UNIVERSITAT DE BARCELONA

ESTIMATION OF DENSITIES AND APPLICATIONS

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Abstract

In this paper we show some estimates for the density of a random variable on the Wiener space that satisfies a nondegeneracy condition using the stochastic calculus of variations. The case of a diffusion process is considered, and an application to the solution of a stochastic partial differential equation is discussed.

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1 Introduction

In this paper we present some estimates for the probability density of a random variable on the Wiener space that verifies a nondegeneracy condition. These estimates are obtained by means of the techniques of the stochastic calculus of variations. First an explicit formula for the density is established in Proposition 2.2. This result is similar to Proposition 2.1.1 of [4], but the hypotheses are slightly more general. From this explicit formula we deduce the basic estimates in lemmas 2.3 and 2.4. In Section 3 we present a martingale inequality that is used in Section 4 for the estimation of the density of a diffusion process. Finally in Section 5 the estimates of densities are applied to solve a nonlinear stochastic partial differential equation with an additive white noise.

2 Estimation of densities using Malliavin Calculus

In this section we will introduce some elements of the stochastic calculus of variations and its application to the estimation of the density of a Brownian functional.

Let $B = \{B_t, t \in [0,T]\}$ be a standard Brownian motion defined on the canonical probability space $(\Omega, \mathcal{F}, P)$. That is, Ω is the space of continuous functions on $[0,T]$ which vanish at zero, $\mathcal{F}$ is the Borel σ-field on Ω, and P is the Wiener measure. Let $H = L^2([0,T])$.

Let us first introduce the derivative operator D. We denote by $C_b^\infty(\mathbb{R}^n)$ the set of all infinitely differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f and all of its partial derivatives are bounded.

Let $\mathcal{S}$ denote the class of random variables of the form

$$F = f(B_{t_1}, \dots, B_{t_n}) \tag{2.1}$$

where f belongs to $C_b^\infty(\mathbb{R}^n)$, and $t_1, \dots, t_n \in [0,T]$. If F has the form (2.1) we define its derivative DF as the stochastic process given by

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B_{t_1}, \dots, B_{t_n}) \mathbf{1}_{[0,t_i]}(t), \quad t \in [0,T]. \tag{2.2}$$

The operator D is closable from $\mathcal{S} \subset L^p(\Omega)$ in $L^p(\Omega; L^2([0,T]))$ for each $p \geq 1$. We will denote by $\mathbb{D}^{1,p}$ the closure of the class of smooth random variables $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p}^p = E(|F|^p) + E\left((\int_0^T |D_t F|^2 dt)^{\frac{p}{2}} \right).$$

We can define the iteration of the operator D in such a way that for a smooth random variable F, the derivative $D^k_{t_1,\dots,t_k} F$ is a k-parameter stochastic process.

Then for every $p \geq 1$ and any natural number k we can introduce the space $\mathbb{D}^{k,p}$ as the completion of the family of smooth random variables $\mathcal{S}$ with respect to the norm:

$$\|F\|_{k,p}^p = E(|F|^p) + \sum_{j=1}^{k} E(\|D^j F\|^p_{L^2([0,T]^j)}). \qquad (2.3)$$

Let V be a real separable Hilbert space. We can also introduce the corresponding Sobolev spaces $\mathbb{D}^{k,p}(V)$ of V-valued random variables.

We will denote by δ the adjoint of the operator D as an unbounded operator from $L^1(\Omega)$ into $L^1(\Omega; L^2([0,T]))$. That is, the domain of δ, denoted by $\operatorname{Dom}\delta$, is the set of processes u in $L^1(\Omega; L^2([0,T]))$ such that there exists an integrable random variable $\delta(u)$ verifying

$$E(F\delta(u)) = E(\int_0^T D_t F u_t dt) \qquad (2.4)$$

for any $F \in \mathcal{S}$. The operator δ is an extension of the Itô stochastic integral, called Skorohod integral (see [8]), in the sense that the set $L^2_a([0,T] \times \Omega)$ of square integrable and adapted processes is included into the domain of δ, and δ restricted to $L^2_a([0,T] \times \Omega)$ coincides with the Itô integral.

The space $\mathbb{D}^{1,p}(H) := L^p(\Omega; L^2([0,T]))$ is included into the domain of δ, for each $p > 1$, and for any process $u \in \mathbb{D}^{1,p}(H)$ there is the following estimate for the L^p-norm of the Skorohod integral:

$$E(|\delta(u)|^p) \leq C_p \left(E((\int_0^T |u_s|^2 ds)^{\frac{p}{2}}) + E((\int_0^T \int_0^T |D_s u_t|^2 ds dt)^{\frac{p}{2}}) \right). \qquad (2.5)$$

We will make use of the following property of the Skorohod integral.

Lemma 2.1 *Fix $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $F \in \mathbb{D}^{1,p}$, $u \in \operatorname{Dom} \delta$, be such that $u \in L^q(\Omega; L^2([0,T]))$ and $\delta(u) \in L^q(\Omega)$. Then $Fu \in \operatorname{Dom} \delta$, and*

$$\delta(Fu) = F\delta(u) - \langle DF, u \rangle_H. \tag{2.6}$$

Proof: Let us denote by A the right-hand side of (2.6), which by hypothesis belongs to $L^1(\Omega)$. Then it suffices to show that for any $G \in \mathcal{S}$ we have $E(GA) = E(\langle DG, Fu \rangle_H)$. We can write

$$E(\langle DG, Fu \rangle_H) = E(\langle D(GF), u \rangle_H) - E(G\langle DF, u \rangle_H).$$

We have to prove that

$$E(\langle D(GF), u \rangle_H) = E(GF\delta(u)). \tag{2.7}$$

This is true by definition of $\delta(u)$ if $FG \in \mathcal{S}$. In order to show (2.7) when $F \in \mathbb{D}^{1,p}$ it suffices to take a sequence of smooth random variables F_n which converge to F in $\mathbb{D}^{1,p}$. QED

