}}{c_{42}\, c_{xy;21}\, c_{xy;23}} \frac{\mathrm{d}\tau}{S_{123;4xy}} \,, \tag{3.40}$$

evaluated on the zeros of the quartic polynomial $S_{123;4xy}(\tau)$. However, since $\omega_U$ can be straightforwardly extended to a meromorphic form on $\mathbb{CP}^1$, we can use the global residue theorem to compute the result in terms of the other poles of $\omega_U$, which correspond to the simple zeros of $c_{42}(\tau)$, $c_{xy;21}(\tau)$ and $c_{xy;23}(\tau)$. Focusing on gluon scattering, the corresponding residues are

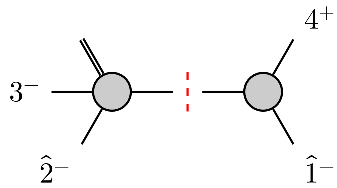
$$\begin{aligned} F_{(42)} &= -\frac{\langle 1\, 3\rangle^4\, q^4}{s_{134}\, \langle 1\, 4\rangle\, \langle 3\, 4\rangle\, \langle 3|q|2]\, \langle 1|q|2]} \,, \\ F_{(xy;21)} &= -\frac{\langle 3|q|4]^3}{s_{124}\, [1\, 2]\, [1\, 4]\, \langle 3|q|2]} \,, \\ F_{(xy;23)} &= -\frac{\langle 1|q|4]^3}{s_{324}\, [3\, 2]\, [3\, 4]\, \langle 1|q|2]} \,, \end{aligned} \tag{3.41}$$

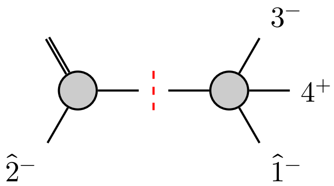
and the complete result is obtained by adding the three terms,

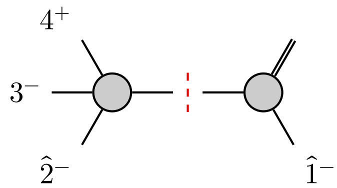
$$F^{---+} = F_{(42)} + F_{(xy;21)} + F_{(xy;23)} \,. \tag{3.42}$$

It is notable that each term in (3.41) depends on $p_x$ and $p_y$ only through the combination $p_x + p_y = q$. Moreover, each term is a rational function of external kinematics. Interestingly, these two properties do not hold for the four terms arising from the solutions of the scattering equation $S_{123;4xy} = 0$, and are only recovered in the sum over the four solutions.

Perhaps more remarkably, each term in (3.41) corresponds to a BCFW diagram for a $[1\ 2\rangle$ shift, analogously to the amplitude case, as discussed in [**43**] (see also [**83**]). Specifically, we have found that the sum in (3.42) corresponds, term by term, to the sum

$F_{(42)} =$  , (3.43)


$F_{(xy;21)} =$  , (3.44)


$F_{(xy;23)} =$  (3.45)


given by the BCFW expansion of the form factor.

## 3.5 Form factors from ambitwistor strings

The result (3.11) bears a close resemblance to the ambitwistor-string formula in (2.232).[4]

In this construction the Parke-Taylor denominator of the measure emerges from a current algebra on the worldsheet, similarly to the standard heterotic string construction. Each vertex operator is dressed with a current $J^a$ built from $N$ free complex fermions $\psi^i$ and $SU(N)$ generators $T^a$. More explicitly, we define

$$J^a(\sigma) = \frac{i}{2}\mathrm{t}^a :\psi^i(\sigma)\bar{\psi}^j(\sigma): \ . \tag{3.46}$$

Recall that the only non-vanishing Wick contraction between complex fermions takes the form

$$\langle \psi^i(\sigma_1)\bar{\psi}^j(\sigma_2)\rangle = \frac{\delta^{ij}}{\sigma_1 - \sigma_2}\,, \tag{3.47}$$

so we may immediately evaluate the correlator of $n$ currents to be

$$\langle J^{a_1}\cdots J^{a_n}\rangle = \frac{\mathrm{tr}\,(\mathrm{t}^{a_1}\cdots \mathrm{t}^{a_n})}{(\sigma_1-\sigma_2)\cdots(\sigma_n-\sigma_1)} + \text{perms} + \cdots\,, \tag{3.48}$$

where we have ignored multiple trace terms, as discussed in Section 2.12. Keeping only the first term corresponds to computing a certain colour-ordered amplitude.

[4]Our superamplitudes have $\eta^0$ for positive helicity and $\eta^4$ for negative helicity gluons, which is the opposite of the convention employed in [**79**].

We may construct the measure of formula (3.11) from ambitwistor strings in a similar way, at least up to an overall factor. We must include two additional vertex operators, corresponding to the punctures $\sigma_{n+1}$ and $\sigma_{n+2}$ on the Riemann sphere. These are dressed with additional currents defined as above. However, in order to obtain the chiral stress tensor super form factor, we now do not require the single trace term. Rather we extract from Wick's theorem the double trace term displayed below,

$$\langle J^{a_1} \cdots J^{a_{n+2}} \rangle = \cdots + \frac{\operatorname{tr}(T^{a_1} \cdots T^{a_n})}{(\sigma_1 - \sigma_2) \cdots (\sigma_n - \sigma_1)} \cdot \frac{\operatorname{tr}(T^{a_{n+1}} T^{a_{n+2}})}{(\sigma_{n+1} - \sigma_{n+2})^2} + \cdots \,, \tag{3.49}$$

providing the appropriate denominator and colour factor for the on-shell state. It would be very interesting to have a complete derivation of (3.11) from ambitwistor strings, also explaining the $\langle x\, y\rangle^2$ prefactor.

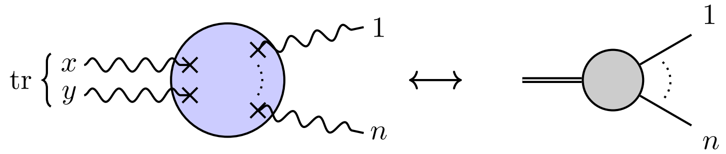



Figure 3.1: The same vertex operators that are used in the amplitude formula can produce a super form factor when one considers terms with a different trace structure. In particular two extra particles can be “merged” together to form the off-shell leg corresponding to the operator insertion.

# 4. Loop recursion relations

In Section 2.9 we discussed introduced the dual space for amplitudes and mentioned its interpretation in terms of the amplitude/Wilson loop duality.

For form factors, two important differences need to be taken into account. First, momentum conservation now reads

$$\sum_{i=1}^{n} p_i = q \; , \tag{4.1}$$

where $q$ is the incoming momentum of the off-shell leg associated with the operator insertion. This implies that the dual Wilson line cannot be drawn as a closed, piecewise light-like polygon. The proposal of [**84**, **85**] at strong coupling is to draw the dual contour as a periodic Wilson line, with period $q$. Furthermore, the inserted local operator is gauge invariant, i.e. a colour singlet, thus making the object inherently non-planar. In [**11**], a similar picture was advocated at weak coupling, and further discussed in [**12**]. In the latter paper, dual MHV rules which crucially involve a periodic configuration in momentum twistor space were also introduced, and applied to the computation of tree-level and one-loop supersymmetric form factors of protected operators.

In this chapter we leave aside a more detailed definition of the form factor/Wilson line duality, and instead give a well-motivated prescription for expressing form factors in terms of region variables living on a periodic contour. Crucially, with such a prescription one can unambiguously define one-loop integrands even for form factors, and hence study loop recursion relations. With this prescription in hand, recursion relations can be formulated straightforwardly. Furthermore, and importantly, this prescription is mandatory in order to define and understand the action of dual conformal symmetry on form factors. In the present chapter we will define this prescription and use it to study loop-level recursion relations for form factor integrands, while the realisation of DCI will be fully studied in Chapter 5. We will consider form factors of the (chiral part of the) stress-tensor multiplet operator, although many of the techniques that we will introduce can potentially be extended to more general operators as well.

This chapter is organised as follows. In Section 4.1 we discuss the assignment of region momenta for form factors and introduce a periodic kinematic configuration inspired by [**11**, **84**, **85**]. This is a key step which then allows us to formulate recursion relations. In Section 4.2 we review NMHV form factors and the particular $R$-invariants used to express them, some of which are novel compared to amplitudes. Section 4.3 is the central section of the chapter. There we introduce the BCFW recursion relation for the loop integrand. Finally, in Section 4.4, various one-loop examples are described in order to illustrate the practical implementation of the recursions and point out important differences compared to recursion relations for amplitudes.

## 4.1 Assignment of region momenta

We begin our discussion by considering a generic form factor diagram, such as that in Figure 4.1, contributing in the planar limit. This could be a Feynman or BCFW diagram or an integral function, and we colour order all external on-shell legs. Because the operator is a gauge singlet, the corresponding line $q$ can be inserted between any pair of lines. Up to one loop one can only have planar diagrams, but starting from two loops, non-planar integrals can appear even at leading order in colour.

Once we have drawn $q$ in a particular position, *e.g.* between legs $i-1$ and $i$ as in Figure 4.1, we label the region variables starting from $q$, moving in a clockwise fashion. We then introduce the region momenta as in (2.181), with the identification

$$x_{i+n} = x_i - q \equiv x_i^- \, . \tag{4.2}$$

When we get back to the leg with momentum $q$, we have moved all the way to $x_i^-$ and this provides a natural way to rewrite $q$ in terms of region variables as[1] $q = x_i - x_i^-$.

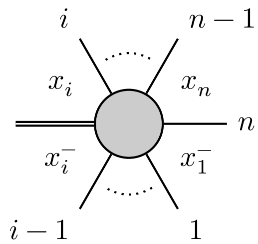



Figure 4.1: Possible assignments of region momenta in a planar form factor diagram. The double line corresponds to the off-shell leg carrying incoming momentum $q$. In our notation $x_i^- \equiv x_i - q$.

We would like to stress that the peculiarity of our prescription is that the definition of $q$ in terms of region variables changes according to the diagram we are considering, since a priori the off-shell leg is not ordered with respect to the on-shell ones. In the next chapter, we will show how this assignment is crucial in defining the action of dual conformal symmetry. In other words, given the infinite sequence of light-like segments in the periodic dual configuration, we associate to every diagram a particular period therein. As an example, in Figure 4.2 we

[1]In our conventions, the momentum $q$ is incoming.

consider the three-leg case and show how the three possible configurations are mapped to three different periods.

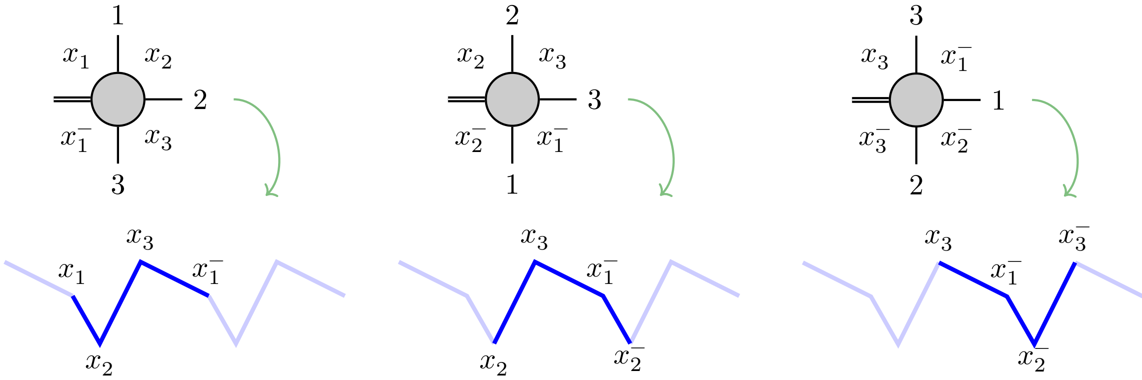
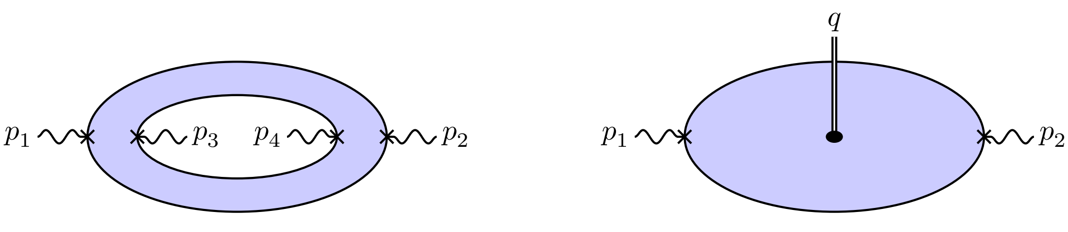



Figure 4.2: Form factor with three external legs and periodic dual configuration. The highlighted region is the one we select.

Notice that our prescription involves the choice of an origin. For instance, in the first diagram of Figure 4.2 we chose to start labelling regions from $x_1$ and then move clockwise around the diagram. It should be clear that we could have labelled region momenta starting from any other vertex. This would have no consequences for the integrated result thanks to translation invariance in dual space. Nevertheless this choice has consequences in the definition of the loop integrand, and the action of the dual conformal generators.

The application of recursion relations to the loop integrand of scattering amplitudes requires the unambiguous definition of the integrand itself. This is obtained in the planar limit by introducing region variables. In a similar way we can introduce region variables for the form factor loop integrands. At one loop, they will involve propagators of the type $1/x_{0i}^2$, where $x_0$ is the region of the loop momentum. It is also clear that an overall shift of the external region variables $x_i \to x_i + mq$ can be compensated by a shift in the loop variable $x_0 \to x_0 + mq$. This feature will be crucial in the derivation of the loop recursion relation presented later.




(a) (b)

Figure 4.3: $(a)$ Worldsheet configuration for a four-point double-trace amplitude. Each $\times$ stands for the insertion of an open string vertex operator. $(b)$ Worldsheet configuration for the Sudakov form factor. The $\bullet$ stands for the closed string vertex operator.

This property of the loop integrand can also be viewed in the light of the recent work [58], where a Wilson loop dual for double-trace contributions to scattering amplitudes is discussed. Their main observation is based on the idea that the string worldsheet for double-trace amplitudes has the topology of a cylinder, or equivalently of an annulus with open string insertions on the two boundaries. The case of form factors can be thought of as a degenerate limit of the double-trace amplitude, where the internal circle of the annulus shrinks to a point corresponding to a closed string insertion (see Figure 4.3). In the large-$N_c$ limit on the gauge theory side only diagrams survive that can be drawn on the punctured disk topology. A neat example is shown in Figure 4.4 and was already mentioned in Section 2.5: a two-loop "non-planar" diagram contributing to the Sudakov form factor is drawn as a planar diagram on the punctured disk.[2] Here, the same property is illustrated from a worldsheet perspective.

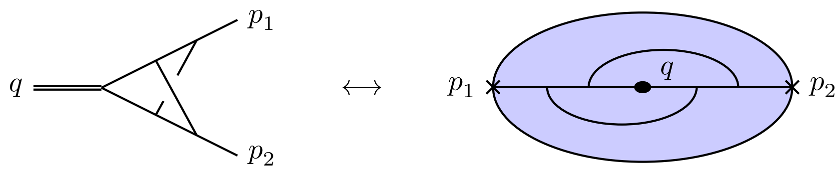



Figure 4.4: A non-planar Feynman diagram which appears as planar when drawn on a punctured disk. All such diagrams contribute to the large-$N_c$ form factor.

As noted above, when trying to naïvely assign region variables on the worldsheet one fails because of the excess momentum flowing from the closed-string insertion inside the worldsheet. The trick is to consider, rather than the disk itself, the surface obtained by cutting the disk from some point on the boundary up to the off-shell closed-string insertion point, which becomes a branch point. Graphically, one can represent the line of cut with the off shell leg which, for the purpose, is drawn as lying on the worldsheet. In Figure 4.5a we show how this works for the two-loop example we considered above. At one loop, however, the prescription becomes simpler, as shown in Figure 4.5b, since one can always arrange the cut in such a way that no loop leg is cut in the process.

This approach is remiscent of the one in [58], where the authors established a correspondence between a double periodic Wilson loop and what they called the *cylinder cut* of the amplitude. We refer to [58] for the precise definition of the cylinder cut. Here we only point out that it depends on an additional momentum $\ell$, which, in the Wilson line picture, parameterises the distance between the two periodic Wilson lines. This momentum $\ell$, as much as our $x_0$ loop variable, is characterised by an ambiguity under shifts by an integer number of periods, i.e. $\ell \mapsto \ell + mq$. In that case, the authors decided to eliminate this ambiguity by summing over all possible shifts. For our purposes, instead of resolving the residual ambiguity of the

[2]The degree of non-planarity of form factors is similar to the one described in [59], since they can be made planar by removing the leg carrying momentum $q$. However $q$ is not light-like, hence the argument of [59] does not apply here. Actually we will show in the next chapter that the full dual conformal symmetry is preserved by form factor diagrams.

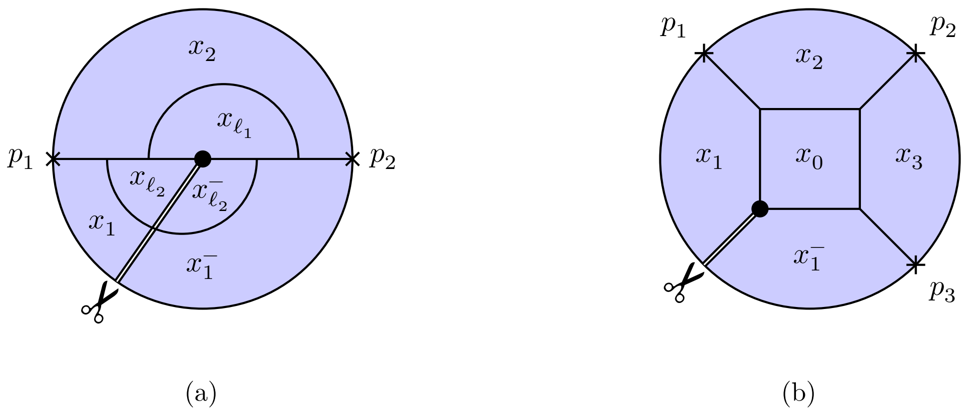



Figure 4.5: The region variables assignement is well defined when performed on a branched covering of the worldsheet disk.

integrand by performing an analogous sum, we just rely on the obvious property

$$\int \mathrm{d}^4 x_0 \; f(x_0) = \int \mathrm{d}^4 x_0 \; f(x_0 + mq) \; , \tag{4.3}$$

and regard different representations of the integrand related by shifts in $x_0$ as different representatives of the same equivalence class of integrands. Although this introduces a level of freedom in defining integrand representations, it allows to re-express the result of the recursion in terms of a more conventional basis of integral functions.

## 4.2 Region variables and $R$-invariants

In order to make our discussion more concrete, and to prepare the ground for some explicit computations that will be presented later in this chapter, in this section we show how the prescription outlined in Section 4.1 can be applied to $R$-invariants coming from NMHV boxes reviewed in Section 2.8. We consider a box diagram as in (2.163) and we decorate it with the assignment of specific (super)region variables as follows: we proceed clockwise and assign the $x$ and $\theta$ variables associated to each one of the four regions starting from the one that comes after the corner where the off-shell leg is inserted. We can represent this for a generic box diagram, as shown in Figure 4.6, without the need to specify where the off-shell leg sits.

The assigned coordinates still form a closed polygon, whose edges, which are in general not light-like, correspond to the momenta and supermomenta flowing out each of the four corners of the box. This is illustrated in Figure 4.7.

By comparison with the diagrams in (2.164) we have that

$$\begin{aligned} x_c &\sim x_r \; , & \theta_c &\sim \theta_r \; , \\ x_a &\sim x_s \; , & \theta_a &\sim \theta_s \; , \\ x_b &\sim x_t \; , & \theta_b &\sim \theta_t \; , \end{aligned} \tag{4.4}$$

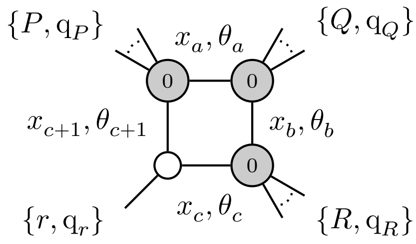



Figure 4.6: Conventions for assigning outgoing momenta and supermomenta as well as region variables for a generic kinematic configuration.

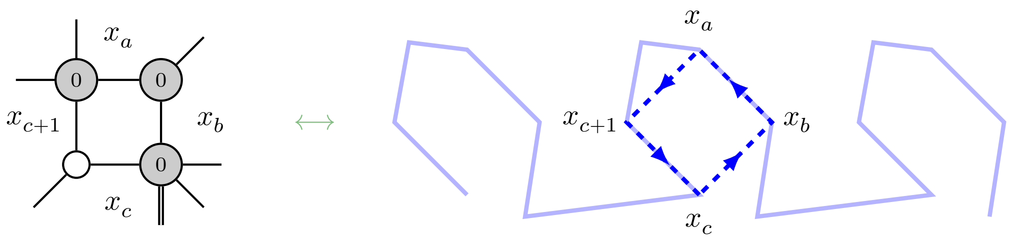



Figure 4.7: An example of a box diagram and its associated polygon in dual space. For simplicity the Grassmann coordinates $\theta$ are omitted.

where the $\sim$ sign indicates that the identity holds up to an appropriate shift by some integer multiple of a period. In terms of region variables, one can rewrite (2.165) and (2.168) as

$$R^{\bullet}_{rst} = \frac{\langle s-1\,s\rangle\langle t-1\,t\rangle\,\delta^{(4)}(\langle r|x_{ca}x_{ab}|\theta_{br}\rangle + \langle r|x_{cb}x_{ba}|\theta_{ar}\rangle)}{x_{ab}^2\langle r|x_{cb}x_{ba}|s-1\rangle\langle r|x_{cb}x_{ba}|s\rangle\langle r|x_{ca}x_{ab}|t-1\rangle\langle r|x_{ca}x_{ab}|t\rangle}\ , \tag{4.5}$$

$$R'_{rss} = -\frac{\langle s-1\,s\rangle\,\delta^{(4)}(\langle r|x_{ca}x_{ab}|\theta_{br}\rangle + \langle r|x_{cb}x_{ba}|\theta_{ar}\rangle)}{x_{ab}^4\langle r|x_{cb}x_{ba}|s-1\rangle\langle r|x_{ca}x_{ab}|s\rangle\langle r|x_{ca}x_{bc}|r\rangle}\ . \tag{4.6}$$

Obtaining the correct expression with the appropriate assignement of region variables will turn out to be crucial, not just for the purpose of establishing recursion relations at loop level discussed in this chapter, but also to associate to each diagram a well defined behaviour under dual conformal transformations, as described in detail in Chapter 5.

