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by

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A Strong Completeness Theorem for the Gentzen systems associated with finite algebras

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Abstract

In this paper we study consequence relations on the set of many sided sequents. We deal with the consequence relations axiomatized by the sequent calculi defined in [2] and associated with arbitrary finite algebras. This consequence relations are examples of what we call Gentzen systems. We define a semantics for these systems and prove a Strong Completeness Theorem, which is an extension of the Completeness Theorem for provable sequents stated in [2]. For the special case of the finite linear MV-algebras, the Strong Completeness Theorem was proved in [11], as a consequence of McNaughton's Theorem. The main tool to prove this result for arbitrary algebras is the deduction-detachment theorem for Gentzen systems.

Mathematics Subject Classification: 03B50, 03F03, 03B22.

Keywords: Many-valued propositional logic, Gentzen system, sequent calculus, deduction theorem, completeness, cut elimination, finite algebra.

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UNIVERSITAT DE BARCELONA CRAI

Introduction and outline of the paper

A class of sequent calculi associated with finite algebras is defined in [2]. In this paper we continue the study of the Gentzen system determined by these sequent calculi. This study started in [11], where, generalizing a result of [14], the m-dimensional algebraizable Gentzen systems were characterized. The Strong Completeness Theorem for the Gentzen systems determined by the sequent calculi associated with the finite linear MV-algebras was proved in [11], by using algebraic methods.

The aim of this paper is to prove the Strong Completeness Theorem (Theorem 4.1) for the Gentzen systems associated with arbitrary finite algebras.

By using the notion of satisfaction of sequents defined in [16] and [2], we define, for any finite algebra, a semantical consequence relation on the set of m-sequents, where m is the cardinality of the algebra. These semantical consequence relations are finitary (Theorem 2.19) and satisfy the same deduction detachment theorem satisfied by the Gentzen systems mentioned above (Theorem 3.6). Then, by also using the already known Completeness Theorem for provable sequents ([2, Theorems 3.1 and 3.2]), we prove the Strong Completeness Theorem.

It is worth noticing (see Theorem 3.7) that for every finite algebra only one Gentzen system is associated with it by means of the sequent calculi defined in [2]. The computer system MULTLOG ([3] and [4]) provides a way to obtain automatically an axiomatization of these consequence relations such that the rules satisfy certain optimality conditions.

1 Preliminary definitions and results

This section contains the basic definitions for this paper and some already known results about deductive systems and Gentzen systems.

Consequence relations and deductive systems

Let $\mathcal{L}$ be a propositional language (i.e. a set of propositional connectives). By an $\mathcal{L}$-algebra we mean a structure $\mathbf{A} = \langle A, \{\Box^{\mathbf{A}} : \Box \in \mathcal{L}\}\rangle$, where A is a non-empty set, called the universe of $\mathbf{A}$, and $\Box^{\mathbf{A}}$ is an operation on A of arity k for each connective $\Box$ of rank k. A **consequence relation** on A is a relation $\vdash$ between subsets of A and elements of A such that the following conditions hold for all $X \cup Y \cup \{a\} \subseteq A$:

(i) $a \in X$ implies $X \vdash a$;

(ii) $X \vdash a$ and $X \subseteq Y$ implies $Y \vdash a$;

(iii) $X \vdash a$ and $Y \vdash b$ for every $b \in X$ implies $Y \vdash a$;

A consequence relation is **finitary** if

(iv) $X \vdash a$ implies $X' \vdash a$ for some finite $X' \subseteq X$.

We denote by $\mathbf{Fm}_{\mathcal{L}}$ the absolutely free algebra of type $\mathcal{L}$ freely generated by a countable infinite set of variables. Its elements are called $\mathcal{L}$-formulas. If $\mathbf{A}$ is an $\mathcal{L}$-algebra, the set of homomorphisms from $\mathbf{Fm}_{\mathcal{L}}$ to $\mathbf{A}$ will be denoted by $Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{A})$.