For any random variable $F \in \mathbb{D}^{1,1}$ and any process $u \in L^1(\Omega; L^2([0,T]))$ we will write

$$D_u F = \int_0^T D_t F u_t dt.$$

Let $F \in \mathbb{D}^{1,1}$ be such that $\|DF\|_H > 0$ a.s. Then we know that F possesses an absolutely continuous distribution (see [2, 6]). On the other hand, we can show the following explicit expression for the density of F, under some additional assumptions:

Proposition 2.2 *Let F be a random variable in the space $\mathbb{D}^{1,1}$. Let u be a process in $L^1(\Omega; L^2([0,T]))$ such that $D_u F \neq 0$ a.s., and $\frac{u}{D_u F}$ belongs to the domain of the operator δ. Then the law of F has a continuous and bounded density given by*

$$p(x) = E\left(\mathbf{1}_{\{F > x\}} \delta\left(\frac{u}{D_u F}\right)\right). \tag{2.8}$$

Proof: Let ψ be a nonnegative smooth function with compact support, and set $\varphi(y) = \int_{-\infty}^{y} \psi(z) dz$, $y \in \mathbb{R}$. We know that $\varphi(F)$ belongs to $\mathbb{D}^{1,1}$, and making the scalar product of its derivative with u yields

$$\langle D(\varphi(F)), u \rangle_H = \psi(F) D_u F.$$

Hence, we obtain

$$E(\psi(F)) = E\left(\left\langle D(\varphi(F)), \frac{u}{D_uF}\right\rangle_H\right).$$

Now, let F_n be a sequence of smooth random variables that converges to F in $\mathbb{D}^{1,1}$. Using the definition of the operator δ we have

$$E\left(\left\langle D(\varphi(F)), \frac{u}{D_uF}\right\rangle_H\right) = \lim_n E\left(\left\langle D(\varphi(F_n)), \frac{u}{D_uF}\right\rangle_H\right)$$
$$= \lim_n E\left(\varphi(F_n)\delta\left(\frac{u}{D_uF}\right)\right) = E\left(\varphi(F)\delta\left(\frac{u}{D_uF}\right)\right).$$

Thus,

$$E(\psi(F)) = E\left(\varphi(F)\delta\left(\frac{u}{D_uF}\right)\right). \tag{2.9}$$

By an approximation argument Eq. (2.9) holds for $\psi(y) = \mathbf{1}_{[a,b]}(y)$. As a consequence, we apply Fubini's theorem to get

$$\begin{aligned} P(a \le F \le b) &= E\left(\left(\int_{-\infty}^F \mathbf{1}_{[a,b]}(x)dx\right)\delta\left(\frac{u}{D_uF}\right)\right) \\ &= \int_a^b E\left(\mathbf{1}_{\{F>x\}}\delta\left(\frac{u}{D_uF}\right)\right)dx, \end{aligned}$$

which implies the desired result. QED

Proposition 2.2 leads to the following estimate for the density $p(x)$ of a random variable F:

$$p(x) \le E\left(\left|\delta\left(\frac{u}{D_uF}\right)\right|\right). \tag{2.10}$$

As a consequence,

$$\int_{\mathbb{R}} p(x)^2 dx \le E\left(\left|\delta\left(\frac{u}{D_uF}\right)\right|\right). \tag{2.11}$$

From Lemma 2.1 and Proposition 2.2 we deduce the following result:

Lemma 2.3 *Let F be a random variable in the space $\mathbb{D}^{1,1}$. Let u be a process in* $\mathrm{Dom}\,\delta$ *verifying the following properties, for some $\frac{1}{p} + \frac{1}{q} = 1$:*

(i) *u belongs to $L^q(\Omega; L^2([0,T]))$.*

(ii) *$\delta(u)$ belongs to $L^q(\Omega)$.*

(iii) *$(D_uF)^{-1}$ belongs to $I\!D^{1,p}$.*

Then F has a density such that

$$p(x) \le E\left(\left|\frac{\delta(u)}{D_uF}\right|\right) + E\left(\left|D_u\left(\frac{1}{D_uF}\right)\right|\right).$$

Proof: From Lemma 2.1 we have

$$\delta\left(\frac{u}{D_uF}\right) = \frac{\delta(u)}{D_uF} - D_u\left(\frac{1}{D_uF}\right).$$

Then we apply Proposition 2.2. QED

The assumptions of the previous lemma can be modified as follows.

Lemma 2.4 *Let F be a random variable in the space $I\!D^{1,1}$. Let u be a process in* Dom δ *verifying the following properties, for some $\frac{1}{p}+\frac{1}{q}=1$:*

(i) *u belongs to $L^q(\Omega; L^2([0,T])$.*

(ii) *$\delta(u)$ belongs to $L^q(\Omega)$.*

(iii) *$(D_uF)^{-1}$ belongs to $L^p(\Omega)$, and*

$$\Phi_u := (D_uF)^{-2}(\|D^2F\|_{H\otimes H}\|u\|_H + \|Du\|_{H\otimes H}\|DF\|_H) \in L^p(\Omega). \tag{2.12}$$

Then F has a density such that

$$p(x) \le E\left(\left|\frac{\delta(u)}{D_uF}\right|\right) \tag{2.13}$$
$$+E\left((D_uF)^{-2}(\|D^2F\|_{H\otimes H}\|u\|_H^2 + \|Du\|_{H\otimes H}\|u\|_H\|DF\|_H)\right).$$

Remarks:

1.- Under the assumptions of Lemma 2.4 we can also write down the following expression for the density of F:

$$p(x) = E\left(\mathbf{1}_{\{F>x\}}\frac{\delta(u)}{D_uF}\right)$$
$$+E\left((D_uF)^{-2}\left(\langle D^2F, u\otimes u\rangle_{H\otimes H} + \int_{[0,T]^2} u_tD_tu_sD_sF dsdt\right)\right).$$

2.- When $u = DF$ the estimate (2.13) leads to

$$p(x) \leq E\left(\left|\frac{\delta(DF)}{\|DF\|_H^2}\right|\right) + 2E\left(\|DF\|_H^{-2}\|D^2F\|_{H\otimes H}\right). \qquad (2.14)$$

3.- Suppose that $F = \int_0^1 H_s dW_s$, where H is an adapted process that verifies:

(i) For each $s \in [0,1]$, $H_s \in \mathbb{D}^{2,2}$, $E\int_0^1 H_s^2 ds < \infty$ and

$$\lambda := \sup_{s,t\in[0,1]} E(|D_s H_t|^p) + \sup_{r,s\in[0,1]} E((\int_0^1 |D_{r,s}^2 H_t|^2 dt)^{p/2}) < \infty,$$

for some $p > 3$.