## 4.3 Recursion relations

Given our prescription for the assignment of region variables in one-loop diagrams, we now proceed to consider the complete one-loop integrand $\boldsymbol{\mathcal{F}}^{(1)}_{n,k}(x_0)$, defined by

$$F^{(1)}_{n,k} = \int \mathrm{d}^d x_0\ \boldsymbol{\mathcal{F}}^{(1)}_{n,k}(\{x_i\};x_0)\ . \tag{4.7}$$

In order to obtain recursion relations we perform particular shifts of the external legs and define $\widehat{\boldsymbol{\mathcal{F}}}^{(1)}_{n,k}(z) \equiv \boldsymbol{\mathcal{F}}^{(1)}_{n,k}(\{\hat{x}_i\};x_0)$ such that

$$0 = \frac{1}{2\pi \mathrm{i}}\oint \frac{\mathrm{d}z}{z}\,\widehat{\boldsymbol{\mathcal{F}}}^{(1)}_{n,k}(z) = \boldsymbol{\mathcal{F}}^{(1)}_{n,k}(\{x_i\};x_0) + \sum_{z_i\neq 0}\operatorname*{Res}_{z=z_i}\frac{\widehat{\boldsymbol{\mathcal{F}}}^{(1)}_{n,k}(z)}{z}\ , \tag{4.8}$$

where the sum is taken over the residues of the integrand occurring at $z_i \neq 0$, and we used $\widehat{\mathcal{F}}^{(1)}_{n,k}(0) = \mathcal{F}^{(1)}_{n,k}(\{x_i\};x_0)$. Just like for amplitudes recursion relations, unitarity and locality guarantee that $\widehat{\mathcal{F}}$ have only first-order poles, and we assume that the chosen deformation preserves the overall momentum conservation and leaves all particle momenta on shell. An important requirement is also that $\widehat{\mathcal{F}}^{(1)}_{n,k}(z) \sim 1/z$ for large $z$. This will be the case for the two-line shift deformation we will consider, i.e. the loop-level generalisation of the familiar tree-level BCFW recursion relation.

We perform a shift of the one-loop integrand that involves the shift of a single region coordinate, together with all its periodic images. To be concrete, we focus on the shift

$$\hat{x}_1^{\bullet} \equiv x_1^{\bullet} - z\lambda_n\tilde{\lambda}_1 \ . \tag{4.9}$$

In terms of spinor variables, the above corresponds to a shift of the form

$$\hat{\lambda}_1 \equiv \lambda_1 - z\lambda_n \ , \qquad \hat{\tilde{\lambda}}_n \equiv \tilde{\lambda}_n + z\tilde{\lambda}_1 \ , \qquad \hat{\eta}_n \equiv \eta_n + z\eta_1 \ . \tag{4.10}$$

Similarly to the case of amplitudes, the residues of the integrand have simple physical origins: they are associated either to factorisation channels or to forward limits of tree-level form factors.

It is useful to introduce the ratios defined by dividing form factors by the corresponding tree-level MHV quantities,

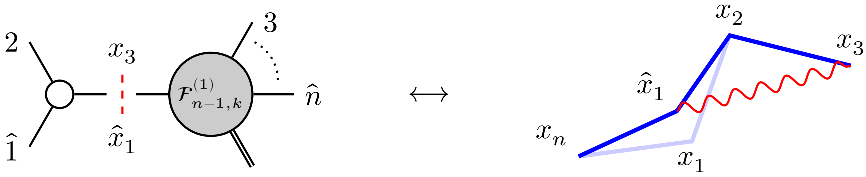
$$\tilde{\mathcal{F}}^{(l)}_{n,k} \equiv \frac{\mathcal{F}^{(l)}_{n,k}}{F^{(0)}_{n,0}} \ . \tag{4.11}$$

We can then propose the following formula for the one-loop integrand:

$$\begin{aligned}
\mathcal{F}^{(1)}_{n,k} = {} & F^{(0)}_{n,0}\, \tilde{\mathcal{F}}^{(1)}_{n-1,k}(\hat{x}_1, x_3, \ldots, x_n, x_0) \\
& - \frac{1}{x_{01}^2} \int \mathrm{d}^4\eta_\ell\, \mathcal{F}^{(0)}_{n+2,k+1}(\hat{x}_1, \ldots, x_n, \hat{x}_1^-, x_0^-) \\
& - \mathrm{i} \sum_{l,i,k_{\mathrm{L}}} \int \mathrm{d}^4\eta_\ell \left[ \mathcal{F}^{(l)}_{i,k_{\mathrm{L}}}(\hat{x}_1, \ldots, x_i)\, \frac{1}{(x_{i1}^+)^2}\, \mathcal{A}^{(1-l)}_{n-i+2,k_{\mathrm{R}}}(\hat{x}_1, x_i, \ldots, x_n) \right. \\
& \left. \qquad + \mathcal{A}^{(l)}_{i,k_{\mathrm{L}}}(\hat{x}_1, \ldots, x_i)\, \frac{1}{(x_{i1})^2}\, \mathcal{F}^{(1-l)}_{n-i+2,k_{\mathrm{R}}}(\hat{x}_1, x_i, \ldots, x_n) \right] ,
\end{aligned} \tag{4.12}$$

where $l = 0, 1$, $i = 2, \ldots, n-1$ and $k_{\mathrm{L}} + k_{\mathrm{R}} = k - 1$ with $k_{\mathrm{L}}, k_{\mathrm{R}} \geq 0$, and $\eta_\ell$ is the Grassmann variable associated to the internal lines. We will now systematically describe the various terms in this formula. Note that for ease of notation we have dropped the dependence on $x_0$ in the last two lines.

The first line of (4.12) originates from the particular factorisation channel




(4.13)

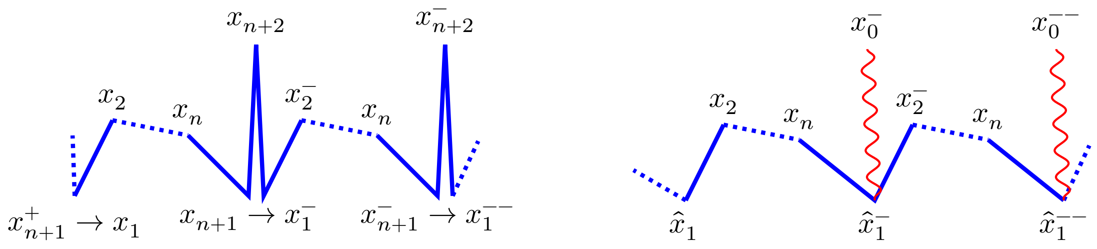
which is the only one associated with the Parke–Taylor prefactor. This diagram is evaluated in the particular kinematics for which $(\hat{x}_1 - x_3)^2 = 0$, as indicated by the light-like wavy red line. According to (4.11), we can write the one-loop integrand in the above diagram as

$$\mathcal{F}^{(1)}_{n-1,k} = F^{(0)}_{n-1,0}\, \tilde{\mathcal{F}}^{(1)}_{n-1,k}(\hat{x}_1, x_3, \dots, x_n; x_0)\,. \tag{4.14}$$

The tree-level prefactor recombines with the $\overline{\text{MHV}}$ amplitude, as in the BCFW recursion at tree level, to give the first line of (4.12). Specifically,

$$\begin{aligned} A^{(0)}_{3,-1}\, \frac{1}{x^2_{13}}\, \mathcal{F}^{(1)}_{n-1,k} &= A^{(0)}_{3,-1}\, \frac{1}{x^2_{13}}\, F^{(0)}_{n-1,0}\, \tilde{\mathcal{F}}^{(1)}_{n-1,k}(\hat{x}_1, x_3, \dots, x_n; x_0) \\ &= F^{(0)}_{n,0}\, \tilde{\mathcal{F}}^{(1)}_{n-1,k}(\hat{x}_1, x_3, \dots, x_n; x_0)\,. \end{aligned} \tag{4.15}$$

The second line of (4.12) contains the contributions from the forward limits. They are evaluated at the value of $z$ for which $(x_0 - \hat{x}_1)^2 = 0$. The geometric interpretation of the forward limit is shown in Figures 4.8a and 4.8b.




(a) Forward limit

(b) Single loop-leg cut

Figure 4.8: Illustration of the forward limits and single cuts on the periodic kinematic configuration. The red wiggly represents the distance $\hat{x}_{01}$ that becomes null at the location of the residue. This also explains the arguments of the $(n+2)$-point form factor appearing in the second line of (4.12).

Compared to the recursion relation of amplitude integrands, there is an important difference arising from diagrams where the shifted region variable $x_i$ appears twice in the expression of the integrand (see Figure 4.9). This occurs when the operator carrying momentum $q$ is located between the shifted region momenta $\hat{x}_1$ and $\hat{x}^-_1$. When taking the sum over the residues, these diagrams will give two contributions: one arising from a pole when $\hat{x}^2_{01} = 0$, and another one from a pole at $(\hat{x}^+_{01})^2 = 0$. These two poles are associated with two different values of $z$. However, as discussed in Section 4.1, we can use the freedom of shifting $x_0$ by a period to find a representation of the integrand with an overall factor $1/x^2_{01}$. Notice that, since $z$ itself depends on $x_0$, this gets shifted as well and the two residues are then associated with the same value of $z$. One may still wonder whether both these contributions are produced in the forward limit of some higher point amplitudes; this is indeed the case and we will demonstrate this in specific examples later on.

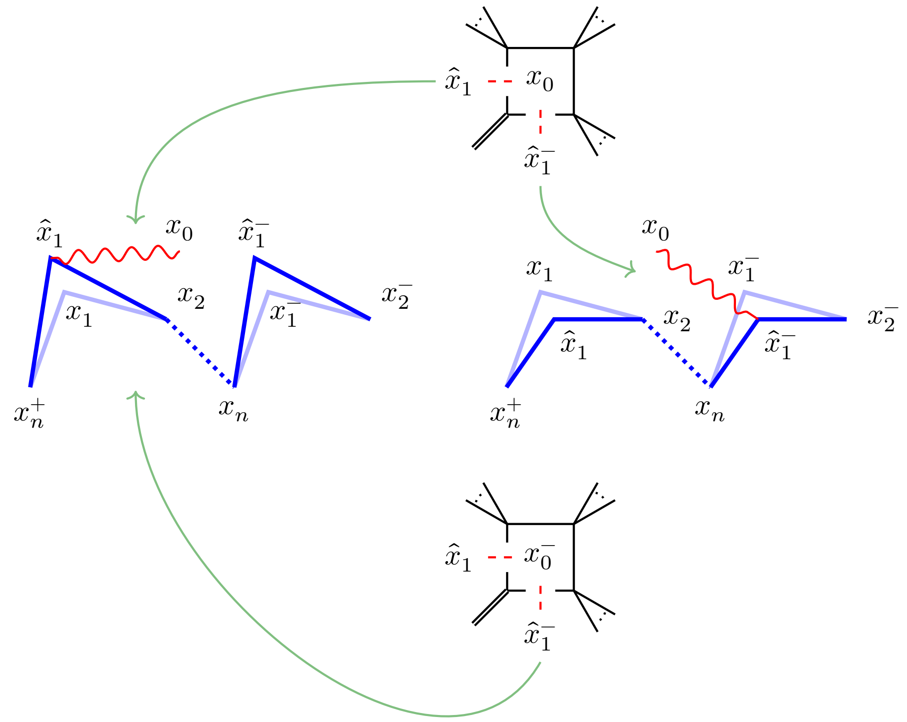
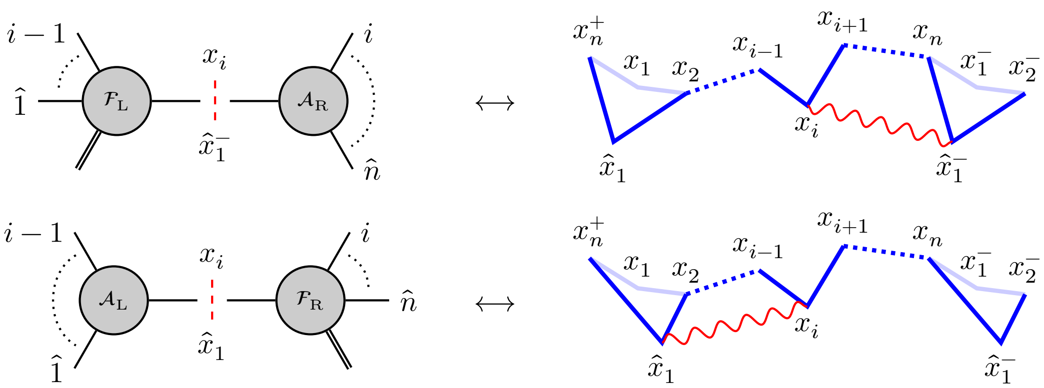



Figure 4.9: The special kinematic configuration where $q$ is located between the shifted region momenta $\hat{x}_1$ and $\hat{x}_1^-$. The two corresponding residues reside on different periods of the periodic kinematic configuration. They can be mapped into each other by a shift of $x_0$. Note that for a generic configuration there is only a single residue as in the case of amplitudes.

Finally, in the last two lines of (4.12) every pole is associated with a standard factorisation channel, thus $z$ is evaluated respectively at $(x_i - \hat{x}_1^-)^2 = 0$ and $(x_i - \hat{x}_1)^2 = 0$ as illustrated below:




(4.16)

Given the one-loop recursion relation (4.12), it is tempting to propose at this point a straightforward all-loop generalisation:

$$\begin{aligned}\boldsymbol{\mathcal{F}}^{(l)}_{n,k} &= F^{(0)}_{n,k}\,\tilde{\boldsymbol{\mathcal{F}}}^{(l)}_{n-1,k}(\hat{x}_1, x_3, \dots, x_n, x_0)\\ &\quad - \frac{1}{x_{01}^2}\int \mathrm{d}^4\eta_\ell\,\boldsymbol{\mathcal{F}}^{(l-1)}_{n+2,k+1}(\hat{x}_1, \dots, x_n, \hat{x}_1^-, x_0^-)\end{aligned}$$

$$
\begin{aligned}
&-\mathrm{i}\sum_{l_\mathrm{L},i,k_\mathrm{L}}\int \mathrm{d}^4\eta_\ell \left[\boldsymbol{\mathcal{F}}^{(l_\mathrm{L})}_{i,k_\mathrm{L}}(\hat{x}_1,\dots,x_i)\,\frac{1}{(x^+_{i1})^2}\,\boldsymbol{\mathcal{A}}^{(l_\mathrm{R})}_{n-i+2,k_\mathrm{R}}(\hat{x}_1,x_i,\dots,x_n)\right.\\
&\qquad\left.+\boldsymbol{\mathcal{A}}^{(l_\mathrm{L})}_{i,k_\mathrm{L}}(\hat{x}_1,\dots,x_i)\,\frac{1}{(x_{i1})^2}\,\boldsymbol{\mathcal{F}}^{(l_\mathrm{R})}_{n-i+2,k_\mathrm{R}}(\hat{x}_1,x_i,\dots,x_n)\right]\,,
\end{aligned}
\tag{4.17}
$$

with $l_\mathrm{L}+l_\mathrm{R}=l$, $i=2,\dots n-1$ and $k_\mathrm{L}+k_\mathrm{R}=k-1$ with $k_\mathrm{L},k_\mathrm{R}\geq 0$. In this expression we have suppressed lower loop variables for ease of notation and we only quote $x_0$ corresponding to the new variable. One of the issues that needs to be clarified at higher loops is the assignment of region variables and the associated ambiguity we discussed in Section 4.1. In the present thesis we will focus exclusively on explicit checks of the one-loop recursion presented in (4.12).

The examples we discuss in the following are one-loop MHV form factors. In this case, the recursion has only two contributions:

$$
\boldsymbol{\mathcal{F}}^{(1)}_{n,k}=F^{(0)}_{n,0}\,\tilde{\boldsymbol{\mathcal{F}}}^{(1)}_{n-1,k}(\hat{x}_1,x_3,\dots,x_n,x_0)-\frac{1}{x^2_{01}}\int\mathrm{d}^4\eta_\ell\,\boldsymbol{\mathcal{F}}^{(0)}_{n+2,k+1}(\hat{x}_1,\dots,x_n,\hat{x}^-_1,x^-_0)\,. \tag{4.18}
$$

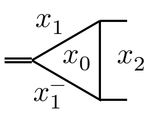
In the next two subsections we provide examples of the BCFW recursion at one loop. We show the validity of our approach by comparing results obtained by using our prescription in (4.12) with integrands obtained from generalised unitarity. To show agreement between the two, we will explicitly check that the result obtained with unitarity methods has residues coming from single loop-leg cuts which are precisely captured by forward limits of tree-level form factors, up to shifts in $x_0$.

## 4.4 Examples

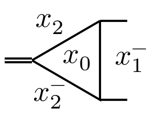
### *4.4.1 The one-loop two-point form factor*

The simplest example is given by the minimal form factor at one loop. As anticipated, we start by considering the one-loop integrand coming from generalised unitarity. In this case only triangles can appear. When summing over cyclic permutations of the external on-shell legs, one obtains

$$
\boldsymbol{\mathcal{F}}^{(1)}_{2,0}(x_1,x_2;x_0)=F^{(0)}_{2,0}\left(s_{12}\right.
$$




$$+\; s_{12}$$

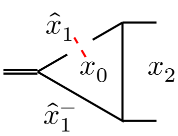



$$\left.\right)\,. \tag{4.19}$$

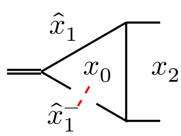
We now consider the BCFW shift

$$
\hat{x}^\bullet_1\equiv x^\bullet_1-z\lambda_2\tilde{\lambda}_1\,, \tag{4.20}
$$

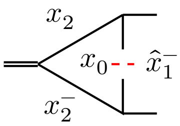
and collect all the residues associated with it. These come from the cuts




$$=-\frac{1}{x^2_{01}x^2_{02}(\hat{x}^+_{01})^2}\,,$$




$$=-\frac{1}{(x^+_{01})^2\hat{x}^2_{01}x^2_{02}}\,,$$




$$= -\frac{1}{(x_{01}^+)^2 x_{02}^2 (x_{02}^+)^2}\ . \tag{4.21}$$

Similarly to the situation depicted in Figure 4.9, the first triangle in (4.19) gives rise to two different cut contributions. Notice also how, in this particular case, the triangle coefficients and the MHV prefactor are insensitive to the deformation. Let us collect the three terms in a single function $\mathcal{I}_{\text{cut}}$. In doing so, we perform the shift $x_0 \mapsto x_0^-$ on the last two terms to obtain a universal prefactor $1/(x_{01})^2$ associated with the cut leg. Hence, we obtain

$$\mathcal{I}_{\text{cut}} = -\frac{F_{2,0}^{(0)}}{x_{01}^2}\left(\frac{s_{12}}{x_{02}^2(\hat{x}_{01}^+)^2} + \frac{s_{12}}{(\hat{x}_{01}^-)^2(x_{02}^-)^2} + \frac{s_{12}}{(x_{02}^-)^2 x_{02}^2}\right)\ . \tag{4.22}$$

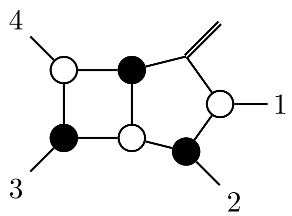
According to (4.12), we can reproduce the results above from the forward limit of $F_{4,1}^{(0)}$. The expression for this, given in (5.14), reads

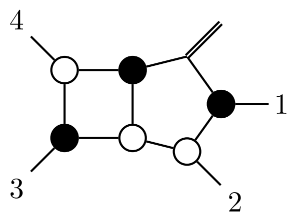
$$\tilde{F}_{4,1}^{(0)} = R'_{133} + R'_{134} + R'_{144} + R''_{131}\ . \tag{4.23}$$

When taking the forward limit, we make the assignments

$$\lambda_4 \to -\lambda_3\ , \qquad \tilde{\lambda}_4 \to \tilde{\lambda}_3\ , \qquad \eta_4 = \eta_3\ . \tag{4.24}$$

By looking at the expressions of the $R$-invariants it is easy to see that some of the denominators vanish under these identifications. In particular, this happens when legs 3 and 4 are attached to the same MHV blob as in the first two diagrams in the second line of (2.169) for $n = 4$, i.e. $R'_{133}$ and $R''_{131}$. Similar diagrams were already considered in the amplitude case [**45**, **46**, **86**]. It turns out that for supersymmetric theories their contribution vanishes in the sum over all the possible external states appearing in the two legs with momenta $p_3$ and $-p_3$ in the forward limit. For $\mathcal{N} = 4$ SYM the sum over the states can be implemented by integrating over the Grassmann variable $\eta_3$. Looking at the expressions for the $R$-invariants $R'_{133}$ and $R''_{131}$ one immediately notices that the dependence on $\eta_4$ disappears in the configuration (4.24). This implies that the integration over $\eta_3$ will always vanish when legs 3 and 4 are attached to the same MHV blob. This provides a systematic and graphical way to isolate the diagrams that contribute to the forward limit of the NMHV amplitude. Represented in terms of on-shell diagrams, these are

$R'_{134} =$ 

, $R'_{144} =$ 

. (4.25)

Therefore, after integrating over $\mathrm{d}^4\eta_3$ we are left with the following contributions:

$$\lim_{p_4 \to -p_3} \int \mathrm{d}^4\eta_3\, R'_{134} = \frac{\delta^{(8)}(\mathrm{q})\, [12]^2}{(p_3 - q)^2\, (p_3 - p_1)^2\, 2p_3 \cdot q}$$

$$= \frac{\delta^{(8)}(\mathrm{q})\,[12]^2}{q^2\,(p_3-p_1)^2\,2p_3\cdot q} + \frac{\delta^{(8)}(\mathrm{q})\,[12]^2}{q^2\,(p_3-q)^2\,(p_3-p_1)^2}\,, \tag{4.26}$$

$$\begin{aligned}\lim_{p_4\to -p_3}\int \mathrm{d}^4\eta_3\,R'_{144} &= -\frac{\delta^{(8)}(\mathrm{q})\,[12]^2}{(p_3+q)^2\,(p_3+p_2)^2\,2p_3\cdot q}\\ &= -\frac{\delta^{(8)}(\mathrm{q})\,[12]^2}{q^2\,(p_3+p_2)^2\,2p_3\cdot q} + \frac{\delta^{(8)}(\mathrm{q})\,[12]^2}{q^2\,(p_3+q)^2\,(p_3+p_2)^2}\,. \end{aligned}\tag{4.27}$$

The sum of these expressions gives

$$\boldsymbol{\mathcal{I}}_{\text{forw}} = -F^{(0)}_{2,0}\left(\frac{s_{12}}{s_{1,-3}s_{1,2,-3}} + \frac{s_{12}}{s_{1,2,3}\,s_{2,3}} + \frac{s_{12}}{s_{2,3}s_{1,-3}}\right)\,, \tag{4.28}$$

where we used the notation $s_{i,\dots,\pm j} = (p_i + \dots \pm p_j)^2$. This is the result for the forward limit. According to (4.12) we need to evaluate this expression on a shifted kinematics. First we express Mandelstam variables in terms of region variables, using the forward kinematics $x_3 = x_1^-$, $x_4 = x_0^-$. Then we simply shift $x_1 \mapsto \hat{x}_1$ as shown in Figure 4.8a.