(ii) $0 < \rho \leq |H_s|$ for all $s \in [0,1]$

Taking $u = DF$ the estimate (2.14) leads to

$$p(x) \leq cP(|F| > |x|)^{1/q},$$

where $q > \frac{p}{p-3}$ and the constant c dpends on λ, ρ, and p (see [5]).

4.- Notice that if u is an adapted process and $D_t F = u_t e^{N_t}$ (this is true if F is the value at time t of a diffusion process), then

$$\left|\frac{\delta(u)}{D_u F}\right| \leq \sup_{0\leq t\leq T} e^{-2N_t} \cdot \frac{|\int_0^T u_t dW_t|}{\int_0^T u_t^2 dt}.$$

Therefore, in order to obtain estimates for the expectation of the above expression we need to estimate the norm p of $\frac{M_t}{\langle M\rangle_t}$, where M_t is a Brownian martingale.

3 Martingale inequalities

The following theorem provides an estimate for the p-norm of a martingale divided by its quadratic variation, where $1 \leq p < \infty$. This norm is bounded by a universal constant times the q-norm of the inverse of the square root of the quadratic variation. A related estimate can be found in Exercise 4.18 of [7].

Theorem 3.1 *Let $\{M_t,\ t \geq 0\}$ be a continuous martingale null at 0. Then for each $1 \leq p < q = p + \varepsilon$, there exists a universal constant $C := C(p, q)$ such that*

$$\left\| \frac{M_t}{\langle M \rangle_t} \right\|_p \leq C \| \langle M \rangle_t^{-\frac{1}{2}} \|_q. \tag{3.1}$$

Proof: We have

$$E\left[\left| \frac{M_t}{\langle M \rangle_t} \right|^p \right] = \int_0^\infty p x^{p-1} P[|M_t| > x \langle M \rangle_t] dx.$$

Notice that

$$P[|M_t| > x\langle M \rangle_t] \leq P[M_t > x\langle M \rangle_t] + P[-M_t > x\langle M \rangle_t].$$

So, it suffices to estimate the term

$$\int_0^\infty x^{p-1} P[M_t > x\langle M \rangle_t] dx.$$

We can assume (by a truncation argument) that M is bounded. Then

$$\begin{aligned} P[M_t > x\langle M \rangle_t] &\leq P[e^{\lambda M_t - \theta \langle M \rangle_t} > e^{(\lambda x - \theta)\langle M \rangle_t}] \\ &\leq E[e^{\lambda M_t - \theta \langle M \rangle_t} e^{-(\lambda x - \theta)\langle M \rangle_t}] \\ &\leq (E[e^{\lambda \alpha M_t - \theta \alpha \langle M \rangle_t}])^{1/\alpha} (E[e^{-\beta(\lambda x - \theta)\langle M \rangle_t}])^{1/\beta}, \end{aligned}$$

where $\frac{1}{\beta} + \frac{1}{\alpha} = 1$. Choose $\lambda, \alpha, \beta, \theta$ such that $\frac{(\lambda\alpha)^2}{2} = \theta\alpha$, that is, $\theta = \frac{\lambda^2 \alpha}{2}$. With these choices we have

$$P[M_t > x\langle M \rangle_t] \leq (E[e^{-\beta(\lambda x - \frac{\lambda^2 \alpha}{2})\langle M \rangle_t}])^{1/\beta}.$$

Optimizing over λ yields $\lambda = \frac{x}{\alpha}$ and we get

$$P[M_t > x\langle M \rangle_t] \leq (E[e^{-\frac{\beta x^2}{2\alpha}\langle M \rangle_t}])^{1/\beta} = (E[e^{-(\beta-1)\frac{x^2}{2}\langle M \rangle_t}])^{1/\beta}.$$

Hence, for each $\varepsilon > 0$, $\delta > 1$:

$$E\left[\left| \frac{M_t}{\langle M \rangle_t} \right|^p \right] \leq 2 \int_0^\infty p x^{p-1} (E[e^{-(\beta-1)\frac{x^2}{2}\langle M \rangle_t}])^{1/\beta} dx$$

$$
\begin{aligned}
&\leq 2\varepsilon^p + 2\int_\varepsilon^\infty px^{p-1}(E[e^{-(\beta-1)\frac{x^2}{2}\langle M\rangle_t}])^{1/\beta}dx \\
&= 2\varepsilon^p + 2p\int_\varepsilon^\infty x^{-\delta}(E[x^{(\delta+p-1)\beta}e^{-(\beta-1)\frac{x^2}{2}\langle M\rangle_t}])^{1/\beta}dx \\
&\leq 2\varepsilon^p + 2p\left(\int_\varepsilon^\infty x^{-\delta}dx\right) \\
&\quad\times\left(E\left[\sup_{x\in\mathbb{R}}[x^{(\delta+p-1)\beta}e^{-(\beta-1)\frac{x^2}{2}\langle M\rangle_t}]\right]\right)^{1/\beta}.
\end{aligned}
$$

Let

$$\Phi(x) = x^{(\delta+p-1)\beta}e^{-(\beta-1)\frac{x^2}{2}\langle M\rangle_t}.$$

Then

$$\Phi'(x) = \left[\beta(\delta+p-1)x^{(\delta+p-1)\beta-1} - x^{\beta(\delta+p-1)+1}(\beta-1)\langle M\rangle_t\right]e^{-(\beta-1)\frac{x^2}{2}\langle M\rangle_t}.$$

The function Φ attains its maximum at

$$x_0 = \sqrt{\frac{\beta(\delta+p-1)}{\beta-1}}\langle M\rangle_t^{-1/2}.$$

Hence,

$$\Phi(x_0) = \left(\frac{\beta(\delta+p-1)}{\beta-1}\right)^{\frac{\delta+p-1}{2}\beta}\langle M\rangle_t^{-\frac{(\delta+p-1)\beta}{2}}e^{-\frac{(\delta+p-1)}{2}\beta}.$$

As a consequence,

$$
\begin{aligned}
&E\left[\left|\frac{M_t}{\langle M\rangle_t}\right|^p\right] \\
&\leq 2\varepsilon^p + \frac{2p}{\delta-1}\varepsilon^{1-\delta}e^{-\frac{\delta+p-1}{2}}\left(\frac{\beta(\delta+p-1)}{\beta-1}\right)^{\frac{\delta+p-1}{2}}\left(E\left[\langle M\rangle_t^{-\frac{(\delta+p-1)\beta}{2}}\right]\right)^{1/\beta} \\
&= 2\varepsilon^p + \varepsilon^{1-\delta}B,
\end{aligned}
$$

where

$$B = \frac{2p}{\delta-1}e^{-\frac{\delta+p-1}{2}}\left(\frac{\beta(\delta+p-1)}{\beta-1}\right)^{\frac{\delta+p-1}{2}}\left(E\left[\langle M\rangle_t^{-\frac{(\delta+p-1)\beta}{2}}\right]\right)^{1/\beta}.$$

Now we optimize over ε. Set $A(\varepsilon) = 2\varepsilon^p + \varepsilon^{1-\delta}B$. Then

$$A'(\varepsilon) = 2p\varepsilon^{p-1} + (1-\delta)\varepsilon^{-\delta}B,$$

and the unique solution of $A'(\varepsilon) = 0$ is given by

$$\varepsilon_0 = \left(\frac{\delta-1}{2p}\right)^{\frac{1}{p+\delta-1}} B^{\frac{1}{p+\delta-1}}$$

and

$$\begin{aligned}
A(\varepsilon_0) &= 2\left(\frac{\delta-1}{2p}\right)^{\frac{p}{p+\delta-1}} B^{\frac{p}{p+\delta-1}} + \left(\frac{\delta-1}{2p}\right)^{\frac{1-\delta}{p+\delta-1}} B^{\frac{1-\delta}{p+\delta-1}+1} \\
&= B^{\frac{p}{p+\delta-1}}\left(2\left(\frac{\delta-1}{2p}\right)^{\frac{p}{p+\delta-1}} + \left(\frac{\delta-1}{2p}\right)^{\frac{1-\delta}{p+\delta-1}}\right) \\
&= \left(\frac{2p}{\delta-1}\right)^{\frac{p}{p+\delta-1}} e^{-p/2}\left(\frac{\beta(\delta+p-1)}{\beta-1}\right)^{p/2}\left(2\left(\frac{\delta-1}{2p}\right)^{\frac{p}{p+\delta-1}} + \left(\frac{\delta-1}{2p}\right)^{\frac{1-\delta}{p+\delta-1}}\right) \\
&\quad \times \left(E\left[\langle M\rangle_t^{-\frac{(\delta+p-1)\beta}{2}}\right]\right)^{\frac{p}{\beta(\delta+p-1)}} \\
&= 2e^{-p/2}\left(\frac{\beta(\delta+p-1)}{\beta-1}\right)^{p/2}\left(1+\frac{p}{\delta-1}\right)\left(E\left[\langle M\rangle_t^{-\frac{(\delta+p-1)\beta}{2}}\right]\right)^{\frac{p}{\beta(\delta+p-1)}}.
\end{aligned}$$

Therefore,

$$\left\|\frac{M_t}{\langle M\rangle_t}\right\|_p \le 2^{1/p}e^{-1/2}\sqrt{\frac{\beta(\delta+p-1)}{\beta-1}}\left(1+\frac{p}{\delta-1}\right)^{1/p}\|\langle M\rangle_t^{-1/2}\|_{(\delta+p-1)\beta}.$$

Take $\delta = 2-\frac{1}{\beta}$ and put $(\delta-1+p)\beta = p+\varepsilon$. Then $\varepsilon = (\beta-1)(p+1)$. Hence, we can introduce the constant

$$C_{p,\varepsilon} = \frac{2^{1/p}}{\sqrt{e}}\left((p+1)\left(1+\frac{p}{\varepsilon}\right)\right)^{\frac{1}{2}+\frac{1}{p}},$$

and we get

$$\left\|\frac{M_t}{\langle M\rangle_t}\right\|_p \le C_{p,\varepsilon}\|\langle M\rangle_t^{-1/2}\|_{p+\varepsilon}.$$

QED

4 Estimation of densities for diffusion processes

Let $B = \{B_t, t \in [0,T]\}$ be a standard Brownian motion defined on the canonical probability space $(\Omega, \mathcal{F}, P)$, as in Section 1. Consider the diffusion process $X = \{X_t, t \in [0,T]\}$ solution of the following stochastic differential equation:

$$X_t = x_0 + \int_0^t \sigma(s, X_s)dB_s + \int_0^t b(s, X_s)ds, \quad t \in [0,T].$$

We will introduce the following hypothesis on the coefficients σ and b:

(H) $\sigma(s,x)$ and $b(s,x)$ are of class C^2 with respect to x, and

$$\begin{aligned} |\sigma(0,x)| &\leq K, \quad |b(0,x)| \leq K, \\ |\sigma'(s,x)| &\leq K, \quad |b'(s,x)| \leq K, \end{aligned}$$

for some constant $K > 0$, and $\sigma''(s,x)$ and $b''(s,x)$ have polynomial growth in x, uniformly in t.

In the sequel we will denote by C a generic constant which may depend on $p > 1$, $T \geq 0$ and the coefficients σ and b.

Theorem 4.1 *Let $\sigma(s,x)$ and $b(s,x)$ functions satisfying hypothesis (H). Suppose that*

$$E\left(|\int_0^t \sigma(s, X_s)^2 ds|^{-\frac{p}{2}}\right) < \infty,$$

for some $p > 1$ and for all $t \in (0,T]$. Then for all $t \in (0,T]$, the random variable X_t possesses a continuous density $p_t(x)$ such that

$$p_t(x) \leq C\|(\int_0^t \sigma(s, X_s)^2 ds)^{-\frac{1}{2}}\|_p, \tag{4.1}$$

for some constant $C > 0$.