With this, we obtain full agreement between the two expressions, as stated in (4.12), i.e. we have

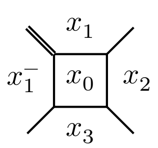
$$\boldsymbol{\mathcal{I}}_{\text{cut}} = \frac{1}{x_{01}^2}\,\hat{\boldsymbol{\mathcal{I}}}_{\text{forw}}\,, \tag{4.29}$$

where in $\hat{\boldsymbol{\mathcal{I}}}_{\text{forw}}$ we performed the identification described above. In this particular case, $\boldsymbol{\mathcal{I}}_{\text{cut}}$ reconstructs the full integrand, since, as noted earlier, the poles we considered are all the poles of the integrand function.

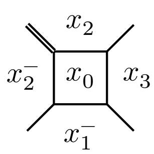
### 4.4.2 The one-loop three-point MHV form factor

The example described in the previous section is very simple because of the small number of diagrams and the absence of boxes. The first case where box diagrams appear is the three-point case, whose one-loop integrand was derived in [**11**]. This result, expressed using the region variable assignment described in Section 4.1, reads

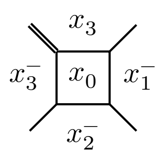
$$\frac{\boldsymbol{\mathcal{F}}^{(1)}_{3,1}(x_0)}{F^{(0)}_{3,0}} = \frac{x_{13}^2(x_{21}^+)^2}{2}$$



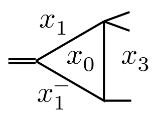

$$+\;\frac{(x_{21}^+)^2(x_{32}^+)^2}{2}$$



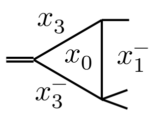

$$+\;\frac{(x_{32}^+)^2x_{13}^2}{2}$$




$$+\;\frac{(x_{21}^+)^2+(x_{32}^+)^2}{2}\Bigg($$

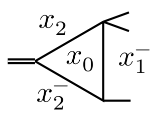



$$+$$

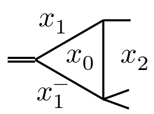



$$\Bigg)$$

$$+\;\frac{(x_{32}^+)^2+x_{13}^2}{2}\Bigg($$

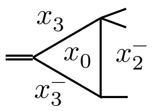



$$+$$

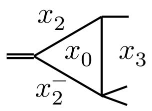



$$\Bigg)$$

$$+\quad \frac{x_{13}^2+(x_{21}^+)^2}{2}\Bigg($$




$$+$$

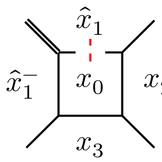



$$\Bigg)\,. \tag{4.30}$$

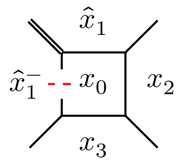
We then consider the BCFW shift

$$\hat{x}_1^{\bullet} \equiv x_1^{\bullet} - z\lambda_3\tilde{\lambda}_1\ , \tag{4.31}$$

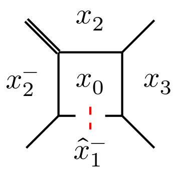
and collect the residues coming from the above expression. These are associated with the cuts



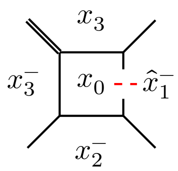

$$= \frac{1}{x_{01}^2 x_{02}^2 x_{03}^2 (\hat{x}_{01}^+)^2}\ ,$$



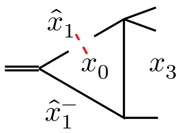

$$= \frac{1}{\hat{x}_{01}^2 x_{02}^2 x_{03}^2 (x_{01}^+)^2}\ ,$$



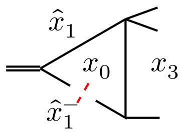

$$= \frac{1}{x_{02}^2 x_{03}^2 (x_{01}^+)^2 (x_{02}^+)^2}\ ,$$



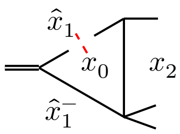

$$= \frac{1}{x_{03}^2 (x_{01}^+)^2 (x_{02}^+)^2 (x_{03}^+)^2}\ ,$$



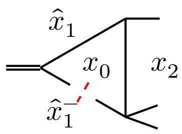

$$= \frac{1}{x_{01}^2 x_{03}^2 (\hat{x}_{01}^+)^2}\ ,$$



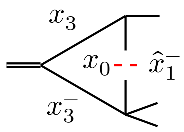

$$= \frac{1}{\hat{x}_{01}^2 x_{03}^2 (x_{01}^+)^2}\ ,$$



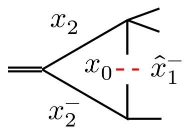

$$= \frac{1}{x_{01}^2 x_{02}^2 (\hat{x}_{01}^+)^2}\ ,$$




$$= \frac{1}{\hat{x}_{01}^2 x_{02}^2 (x_{01}^+)^2}\ ,$$




$$= \frac{1}{x_{03}^2 (x_{01}^+)^2 (x_{03}^+)^2}\ ,$$




$$= \frac{1}{x_{02}^2 (x_{01}^+)^2 (x_{02}^+)^2}\ . \tag{4.32}$$

As done in the previous section, we shift $x_0$ appropriately on each term to collect an overall $1/x_{01}^2$ factor. The sum of all the residues reads

$$\begin{aligned} \boldsymbol{\mathcal{I}}_{\rm cut} = -\frac{F_{3,0}^{(0)}}{2x_{01}^2}\Bigg[ & \frac{(\hat{x}_{21}^+)^2\hat{x}_{13}^2}{x_{02}^2 x_{03}^2 (\hat{x}_{01}^+)^2} + \frac{(\hat{x}_{21}^+)^2\hat{x}_{13}^2}{(x_{03}^-)^2 (x_{02}^-)^2 (\hat{x}_{01}^-)^2} + \frac{(\hat{x}_{21}^+)^2 (x_{32}^+)^2}{(x_{02}^-)^2 x_{02}^2 (x_{03}^-)^2} + \frac{\hat{x}_{13}^2 (x_{32}^+)^2}{x_{03}^2 x_{02}^2 (x_{03}^-)^2} \\ & + \frac{(x_{32}^+)^2 + (\hat{x}_{21}^+)^2}{(x_{03}^-)^2 x_{03}^2} + \frac{\hat{x}_{13}^2 + (x_{32}^+)^2}{x_{02}^2 (\hat{x}_{01}^+)^2} + \frac{(x_{32}^+)^2 + (\hat{x}_{21}^+)^2}{x_{03}^2 (\hat{x}_{01}^+)^2} + \frac{(x_{32}^+)^2 + \hat{x}_{13}^2}{x_{02}^2 (x_{02}^-)^2} \\ & + \frac{(\hat{x}_{21}^+)^2 + (x_{32}^+)^2}{(x_{03}^-)^2 (\hat{x}_{01}^-)^2} + \frac{(x_{32}^+)^2 + \hat{x}_{13}^2}{(x_{02}^-)^2 (\hat{x}_{01}^-)^2}\Bigg]\ . \end{aligned} \tag{4.33}$$

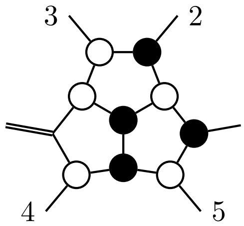
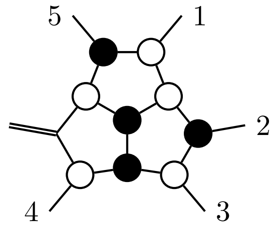
We will now show that the above can be obtained through the forward limit of the five-point NMHV form factor. We start from the general expression for the NMHV form factor (5.14) and we consider the five-point case

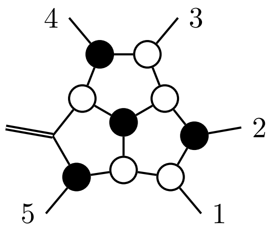
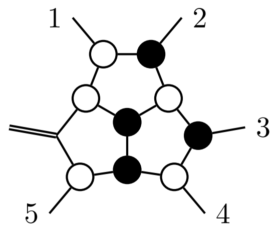
$$\tilde{F}_{5,1}^{(0)} = R'_{135} + R'_{145} + R'_{155} + R''_{135} + R'_{134} + R'_{133} + R'_{144} + R''_{131} + R''_{141}\ . \tag{4.34}$$

We then consider the forward limit of legs 4 and 5 by setting

$$\lambda_5 \to -\lambda_4\,, \qquad \tilde{\lambda}_5\,, \to \tilde{\lambda}_4 \qquad \eta_5 = \eta_4\,. \tag{4.35}$$

Analogously to the previous case, only some $R$-invariants, namely

$R'_{135} =$  ,  $R'_{145} =$  ,

$R'_{155} =$  ,  $R''_{135} =$  ,  (4.36)

give a non-vanishing contribution after the fermionic integration,

$$\begin{aligned}
\lim_{p_5\to -p_4} \int \mathrm{d}^4\eta_4\, R'_{135} &= \frac{\delta^{(8)}(\mathrm{q})\,[1\,2]^2}{(p_{12}-p_4)^2\, p_{14}^2\,[4|q|3\rangle\langle 3\,4\rangle}\,,\\
\lim_{p_5\to -p_4} \int \mathrm{d}^4\eta_4\, R'_{145} &= -\frac{\delta^{(8)}(\mathrm{q})\,q^4}{(q-p_4)^2\langle 1\,2\rangle\langle 2\,3\rangle[4|q|3\rangle[4|q|4\rangle\langle 1\,4\rangle}\,,\\
\lim_{p_5\to -p_4} \int \mathrm{d}^4\eta_4\, R'_{155} &= -\frac{\delta^{(8)}(\mathrm{q})\,q^4}{(q+p_4)^2\langle 1\,2\rangle\langle 2\,3\rangle[4|q|1\rangle[4|q|4\rangle\langle 3\,4\rangle}\,,\\
\lim_{p_5\to -p_4} \int \mathrm{d}^4\eta_4\, R''_{135} &= \frac{\delta^{(8)}(\mathrm{q})\,[2\,3]^2}{(p_{23}+p_4)^2\, p_{34}^2\,[4|q|1\rangle\langle 4\,1\rangle}\,.
\end{aligned} \tag{4.37}$$

As usual, the result of BCFW recursion relations contains spurious poles. By making use of the kinematic identities

$$\begin{aligned}
\langle 2\,4\rangle[4|q|3\rangle[3\,2] &= s_{24}s_{23} + \tfrac{1}{2}(s_{13}s_{24} - s_{12}s_{34} + s_{14}s_{23})\,,\\
\langle 1\,4\rangle[4|q|3\rangle[3\,1] &= s_{14}s_{13} + \tfrac{1}{2}(s_{13}s_{24} - s_{12}s_{34} + s_{14}s_{23})\,,\\
\langle 2\,4\rangle[4|q|1\rangle[1\,2] &= s_{12}s_{24} + \tfrac{1}{2}(s_{12}s_{34} - s_{14}s_{23} + s_{13}s_{24})\,,\\
\langle 3\,4\rangle[4|q|1\rangle[1\,2] &= s_{13}s_{34} + \tfrac{1}{2}(s_{12}s_{34} - s_{14}s_{23} + s_{13}s_{24})\,,
\end{aligned} \tag{4.38}$$

and after some partial fractioning, we can write the sum of the four terms above as

$$\begin{aligned}
\boldsymbol{\mathcal{I}}_{\text{forw}} = -\frac{F^{(0)}_{3,0}}{2}\Bigg[ &\frac{s_{12}s_{13}}{s_{1,2,-4}s_{1,-4}s_{34}} + \frac{s_{23}s_{12}}{s_{1,-4}s_{1,2,-4}s_{1,2,3,-4}} + \frac{s_{23}s_{13}}{s_{2,3,4}s_{1,-4}s_{34}} + \frac{s_{23}s_{12}}{s_{34}s_{2,3,4}s_{1,2,3,4}}\\
&+\frac{s_{13}+s_{23}}{s_{34}s_{1,2,-4}} + \frac{s_{12}+s_{13}}{s_{1,-4}s_{1,2,3,-4}} + \frac{s_{13}+s_{23}}{s_{1,2,-4}s_{1,2,3,-4}} + \frac{s_{13}+s_{12}}{s_{1,-4}s_{2,3,4}} + \frac{s_{23}+s_{13}}{s_{34}s_{1,2,3,4}} + \frac{s_{13}+s_{12}}{s_{2,3,4}s_{1,2,3,4}}\Bigg]\,.
\end{aligned} \tag{4.39}$$

If we now identify $x_5 = x_0^-$, $x_4 = x_1^-$ and perform the shift $x_1 \mapsto \hat{x}_1$, i.e. if we set

$$\begin{aligned} s_{1,-4} &= x_{02}^2\,, & s_{1,2,-4} &= x_{03}^2\,, & s_{1,2,3,-4} &= (\hat{x}_{01}^+)^2\,,\\ s_{34} &= (x_{03}^-)^2\,, & s_{2,3,4} &= (x_{02}^-)^2\,, & s_{1,2,3,4} &= (\hat{x}_{01}^-)^2\,,\\ s_{12} &= \hat{x}_{13}^2\,, & s_{23} &= (\hat{x}_{21}^+)^2\,, & s_{13} &= (x_{32}^+)^2\;, \end{aligned} \tag{4.40}$$

we arrive at

$$\boldsymbol{\mathcal{I}}_{\text{cut}} = \frac{1}{x_{01}^2}\,\hat{\boldsymbol{\mathcal{I}}}_{\text{forw}}\,. \tag{4.41}$$

The complete integrand is then obtained by including the contribution from the first line in (4.12), where the corresponding residue is due to the overall tree-level MHV form factor $F_{3,0}^{(0)}$ which leads to the factorisation depicted in (4.13).

# 5. Dual conformal symmetry

In this chapter, we extend the notion of dual conformal symmetry to form factors of the stress tensor multiplet operator in $\mathcal{N}=4$ sYM. Form factors of half-BPS operator are by now very well studied, both at weak [11, 12, 24, 75, 87–92] and strong coupling [84, 85]. The extension to form factors of the on-shell diagram formalism and their formulation in terms of twistor variables, exhibiting an underlying Grassmannian geometry, have also been studied [1, 34–37, 74, 83]. Yet, despite the availability of many perturbative results, the dual conformal symmetry properties of form factors of protected operators have not yet been investigated (see [93] for comments regarding the $q^2 = 0$ case). One reason why this question was not addressed is the presence of triangle integrals in the expressions for one-loop form factors.

Triangles, unlike boxes, are expected to break dual conformal invariance explicitly, as one can see easily. Consider first a one-loop box integral in dual variables, which is given by

$$I_4 \propto \int \mathrm{d}^4 x_0 \, \frac{1}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} \; . \tag{5.1}$$

Performing an inversion $x_i \to x_i/x_i^2$ and a compensating change of variables $x_0 \to x_0/x_0^2$ (which implies $\mathrm{d}^4x_0 \to \mathrm{d}^4x_0/x_0^8$) one gets

$$I_4 \to I_4 \, x_1^2 \, x_2^2 \, x_3^2 \, x_4^2 \; , \tag{5.2}$$

which can be compensated by a numerator $x_{13}^2 x_{24}^2$. This is not possible for the triangle integral

$$I_3 \propto \int \mathrm{d}^4 x_0 \, \frac{1}{x_{01}^2 x_{02}^2 x_{03}^2} \; , \tag{5.3}$$

whose integrand variation depends explicitly on the loop variable $x_0$, preventing a covariant transformation. This led to the expectation that any quantity involving triangle integrals cannot be dual conformal invariant. We will show in the following that this expectation is naive, and our careful analysis of the form factors at tree (one-loop) level will reveal the presence of (anomalous) dual conformal symmetry in complete analogy to the case of amplitudes. We will start from tree level, where dual conformal invariance descends from the invariance of certain $R$-invariants appearing in tree-level form factors. We then move to one loop, where we present a derivation of the dual conformal anomaly along the lines of [53], and importantly also explicitly check the dual conformal anomaly for the MHV and NMHV cases.

A key aspect of our investigation is the appropriate assignment of region variables for form factors introduced in Chapter 4. For the case of scattering amplitudes, the sum of external on-shell momenta vanishes and dual momenta are the vertices of a light-like polygon. For form factors, the presence of the operator insertion leads one to consider a periodic configuration of region variables [**11**, **84**]. In the following, we will recast the prescription for the assignement of region variables in terms of momentum twistors. Note however that special conformal transformations do not preserve distances, and consequently do not preserve periodicity under translations. In general, a periodic configuration of the dual variables is invariant under a discrete translation by a period $q$. We denote by $\mathbb{P}$ the action of such a translation. After a dual special conformal transformation $\boldsymbol{\mathcal{K}}$, the configuration will be invariant under the action of twisted periodicity

$$\tilde{\mathbb{P}} = \boldsymbol{\mathcal{K}} \cdot \mathbb{P} \cdot \boldsymbol{\mathcal{K}}^{-1} \,. \tag{5.4}$$

This subtlety was already noticed in [**58**], where the authors looked at double-trace scattering amplitudes and argued that the original Wilson line correlator and the twisted one correspond to the same scattering amplitude. Here we find a very similar picture: a dual conformal transformation maps a configuration of region variables, which is periodic under translations, to a configuration that obeys twisted periodicity; nevertheless, we will show that this does not change the final result of the tree-level form factor (or to be more precise the appropriate ratio), and at one loop induces an anomaly that is completely analogous to that of amplitudes.

The rest of the chapter is organised as follows. In Section 5.1, we review the tree-level results of [**74**], with a particular focus on dual conformal symmetry, made manifest by the formulation in terms of twistor variables. In Section 5.2, we provide a unitarity-based derivation of the anomalous dual conformal symmetry at one loop. We then test our findings in Section 5.3, where we show explicitly that MHV and NMHV one-loop form factors obey the same anomalous dual-conformal Ward identity as amplitudes. Several technical details and definitions are included in four appendices.

## 5.1 Dual conformal symmetry at tree level

As for the case of scattering amplitudes, it is convenient to analyse the properties of the ratio

$$\tilde{F}^{(0)}_{n,k} = \frac{F^{(0)}_{n,k}}{F^{(0)}_{n,0}} \,. \tag{5.5}$$

We will show that the ratio $\tilde{F}^{(0)}_{n,k}$ is invariant under dual conformal transformations. This feature was already mentioned in [**74**], and here we review some of the results of that paper, focusing on the properties under dual conformal transformations.

An important difference between the amplitude and the form factor computation is that there is no momentum conservation for the external legs, i.e.

$$\sum_{i=1}^{n} p_i^{\alpha\dot{\alpha}} = q^{\alpha\dot{\alpha}} \,, \qquad\qquad \sum_{i=1}^{n} \mathrm{q}_i^{A\alpha} = \gamma^{A\alpha} \,, \tag{5.6}$$

and

$$p_i^{\alpha\dot{\alpha}} = \lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}} \,, \qquad\qquad \mathrm{q}_i^{A\alpha} = \eta_i^{A}\lambda_i^{\alpha} \,. \tag{5.7}$$

Consequently, region supermomenta are defined on a periodic contour [11, 12, 84]

$$x_i^{\alpha\dot{\alpha}} \sim x_i^{\alpha\dot{\alpha}} + mq^{\alpha\dot{\alpha}} \,, \qquad\qquad \theta_i^{A\alpha} \sim \theta_i^{A\alpha} + m\gamma^{A\alpha} \,, \tag{5.8}$$

for $m \in \mathbb{Z}$. This introduces a redundancy in the assignment of dual variables and one has to establish a consistent convention. This issue was already discussed in [2, 74]. Here, we follow the convention of [2], which can be summarised as follows. We choose one particular period, whose points are called $(x_i, \theta_i)$. Image points belonging to the other periods are indicated using the notation

$$x_i^{[m]} = x_i + m\, q \,, \qquad\qquad \theta_i^{[m]} = \theta_i + m\, \gamma \,. \tag{5.9}$$

For the specific case $m = \pm 1$, we also use $x_i^{\pm} = x_i \pm q$ and $\theta_i^{\pm} = \theta_i \pm \gamma$. Notice that, for any $m \in \mathbb{Z}$,

$$p_i = x_i^{[m]} - x_{i+1}^{[m]} \,, \qquad\qquad \mathrm{q}_i = \theta_i^{[m]} - \theta_{i+1}^{[m]} \,. \tag{5.10}$$

As done in Chapter 4, we use the position of the off-shell leg to start assigning region momenta and we ask that the first region we encounter always sits in the particular period we selected (i.e. that with regions $x_i$). In the case of $R$-invariants, it is easy to understand how this works looking at Figure 5.1, where we selected two specific $R$-invariants with $r = 1$, and we assigned region variables accordingly. In Section 5.2 we will use the same prescription for the case of one-loop form factors.