Proof: Fix $t \in (0,T]$. Under hypothesis **(H)** we know that $X_t \in \mathbb{D}^{2,p}$ for all $p \geq 2$, and, if $s \leq t$ we have

$$D_s X_t = \sigma(s, X_s) + \int_s^t \sigma'(r, X_r) D_s X_r dB_r + \int_s^t b'(r, X_r) D_s X_r dr.$$

Hence,

$$D_sX_t = \sigma(s, X_s)\exp\left(\int_s^t \sigma'(r, X_r)dB_r + \int_s^t (b' - \frac{1}{2}(\sigma')^2)(r, X_r)dr\right).$$

Using the notation

$$M_{s,t} = \exp\left(\int_s^t \sigma'(r, X_r)dB_r + \int_s^t (b' - \frac{1}{2}(\sigma')^2)(r, X_r)dr\right),$$

we can write

$$D_sX_t = \sigma(s, X_s)M_{s,t},$$

for all $s \leq t$. For $s_2 < s_1 \leq t$ we have

$$\begin{aligned} D^2_{s_1,s_2}X_t &= \sigma'(s_1, X_{s_1})D_{s_2}X_{s_1}M_{s_1,t} + \sigma(s_1, X_{s_1})M_{s_1,t} \\ &\quad \times \left(\int_{s_1}^t \sigma''(r, X_r)D_{s_2}X_r dB_r + \int_{s_1}^t (b'' - \sigma'\sigma'')(r, X_r)D_{s_2}X_r dr\right) \\ &= \sigma'(s_1, X_{s_1})\sigma(s_2, X_{s_2})M_{s_1,t}M_{s_2,s_1} \\ &\quad + \sigma(s_1, X_{s_1})\sigma(s_2, X_{s_2})M_{s_1,t} \\ &\quad \times \left(\int_{s_1}^t \sigma''((r, X_r)M_{s_2,r}dB_r + \int_{s_1}^t (b'' - \sigma'\sigma'')(r, X_r)M_{s_2,r}dr\right). \end{aligned}$$

We will apply Lemma 2.4 to the random variable X_t and to the process $u_s = \sigma(s, X_s)\mathbf{1}_{[0,t]}(s)$. Clearly u_s belongs to the domain of δ, and

$$\delta(\sigma(s, X_s)\mathbf{1}_{[0,t]}(s)) = \int_0^t \sigma(s, X_s)dB_s.$$

As a consequence, the process u satisfies the assumptions (i) and (ii) of Lemma 2.4 for all $q > 1$. On the other hand, $D_u(X_t) = \int_0^t \sigma(s, X_s)^2 M_{s,t}ds$. Consider the random variables

$$R_t = \int_0^t \sigma(s, X_s)^2 ds,$$

and

$$S_t = \left(\sup_{s\in[0,t]} M_{0,t}^{-1}\right)\left(\sup_{s\in[0,t]} M_{0,t}\right).$$

Notice that $\sup_{s\in[0,t]} M_{s,t}^{-1} \leq S_t$, and $E(S_t^m) < \infty$ for all $m \geq 2$. Then, we have the following estimate

$$|D_u(X_t)|^{-1} \leq (R_t)^{-1}S_t.$$

Hence, $(D_u(X_t))^{-1}$ belongs to $L^{p'}(\Omega)$ for any $1 < p' < p$. In order to show that the random variable Φ_u given by (2.12) belongs to $L^{p'}(\Omega)$ for any $1 < p' < p$ we will make use of the following estimates:

$$\|D^2X_t\|_{H\otimes H} \leq (S_t)^2\sqrt{R_t}\|\sigma'\|_\infty$$
$$+2(S_t)^2R_t \sup_{0\leq s\leq t}\left|\int_0^s \sigma''(r,X_r)M_{0,r}dB_r + \int_0^s (b''-\sigma'\sigma'')(r,X_r)M_{0,r}dr\right|,$$

$$\begin{aligned}\|u\|_H^2 &= R_t,\\ \|Du\|_{H\otimes H} &\leq \|\sigma'\|_\infty\sqrt{R_t}S_t,\\ \|DX_t\|_H &\leq \sqrt{R_t}S_t.\end{aligned}$$

Thus,

$$\Phi_u \leq (R_t)^{-\frac{1}{2}}\Bigg((S_t)^4\|\sigma'\|_\infty$$
$$+2(S_t)^4 \sup_{0\leq s\leq t}\left|\int_0^s \sigma''(r,X_r)M_{0,r}dB_r + \int_0^s (b''-\sigma'\sigma'')(r,X_r)M_{0,r}dr\right|$$
$$+\|\sigma'\|_\infty(S_t)^4\Bigg) := (R_t)^{-\frac{1}{2}}\Psi_u,$$

and Φ_u belongs to $L^{p'}(\Omega)$ for any $1 < p' < p$ because $(R_t)^{-\frac{1}{2}}$ is in $L^p(\Omega)$, and Ψ_u has moments of all orders.

By Theorem 3.1 we have

$$E\left(\left|\frac{\delta(u)}{D_uX_t}\right|\right) \leq E\left(\frac{\left|\int_0^t \sigma(s,X_s)dB_s\right|}{\int_0^t \sigma(s,X_s)^2ds}S_t\right)$$
$$\leq C\|(R_t)^{-\frac{1}{2}}\|_p.$$

Again by Hölder's inequality we obtain

$$E(\Phi_u) \leq C\|(R_t)^{-\frac{1}{2}}\|_p,$$

amd, therefore,

$$p_t(x) \leq C\|(R_t)^{-\frac{1}{2}}\|_p.$$

QED

Let us suppose that $\sigma : I\!R \to I\!R$ is a twice continuously differentiable function such that σ' is bounded, and $\sigma(x) > 0$ for all $x > 0$, and $\sigma(0) = 0$. Fix $x > 0$. Let X_t be the solution of the stochastic differential equation

$$X_t = x + \int_0^t \sigma(X_s)dB_s + \int_0^t b(s, X_s)ds, \quad t \in [0, T]. \tag{4.2}$$