It should be clear that this is just one specific choice, we may well choose any other period but the result for any $R$-invariant would be unchanged. We stress that, as discussed in the Introduction, dual special conformal transformations act differently for different periods, and this causes ambiguities in the action on an MHV prefactor – which is why we prefer to divide it out and work with quantities written in the form of $R$-invariants (see Section 5.2 for a discussion of the loop level case), and translating them in twistor variables as was done in [74]. Also twistor variables are arranged in periodic configurations

$$\mathcal{Z}_i^{[m]M} = \begin{pmatrix} Z_i^{[m]\hat{A}} \\ \chi_i^{[m]A} \end{pmatrix} , \qquad Z_i^{[m]\hat{A}} = \begin{pmatrix} \lambda_i^{\alpha} \\ (x_i^{[m]})^{\dot{\alpha}\alpha}\lambda_{i\alpha} \end{pmatrix} , \qquad \chi_i^{[m]A} = (\theta_i^{[m]})^{A\alpha}\lambda_{i\alpha} \,, \tag{5.11}$$

but this does not affect the invariance of (2.218), which holds for five arbitrary twistors. This implies that whenever a result can be written in terms of five-brackets (2.218), it is automatically invariant. Notice also that under rescaling, for any $m \in \mathbb{Z}$,

$$\mathcal{Z}_i^{[m]M} \to \zeta_i\, \mathcal{Z}_i^{[m]M} \,. \tag{5.12}$$

This can be understood by thinking of the rescaling as a freedom in the definition of $\lambda_i$. Since $\lambda_i$ is not affected by the shifts (5.11), all the image twistors should be rescaled by the same factor.

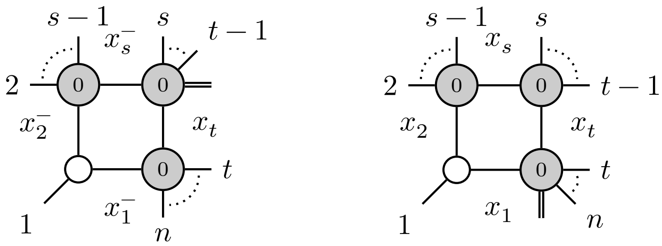
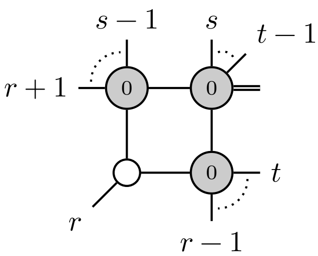
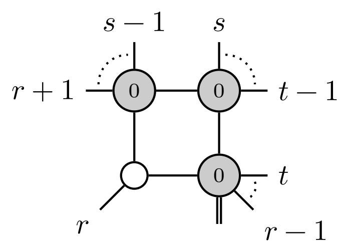



Figure 5.1: Examples of region variables assignment for two $R$-invariants. We label region momenta starting from the region adjacent to the corner containing the off-shell leg in clockwise order.

In [74] it was shown that two different configurations are needed to compute the NMHV form factor. They are represented by

$R'_{rst} =$  , $\quad R''_{rst} =$  , (5.13)

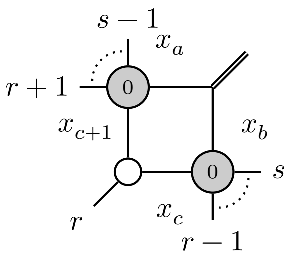


and the expression of the $n$-point NMHV form factor is

$$\tilde{F}^{(0)}_{n,1} \;=\; \sum_{j=3}^{n}\sum_{i=3}^{j} R'_{1ij} + \sum_{j=5}^{n+1}\sum_{i=3}^{j-2} R''_{1ij} \,, \tag{5.14}$$

where the sum is performed with a periodic identification $n{+}1 \sim 1$. This expression was derived using a $[1\,2\rangle$ shift, and as a consequence all of the $R$-invariants involved have $r = 1$, and one can simply use the region momenta assignment shown in Figure 5.1. Using BCFW recursion relations it is possible to show that, for arbitrary helicity configuration, the tree-level form factor can be written in terms of $R'$ and $R''$. Therefore, one simply needs to show that these two functions are dual conformal invariant.

It turns out that, for $s \neq t$, $R'$ and $R''$ are given by (2.165), with the region variables assignment described below (5.10) (see also Figure 5.1). There is however a limiting case that needs to be discussed separately. For the specific configuration $R'_{rss}$, (2.165) does not apply and one has instead

$R'_{rss} =$ 


$$= -\frac{\langle s-1\, s\rangle\, \delta^{(4)}(\langle r|x_{ca}x_{ab}|\theta_{bc}\rangle + \langle r|x_{cb}x_{ba}|\theta_{ac}\rangle)}{x_{ab}^4 \langle r|x_{cb}x_{ba}|s-1\rangle\langle r|x_{ca}x_{ab}|s\rangle\langle r|x_{ca}x_{bc}|r\rangle}\ . \tag{5.15}$$

Notice that in this case $x_a = x_b^-$ and $x_{ab} = -q$. Taking the ratio with the limiting case of (2.165), one can rewrite (5.15) as

$$R'_{rss} = -\frac{\langle r|x_{ca}x_{ab}|s-1\rangle\langle r|x_{cb}x_{ba}|s\rangle}{x_{ab}^2\langle s-1\, s\rangle\langle r|x_{ca}x_{bc}|r\rangle}\,[(s-1)^-, s^-, s-1, s, r]\ . \tag{5.16}$$

As was shown in [**74**], the prefactor in (5.16) can be written as a ratio of four-brackets (2.216). Since the four-bracket (2.216) is invariant under dual conformal transformations, once the prefactor is written in that form, we just need to check that it is also invariant under the little group scaling (5.12). To this end, we first note that one can recast $R'_{rss}$ as

$$R'_{rss} = \frac{\langle r, (s-1)^-, s^-, s-1\rangle\langle r, s-1, s, s^-\rangle}{\langle r^+, s-1, s, r\rangle\langle s, s^-, s-1, (s-1)^-\rangle}\,[(s-1)^-, s^-, s-1, s, r]\ . \tag{5.17}$$

The novel feature of (5.17) is that the prefactor contains brackets involving one region variable as well as its image after one period. To see how this happens consider the expression $\langle r|x_{ca}x_{bc}|r\rangle$, which can be rewritten as

$$\langle r|x_{ca}x_{bc}|r\rangle = \langle r|(x_c^+ - x_b)x_{bc}|r\rangle = \frac{\langle r^+, s-1, s, r\rangle}{\langle s-1\, s\rangle}\ . \tag{5.18}$$

Notice also that, by using a similar argument, it is easy to show that the four-bracket is invariant under an overall translation by a period:

$$\langle r^+, s-1, s, r\rangle = \langle r, (s-1)^-, s^-, r^-\rangle\ . \tag{5.19}$$

This is actually a trivial statement since we know that the four-bracket is invariant under the full dual conformal group and dual translations are just a subgroup. Furthermore, since the little group transformation (5.12) does not depend on the specific period, we conclude that $R'_{rss}$ is invariant under little group scaling, and consequently is a good dual conformal invariant.

## 5.2 One-loop anomaly: a general proof

In [**53**] a deep connection between IR divergences of one-loop scattering amplitudes and the dual conformal anomaly was established. The argument of [**53**] is based on the fact that only unitarity cuts in two-particle channels contribute to the discontinuity of the IR-divergent part of an amplitude. Therefore, in the multiparticle case, the phase space integration can be performed strictly in four dimensions, and dual conformal symmetry of the discontinuity essentially descends from the covariance of the tree-level ingredients. A careful analysis shows that the invariance of the discontinuity is sufficient to prove that no multiparticle invariant can be present in the dual conformal anomaly, confirming the structure previously conjectured in [**49**] (see [**53**] for additional details of this derivation).

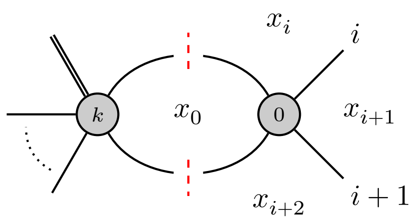



Figure 5.2: The only two-particle cut contributing to the IR divergent part of the form factor as well as to the dual conformal anomaly.

The argument can be extended to the case of form factors without any modification. Indeed, we know that the IR structure of the one-loop form factor is analogous to that of scattering amplitudes – it depends only on two-particle invariants (see (5.25)). Therefore, the IR behaviour of one-loop form factors should be fully reproduced by the two-particle cut in Figure 5.2, which reads

$$F^{(1)}_{n,k}\Big|_{x^2_{i,i+1}\text{cut}} = \int \mathrm{dLIPS}(\ell_1,\ell_2) \int \mathrm{d}^4\eta_{\ell_1}\, \mathrm{d}^4\eta_{\ell_2}\, A^{(0)}_{4,0}(i,i+1,\ell_2,\ell_1) F^{(0)}_{n,k}(-\ell_1,-\ell_2,i+2,\dots,i-1)\ . \tag{5.20}$$

The integration over fermionic variables can be immediately performed using the fermionic delta function of $A_{4,0}$, yielding

$$F^{(1)}_{n,k}\Big|_{x^2_{i,i+1}\text{cut}} = \int \mathrm{dLIPS}(\ell_1,\ell_2)\ \frac{\langle \ell_1\ell_2\rangle^3 F^{(0)}_{n,k}(-\ell_1,-\ell_2,i+2,\dots,i-1)}{\langle i,i+1\rangle\langle i+1,\ell_2\rangle\langle \ell_1,i\rangle}\ . \tag{5.21}$$

Furthermore, using some spinor variable manipulations, we can rewrite (5.21) as

$$F^{(1)}_{n,k}\Big|_{x^2_{i,i+1}\text{cut}} = \int \mathrm{dLIPS}(\ell_1,\ell_2)\ \frac{\langle \ell_1\ell_2\rangle^2 F^{(0)}_{n,k}(-\ell_1,-\ell_2,i+2,\dots,i-1)}{\langle i,i+1\rangle^2}\,\frac{x^2_{i,i+2}}{x^2_{0,i+1}}\ . \tag{5.22}$$

The crucial observation is that the IR-singular region of this integral is related to the collinear kinematic configuration

$$\ell_1 = -p_i\ , \qquad \ell_2 = -p_{i+1}\ , \qquad x_0 = x_{i+1}\ . \tag{5.23}$$

The divergence in the integral (5.22) is clearly related to the propagator $x^2_{0,i+1}$. The rest of the integrand can be evaluated in the configuration (5.23), and the cut can be uplifted to the corresponding integral, leading to

$$F^{(1)}_{n,k}\Big|_{\text{IR}} = F^{(0)}_{n,k}\int \mathrm{d}^d x_0\ \frac{x^2_{i,i+2}}{x^2_{0i}x^2_{0,i+1}x^2_{0,i+2}}\ , \tag{5.24}$$

which evaluates to

$$F_{n,k}^{(1)}\Big|_{\mathrm{IR}} = -F_{n,k}^{(0)} \sum_{i=1}^{n} \frac{(-x_{ii+2}^2)^{-\epsilon}}{\epsilon^2} \ . \tag{5.25}$$

This reproduces the correct IR behaviour of the form factor.

The argument used in [53] to relate the IR behaviour of scattering amplitudes to the expression of the dual conformal anomaly is based on the idea of applying a dual conformal transformation in the very first step of the above derivation, i.e. on the two-particle cut. The covariance of the tree-level ingredients allows to show that the anomaly is related to the variation of the integration measure, which needs to be $d$-dimensional since the integral diverges (all the other two-particle cuts are finite and do not contribute to the anomaly). In particular, using the definition of the generator of dual special conformal transformations

$$\mathsf{K}^{\mu} = \sum_{i=1}^{n} \left[ -2x_i^{\mu} x_i^{\nu} \frac{\partial}{\partial x_i^{\nu}} + x_i^2 \frac{\partial}{\partial x_{i\,\mu}} \right] \ , \tag{5.26}$$

the fact, proven in the previous section, that tree-level form factors transform covariantly, and following steps similar to those of [53], we arrive at

$$\mathsf{K}^{\mu} F_{n,k}^{(1)}\Big|_{x_{i,i+1}^2 \mathrm{cut}} = 4\epsilon \int \mathrm{dLIPS}(\ell_1,\ell_2) \int \mathrm{d}^4\eta_{\ell_1}\ \mathrm{d}^4\eta_{\ell_2}\ x_0^{\mu}\ A_{4,0}^{(0)}(i,i+1,\ell_2,\ell_1)\ F_{n,k}^{(0)}(-\ell_1,-\ell_2,\dots)\,, \tag{5.27}$$

with $\epsilon = 2 - d/2$. After this observation we can simply follow all the steps leading to (5.25), and hence we conclude that the one-loop anomaly has the form

$$\mathsf{K}^{\mu} \tilde{F}_{n,k}^{(1)} = -4\,\tilde{F}_{n,k}^{(0)} \sum_{i=1}^{n} \frac{x_{i+1}^{\mu}(-x_{ii+2}^2)^{-\epsilon}}{\epsilon} \ . \tag{5.28}$$

Note that the right-hand side of (5.28) depends on the region momenta of the particles (and not just the momenta).

Although the form of the anomaly resembles that of the amplitude case, the consequences for the one-loop expansion of the form factor in terms of scalar integrals are rather different. Indeed, one-loop form factors may contain three-mass triangles, which are finite in four dimensions and, in view of the previous arguments, cannot contribute to the anomaly. On the other hand, we showed at the beginning of this section that triangle integrals cannot be dual conformal invariant on their own. Therefore, two things can happen: either the variation of the finite triangles cancels some other variation arising from the finite part of other integrals (in this case boxes); or the variation vanishes after summing over permutations. Notice also that, in the NMHV example, the three-mass triangle comes with a complicated coefficient, and its variation needs to be taken into account as well (see Section 5.3.2).

To understand how the anomaly emerges, in the following we will explicitly check its form for MHV and NMHV form factors at one loop. Before doing that, we first elaborate on the consequences of (5.28) for the finite part of one-loop form factors. The universal IR-divergent part of a generic one-loop form factor has the form (5.25). Using

$$\mathsf{K}^\mu x_{ab}^2 = -2(x_a + x_b)^\mu \, x_{ab}^2 \ , \tag{5.29}$$

we can separate out the anomaly of the finite part. Doing so, one quickly arrives at

$$\begin{aligned}\mathsf{K}^\mu \ \tilde{F}^{(1)}_{n,k}\Big|_{\text{fin}} = -\tilde{F}^{(0)}_{n,k} \Bigg[ \frac{2}{\epsilon} \sum_{i=1}^{n} \Big( 2x^\mu_{i+1} - (x^\mu_i + x^\mu_{i+2}) \Big) \\ - 2 \sum_{i=1}^{n} \Big( 2x^\mu_{i+1} - (x^\mu_i + x^\mu_{i+2}) \Big) \log\left(-x^2_{i i+2}\right) \Bigg] \ .\end{aligned} \tag{5.30}$$

The first sum evaluates to zero, thus we obtain

$$\mathsf{K}^\mu \ \tilde{F}^{(1)}_{n,k}\Big|_{\text{fin}} = -2 \, \tilde{F}^{(0)}_{n,k} \sum_{i=1}^{n} p^\mu_i \log\left( \frac{x^2_{i i+2}}{x^2_{i-1\, i+1}} \right) \ , \tag{5.31}$$

which, importantly, only depends on differences of region momenta (i.e. momenta) and Mandelstam invariants of the particles. We now show the validity of this formula for the MHV and NMHV form factor at one loop.

## 5.3 Examples

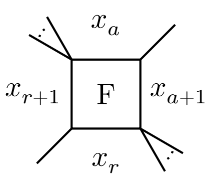
Having presented a general derivation of the dual conformal anomaly, we now analyse a number of specific examples, namely the one-loop MHV and NMHV form factors. The latter are particularly interesting due to the presence of a three-mass triangle, whose variation requires a novel cancellation mechanism to be consistent with our general result (5.28) and (5.31).

There is an important preliminary observation to be made – in order to find the correct anomaly, it is crucial to assign region variables according to the prescription described in Section 5.1 and illustrated in Figure 5.1. In particular, this has to be done diagram by diagram in the expansion of the result in terms of scalar integrals; crucially, the definition of the period $q$ in terms of region variables, and consequently its variation under special conformal transformations, is different for each of the diagrams involved in the computation. Let us now see how this works in practice.

### 5.3.1 *One-loop n-point MHV form factor*

The generic one-loop MHV super form factor can be written compactly as [11]:

$$F^{(1)}_{n,0} = F^{(0)}_{n,0} \left( -\sum_{i=1}^{n} \frac{(-x^2_{i\,i+2})^{-\epsilon}}{\epsilon^2} + \sum_{r,a} \right. $$




$$\left. \vphantom{\sum_{i=1}^{n}} \right) \ . \tag{5.32}$$

where the label “F” inside the box indicates the finite part of the reduced box integral (D.4). The sum is over all possible boxes; the off-shell leg can appear in both massive corners of the box function. The recipe to write the previous expression in terms of region variables depends as usual on the position of the off-shell legs, and an example is shown in Figure 5.3. In that case the leg with momentum $p_1$ is associated to one of the massless legs and the region variables are assigned according to the two possible locations of the off-shell leg. A similar recipe can be applied to the other cyclic permutations.

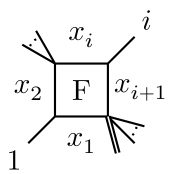
In the following we will act with dual conformal generators on the finite part of a generic one-loop MHV super form factor. We will use the following two general formulae, obtained as repeated applications of (5.29):

$$\mathsf{K}^\mu \,\mathrm{Li}_2\left(1-\frac{x_{ab}^2}{x_{ac}^2}\right) = 2x_{ab}^2\,\frac{\log(x_{ab}^2/x_{ac}^2)}{x_{ab}^2-x_{ac}^2}\,x_{bc}^\mu\,, \tag{5.33}$$

$$\mathsf{K}^\mu \frac{1}{2}\log^2\left(\frac{x_{ab}^2}{x_{a+1\,b+1}^2}\right) = -2\log\left(\frac{x_{ab}^2}{x_{a+1\,b+1}^2}\right)\left(x_{a\,a+1}^\mu + x_{b\,b+1}^\mu\right)\,. \tag{5.34}$$

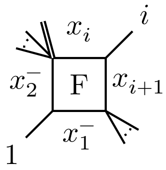

Figure 5.3: The two possible types of two-mass easy box functions. The double line represents the incoming momentum $q$ of the operator. Note the two different assignments of region momenta in the two cases.

Without loss of generality, we will now compute the term in the anomaly of the finite part of the $n$-point MHV form factor that is proportional to the momentum $p_1$. It is easy to realise that such terms can only arise from box functions where $p_1$ is one of the two massless legs. To perform the calculation we need to distinguish terms where the form factor momentum is inserted in the two possible massive corners of a two-mass easy box. These two situations are depicted in Figure 5.3. The term proportional to $p_1$ in the variation of the first type of box gives

$\mathsf{K}^\mu$ 


$$\sim -2p_1^\mu\left[\frac{x_{i+1\,1}^2}{x_{i+1\,1}^2-x_{i+1\,2}^2}\log\frac{x_{i+1\,1}^2}{x_{i+1\,2}^2} - \frac{x_{2\,i}^2}{x_{2\,i}^2-x_{1\,i}^2}\log\frac{x_{2\,i}^2}{x_{1\,i}^2} - \log\frac{x_{1\,i}^2}{x_{2\,i+1}^2}\right]\,, \tag{5.35}$$

while for the second type of box we have

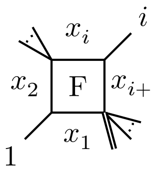
$\mathsf{K}^\mu$ 

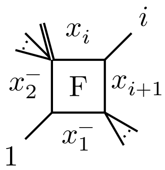

$$\sim -2p_1^\mu \left[ \frac{(x^+_{i+1,1})^2}{(x^+_{i+1,1})^2-(x^+_{i+1,2})^2} \log \frac{(x^+_{i+1,1})^2}{(x^+_{i+1,2})^2} - \frac{(x^-_{2i})^2}{(x^-_{2i})^2-(x^-_{1i})^2} \log \frac{(x^-_{2i})^2}{(x^-_{1i})^2} - \log \frac{(x^-_{1i})^2}{(x^-_{2i+1})^2} \right] . \tag{5.36}$$

Combining the variations and performing the sums

$$-\sum_{i=2}^{n-2} \log \frac{(x^+_{i+1,1})^2}{(x^+_{i+1,2})^2} + \sum_{i=3}^{n-1} \log \frac{x^2_{i+1,1}}{x^2_{i+1,2}} + \sum_{i=2}^{n-1} \log \frac{(x^-_{1i})^2}{(x^-_{2,i+1})^2} + \sum_{i=3}^{n} \log \frac{x^2_{1i}}{x^2_{2,i+1}} , \tag{5.37}$$

we obtain

$\sum_i \mathsf{K}^\mu \Bigg($ 

$+$ 


$$\Bigg) \sim -2\, p_1^\mu \log \left( \frac{x^2_{13}}{(x^-_{2\,n})^2} \right) , \tag{5.38}$$

in agreement with the term proportional to $p_1$ on the right-hand side of (5.31). Summarising, we have shown that the finite part of the dual conformal anomaly of an $n$-point MHV form factor is exactly reproduced by our general formula (5.31). Next, we move on to consider NMHV form factors.

### *5.3.2 One-loop NMHV form factor*

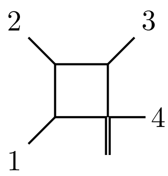
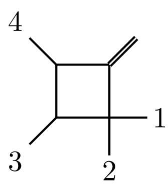
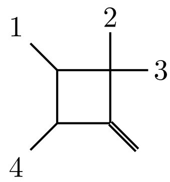
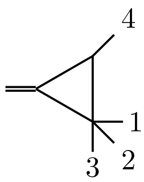
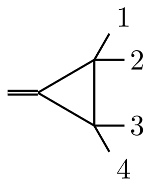
The one-loop NMHV form factor can be computed using generalised unitarity as a combination of boxes and triangles [**75**]. The presence of the latter constitutes an important difference compared to amplitudes. In particular, for amplitudes the box integrals are invariant on their own[1], and in addition their coefficients are invariant as well.

For form factors one may expect dual conformal symmetry to be broken. However, in the following we will discover a new cancellation mechanism that ensures that the final result is invariant up to the expected anomaly. The three-point NMHV form factor coincides with the $\overline{\text{MHV}}$ result, and therefore can be extracted from the MHV case considered earlier by conjugation (this is analogous to the case of the five-point amplitude). The first interesting case is that of a four-point NMHV form factor, as this is the first example which has a three-mass triangle. Since two-mass and one-mass triangles are IR divergent with vanishing finite part, their coefficient can be fixed by requiring a consistent divergent part for the final form

[1]To be precise they are anomalous as we will discuss in Section 5.3.2.

factor, i.e. (5.25). On the other hand, the three-mass triangle is finite, and its coefficient has to be determined independently.

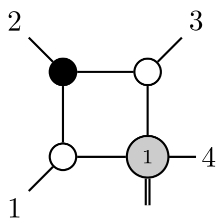
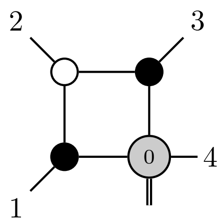
We start by writing $\tilde{F}^{(1)}_{4,1}$ as a linear combination of reduced scalar integrals:

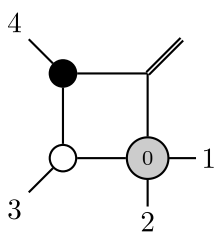
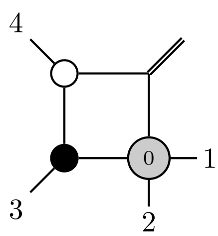
$$\tilde{F}^{(1)}_{4,1} = b^{1\mathrm{m}}$$

$$+\, b_1^{2\mathrm{mh}}$$

$$+\, b_2^{2\mathrm{mh}}$$

$$+\, c^{2\mathrm{m}}$$

$$+\, c^{3\mathrm{m}}$$

$$+\,\text{cyclic}\,, \tag{5.39}$$

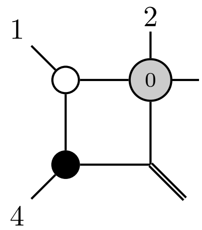
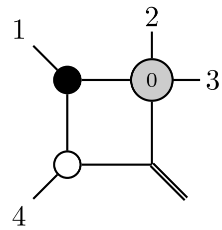
where the sum is performed over cyclic permutations of the external legs. Notice that the dependence of the coefficients on the external momenta is understood and must be permuted accordingly. This is an expansion in terms of reduced scalar integrals, i.e. where a dimensionful constant in the integral has been reabsorbed in the coefficient (see Appendix D for details). The coefficients of this linear combination have been determined in [75]. Here we review that derivation and consider the transformation of the result under dual conformal symmetry. We start by the contribution of boxes and divergent triangles.

### Boxes and divergent triangles

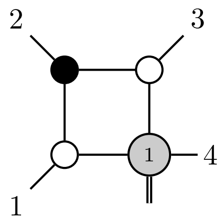
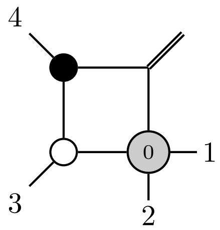
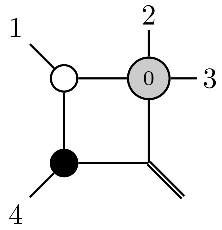
The contribution of boxes is easily computed using the maximal cuts. Each of the diagrams receives a contribution from two different cuts. In particular

$$b^{1\mathrm{m}} = \frac{1}{2}$$

$$+\,\frac{1}{2}$$

$$\tag{5.40}$$

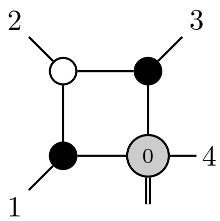
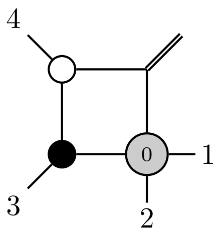
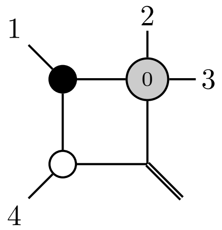
$$b_1^{2\mathrm{mh}} = \frac{1}{2}$$

$$+\,\frac{1}{2}$$

$$\tag{5.41}$$

$$b_2^{\rm 2mh} = \frac{1}{2}$$  $$+\frac{1}{2}$$  (5.42)

However, using the non-trivial identities [2, 75]

 $=$  $=$  $$= R'_{144} = R'_{311}\,, \qquad (5.43)$$

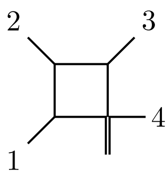
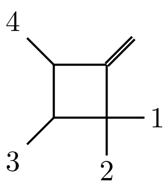
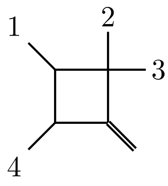
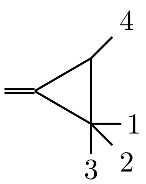
and

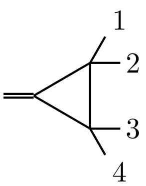
 $=$  $=$  $$= R'_{241} = R''_{424}\,, \qquad (5.44)$$

and noticing that, by IR consistency, $c^{\rm 2m}$ is fixed to

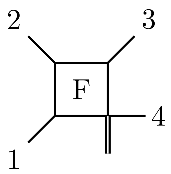
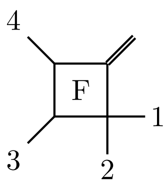
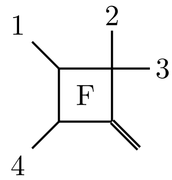
$$c^{\rm 2m} = R'_{144} + R'_{241}\,, \qquad (5.45)$$

we arrive at the following compact expression for the NMHV four-point form factor:

$$\tilde{F}^{(1)}_{4,1} = \frac{c^{\rm 2m}}{2}\Bigg($$  $+$  $+$  $+\,2$  $\Bigg)$

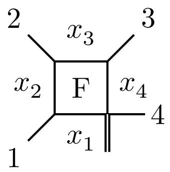
$+\, c^{\rm 3m}$  $+\,\text{cyclic}\,.$ (5.46)

We focus here on the first line of (5.46), and compute its variation under dual conformal transformations, while the three-mass triangle is discussed later. The overall coefficient $c^{\rm 2m}$ is expressed in terms of $R$-invariants (see (5.45)) and therefore is explicitly dual conformal invariant as shown in Section 5.1. Furthermore, in light of (5.31), we are interested in the finite part of the result and we can neglect the two-mass triangles, which are purely divergent. We then look at the particular combination

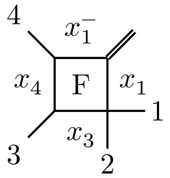
$V =$  $+$  $+$  , (5.47)

where again the letter F indicates the finite part of the integral.

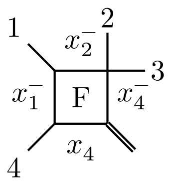
The variation of the scalar box integrals can be computed in two different ways: either one takes the variation of the integrands and then uses some reduction techniques to recast the result in terms of scalar triangles as was done in [51], or one just takes the variation of the finite part of the integrated result (explicit expressions can be found in Appendix D). Either way, the result is

$\mathsf{K}^\mu$ 

$$= 2p_1^\mu\left(\frac{x_{14}^2}{x_{14}^2-x_{24}^2}\log\frac{x_{14}^2}{x_{13}^2}+\frac{x_{24}^2}{x_{14}^2-x_{24}^2}\log\frac{x_{13}^2}{x_{24}^2}\right) + 2p_3^\mu\left(\frac{x_{13}^2}{x_{13}^2-x_{14}^2}\log\frac{x_{24}^2}{x_{13}^2}+\frac{x_{14}^2}{x_{13}^2-x_{14}^2}\log\frac{x_{14}^2}{x_{24}^2}\right)\,, \tag{5.48}$$

$\mathsf{K}^\mu$ 

$$= -(p_1^\mu+p_2^\mu)\log\frac{(x_{13}^-)^2}{x_{13}^2} - q^\mu\log\frac{(x_{13}^-)^2}{q^2} + 2(p_1^\mu+p_2^\mu+p_4^\mu)\log\frac{(x_{13}^-)^2}{x_{14}^2} + 2p_3^\mu\frac{x_{13}^2}{x_{13}^2-x_{14}^2}\log\frac{x_{13}^2}{x_{14}^2} - 2p_4^\mu\frac{q^2}{q^2-x_{14}^2}\log\frac{q^2}{x_{14}^2}\,, \tag{5.49}$$

$\mathsf{K}^\mu$ 

$$= (p_2^\mu+p_3^\mu)\log\frac{(x_{24}^-)^2}{x_{24}^2} + q^\mu\log\frac{(x_{24}^-)^2}{q^2} - 2(p_2^\mu+p_3^\mu+p_4^\mu)\log\frac{(x_{24}^-)^2}{x_{14}^2} - 2p_1^\mu\frac{x_{24}^2}{x_{24}^2-x_{14}^2}\log\frac{x_{24}^2}{x_{14}^2} + 2p_4^\mu\frac{q^2}{q^2-x_{14}^2}\log\frac{q^2}{x_{14}^2}\,. \tag{5.50}$$

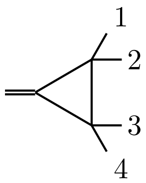
Notice that, in computing these variations, the correct assignment of region variables is essential. As in our previous examples, we start assigning region variables from the position of the off-shell leg and then follow the ordering along the periodic configuration. The variations above are then obtained by writing each integral using their particular region variable assignment, and acting with the generator $\mathsf{K}^\mu$ in (5.26). For the particular combination in (5.47), this gives

$$\mathsf{K}^\mu V = p_1^\mu \log \frac{(x_{24}^-)^2}{x_{13}^2} + p_2^\mu \log \frac{x_{13}^2}{x_{24}^2} + p_3^\mu \log \frac{x_{24}^2}{(x_{13}^-)^2} + p_4^\mu \log \frac{(x_{13}^-)^2}{(x_{24}^-)^2} \,. \tag{5.51}$$

This surprisingly simple combination is invariant under cyclic permutations. Therefore, using (5.46) we can write

$$\mathsf{K}^\mu \tilde{F}_{4,1}^{(1)}\Big|_{\text{fin}} = \frac{1}{2} \mathsf{K}^\mu V \sum_{\text{cyclic}} c^{2\text{m}} + \mathsf{K}^\mu T^{3\text{m}} \,, \tag{5.52}$$

where $T^{3\text{m}}$ is the contribution of the three-mass triangles

$$T^{3\text{m}} = c^{3\text{m}}$$  $$+ \,\text{cyclic}\,. \tag{5.53}$$


The sum over cyclic permutations of $c^{2\text{m}}$ reads

$$\sum_{\text{cyclic}} c^{2m} = R'_{144} + R'_{241} + R'_{211} + R'_{312} + R'_{322} + R'_{423} + R'_{433} + R'_{134} = 4\tilde{F}_{4,1}^{(0)} \,, \tag{5.54}$$

where for the last equality we used (5.14) combined with the identities (5.43), (5.44) and permutations thereof. Expressing (5.51) in terms of region variables we have

$$\mathsf{K}^\mu \tilde{F}_{4,1}^{(1)}\Big|_{\text{fin}} = -2\, \tilde{F}_{4,1}^{(0)} \sum_{i=1}^{4} p_i^\mu \log\left(\frac{x_{i\,i+2}^2}{x_{i-1\,i+1}^2}\right) + \mathsf{K}^\mu T^{3\text{m}} \,. \tag{5.55}$$

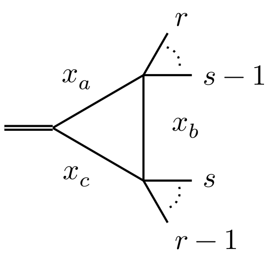
This result implies that the boxes already account for the full anomaly (5.31). As a consequence, the necessary and sufficient condition for dual conformal invariance is

$$\mathsf{K}^\mu T^{3\text{m}} = 0 \,. \tag{5.56}$$

We will check this surprising relation in the next section.

### Three-mass triangles

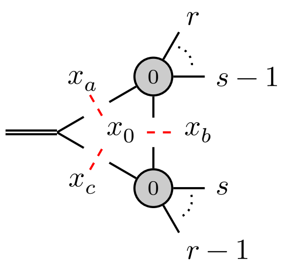
In this section we show that the contribution of the three-mass triangles is dual conformal invariant. We start by reviewing the computation of $c^{3\text{m}}$. This coefficient is harder than the boxes' since it requires looking at non-maximal cuts. Nevertheless, a prescription for the direct extraction of this coefficient was given in [**94**] and applied to the case of form factors in [**75**]. Let us consider the general configuration




(5.57)

which contains an arbitrary number of legs, but no external momentum in the massive corner containing the off-shell leg. In [75] it was shown that only this type of diagrams arise in the computation of the one-loop NMHV form factor. Here, we will show that this structure is crucial for the dual conformal invariance of the coefficient $c^{3\mathrm{m}}$, which would be spoiled by the presence of an external leg in the same corner of the off-shell leg. The four-point case can be immediately recovered by setting $r = 1$ and $s = 3$. Notice also that, for this particular configuration, $x_c = x_a^-$.

The starting point for the computation of $c^{3\mathrm{m}}$ is the triple cut




$$= \int \prod_{i=1}^{3} \mathrm{d}^4\eta_{\ell_i}\, F_{2,0}^{(0)}(-\ell_3, \ell_1)\, A_{n_1,0}^{(0)}(-\ell_1, \dots, \ell_2)\, A_{n_2,0}^{(0)}(-\ell_2, \dots, \ell_3) \tag{5.58}$$

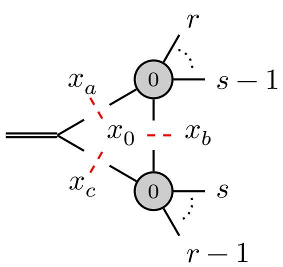
with

$$\ell_1 = x_{a0}\,, \qquad \ell_2 = x_{b0}\,, \qquad \ell_3 = x_{c0}\,. \tag{5.59}$$

The integration over the fermionic variables yields

$$\begin{aligned}
&\int \prod_{i=1}^{3} \mathrm{d}^4\eta_{\ell_i}\, \delta^8(\eta_{\ell_1}\lambda_{\ell_1} - \eta_{\ell_3}\lambda_{\ell_3} + \theta_{ca})\, \delta^8(\eta_{\ell_2}\lambda_{\ell_2} - \eta_{\ell_1}\lambda_{\ell_1} + \theta_{ab})\, \delta^8(\eta_{\ell_3}\lambda_{\ell_3} - \eta_{\ell_2}\lambda_{\ell_2} + \theta_{bc}) \\
&= \delta^8(\mathrm{q}_{\mathrm{tot}}) \int \prod_{i=1}^{3} \mathrm{d}^4\eta_{\ell_i}\, \delta^{(4)}(\langle \ell_1\, \ell_2\rangle \eta_{\ell_2} + \langle \ell_1\, \theta_{ab}\rangle)\, \delta^{(4)}(\langle \ell_1\, \ell_2\rangle \eta_{\ell_1} + \langle \ell_2\, \theta_{ab}\rangle)\, \frac{1}{\langle \ell_1\, \ell_2\rangle^4} \\
&\qquad\qquad \times\ \delta^{(4)}(\langle \ell_2\, \ell_3\rangle \eta_{\ell_3} + \langle \ell_2\, \theta_{bc}\rangle)\, \delta^{(4)}(\langle \ell_2\, \ell_3\rangle \eta_{\ell_2} + \langle \ell_3\, \theta_{bc}\rangle)\, \frac{1}{\langle \ell_2\, \ell_3\rangle^4} \\
&= \delta^8(\mathrm{q}_{\mathrm{tot}})\, \delta^{(4)}(\langle \ell_1\, \ell_2\rangle\langle \ell_3\, \theta_{bc}\rangle - \langle \ell_2\, \ell_3\rangle\langle \ell_1\, \theta_{ab}\rangle)\,.
\end{aligned} \tag{5.60}$$

After these manipulations the three-particle cut reads




$$= F_{n,0}^{(0)}\, \frac{\langle s-1\, s\rangle\langle r-1\, r\rangle\, \delta^{(4)}(\langle \ell_1\, \ell_2\rangle\langle \ell_3\, \theta_{bc}\rangle - \langle \ell_2\, \ell_3\rangle\langle \ell_1\, \theta_{ab}\rangle)}{\langle r\, \ell_1\rangle\langle s-1\, \ell_2\rangle\langle s\, \ell_2\rangle\langle r-1\, \ell_3\rangle\langle \ell_1\, \ell_2\rangle\langle \ell_2\, \ell_3\rangle\langle \ell_1\, \ell_3\rangle^2}\,, \tag{5.61}$$

and the associated coefficient is

$$c^{3\mathrm{m}} = \frac{\langle s-1\, s\rangle\langle r-1\, r\rangle\, \delta^{(4)}(\langle \ell_1\, \ell_2\rangle\langle \ell_3\, \theta_{bc}\rangle - \langle \ell_2\, \ell_3\rangle\langle \ell_1\, \theta_{ab}\rangle)}{\Delta_{abc}\langle r\, \ell_1\rangle\langle s-1\, \ell_2\rangle\langle s\, \ell_2\rangle\langle r-1\, \ell_3\rangle\langle \ell_1\, \ell_2\rangle\langle \ell_2\, \ell_3\rangle\langle \ell_1\, \ell_3\rangle^2}\,, \tag{5.62}$$

where $\Delta_{abc}$ is defined in (D.7) and originates from expanding the form factor in a basis of reduced triangles, see (D.6). A similar factor would appear for the case of boxes, but it always cancels after evaluating the quadruple cut on the corresponding solution. Here a similar cancellation does not seem to happen and we will have to deal with this additional factor. Furthermore, the MHV factor in (5.61) has been removed because the expansion (5.53) refers to the ratio $\tilde{F}^{(0)}_{4,1}$.

As usual, in (5.62) as well as in (5.58), the loop legs are evaluated on the solution of the on-shell conditions for the cut legs. Since the three-particle cut is not maximal in four dimensions, the on-shell constraints fix a one-parameter family of solutions and do not allow to fix immediately the coefficient of the three-mass triangle. Geometrically, this corresponds to a curve of allowed values for the internal region variable $x_0$. This is the curve of points that are light-like separated from the three points $x_a$, $x_b$ and $x_c$.

The construction of [**94**] showed that there is a particular value on this curve that isolates the triangle coefficient. Furthermore, since the constraint is quadratic, there are two solutions and, as dictated by generalised unitarity, one has to take an average. Details on this procedure are provided in Appendix E.1. To simplify the final result, it is convenient to introduce the variables

$$\frac{x_{ab}^2}{x_{ac}^2} = u = z\bar{z}\,, \qquad\qquad \frac{x_{bc}^2}{x_{ac}^2} = v = (1-z)(1-\bar{z})\,. \tag{5.63}$$

In terms of these variables, the coefficient of the triangle can be cast in the form

$$c^{3\mathrm{m}} = \frac{1}{\Delta_{abc}}\left[\frac{\langle r-1\,r\rangle\langle s-1\,s\rangle\delta^{(4)}((z-1)\langle K^\flat\,\theta_{ab}\rangle + z\langle K^\flat\,\theta_{bc}\rangle)}{z(1-z)\langle r\,K^\flat\rangle\langle s-1\,K^\flat\rangle\langle s\,K^\flat\rangle\langle t\,K^\flat\rangle} + (z\leftrightarrow\bar{z})\right], \tag{5.64}$$

with[2]

$$K^{\flat\mu} = x^\mu_{ab}(z-1) + x^\mu_{bc}z. \tag{5.65}$$

Notice that $(K^\flat)^2 = 0$, which allows us to use it inside the spinor brackets. The sum over the exchange of $z$ and $\bar{z}$ in (5.64) corresponds to the average over the two solutions discussed earlier and it involves also the definition of $K^\flat$.

The exchange of $z$ and $\bar{z}$ is not the only symmetry of $c^{3\mathrm{m}}$. It is easy to see that (5.64) is symmetric under the exchange

$$\begin{cases} x_{ab}\leftrightarrow x_{bc}\,, \\ \;\; u\leftrightarrow v\,. \end{cases} \tag{5.66}$$

This particular feature will be important in the following.

The form (5.64) is not ideal to test dual conformal invariance. We will find an alternative expression which makes this symmetry more manifest. In order to achieve this, we start from

[2]Compared to [**75**], our definition of $K^\flat$ is rescaled for convenience, taking advantage of cancellations between numerator and denominator (see also (E.9)).