Consider the stopping time

$$\tau = \inf\{s > 0 : \sigma(X_s) = \frac{1}{2}\sigma(x)\}.$$

Let $p > 1$. Then we have

$$\begin{aligned} E\left(\left(\int_0^t \sigma(X_s)^2 ds\right)^{-\frac{p}{2}}\right) &= E\left(\mathbf{1}_{\{\tau > t\}}\left(\int_0^t \sigma(X_s)^2 ds\right)^{-\frac{p}{2}}\right) \\ &+ E\left(\mathbf{1}_{\{\tau \le t\}}\left(\int_0^t \sigma(X_s)^2 ds\right)^{-\frac{p}{2}}\right) \\ &\le \left(\frac{\sigma(x)}{2}\right)^{-p} t^{-\frac{p}{2}} + \left(\frac{\sigma(x)}{2}\right)^{-p} E\left(\mathbf{1}_{\{\tau \le t\}}\tau^{-\frac{p}{2}}\right). \end{aligned}$$

We have the following estimates

$$E\left(\mathbf{1}_{\{\tau \le t\}}\tau^{-\frac{p}{2}}\right) = \frac{p}{2}\int_{\frac{1}{t}}^{\infty} y^{\frac{p}{2}-1} P(\tau < \frac{1}{y})dy. \tag{4.3}$$

On the other hand,

$$\begin{aligned} P(\tau < \frac{1}{y}) &= P\left(\inf_{0\le s\le \frac{1}{y}} \sigma(X_s) \le \frac{1}{2}\sigma(x)\right) \\ &\le P\left(\sup_{0\le s\le \frac{1}{y}} |\sigma(X_s) - \sigma(x)| \ge \frac{1}{2}\sigma(x)\right) \\ &\le \left(\frac{2}{\sigma(x)}\right)^q E\left(\sup_{0\le s\le \frac{1}{y}} |\sigma(X_s) - \sigma(x)|^q\right) \\ &\le \left(\frac{2}{\sigma(x)}\right)^q \|\sigma'\|_\infty^q E\left(\sup_{0\le s\le \frac{1}{y}} |X_s - x|^q\right) \\ &\le C\left(\frac{2}{\sigma(x)}\right)^q y^{-\frac{q}{2}}, \end{aligned}$$

for some constant $C > 0$. Hence, if $p < q$ we obtain from (4.3)

$$E\left(\mathbf{1}_{\{\tau \le t\}} \tau^{-\frac{p}{2}}\right) \le C\sigma(x)^{-q} t^{-\frac{p-q}{2}}. \tag{4.4}$$

Therefore

$$E\left(\left(\int_0^t \sigma(X_s)^2 ds\right)^{-\frac{p}{2}}\right) \le C\left(\frac{1}{\sigma(x)\sqrt{t}} + \sigma(x)^{-1-\frac{q}{p}} t^{-\frac{p-q}{2p}}\right).$$

Finally, taking $q = p(1+\epsilon)$ we get

$$p_t(y) \le C\left(\sigma(x)^{-1} t^{-\frac{1}{2}} + \sigma(x)^{-2-\epsilon} t^{\frac{\epsilon}{2}}\right). \tag{4.5}$$

When the drift is nonnegative we can obtain an estimate which is better than (4.5):

Proposition 4.2 *Let us suppose that $\sigma : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function such that $\sigma(0) = 0$, $\sigma(x) > 0$ and $\sigma'(x) > 0$ if $x > 0$ and σ' is bounded. Let $b : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a function satisfying hypothesis (H1) and such that $b(s,x) \ge 0$. Then, the density $p_t(y)$ of the solution X_t of Eq. (4.2) satisfies*

$$p_t(y) \le \frac{C}{\sigma(x)\sqrt{t}}, \tag{4.6}$$

for all $x > 0$, $t \in (0,T]$.

Proof: Applying Itô's formula leads to

$$\frac{\sigma(X_s)}{\sigma(x)} = 1 + \int_0^s \frac{(\sigma'\sigma)(X_r)}{\sigma(x)} dB_r + \int_0^s \frac{(\sigma''\sigma^2)(X_r)}{2\sigma(x)} dr + \int_0^s \frac{b(r,X_r)\sigma'(X_r)}{\sigma(x)} dr.$$

Hence,

$$\frac{\sigma(X_s)}{\sigma(x)} = e^{N_s}\left(1 + \int_0^s \frac{e^{-N_r}}{\sigma(x)} b(r,X_r)\sigma'(X_r) dr\right),$$

where

$$N_s = \int_0^s \sigma'(X_r) dB_r - \frac{1}{2}\int_0^s (\sigma'(X_r))^2 dr + \frac{1}{2}\int_0^s (\sigma''\sigma)(X_r) dr.$$

Then, using the estimate (4.1) we obtain

$$\begin{aligned}\sigma(x)p_t(y) &\leq C\left\|\left(\int_0^t\left(\frac{\sigma(X_s)}{\sigma(x)}\right)^2 ds\right)^{-\frac{1}{2}}\right\|_p \\ &\leq C\left\|\sup_{0\leq s\leq t} e^{-N_s}\left(\int_0^t\left(1+\int_0^s \frac{e^{-N_r}}{\sigma(x)}b(r,X_r)\sigma'(X_r)dr\right)^2 ds\right)^{-\frac{1}{2}}\right\|_p \\ &\leq \frac{C}{\sqrt{t}}.\end{aligned}$$