(5.62). Importantly, we will not need the particular form of the solution to prove dual conformal symmetry. In other words, our derivation applies for any $x_0$ sitting on the curve of solutions to the on-shell conditions for the three cut legs. As a bonus, we will see that this derivation allows an easier evaluation on the kinematic solution with respect to (5.62). First we rewrite (5.62) using the identities

$$\langle\ell_2\,\ell_1\rangle[\ell_1\,\ell_3]\langle\ell_3\,r-1\rangle = \langle\ell_2|x_{0a}x_{ac}|r-1\rangle\,, \qquad \langle\ell_2\,\ell_3\rangle[\ell_3\,\ell_1]\langle\ell_1\,r\rangle = \langle\ell_2|x_{0c}x_{ca}|r\rangle\,, \tag{5.67}$$

$$\langle\ell_2\,\ell_1\rangle[\ell_1\,\ell_2]\langle\ell_2\,s-1\rangle = \langle\ell_2|x_{0a}x_{ab}|s-1\rangle\,, \qquad \langle\ell_2\,\ell_3\rangle[\ell_3\,\ell_2]\langle\ell_2\,s\rangle = \langle\ell_2|x_{0c}x_{cb}|s\rangle\,, \tag{5.68}$$

$$\langle\ell_2\,\ell_1\rangle[\ell_1\,\ell_3]\langle\ell_3\,\theta_{bc}\rangle = -\langle\ell_2|x_{0a}x_{ac}|\theta_{cb}\rangle\,, \qquad \langle\ell_2\,\ell_3\rangle[\ell_3\,\ell_1]\langle\ell_1\,\theta_{ab}\rangle = \langle\ell_2|x_{0c}x_{ca}|\theta_{ab}\rangle\,, \tag{5.69}$$

where we used momentum conservation at the three vertices and the on-shell condition for the loop legs. Furthermore, the loop leg $\ell_2$ is adjacent both to $x_0$ and $x_b$, therefore

$$\langle\ell_2|x_0 = \langle\ell_2|x_b\ . \tag{5.70}$$

This gives

$$c^{3\mathrm{m}} = \frac{\langle s-1\,s\rangle\langle r-1\,r\rangle\,\delta^{(4)}(\langle\ell_2|x_{ba}x_{ac}|\theta_{cb}\rangle+\langle\ell_2|x_{bc}x_{ca}|\theta_{ab}\rangle)}{x_{ac}^2\langle\ell_2|x_{ba}x_{ab}|s-1\rangle\langle\ell_2|x_{bc}x_{cb}|s\rangle\langle\ell_2|x_{ba}x_{ac}|r-1\rangle\langle\ell_2|x_{bc}x_{ca}|r\rangle}\,\frac{uv}{\Delta}\,, \tag{5.71}$$

where we introduced the quantity

$$\Delta = \sqrt{(1-u-v)^2-4uv} = |z-\bar{z}|\ . \tag{5.72}$$

Using momentum supertwistors and the identities

$$\langle s-1\,s\rangle x_{ab}^2\langle r-1\,r\rangle = -\langle s-1,s,r-1,r\rangle\ , \tag{5.73}$$

$$\langle s-1\,s\rangle x_{bc}^2\langle r-1\,r\rangle = -\langle s-1,s,(r-1)^-,r^-\rangle\ , \tag{5.74}$$

$$\langle r-1\,r\rangle^2 x_{ac}^2 = -\langle r-1,r,(r-1)^-,r^-\rangle\ , \tag{5.75}$$

we can rewrite (5.71) as

$$c^{3\mathrm{m}} = \mathcal{R}_{r,s}(\ell_2)\frac{\sqrt{uv}}{\Delta}\ , \tag{5.76}$$

with

$$\begin{aligned}\mathcal{R}_{r,s}(\ell_2) = [\ell_2,r,r-1,r^-,(r-1)^-]\ &\frac{\langle\ell_2,r,r-1,r^-\rangle\langle\ell_2,r^-,(r-1)^-,r-1\rangle}{\langle\ell_2,r,r-1,s-1\rangle\langle\ell_2,r^-,(r-1)^-,s\rangle}\\ \times&\frac{\langle s-1,s,r-1,r\rangle^{\frac{1}{2}}\langle s-1,s,(r-1)^-,r^-\rangle^{\frac{1}{2}}}{\langle r-1,r,(r-1)^-1,r^-\rangle}\ .\end{aligned} \tag{5.77}$$

To arrive at this expression in terms of dual conformal invariant five- and four-brackets, we introduced the new supertwistor

$$\mathcal{Z}_{\ell_2}^M = \begin{pmatrix} Z_{\ell_2}^{\hat{A}} \\ \theta_b^{A\alpha}\lambda_{\ell_2\alpha}\end{pmatrix}\ , \qquad Z_{\ell_2}^{\hat{A}} = \begin{pmatrix}\lambda_{\ell_2}^{\alpha} \\ x_b^{\dot{\alpha}\alpha}\lambda_{\ell_2\alpha}\end{pmatrix}\ . \tag{5.78}$$

One can easily check that (5.77) is invariant under the little group scaling (5.12) as well.

The emergence of dual conformal invariant structures in a three-particle cut is a pleasant surprise and a strong hint of dual conformal invariance. As we already stressed, (5.76) is to be evaluated at a specific value of the loop momenta. Notice, however, that in this version of $c^{3\mathrm{m}}$ the whole dependence on the loop momenta is through $\lambda_{\ell_2}$. Therefore it is extremely simple to evaluate it on the explicit solution. Indeed, as we review in Appendix E.1, in the limit corresponding to the direct extraction of the triangle coefficient one can effectively replace

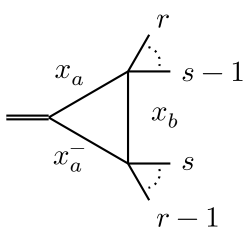
$$\lambda_{\ell_2} \;\to\; \lambda_{K^\flat}\,, \tag{5.79}$$

with $K^\flat$ given in (5.65). With this insight, we can finally write

$$c^{3\mathrm{m}} = \frac{1}{2}\left(\boldsymbol{\mathcal{R}}_{r,s}(K^\flat) + \boldsymbol{\mathcal{R}}_{r,s}(\bar{K}^\flat)\right)\frac{\sqrt{uv}}{\Delta}\,, \tag{5.80}$$

where $\bar{K}^\flat$ is obtained from $K^\flat$ after the replacement $z \to \bar{z}$. $K^\flat$ and $\bar{K}^\flat$ correspond to the two solutions of the on-shell constraints. Although it is not immediately obvious, (5.80) and (5.64) are identical.

After fixing this coefficient, we are left with

$c^{3\mathrm{m}}$ 


$$= \frac{1}{2}\left(\boldsymbol{\mathcal{R}}_{r,s}(K^\flat) + \boldsymbol{\mathcal{R}}_{r,s}(\bar{K}^\flat)\right) g(u,v)\,, \tag{5.81}$$

where

$$g(u,v) = \frac{\sqrt{uv}}{\Delta} F^{3\mathrm{m}}(z,\bar{z})\,, \tag{5.82}$$

and $F^{3\mathrm{m}}(z,\bar{z})$ is the explicit result of the reduced three-mass triangle (see Appendix D)

$$F^{3\mathrm{m}}(z,\bar{z}) = \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \tfrac{1}{2}\log(z\bar{z})\log\left(\frac{1-z}{1-\bar{z}}\right)\,. \tag{5.83}$$

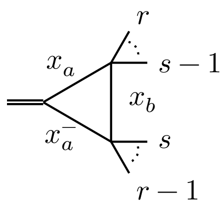
What remains to be proven is the invariance of the function $g(u,v)$. However it is not hard to see, by acting with the generator $\mathsf{K}^\mu$ in (5.26), that the variation of $g(u,v)$ is non-vanishing. On the other hand, we will now show that this variation cancels in the sum over all possible triangles. To begin with, one can show that $F^{3\mathrm{m}}(z,\bar{z}) = F^{3\mathrm{m}}(1-z,1-\bar{z})$ as a consequence of the identity

$$\mathrm{Li}_2(z) = -\mathrm{Li}_2(1-z) - \log(1-z)\log(z) + \frac{\pi^2}{6}\,, \tag{5.84}$$

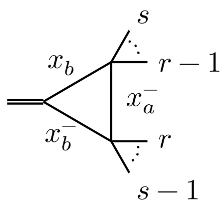
thus implying

$$g(u,v) = g(v,u)\,. \tag{5.85}$$

Therefore $g(u,v)$ is a symmetric function under the exchange (5.66). Notice that, in the sum over all possible three-mass triangles one always has a contribution where $u$ and $v$ are swapped. These are

$c^{3\mathrm{m}}$



$$= \frac{1}{2}\left(\mathcal{R}_{r,s}(K^\flat) + \mathcal{R}_{r,s}(\bar{K}^\flat)\right) g(u,v) \tag{5.86}$$

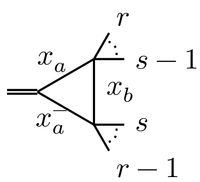
$c^{3\mathrm{m}}$


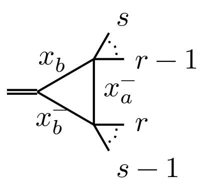

$$= \frac{1}{2}\left(\mathcal{R}_{r,s}(K^\flat) + \mathcal{R}_{r,s}(\bar{K}^\flat)\right) g(v,u) \tag{5.87}$$

where we used the property $\mathcal{R}_{r,s} = \mathcal{R}_{s,r}$, which we mentioned around (5.66). Crucially these two configurations are identical when written in terms of Mandelstam invariants, but it is immediate to see that their region variables assignments are different and consequently also their variation under dual special conformal transformation. In particular, we will show that

$$\mathsf{K}^\mu g(u,v) = -\mathsf{K}^\mu g(v,u)\,, \tag{5.88}$$

thus providing the cancellation

$\mathsf{K}^\mu \Bigg( c^{3\mathrm{m}}$



$+ \; c^{3\mathrm{m}}$



$\Bigg) = 0\,. \quad (5.89)$

In order to prove our crucial result (5.88), we start from the variation of the basic ingredients

$$\mathsf{K}^\mu u = -2u\, x_{bc}^\mu\,, \qquad \mathsf{K}^\mu v = -2v\, x_{ba}^\mu\,, \tag{5.90}$$

from which we derive

$$\mathsf{K}^\mu g(u,v) = -2u\partial_u g(u,v) x_{bc}^\mu + 2v\partial_v g(u,v) x_{ab}^\mu\,. \tag{5.91}$$

Now we apply to this equation the exchange (5.66), leading to

$$\mathsf{K}^\mu g(v,u) = -2v\partial_v g(v,u) x_{ab}^\mu + 2u\partial_u g(v,u) x_{bc}^\mu\,. \tag{5.92}$$

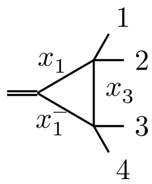
Then, we can simply use the identities

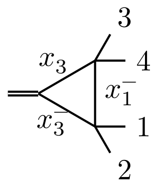
$$\partial_u g(v,u) = \partial_u g(u,v)\,, \qquad\qquad \partial_v g(v,u) = \partial_v g(u,v)\,, \tag{5.93}$$

which are trivial consequences of (5.85), to see that (5.88) holds for any symmetric function of $u$ and $v$.

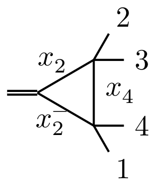
In summary, we have proven that, given a symmetric function of $u$ and $v$, its variation under dual conformal transformation is antisymmetric in $u$ and $v$. In particular this applies to $g(u, v)$ defined in (5.82) (for completeness we have written its explicit variation in Appendix E.2). Therefore, we conclude that the variation of the three-mass triangle contributions cancels out in the sum over all the possible triangles. We stress how non-trivial this result is – quantities involving triangle functions can therefore be dual conformal invariant.

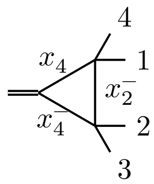
As an example, let us discuss in detail the four-point case. In that case one simply has four possible permutations, and the cancellation is

$\mathsf{K}^\mu \Big( c^{3\mathrm{m}}$ 

$+ \; c^{3\mathrm{m}}$ 

$\Big) = 0$ (5.94)

$\mathsf{K}^\mu \Big( c^{3\mathrm{m}}$ 

$+ \; c^{3\mathrm{m}}$ 

$\Big) = 0$ (5.95)

which can be checked explicitly.

# 6. Conclusions and Outlook

In this thesis, we showed that various properties and techniques that were originally observed and developed in the context of scattering amplitudes, have their analogues for form factors.

In Chapter 3 we showed how the same twistor- and ambitwistor-string worldsheet formulas derived for $\mathcal{N} = 4$ amplitudes can be used to compute form factors from the stress-tensor multiplet at tree level. This is done by picking terms with a different trace structure in the vertex-operator correlators. The reason why these terms give rise precisely to form factors is not yet fully understood. From a string theory perspective, these correspond to the exchange of conformal supergravity states.

A natural extension of these results would be to study if the same string models can be used to generate results for form factors of other supermultiplets, or alternatively, if other trace structures lead to results that can have an interpretation on the gauge-theory side.

Another interesting line of inquiry would be to try to extend these results to loop level, given the recent successes obtained for ambitwistor-string models [**95**–**98**].

Finally, one might try to generate, in a similar fashon, form factors for other field theories which have a known formulation in the ambitwistor-string framework [**99**].

In Chapters 4 and 5 we studied form factors at loop level. A crucial ingredient in obtaining these results was the introduction of a particular prescription for the assignment of dual-space variables.

In Chapter 4 we derived loop-level recursion relations along the lines of the ones presented in [**45**] for $\mathcal{N} = 4$ superamplitudes. We studied a BCFW-like deformation, leading to the formula in (4.12) for integrands at one-loop, but one could easily obtain other recursion relations starting from different complex deformations. In [**2**], for example, we also discuss a deformation which leads to an expansion in terms of MHV diagrams.

Although no explicit examples are provided, following the same principles outlined in Section 4.1 one should be able to generate integrands at all loops.

With the same prescription for the assignment of region variables, in Chapter 5 we provided strong evidence for the invariance of quantum form factors under dual conformal symmetry. At tree level, this was partly understood in [**74**] using a formulation in terms of twistor variables. The extension of these results to loop level seemed to be obstructed by the appearance of scalar triangles in the loop expansion. Here, we presented a general argument for one-loop dual conformal invariance and explicitly analysed the cancellation mechanism leading to a vanishing variation for finite triangles.

Our observation opens the way to many future developments. One obvious question is whether dual conformal invariance survives for higher loops and, if so, which constraints can be put on the allowed scalar integrals and their coefficients. At one loop we already noticed interesting features. In (5.46) the box integrals organise themselves in a simple combination, whose variation under dual conformal symmetry yields exactly the correct anomaly (5.51). Conversely, one could say that dual conformal invariance constrains the box coefficients such that the combination of box functions leads to the correct dual conformal anomaly. A similar argument allows to exclude the presence of three-mass triangles different from (5.57). Indeed, while cancellations like (5.89) do not rely on having only the off-shell leg in one corner, the possibility of recasting the three-mass coefficient in a dual conformal invariant form (such as (5.77)), is linked to the specific configuration (5.57) where the off-shell leg sits alone at one corner, and would be spoiled in a more general case.

Another interesting question is whether dual conformal invariance survives for form factors of different operators. One could start looking at protected longer operators, for which some loop results are already available [**28**, **100**]. Afterwards, one would naturally move to unprotected operators [**30**–**32**, **101**–**103**]. In that case the presence of ultraviolet divergences makes things more subtle and the argument of Section 5.2 would have to be revisited.

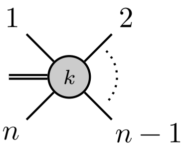
Since our method for showing dual conformal invariance applies to the expansion of the result in terms of scalar integrals, it would be important to develop a general method to test dual conformal symmetry on the final result in terms of Mandelstam invariants. In particular, while there is an unambiguous map between Mandelstam invariants and region variables, the definition of $q^2$ (and in particular its variation under dual conformal invariance) changes according to the specific scalar integral. Rewriting Mandelstam variables in terms of twistors may potentially help in finding new dual conformal invariants on the periodic configuration.

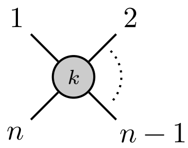
It would also be exciting to better understand how the duality with Wilson loops is realised at weak coupling. In the dual picture, dual conformal invariance is simply the ordinary conformal invariance of the Wilson loop expectation value and this would provide new important insights. In particular, given the latest developments in the computation of exact scattering amplitudes, a Wilson loop dual would allow to access the non-perturbative regime, thus gaining a deeper understanding of the symmetries.

As mentioned, the authors of [**57**] developed a dual conformal invariant regularisation for scattering amplitudes. This led to the formulation of new unitarity-based techniques which allow to compute the integrand of scattering amplitudes for arbitrary helicity configurations and number of external legs up to three loops [**104**]. A similar technique for the case of form factors would allow to notably increase the amount of perturbative data at our disposal.

# A. Notation and conventions

Throughout this thesis we use the following notation to indicate the N$^k$MHV tree-level amplitude and form factor

$$F^{(0)}_{n,k} =$$ 

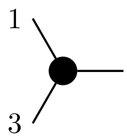
$$A^{(0)}_{n,k} =$$ 

(A.1)

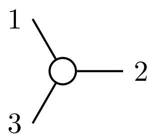
Our conventions for the MHV cases are

$$F^{(0)}_{n,0} = \frac{\delta^{(8)}(\gamma - \sum_{i=1}^{n} \lambda_i \eta_i)}{\langle 1\,2\rangle \cdots \langle n\,1\rangle}\,, \qquad\qquad A^{(0)}_{n,0} = \mathrm{i}\frac{\delta^{(8)}(\sum_{i=1}^{n} \lambda_i \eta_i)}{\langle 1\,2\rangle \cdots \langle n\,1\rangle} \tag{A.2}$$

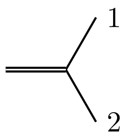
The usual delta function for momentum conservation is not indicated. For the simplest cases of three-point amplitude and two-point form factor we use the notation

$$A^{(0)}_{3,0} =$$ 

$$= \mathrm{i}\,\frac{\delta^{(8)}(\lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3)}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}\,,$$

$$A^{(0)}_{3,-1} =$$ 

$$= -\mathrm{i}\,\frac{\delta^{(4)}([2\,3]\eta_1 + [3\,1]\eta_2 + [1\,2]\eta_3)}{[1\,2][2\,3][3\,1]}\,,$$

$$F^{(0)}_{2,0} =$$ 

$$= \frac{\delta^{(8)}(\lambda_1\eta_1 + \lambda_2\eta_2 - \gamma)}{\langle 1\,2\rangle\langle 2\,1\rangle}\,. \tag{A.3}$$

All the external legs are outgoing, except for the off-shell leg. The latter has incoming momentum $q$ and supermomentum $\gamma$, with

$$q = \sum_{i=1}^{n} p_i\,, \qquad \gamma = \sum_{i=1}^{n} \mathrm{q}_i\,. \tag{A.4}$$

Note that $\mathcal{F}_2^{\mathrm{MHV}}$ is the minimal supersymmetric form factor of the chiral half of the protected stress-tensor multiplet (for details see [12]) and $\gamma$ labels different components of this multiplet.

We use supersymmetric region variables according to the convention

$$x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}} = p_i^{\alpha\dot{\alpha}} = \lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}}\,, \tag{A.5}$$

$$\theta_i^{A\alpha} - \theta_{i+1}^{A\alpha} = \mathrm{q}_i^{A\alpha} = \eta_i^A \lambda_i^\alpha \ . \tag{A.6}$$

More generally one has, for $i < j$,

$$p_i + p_{i+1} + \cdots + p_j = x_i - x_{j+1} \equiv x_{i\,j+1} \ , \tag{A.7}$$

and similarly for the $\theta_i$ variables. If $q \neq 0$ the dual coordinates do not describe a closed polygon. However they are still arranged in periodic configurations, where the image variables are defined as

$$x_i^{[m]} = x_i + mq \ , \qquad\qquad \theta_i^{[m]} = \theta_i + m\gamma \ , \tag{A.8}$$

with $m \in \mathbb{Z}$. For $m = \pm 1$ we use the notation

$$x_i^{\pm} = x_i \pm q \, , \qquad\qquad \theta_i^{\pm} = \theta_i \pm \gamma \ . \tag{A.9}$$

The same kinematic configuration can be encoded in terms of momentum-twistor variables since edges of the periodic line are light rays in dual space. The incidence relation

$$\mu_i^{\dot\alpha} = \mathrm{i} x_i^{\alpha\dot\alpha} \lambda_{i\,\alpha} = \mathrm{i} x_{i+1}^{\alpha\dot\alpha} \lambda_{i\,\alpha} \ , \tag{A.10}$$

fixes the components of the twistor

$$Z_i^{\hat A} = \begin{pmatrix} \lambda_i^\alpha \\ \mu_i^{\dot\alpha} \end{pmatrix} \ , \tag{A.11}$$

and the ambiguity in the choice of the spinor-helicity variables $(\lambda_i, \tilde\lambda_i)$ now translates to the fact that $Z_i$ are interpreted as projective coordinates in twistor space $\mathbb{T} \simeq \mathbb{CP}^3$. The supersymmetric version is simply

$$\mathcal{Z}_i^M = \begin{pmatrix} Z_i^{\hat A} \\ \chi_i^A \end{pmatrix} \ , \qquad\qquad \chi_i^A = \theta_i^{A\,\alpha} \lambda_{i\alpha} \ . \tag{A.12}$$

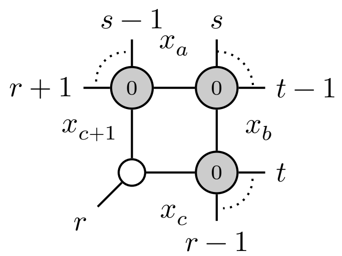
Periodicity, as in (A.8), is implemented by the condition

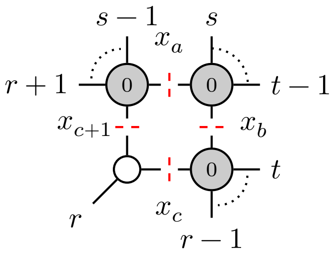
$$\mathcal{Z}_i^{[m]M} = \begin{pmatrix} Z_i^{[m]\hat A} \\ \chi_i^{[m]A} \end{pmatrix} \ , \qquad Z_i^{[m]\hat A} = \begin{pmatrix} \lambda_i^\alpha \\ (x_i^{[m]})^{\dot\alpha\alpha} \lambda_{i\alpha} \end{pmatrix} \ , \qquad \chi_i^{[m]A} = (\theta_i^{[m]})^{A\alpha} \lambda_{i\alpha} \ . \tag{A.13}$$

This can be seen as the finite translation generated by

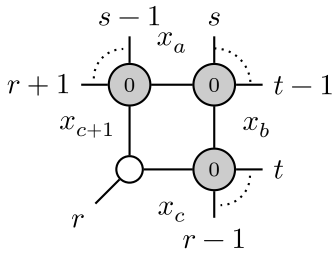
$$\mathsf{P}_{\alpha\dot\alpha} = \lambda_\alpha \frac{\partial}{\partial \mu^{\dot\alpha}} \ . \tag{A.14}$$

In Section 2.8 we introduced the following notation for $R$-invariants:

$R_{rst} =$  (A.15) hinting at their connection to a quadruple cut. The precise relation is the following




$$= \mathrm{i}\Delta_{abc\,c+1} A^{(0)}_{n,0} \tag{A.16}$$




with

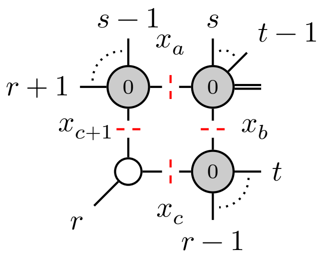
$$\Delta_{abcd} = \sqrt{(x^2_{ac}x^2_{bd} - x^2_{bc}x^2_{ad} + x^2_{ab}x^2_{cd})^2 - 4x^2_{ac}x^2_{bd}x^2_{ab}x^2_{cd}}\ . \tag{A.17}$$

If $x^2_{cd} = 0$, as it happens in (A.16), this factor reduces to

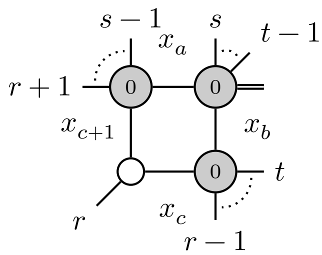
$$\Delta_{abc\,c+1} = x^2_{ac}x^2_{bc+1} - x^2_{ac+1}x^2_{bc}\ . \tag{A.18}$$

Notice in particular that this is the form of $\Delta_{abcd}$ for all the IR divergent boxes. The four-mass box is the only one for which one needs to use (A.17) and it is IR finite and dual conformal invariant by itself.

For the case of form factors we have a similar relation between cuts and $R$-invariants




$$= \mathrm{i}\Delta_{abc\,c+1} F^{(0)}_{n,0} \tag{A.19}$$




and similarly for $R''_{rst}$.

It is well-known that the quadruple cut in four dimensions computes the coefficient of the boxes.

# B. Delta functions

## B.1 Grassmann deltas

Let $a$ be some degree-1 polynomial in the Grassmann algebra. Then

$$\delta(a_i) = a_i \ . \tag{B.1}$$

Under a linear transformation

$$b_i = \sum_{j=i}^{k} m_{ij}\, a_j \tag{B.2}$$

we have

$$\prod_{i=1}^{k} \delta(b_i) = (\det m) \prod_{i=1}^{k} \delta(a_i) \ . \tag{B.3}$$

Introducing "R-symmetry" indices, we define

$$\delta^{(N)}(a) \equiv \prod_{A=1}^{N} \delta(a^A) \ . \tag{B.4}$$

An important technical result is

$$\prod_{i=1}^{k} \prod_{A=1}^{N} \delta(a_i) = (-1)^{\lfloor N/2 \rfloor \lfloor k/2 \rfloor} \prod_{A=1}^{N} \prod_{i=1}^{k} \delta(a_i) \ . \tag{B.5}$$

From the above one obtains

$$\begin{aligned} \prod_{i=1}^{k} \delta^{(N)}(b_i) &= (-1)^{\lfloor N/2 \rfloor \lfloor k/2 \rfloor} \prod_{A=1}^{N} \prod_{i=1}^{k} \delta(b_i^A) = (-1)^{\lfloor N/2 \rfloor \lfloor k/2 \rfloor} \prod_{A=1}^{N} \left[ (\det m) \prod_{i=1}^{k} \delta(a_i^A) \right] \\ &= (\det m)^N \prod_{i=1}^{k} \delta^{(N)}(a_i) \ . \end{aligned} \tag{B.6}$$

As in the "bosonic" case one can determine the support of the delta to manipulate expressions. In fact, for some $k \in \mathbb{C}$,

$$\int \mathrm{d}a \ \delta(ka - b)\, f(a) = k\, f(b/k) = \int \mathrm{d}a \ \delta(ka - b)\, f(b/k) \ . \tag{B.7}$$

Let us now deal with spinor indices. Let $\mathrm{q}^{A\alpha}$ be a Grassmann-odd spinor and $x^\alpha$, $y^\alpha$ Grassmann-even spinors. Then, if $\langle x\,y\rangle = x^2y^1 - x^1y^2$,

$$\begin{aligned}\delta^{(2N)}(\mathrm{q}) \equiv \prod_{A=1}^{N}\delta^{(2)}(\mathrm{q}^A) &= \langle y\,x\rangle^{-N}\prod_{A=1}^{N}\delta(\langle x\,\mathrm{q}^A\rangle)\,\delta(\langle y\,\mathrm{q}^A\rangle)\\ &= (-1)^{\lfloor N/2\rfloor}\langle y\,x\rangle^{-N}\delta^{(N)}(\langle x\,\mathrm{q}\rangle)\,\delta^{(N)}(\langle y\,\mathrm{q}\rangle)\;.\end{aligned} \tag{B.8}$$

From the above it follows that, for $N=4$, one has

$$\delta^{(2N)}(\mathrm{q}_1+\ldots+\mathrm{q}_n) = \prod_{i<j}\langle i\,j\rangle\prod_{A=1}^{4}\eta_i^A\eta_j^A\;. \tag{B.9}$$

## B.2 Complex deltas

The complex delta function is the $(0,1)$-form defined as

$$\delta(f(z)) = \bar\partial\frac{1}{2\pi\mathrm{i}\,f(z)} = \delta(\mathrm{Re}\,f)\,\delta(\mathrm{Im}\,f)\,\mathrm{d}\overline{f(z)}\;. \tag{B.10}$$

From the definition it follows that

$$\delta(\lambda z) = \lambda^{-1}\delta(z)\;. \tag{B.11}$$

Let us consider an $n$-dimensional complex integral of some function $g$. If one localises the integral with $n$ complex delta functions with arguments $f_i$, the result is given in terms of

$$\int \mathrm{d}z_1\ldots\mathrm{d}z_m\;g(z)\;\delta(f_1(z))\ldots\delta(f_m(z)) = \sum_{z_0\in f^{-1}(0)}\operatorname*{Res}_{z=z_0}\frac{g(z)\,\mathrm{d}z_1\wedge\ldots\wedge\mathrm{d}z_m}{f_1(z)\ldots f_m(z)}\;. \tag{B.12}$$

Let $z_*$ be an isolated common zero of the $f$'s. Then this is nondegenerate if the Jacobian

$$\mathrm{Jac}_f(z_*) = \det\frac{\partial(f_1,\ldots,f_n)}{\partial(z_1,\ldots,z_n)}\Big|_{z_*}\neq 0\;. \tag{B.13}$$

In this case

$$\operatorname*{Res}_{z=z_*}\frac{g(z)\,\mathrm{d}z_1\wedge\ldots\wedge\mathrm{d}z_m}{f_1(z)\ldots f_m(z)} = \frac{g(z_*)}{\mathrm{Jac}_f(z_*)}\;. \tag{B.14}$$

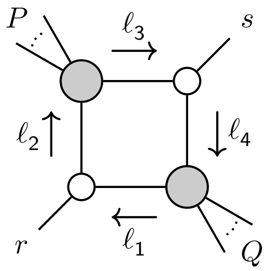
# C. $R$-invariants

In this appendix we provide a streamlined derivation of R-invariants from box diagrams. The computation applies both to amplitudes and form factors.

## C.1 Grassmann structure

In this first section we derive general results for the Grassmann structure of box diagrams. Namely, we perform the integration of the Grassmann deltas coming from the four corners. Grey blobs represent generic amplitudes or form factors of positive MHV degree.

Let us start by considering




$$\boldsymbol{\mathcal{I}}_{\mathrm{B}} = \int \mathrm{d}^4\eta_{\ell_1} \cdots \mathrm{d}^4\eta_{\ell_4}\ \delta^{(4)}(G_1)\ \delta^{(8)}(G_2)\ \delta^{(4)}(G_3)\ \delta^{(8)}(G_4)\ f(\boldsymbol{\eta})\ , \tag{C.1}$$

where

$$\begin{aligned}
G_1 &= [\ell_2\,\ell_1]\eta_r + [r\,\ell_2]\eta_{\ell_1} + [\ell_1\,r]\eta_{\ell_2}\ ,\\
G_2 &= \mathrm{q}_P - \mathrm{q}_{\ell_2} + \mathrm{q}_{\ell_3}\ ,\\
G_3 &= [\ell_4\,\ell_3]\eta_s + [s\,\ell_4]\eta_{\ell_3} + [\ell_3\,s]\eta_{\ell_4}\ ,\\
G_4 &= \mathrm{q}_Q - \mathrm{q}_{\ell_4} + \mathrm{q}_{\ell_1}\ ,
\end{aligned} \tag{C.2}$$

and $f$ is some function coming from the two corners of positive MHV degree. The integral can be performed over

$$\begin{aligned}
\delta^{(8)}(G_2) &= \langle\ell_2\,\ell_3\rangle^4\,\delta^{(4)}\left(\eta_{\ell_3} - \frac{\langle\ell_2\,\mathrm{q}_P\rangle}{\langle\ell_3\,\ell_2\rangle}\right)\ \delta^{(4)}\left(\eta_{\ell_2} - \frac{\langle\ell_3\,\mathrm{q}_P\rangle}{\langle\ell_3\,\ell_2\rangle}\right)\ ,\\
\delta^{(8)}(G_4) &= \langle\ell_1\,\ell_4\rangle^4\,\delta^{(4)}\left(\eta_{\ell_1} - \frac{\langle\ell_4\,\mathrm{q}_Q\rangle}{\langle\ell_1\,\ell_4\rangle}\right)\ \delta^{(4)}\left(\eta_{\ell_4} - \frac{\langle\ell_1\,\mathrm{q}_Q\rangle}{\langle\ell_1\,\ell_4\rangle}\right)\ .
\end{aligned} \tag{C.3}$$

Combining the above solutions for the deltas with the kinematical constraints

$$\lambda_{\ell_1} = \frac{[r\,\ell_2]}{[\ell_1\,\ell_2]}\lambda_r\,, \qquad \lambda_{\ell_2} = \frac{[r\,\ell_1]}{[\ell_1\,\ell_2]}\lambda_r\,, \qquad \lambda_{\ell_3} = \frac{[s\,\ell_4]}{[\ell_3\,\ell_4]}\lambda_s\,, \qquad \lambda_{\ell_4} = \frac{[s\,\ell_3]}{[\ell_3\,\ell_4]}\lambda_s\,, \tag{C.4}$$

coming from the $\overline{\text{MHV}}$ vertices, one finds

$$\delta^{(4)}(G_1)\;\delta^{(4)}(G_3) = \left(\frac{[\ell_1\,\ell_2][\ell_3\,\ell_4]}{\langle r\,s\rangle}\right)^4 \delta^{(8)}(\mathrm{q}_r + \mathrm{q}_P + \mathrm{q}_s + \mathrm{q}_Q)\,. \tag{C.5}$$

Therefore

$$\boldsymbol{\mathcal{I}}_{\mathrm{B}} = \left(\frac{\langle \ell_1\,\ell_4\rangle\langle \ell_2\,\ell_3\rangle[\ell_1\,\ell_2][\ell_3\,\ell_4]}{\langle r\,s\rangle}\right)^4 \delta^{(8)}(\mathrm{q}_r + \mathrm{q}_P + \mathrm{q}_s + \mathrm{q}_Q)\; f(\boldsymbol{\eta})\,, \tag{C.6}$$

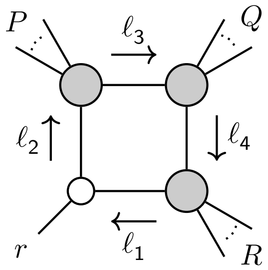
where $f$ is now evaluated on the solutions given above. In particular one finds, for internal super-momenta,

$$\mathrm{q}_{\ell_1} = \frac{\langle s\,\mathrm{q}_Q\rangle}{\langle r\,s\rangle}\lambda_r\,, \qquad \mathrm{q}_{\ell_2} = \frac{\langle s\,\mathrm{q}_P\rangle}{\langle s\,r\rangle}\lambda_r\,, \qquad \mathrm{q}_{\ell_3} = \frac{\langle r\,\mathrm{q}_P\rangle}{\langle s\,r\rangle}\lambda_s\,, \qquad \mathrm{q}_{\ell_4} = \frac{\langle r\,\mathrm{q}_Q\rangle}{\langle r\,s\rangle}\lambda_s\,. \tag{C.7}$$

One can rearrange the coefficient as

$$\left(\frac{\langle \ell_1\,\ell_4\rangle\langle \ell_2\,\ell_3\rangle[\ell_1\,\ell_2][\ell_3\,\ell_4]}{\langle r\,s\rangle}\right)^4 = [r|\ell_1\,\ell_3|s]^4 = [r|\ell_2\,\ell_4|s]^4\,. \tag{C.8}$$

The second case we consider is




$$\boldsymbol{\mathcal{I}}_{\mathrm{A}} = \int \mathrm{d}^4\eta_{\ell_1}\cdots\mathrm{d}^4\eta_{\ell_4}\;\delta^{(4)}(G_1)\;\delta^{(8)}(G_2)\;\delta^{(8)}(G_3)\;\delta^{(8)}(G_4)\;f(\boldsymbol{\eta})\,, \tag{C.9}$$

where

$$\begin{aligned}
G_1 &= [\ell_2\,\ell_1]\eta_r + [r\,\ell_2]\eta_{\ell_1} + [\ell_1\,r]\eta_{\ell_2}\,,\\
G_2 &= \mathrm{q}_P - \mathrm{q}_{\ell_2} + \mathrm{q}_{\ell_3}\,,\\
G_3 &= \mathrm{q}_Q - \mathrm{q}_{\ell_3} + \mathrm{q}_{\ell_4}\,,\\
G_4 &= \mathrm{q}_R - \mathrm{q}_{\ell_4} + \mathrm{q}_{\ell_1}\,,
\end{aligned} \tag{C.10}$$

and $f$ is a generic function coming from the three corners where the MHV degree is unspecified. $\overline{\text{MHV}}$ kinematics implies

$$\lambda_{\ell_1} = \lambda_r\frac{[r\,\ell_2]}{[\ell_1\,\ell_2]}\,, \qquad \lambda_{\ell_2} = \lambda_r\frac{[r\,\ell_1]}{[\ell_1\,\ell_2]} \tag{C.11}$$

which, in turn, on the support of $\Delta_1$, imposes the constraint

$$-\,\mathrm{q}_{\ell_1} + \mathrm{q}_{\ell_2} + \mathrm{q}_r = 0\ , \tag{C.12}$$

from which

$$G_2 + G_3 + G_4 = \mathrm{q}_r + \mathrm{q}_P + \mathrm{q}_Q + \mathrm{q}_R \tag{C.13}$$

follows. We can then massage the integral into

$$\boldsymbol{\mathcal{I}}_{\mathrm{A}} = \int \mathrm{d}^4\eta_{\ell_1} \cdots \mathrm{d}^4\eta_{\ell_4}\ \delta^{(8)}(G_2)\ \delta^{(8)}(G_4)\ \delta^{(8)}(G_2 + G_3 + G_4)\ \delta^{(4)}(G_1)\ f(\boldsymbol{\eta})\ , \tag{C.14}$$

and perform the integral over the first two deltas

$$\begin{aligned} \delta^{(8)}(G_2) &= \langle \ell_2\, \ell_3 \rangle^4\, \delta^{(4)}\left(\eta_{\ell_3} - \frac{\langle \ell_2\, \mathrm{q}_P \rangle}{\langle \ell_3\, \ell_2 \rangle}\right)\ \delta^{(4)}\left(\eta_{\ell_2} - \frac{\langle \ell_3\, \mathrm{q}_P \rangle}{\langle \ell_3\, \ell_2 \rangle}\right)\ , \\ \delta^{(8)}(G_4) &= \langle \ell_1\, \ell_4 \rangle^4\, \delta^{(4)}\left(\eta_{\ell_1} - \frac{\langle \ell_4\, \mathrm{q}_R \rangle}{\langle \ell_1\, \ell_4 \rangle}\right)\ \delta^{(4)}\left(\eta_{\ell_4} - \frac{\langle \ell_1\, \mathrm{q}_R \rangle}{\langle \ell_1\, \ell_4 \rangle}\right)\ . \end{aligned} \tag{C.15}$$

This fixes

$$\mathrm{q}_{\ell_1} = \frac{\langle \ell_4\, \mathrm{q}_R \rangle}{\langle r\, \ell_4 \rangle}\lambda_r\ , \qquad \mathrm{q}_{\ell_2} = \frac{\langle \ell_3\, \mathrm{q}_P \rangle}{\langle \ell_3\, r \rangle}\lambda_r\ , \qquad \mathrm{q}_{\ell_3} = \frac{\langle r\, \mathrm{q}_P \rangle}{\langle \ell_3\, r \rangle}\lambda_{\ell_3}\ , \qquad \mathrm{q}_{\ell_4} = \frac{\langle r\, \mathrm{q}_R \rangle}{\langle r\, \ell_4 \rangle}\lambda_{\ell_4}\ , \tag{C.16}$$

over which $f$ will be evaluated, and

$$\delta^{(4)}(G_1) = \frac{1}{\langle \ell_1\, \ell_4 \rangle^4 \langle \ell_2\, \ell_3 \rangle^4} \left(\frac{[r\, \ell_1][r\, \ell_2]}{[\ell_1\, \ell_2][\ell_3\, \ell_4]}\right)^4 \delta^{(4)}(\langle \mathrm{q}_r + \mathrm{q}_P | QR | r \rangle - \langle \mathrm{q}_R | QP | r \rangle)\,, \tag{C.17}$$

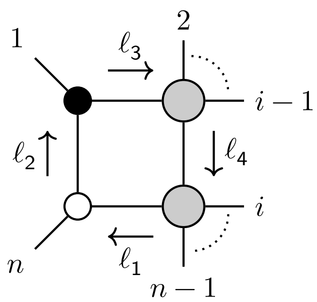
which gives

$$\boldsymbol{\mathcal{I}}_{\mathrm{A}} = \left(\frac{[r\, \ell_1][r\, \ell_2]}{[\ell_1\, \ell_2][\ell_3\, \ell_4]}\right)^4 \delta^{(8)}(\mathrm{q}_r + \mathrm{q}_P + \mathrm{q}_Q + \mathrm{q}_R)\ \delta^{(4)}(\langle \mathrm{q}_r + \mathrm{q}_P | QR | r \rangle - \langle \mathrm{q}_R | QP | r \rangle)\ f(\boldsymbol{\eta})\ . \tag{C.18}$$

## C.2 BCFW boxes

Let us look at the configuration

$$C_{\mathrm{BCFW}} =$$




(C.19)

MHV kinematics and momentum conservation impose

$$\ell_1 = \lambda_n (z\tilde{\lambda}_1 + \tilde{\lambda}_n)\ , \qquad \ell_2 = z\lambda_n \tilde{\lambda}_1\ , \qquad \ell_3 = -(\lambda_1 - z\lambda_n)\tilde{\lambda}_1\ , \tag{C.20}$$

where $z$ is such that $\ell_4$ is on-shell. The expression for $z$ can be obtained identifying channels

$$s \equiv (p_1 + p_n)^2 = (\ell_1 - \ell_3)^2 = \langle n\,1\rangle[1\,n]\ ,$$
$$t \equiv P^2 \equiv (p_1 + \ldots + p_{i-1})^2 = (\ell_2 - \ell_4)^2 = -z\langle n\,\ell_4\rangle[\ell_4\,1] = z\langle n|P|1]\ . \tag{C.21}$$

With the above, one finds

$$\int \mathrm{d}^4\eta_{\ell_1}\,\mathrm{d}^4\eta_{\ell_2}\,\mathrm{d}^4\eta_{\ell_3}\ \frac{\delta^{(4)}(G_1)}{[n\,\ell_2][\ell_2\,\ell_1][\ell_1\,n]}\ \frac{\delta^{(8)}(G_2)}{\langle 1\,\ell_3\rangle\langle\ell_3\,\ell_2\rangle\langle\ell_2\,1\rangle} = s = -\mathrm{i}(\mathrm{i}\,\Delta)P^{-2}\ , \tag{C.22}$$

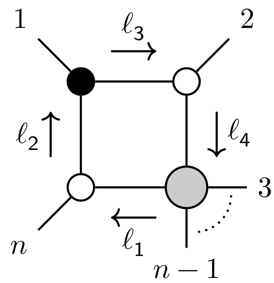
which also fixes

$$\eta_{\ell_3} = \eta_1\ , \qquad \eta_{\ell_1} = \eta_n + z\eta_1\ . \tag{C.23}$$

But this gives precisely the result of a BCFW term with a $[n\,1\rangle$ shift, where the two generic corners are glued together. This establishes a correspondence between this particular type of box diagrams and BCFW terms.