QED

5 Nonlinear stochastic partial differential equation with additive white noise

In this section we will apply the estimates for the density of a diffusion process obtained in the previous section to solve parabolic stochastic equations perturbed by an additive white noise. Consider the second order differential operator given by

$$Lf = \frac{1}{2}\sigma(x)^2 f'' + b(x)f',$$

where σ and b are twice continuously differentiable functions with bounded first derivative, and such that $\sigma(x) > 0$ and $\sigma'(x) > 0$ for all $x > 0$, $\sigma(0) = 0$, and $b(x) \geq 0$. Let $W = \{W(A), A \in \mathcal{B}([0,\infty) \times I\!R), |A| < \infty\}$ be a white noise on the parameter space $[0,\infty) \times I\!R$ with intensity equals to the Lebesgue measure. We are interested in the following stochastic partial differential equation

$$\frac{\partial v}{\partial t} = Lv + \frac{\partial^2 W}{\partial t \partial x} + g(v(t,x)), \tag{5.1}$$

where $t \geq 0$, $x \geq 0$, and with an initial condition $v(0,x) = u_0(x)$. We assume that $u_0 \in C_b([0,\infty))$. Equation (5.1) is formal, and, as usual, we will replace it by the following evolution equation

$$\begin{aligned}v(t,x) = \int_0^t p_t(x,y)u_0(y)dy + \int_0^t\int_0^\infty p_{t-s}(x,y)W(ds,dy) \\ + \int_0^t\int_0^\infty p_{t-s}(x,y)g(v(s,y))dsdy,\end{aligned} \tag{5.2}$$

where $p_t(x,y)$ is the fundamental solution of

$$\frac{\partial p}{\partial t} = Lp.$$

Let us first consider the case $g \equiv 0$ and $u_0 \equiv 0$. Then the solution to Equation (5.2) will be a zero mean Gaussian process given by

$$u(t,x) = \int_0^t \int_0^\infty p_{t-s}(x,y) W(ds,dy). \tag{5.3}$$

Proposition 5.1 *The random field $u = \{u(t,x), t \geq 0, x > 0\}$ given by (5.3) is a solution to Equation (5.1) in the sense of distributions, and moreover,*

$$E(|u(t,x)|^2) \leq C\sigma(x)^{-1}. \tag{5.4}$$

Proof: We will only show the estimate (5.4). Using the estimate (4.6) we obtain

$$E(|u(t,x)|^2) = \int_0^t \int_0^\infty p_{t-s}(x,y)^2 ds dy \leq \int_0^t (\sup_{y>0} p_{t-s}(x,y)) ds$$
$$\leq \int_0^t \frac{C}{\sigma(x)\sqrt{s}} ds = \frac{C\sqrt{t}}{\sigma(x)}.$$

QED

Proposition 5.2 *The random field $u = \{u(t,x), t \geq 0, x > 0\}$ given by (5.3) possesses a version which is continuous in $[0,\infty) \times (0,\infty)$, and satisfies for each $T > 0$ and $\frac{1}{2} < \gamma < 1$,*

$$\lim_{x \downarrow 0} \sup_{0 \leq t \leq T} \sigma(x)^\gamma |u(t,x)| = 0.$$

Proof: Fix $\frac{1}{2} < \gamma < 1$ and define

$$\begin{aligned} \tilde{u}(t,x) &= \sigma(x)^\gamma u(t,x), \quad x > 0, t \geq 0, \\ \tilde{u}(t,0) &= 0. \end{aligned}$$

We are going to show that $\bar{u}(t,x)$ has a continuous version in $[0,\infty)^2$ by means of the Kolmogorov's continuity criterion. Suppose that $t \geq s$ and $x > y$. Applying the estimate (5.4) we deduce:

$$\begin{aligned}
&E(|\sigma(x)^\gamma \bar{u}(t,x) - \sigma(y)^\gamma \bar{u}(s,y)|^2) \leq C\{E(|\sigma(x)^\gamma \bar{u}(t,x) - \sigma(y)^\gamma \bar{u}(t,y)|^2) \\
&\quad + E(|\sigma(y)^\gamma (\bar{u}(t,x) - \bar{u}(s,x))|^2) + E(|\sigma(y)^\gamma (\bar{u}(s,x) - \bar{u}(s,y))|^2)\} \\
&\leq C\left\{ \frac{C}{\sigma(x)} (\sigma(x)^\gamma - \sigma(y)^\gamma)^2 + \sigma(y)^{2\gamma} \int_s^t \int_0^\infty p_{t-\theta}(x,z)^2 dz d\theta \right. \\
&\quad + \sigma(y)^{2\gamma} \int_0^s \int_0^\infty |p_{t-\theta}(x,z) - p_{s-\theta}(x,z)|^2 dz d\theta \\
&\quad \left. + \sigma(y)^{2\gamma} \int_0^s \int_0^\infty |p_{s-\theta}(x,z) - p_{s-\theta}(y,z)|^2 dz d\theta \right\}. \\
&= C\{A_1 + A_2 + A_3 + A_4\}.
\end{aligned}$$

Clearly

$$A_1 \leq C|x-y|^{2\gamma-1}.$$

For the term A_2 using the estimate (4.6) we obtain

$$A_2 \leq \sigma(y)^{2\gamma} \int_s^t \frac{C}{\sigma(x)\sqrt{t-\theta}} d\theta \leq C|t-s|^{\frac{1}{2}}.$$

The estimation of the terms A_3 and A_4 is more involved. Define $u_r^s = \sigma(X_r)\mathbf{1}_{[0,s]}(r)$, where X_r is the solution to Eq. (4.2). We can write

$$\begin{aligned}
&\int_0^\infty |p_{t-\theta}(x,z) - p_{s-\theta}(x,z)|^2 dz \\
&= E\left[p_{t-\theta}(x, X_{t-\theta}) - p_{t-\theta}(x, X_{s-\theta}) - p_{s-\theta}(x, X_{t-\theta}) + p_{s-\theta}(x, X_{s-\theta})\right] \\
&= E\left[\left(\mathbf{1}_{\{X_{t-\theta} > \bar{X}_{t-\theta}\}} - \mathbf{1}_{\{X_{t-\theta} > \bar{X}_{s-\theta}\}}\right) \delta\left(\frac{u^{t-\theta}}{DX_{u^{t-\theta}} X_{t-\theta}}\right)\right] \\
&\quad - E\left[\left(\mathbf{1}_{\{X_{s-\theta} > \bar{X}_{t-\theta}\}} - \mathbf{1}_{\{X_{s-\theta} > \bar{X}_{s-\theta}\}}\right) \delta\left(\frac{u^{s-\theta}}{DX_{u^{s-\theta}} X_{s-\theta}}\right)\right],
\end{aligned}$$