If we consider in particular a box of the type

$$C'_{\mathrm{BCFW}} = \tag{C.24}$$




we find, on top of identities (2.139)

$$\ell_2 = (\lambda_1 + k\lambda_2)\tilde\lambda_1\ , \qquad \ell_3 = k\lambda_2\tilde\lambda_1\ , \qquad \ell_4 = \lambda_2(k\tilde\lambda_1 - \tilde\lambda_2)\ , \tag{C.25}$$

with

$$z = \frac{\langle 1\,2\rangle}{\langle n\,2\rangle}\ , \qquad k = \frac{\langle n\,1\rangle}{\langle 2\,n\rangle}\ . \tag{C.26}$$

We find

$$\boldsymbol{\mathcal{I}}_{\mathrm{B}} = \left(\frac{st}{\langle 2\,n\rangle}\right)^4 \delta^{(8)}(\mathfrak{q})\ f(\boldsymbol{\eta})\ , \tag{C.27}$$

$$-\mathrm{i}\,\frac{1}{[n\,\ell_2][\ell_2\,\ell_1][\ell_1\,n]}\,\frac{1}{\langle 1\,\ell_3\rangle\langle\ell_3\,\ell_2\rangle\langle\ell_2\,1\rangle}\,\frac{1}{[2\,\ell_4][\ell_4\,\ell_3][\ell_3\,2]} = -\mathrm{i}\,\frac{1}{zk}\left(\frac{\langle n\,2\rangle}{st}\right)^3\ . \tag{C.28}$$

Therefore

$$X^{(0)}_{n,0}(1,\ldots,n)\ C'_{\mathrm{BCFW}} = \frac{\langle n\,2\rangle}{\langle 1\,2\rangle\langle n\,1\rangle}\ \mathbb{X}^{(0)}_{n-1}(a,3,\ldots,n-1,b)\ , \tag{C.29}$$

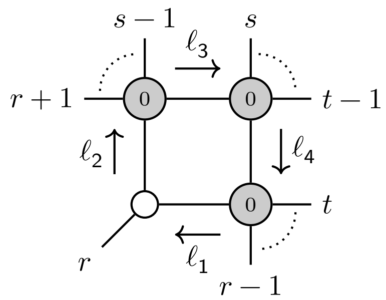
with

$$\begin{aligned}
\lambda_a &= \lambda_2\ , & \tilde\lambda_a &= \tilde\lambda_2 + \frac{\langle 1\,n\rangle}{\langle 2\,n\rangle}\tilde\lambda_1\ , & \eta_a &= \eta_2 + \frac{\langle 1\,n\rangle}{\langle 2\,n\rangle}\eta_1\ ,\\
\lambda_b &= \lambda_n\ , & \tilde\lambda_b &= \tilde\lambda_n + \frac{\langle 1\,2\rangle}{\langle n\,2\rangle}\tilde\lambda_1\ , & \eta_b &= \eta_n + \frac{\langle 1\,2\rangle}{\langle n\,2\rangle}\eta_1\ .
\end{aligned} \tag{C.30}$$

This term alone determines the results for MHV quantities.

## C.3 Generic R-invariants

Let us start by fist looking at amplitudes. We define

$$R_{r,s,t} :=$$  (C.31)

In our pictures, corners are labeled with their MHV degree, i.e. we label them with the value of $k$ as in N$^k$MHV.

Let us first write down the product of the four corners, but only their bosonic part:

$$\begin{aligned} B_{r,s,t} &= -\frac{1}{\text{P.T.}} \\ &\times \frac{\langle r\, r+1\rangle\langle s-1\, s\rangle\langle t-1\, t\rangle\langle r-1\, r\rangle}{[r\,\ell_2][\ell_2\,\ell_1][\ell_1\, r]\langle \ell_2\, r+1\rangle\langle s-1\,\ell_3\rangle\langle \ell_3\,\ell_2\rangle\langle \ell_3\, s\rangle\langle t-1\,\ell_4\rangle\langle \ell_4\,\ell_3\rangle\langle \ell_4\, t\rangle\langle r-1\,\ell_1\rangle\langle \ell_1\,\ell_4\rangle} \end{aligned} \tag{C.32}$$

Combining with the result of Grassmann integration we get

$$B_{r,s,t}\,\boldsymbol{\mathcal{I}}_{\text{A}} = \mathrm{i}\Delta_{r,s,t}\, A^{(0)}_{n,0}\, c_{r,s,t}\, \delta^{(4)}(\langle \mathrm{q}_r + \mathrm{q}_P|QR|r\rangle - \langle \mathrm{q}_R|QP|r\rangle)\ , \tag{C.33}$$

where

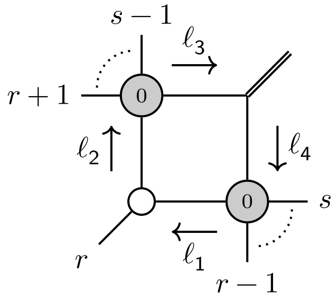
$$\begin{aligned} \Delta_{r,s,t} &= \frac{[\ell_1\, r][\ell_2\, r]\langle \ell_3\, r\rangle\langle \ell_4\, r\rangle[\ell_3\,\ell_4]}{[\ell_1\,\ell_2]} \\ &= s_{\ell_1\ell_3}s_{\ell_2\ell_4} - s_{\ell_1\ell_4}s_{\ell_2\ell_3} \\ &= (p_r+P)^2(p_r+R)^2 - P^2R^2\ , \end{aligned} \tag{C.34}$$

$$\begin{aligned} c_{r,s,t} &= -\frac{\langle s-1\, s\rangle\langle t-1\, t\rangle}{\langle \ell_3\,\ell_4\rangle[\ell_4\,\ell_3]\langle r|\ell_4\ell_3|s-1\rangle\langle r|\ell_4\ell_3|s\rangle\langle r|\ell_3\ell_4|t-1\rangle\langle r|\ell_3\ell_4|t\rangle} \\ &= \frac{\langle s-1\, s\rangle\langle t-1\, t\rangle}{Q^2\langle r|RQ|s-1\rangle\langle r|RQ|s\rangle\langle r|PQ|t-1\rangle\langle r|PQ|t\rangle}\ . \end{aligned} \tag{C.35}$$

This means that

$$R_{r,s,t} = \frac{\langle s-1\, s\rangle\langle t-1\, t\rangle\,\delta^{(4)}(\langle \mathrm{q}_r + \mathrm{q}_P|QR|r\rangle - \langle \mathrm{q}_R|QP|r\rangle)}{Q^2\langle r|RQ|s-1\rangle\langle r|RQ|s\rangle\langle r|PQ|t-1\rangle\langle r|PQ|t\rangle}\ . \tag{C.36}$$

Let us now see what happens for form factors. One can freely insert the off-shell leg in any of the three massive corners, provided that the corner contains at least one on shell legs. The cases where a corner is a $F^{(0)}_{2,0}$ need special care and we will review them below. We start by considering

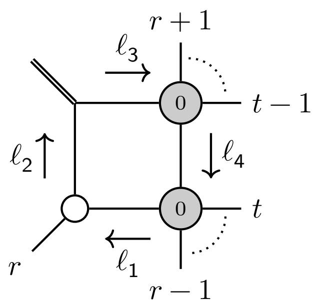



(C.37)

where one simply needs to modify $c$ as

$$\begin{aligned} c \mapsto & \left( c\, \frac{\langle \ell_3\, s\rangle \langle t-1\, \ell_4\rangle}{\langle \ell_3\, \ell_4\rangle \langle t-1\, t\rangle} \right)_{t=s} \\ &= - \frac{\langle s-1\, s\rangle}{s^2_{\ell_3 \ell_4} \langle r|\ell_4 \ell_3|s-1\rangle \langle r|\ell_3\ell_4|s\rangle \langle r|\ell_3\ell_4|r\rangle} \\ &= - \frac{\langle s-1\, s\rangle}{Q^4 \langle r|RQ|s-1\rangle \langle r|PQ|s\rangle \langle r|PQ|r\rangle} \, . \end{aligned} \tag{C.38}$$

Another case is given by

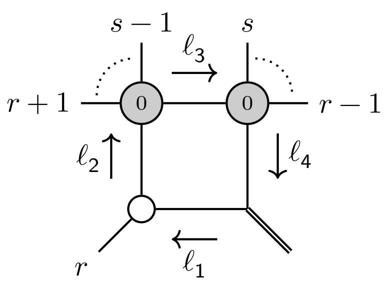



(C.39)

where now

$$\begin{aligned} c \mapsto & \left( c\, \frac{\langle \ell_2\, r+1\rangle \langle s-1\, \ell_3\rangle}{\langle \ell_2\, \ell_3\rangle \langle s-1\, s\rangle} \right)_{s=r+1} \\ &= - \frac{\langle r\, r+1\rangle \langle t-1\, t\rangle}{s_{\ell_3 \ell_4} \langle r|\ell_4 \ell_3|r\rangle \langle r|\ell_4\ell_3|r+1\rangle \langle r|\ell_3\ell_4|t-1\rangle \langle r|\ell_3\ell_4|t\rangle} \\ &= \frac{\langle r\, r+1\rangle \langle t-1\, t\rangle}{Q^2 \langle r|RQ|r\rangle \langle r|RQ|r+1\rangle \langle r|PQ|t-1\rangle \langle r|PQ|t\rangle} \, . \end{aligned} \tag{C.40}$$

Lastly for



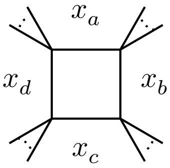

(C.41)

we have

$$
\begin{aligned}
c \mapsto & \left( c \, \frac{\langle \ell_4 \, t\rangle \langle r-1 \, \ell_1 \rangle}{\langle \ell_4 \, \ell_1 \rangle \langle t-1 \, t\rangle} \right)_{t=r} \\
&= -\frac{\langle s-1 \, s\rangle \langle r-1 \, r\rangle}{s_{\ell_3 \ell_4} \langle r|\ell_4 \ell_3|s-1\rangle \langle r|\ell_4 \ell_3|s\rangle \langle r|\ell_3 \ell_4|r-1\rangle \langle r|\ell_3 \ell_4|r\rangle} \\
&= \frac{\langle s-1 \, s\rangle \langle r-1 \, r\rangle}{Q^2 \langle r|RQ|s-1\rangle \langle r|RQ|s\rangle \langle r|PQ|r-1\rangle \langle r|PQ|r\rangle} \, .
\end{aligned} \tag{C.42}
$$

# D. Reduced scalar integrals

In this thesis we expand one-loop results in terms of reduced scalar integrals, i.e. conveniently defined dimensionless quantities that are simply related to the original scalar integral. For the boxes we have

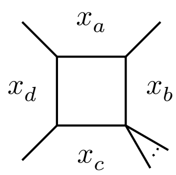
$$\frac{1}{2\pi^{2-\epsilon} r_\Gamma} \int \mathrm{d}^{4-2\epsilon} x_0 \, \frac{1}{x_{0a}^2 x_{0b}^2 x_{0c}^2 x_{0d}^2} = \frac{1}{\mathrm{i}\Delta_{abcd}}$$  (D.1)



where the picture represents the reduced box integral, and $\Delta_{abcd}$ is given in (A.17). The fact that this factor cancels in the product of the box coefficient given by the quadruple cut (A.16) and the scalar integral is the main reason why we find convenient to use this basis. The factors on the left-hand side appear in front of any one-loop diagram and can be reabsorbed in the definition of the coupling. For completeness we remind the reader that

$$r_\Gamma = \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \ . \tag{D.2}$$

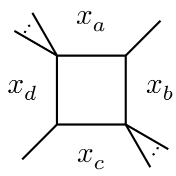
We also list the expression of the reduced box integrals that are needed for our computations:





$$= -\frac{1}{\epsilon^2}\left((-x_{ac}^2)^{-\epsilon} + (-x_{bd}^2)^{-\epsilon} - (-x_{bc}^2)^{-\epsilon}\right) \tag{D.3}$$

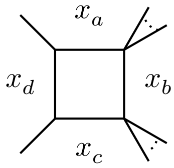
$$+ \mathrm{Li}_2\left(1 - \frac{x_{bc}^2}{x_{ac}^2}\right) + \mathrm{Li}_2\left(1 - \frac{x_{bc}^2}{x_{bd}^2}\right) + \frac{1}{2}\log^2\left(\frac{x_{ac}^2}{x_{bd}^2}\right) + \frac{\pi^2}{6}$$





$$= -\frac{1}{\epsilon^2}\left((-x_{ac}^2)^{-\epsilon} + (-x_{bd}^2)^{-\epsilon} - (-x_{bc}^2)^{-\epsilon} - (-x_{ad}^2)^{-\epsilon}\right) \tag{D.4}$$

$$+ \mathrm{Li}_2\left(1 - \frac{x_{ad}^2}{x_{ac}^2}\right) + \mathrm{Li}_2\left(1 - \frac{x_{ad}^2}{x_{db}^2}\right) + \mathrm{Li}_2\left(1 - \frac{x_{bc}^2}{x_{ac}^2}\right) + \mathrm{Li}_2\left(1 - \frac{x_{bc}^2}{x_{db}^2}\right)$$
$$- \mathrm{Li}_2\left(1 - \frac{x_{ad}^2 x_{bc}^2}{x_{ac}^2 x_{db}^2}\right) + \frac{1}{2}\log^2\left(\frac{x_{ac}^2}{x_{db}^2}\right) \ ,$$



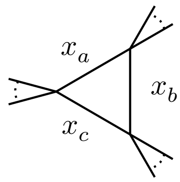

$$= -\frac{1}{\epsilon^2}\left(\frac{1}{2}(-x_{ac}^2)^{-\epsilon} + (-x_{bd}^2)^{-\epsilon} - \frac{1}{2}(-x_{bc}^2)^{-\epsilon} - \frac{1}{2}(-x_{ab}^2)^{-\epsilon}\right) \tag{D.5}$$

$$+ \mathrm{Li}_2\left(1 - \frac{x_{bc}^2}{x_{bd}^2}\right) + \mathrm{Li}_2\left(1 - \frac{x_{ab}^2}{x_{db}^2}\right) + \frac{1}{2}\log^2\left(\frac{x_{ac}^2}{x_{bd}^2}\right)$$
$$- \frac{1}{2}\log\left(\frac{x_{ac}^2}{x_{bc}^2}\right)\log\left(\frac{x_{ac}^2}{x_{ab}^2}\right)$$

In the main text we also use a F inside the diagram to indicate that we consider only the finite part of the one-loop integrals. By finite part we mean the previous expressions where the first line has been removed.

For triangles, we use a notation that is analogous to the box case

$$\frac{1}{2\pi^{2-\epsilon} r_\Gamma}\int \mathrm{d}^{4-2\epsilon}x_0 \, \frac{1}{x_{0a}^2 x_{0b}^2 x_{0c}^2} = \frac{1}{\mathrm{i}\Delta_{abc}} \tag{D.6}$$



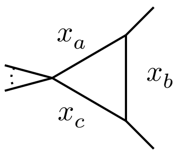

with

$$\Delta_{abc} = \sqrt{(x_{ac}^2 - x_{bc}^2 + x_{ab}^2)^2 - 4x_{ab}^2 x_{ac}^2}\,. \tag{D.7}$$

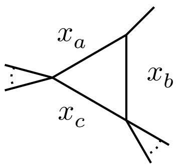
Notice that, for $x_{ab}^2 = 0$, this factor reads

$$\Delta_{a\,a+1\,c} = x_{ac}^2 - x_{a+1,c}^2\,. \tag{D.8}$$

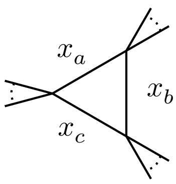
The three possible cases are given by




$$= \frac{(-x_{ac}^2)^{-\epsilon}}{2\epsilon^2} \tag{D.9}$$




$$= \frac{(-x_{bc}^2)^{-\epsilon} - (-x_{ac}^2)^{-\epsilon}}{2\epsilon^2} \tag{D.10}$$




$$= \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \tfrac{1}{2}\log(z\bar{z})\log\left(\frac{1-z}{1-\bar{z}}\right) \tag{D.11}$$

where, for the last integral, we used the variables (5.63). One may be worried that the two-mass triangle is odd under the exchange of the two massive corners. In fact, this sign is compensated by the $\Delta$ factor (D.8). Since we are expanding in terms of reduced integrals, we need to choose a convention and fix the sign of the coefficient accordingly. Using the convention (D.10), one can check that the coefficient (5.45), which we determined by IR consistency, has the right sign to cancel the unwanted three-particle invariants in the IR divergent part of the form factor.

# E. Triple cuts

### E.1 Cut solutions

In this appendix we review some results of [94], adapting them to our notation. In the conventions of section 5.3.2 we set $x_{bc} = K_1$ and $x_{ac} = K_2 = q$. We define also the two massless projections

$$K_1^{\flat,\mu} = \frac{K_1^\mu - \frac{K_1^2}{\gamma_\pm} K_2^\mu}{1 - \frac{K_1^2 K_2^2}{\gamma_\pm^2}}\,, \qquad K_2^{\flat,\mu} = \frac{K_2^\mu - \frac{K_2^2}{\gamma_\pm} K_1^\mu}{1 - \frac{K_1^2 K_2^2}{\gamma_\pm^2}}\,, \tag{E.1}$$

where, using the variables (5.63),

$$\gamma_+ = q^2(1-\bar{z})\,, \qquad \gamma_- = q^2(1-z)\,. \tag{E.2}$$

The two different values are associated to the two solutions of the kinematics constraints. In general the mapping between the two solutions is achieved by $z \leftrightarrow \bar{z}$. Consequently,

$$\frac{K_1^2}{\gamma_+} = (1-z)\,, \qquad \frac{K_1^2}{\gamma_-} = (1-\bar{z})\,, \qquad \frac{K_2^2}{\gamma_+} = \frac{1}{1-\bar{z}}\,, \qquad \frac{K_2^2}{\gamma_-} = \frac{1}{1-z}\,. \tag{E.3}$$

We can now express the loop momenta in terms of these massless projections and their associated spinor variables $\lambda^\alpha_{K_i^\flat}$ and $\tilde{\lambda}^{\dot{\alpha}}_{K_i^\flat}$. Explicitly

$$\lambda^\alpha_{\ell_i} = t\lambda^\alpha_{K_1^\flat} + \alpha_{i1}\lambda^\alpha_{K_2^\flat}\,, \tag{E.4}$$

$$\tilde{\lambda}^{\dot{\alpha}}_{\ell_i} = \frac{\alpha_{i2}}{t}\tilde{\lambda}^{\dot{\alpha}}_{K_1^\flat} + \tilde{\lambda}^{\dot{\alpha}}_{K_2^\flat}\,, \tag{E.5}$$

with coefficients

$$\alpha_{11}^+ = \frac{z(\bar{z}-1)}{z-\bar{z}}\,, \qquad \alpha_{12}^+ = \frac{\bar{z}(z-1)}{(z-\bar{z})(\bar{z}-1)}\,, \tag{E.6}$$

$$\alpha_{21}^+ = \frac{z(z-1)}{z-\bar{z}}\,, \qquad \alpha_{22}^+ = \frac{\bar{z}}{z-\bar{z}}\,, \tag{E.7}$$

$$\alpha_{31}^+ = \frac{\bar{z}(z-1)}{z-\bar{z}}\,, \qquad \alpha_{32}^+ = \frac{z}{z-\bar{z}}\,. \tag{E.8}$$

The coefficients associated to the other solution can be found by exchanging $z \leftrightarrow \bar{z}$.

Notice that in the limit $t \to \infty$ all the $\lambda_{\ell_i}$ go to $\lambda_{K_1^\flat}$. Since the limit $t \to \infty$ is the one leading to the direct extraction of the three-mass triangle coefficient, the final result depends only on $K_1^\flat$. In particular, in (5.65) we used a rescaled version of it

$$K^\flat = K_1^\flat \left(1 - \frac{1-z}{1-\bar{z}}\right) . \tag{E.9}$$

The two are not equal, but all our results depend only on $\lambda_{K_1^\flat}$ and we can use the rescaling freedom to replace $\lambda_{K_1^\flat} \to \lambda_{K^\flat}$.

Nevertheless, one should be careful because (5.62) depends also on the contractions $\langle \ell_i \ell_j \rangle$ and the subleading order as $t \to \infty$ becomes relevant in that case,

$$\langle \ell_1 \, \ell_2 \rangle_+ = tz \langle K_1^\flat \, K_2^\flat \rangle \,, \tag{E.10}$$

$$\langle \ell_1 \, \ell_3 \rangle_+ = t \langle K_1^\flat \, K_2^\flat \rangle \,, \tag{E.11}$$

$$\langle \ell_2 \, \ell_3 \rangle_+ = t(1-z) \langle K_1^\flat \, K_2^\flat \rangle \,. \tag{E.12}$$

Once more, the other solution is obtained with the replacement $z \to \bar{z}$. Using these expressions it is easy to go from (5.62) to (5.64). In our alternative expression for the coefficient, (5.71), as well as (5.77), depends on the loop momenta only through $\lambda_{\ell_2}$ and this allows to use straightforwardly the replacement (5.79).

## E.2 Dual conformal variations

Here we consider explicit variations under dual conformal transformations of the function $g(u,v)$ defined in (5.82). We start from (5.90) and we derive

$$\begin{aligned}
\mathsf{K}^\mu z &= \frac{2(z-1)z}{z-\bar{z}} \left((1-\bar{z}) x_{ab}^\mu - \bar{z} x_{bc}^\mu\right) \,, \\
\mathsf{K}^\mu \bar{z} &= \frac{2(1-\bar{z})\bar{z}}{z-\bar{z}} \left((1-z) x_{ab}^\mu - z x_{bc}^\mu\right) \,.
\end{aligned} \tag{E.13}$$

The variation of $\Delta = |z - \bar{z}|$ follows immediately

$$\mathsf{K}^\mu \Delta = \frac{2[v(1+u-v)x_{ab}^\mu - u(1-u+v)x_{bc}^\mu]}{\Delta} \,, \tag{E.14}$$

and it is clearly antisymmetric under the exchange (5.66). Also the variation of $F^{3\mathrm{m}}$ in (5.83) is easily computed

$$\mathsf{K}^\mu F^{3\mathrm{m}}(z,\bar{z}) = -\frac{\log u}{\Delta} \left((u+v-1)\, x_{ab}^\mu + 2u\, x_{bc}^\mu\right) + \frac{\log v}{\Delta} \left((u+v-1)\, x_{bc}^\mu + 2v\, x_{ab}^\mu\right) \,, \tag{E.15}$$

and is antisymmetric as expected. The last ingredient in $g(u,v)$ is $\sqrt{uv}$, whose variation is simply

$$\mathsf{K}^\mu \sqrt{uv} = (x_{ab}^\mu - x_{bc}^\mu)\sqrt{uv} \,. \tag{E.16}$$

Therefore we have shown with an explicit computation that the variation of $g(u,v)$ under dual special conformal transformations is antisymmetric under the exchange (5.66).

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