where $\{\bar{X}_t^x\}$ denotes an independent copy of $\{X_t^x\}$. Thus it suffices to estimate a term of the form

$$\begin{aligned}
A_3^1 &= \sigma(y)^{2\gamma} \int_0^s E\left[\mathbf{1}_{\{\bar{X}_{t-\theta} < X_{t-\theta} \leq \bar{X}_{s-\theta}\}} \left|\delta\left(\frac{u^{t-\theta}}{DX_{u^{t-\theta}} X_{t-\theta}}\right)\right|\right] d\theta \\
&\leq \sigma(y)^{2\gamma} \int_0^s \bar{E}\left[\left(\int_{\bar{X}_{t-\theta}}^{\bar{X}_{s-\theta}} p_{t-\theta}(x,z) dz\right)^{\frac{1}{\alpha}}\right] \|\delta\left(\frac{u^{t-\theta}}{DX_{u^{t-\theta}} X_{t-\theta}}\right)\|_\beta d\theta,
\end{aligned}$$

with $\frac{1}{\alpha}+\frac{1}{\beta}=1$. As a consequence,

$$A_3^1 \le \sigma(y)^{2\gamma}\int_0^s \left(\frac{C}{\sqrt{\theta}\sigma(x)}\right)^{1+\frac{1}{\alpha}} \left(\bar{E}((\bar{X}_{s-\theta}-\bar{X}_{t-\theta})^+)\right)^{\frac{1}{\alpha}} d\theta$$
$$\le C|s-t|^{\frac{1}{2\alpha}},$$

if $\alpha=\frac{1}{2\gamma-1}$.

The estimation of the term A_4 can be done similarly as follows:

$$\int_0^s\int_0^\infty |p_{s-\theta}(x,z)-p_{s-\theta}(y,z)|^2 dz d\theta$$
$$=\int_0^s E\left(p_\theta(x,X_\theta^x)-p_\theta(x,X_\theta^y)-p_\theta(y,X_\theta^x)+p_\theta(y,X_\theta^y)\right)d\theta,$$

where $\{X_t^x\}$ denotes the solution to Eq. (4.2) with starting point x. Hence, we can write

$$A_4=\sigma(y)^{2\gamma}\left\{\int_0^s E\left[\left(\mathbf{1}_{\{X_\theta^x>\bar{X}_\theta^x\}}\mathbf{1}_{\{X_\theta^x>\bar{X}_\theta^y\}}\right)\delta\left(\frac{u^\theta}{D_{u^\theta}X_\theta^x}\right)\right]d\theta\right.$$
$$\left.-\int_0^s E\left[\left(\mathbf{1}_{\{X_\theta^y>\bar{X}_\theta^x\}}\mathbf{1}_{\{X_\theta^y>\bar{X}_\theta^y\}}\right)\delta\left(\frac{u^\theta}{D_{u^\theta}X_\theta^y}\right)\right]d\theta,\right.$$

where $\{\bar{X}_t^x\}$ denotes an independent copy of $\{X_t^x\}$. Thus, it suffices to study a term of the form

$$A_4^1=\sigma(y)^{2\gamma}\int_0^s E\left[\mathbf{1}_{\{\bar{X}_\theta^x<X_\theta^x\le\bar{X}_\theta^y\}}\left|\delta\left(\frac{u^\theta}{D_{u^\theta}X_\theta^x}\right)\right|\right]d\theta$$
$$\le\sigma(y)^{2\gamma}\int_0^s \bar{E}\left[\left(\int_{\bar{X}_\theta^x}^{\bar{X}_\theta^y} p_\theta(x,z)dz\right)^{\frac{1}{\alpha}}\right]\left\|\delta\left(\frac{u^\theta}{D_{u^\theta}X_\theta^x}\right)\right\|_\beta d\theta,$$

with $\frac{1}{\alpha}+\frac{1}{\beta}=1$. As a consequence,

$$A_4^1\le\sigma(y)^{2\gamma}\int_0^s\left(\frac{C}{\sqrt{\theta}\sigma(x)}\right)^{1+\frac{1}{\alpha}}\left(\bar{E}((\bar{X}_\theta^y-\bar{X}_\theta^x)^+)\right)^{\frac{1}{\alpha}}d\theta$$
$$\le C|x-y|^{\frac{1}{\alpha}},$$

if $\alpha=\frac{1}{2\gamma-1}$. QED

Suppose now that $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz and bounded function. Define $z(t,x) = v(t,x) - u(t,x)$ where u is the solution to Eq. (5.3). Then v solves Eq. (5.1) if and only if z is a solution of the pathwise equation

$$z(t,x) = \int_0^t p_t(x,y)u_0(y)dy + \int_0^t \int_0^\infty p_{t-s}(x,y)g(u(s,y)+z(s,y))dsdy. \quad (5.5)$$

Fix $\frac{1}{2} < \gamma < 1$. Let us denote by C_γ the class of continuous functions $\varphi : [0,T] \times (0,\infty) \to \mathbb{R}$ such that

$$\lim_{x \downarrow 0} \sup_{0 \le t \le T} \sigma(x)^\gamma |\varphi(t,x)| = 0.$$

Let C_γ^0 the class of functions in C_γ such that $\varphi(0,x) = 0$. Using standard arguments, and the Lipschtiz and boundedness properties of g, one can show that for any function $\varphi \in C_\gamma^0$, there is a unique bounded function T continuous in $[0,T] \times (0,\infty)$ such that

$$\begin{aligned}(T\varphi)(t,x) &= \int_0^t p_t(x,y)u_0(y)dy \\ &+ \int_0^t \int_0^\infty p_{t-s}(x,y)g(\varphi(s,y) + (T\varphi)(s,y))dsdy.\end{aligned}$$

By Proposition 5.2 the stochastic process $u(t,x)$ has its trajectories in C_γ^0 a.s. Consequently, Eq. (5.1) has a unique solution with trajectories continuous in $[0,T] \times (0,\infty)$ given by $v = Tu$.

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