Example 1.1 Different consequence relations have been considered in the literature; for instance:

(i) A **deductive system** is a pair $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, where $\vdash_{\mathcal{S}}$ is a finitary consequence relation on the set of $\mathcal{L}$-formulas, $Fm_{\mathcal{L}}$, which is structural in the following sense: $\Gamma \vdash \varphi$ implies $h(\Gamma) \vdash h(\varphi)$ for every $h \in Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{Fm}_{\mathcal{L}})$, where $h(\Gamma)$ stands for $\{h(\gamma) : \gamma \in \Gamma\}$. Deductive systems have been studied, among other places, in [6] (where the concept of an *algebraizable* deductive system is defined), [5] and [9].

(ii) Let $1 \leq k$. A **k-dimensional deductive system** $\mathcal{S}$ over $\mathcal{L}$ is a pair $\langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ where $\vdash_{\mathcal{S}}$ is a finitary consequence relation over $Fm^k_{\mathcal{L}}$, ($Fm^k_{\mathcal{L}} = \{\langle \varphi_0, \ldots, \varphi_{k-1} \rangle : \varphi_i \in Fm_{\mathcal{L}}\}$), which is structural in the following sense: for all $\{\langle \gamma^i_0, \ldots, \gamma^i_{k-1} \rangle : i < n\} \cup \{\langle \varphi_0, \ldots, \varphi_{k-1} \rangle\} \subseteq Fm^k_{\mathcal{L}}$, and all $h \in Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{Fm}_{\mathcal{L}})$,

$$\{\langle \gamma^i_0, \ldots, \gamma^i_{k-1} \rangle : i < n\} \vdash_{\mathcal{S}} \langle \varphi_0, \ldots, \varphi_{k-1} \rangle \Rightarrow$$
$$\{\langle h(\gamma^i_0), \ldots, h(\gamma^i_{k-1}) \rangle : i < k\} \vdash_{\mathcal{S}} \langle h(\varphi_0), \ldots, h(\varphi_{k-1}) \rangle.$$

The set $Fm^k_{\mathcal{L}}$ is called the set of k-formulas.

k-dimensional deductive systems have been studied in [7].

(iii) The theory of (2-dimensional) Gentzen systems was developed in [14], where sequents are defined to be pairs of finite sequences of formulas, possibly with some limitations on the length of the sequences given by the *type* of the sequents. Let us recall some of the definitions given in [14]:

Let α and β be subsets of the set ω of natural numbers. An $\mathcal{L}$-**sequent of type** (α, β) is a pair of finite sequences of $\mathcal{L}$-formulas such that the length of Γ belongs to α and the length of Δ belongs to β. A **(2-)Gentzen system of type** (α, β) is a pair $\mathcal{G} = \langle \mathcal{L}, \vdash_{\mathcal{G}} \rangle$, where $\vdash_{\mathcal{G}}$ is a finitary and structural consequence relation on the set $Seq_{\mathcal{L}}^{(\alpha,\beta)}$ of $\mathcal{L}$-sequents of type (α, β). In [14] the consequence relation is denoted by the symbol $\mid\!\sim_{\mathcal{G}}$. A consequence relation $\vdash_{\mathcal{G}}$ on the set $Seq_{\mathcal{L}}^{(\alpha,\beta)}$ is said to be **structural** if $\{(\Gamma_i, \Delta_i) : i \in I\} \vdash_{\mathcal{G}} (\Gamma, \Delta)$ implies $\{h(\Gamma_i, \Delta_i) : i \in I\} \vdash_{\mathcal{G}} h(\Gamma, \Delta)$ for every $h \in Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{Fm}_{\mathcal{L}})$, where $h((\varphi_0, \dots, \varphi_{m-1}), (\psi_0, \dots, \psi_{n-1}))$ stands for the sequent

$$((h(\varphi_0), \dots, h(\varphi_{m-1})), (h(\psi_0), \dots, h(\psi_{n-1}))).$$

∎

Matrices for deductive systems

The notion of a *matrix* allows the introduction of a very general concept of a semantics for a deductive system. Let us recall the definitions of a matrix, a matrix model of a deductive system and a matrix semantics of a deductive system (cf. [17], [6] and [7]).

An $\mathcal{L}$**-matrix** is a pair $\langle \mathbf{A}, F \rangle$, where $\mathbf{A}$ is an $\mathcal{L}$-algebra and F is a subset of A.

Every matrix $\langle \mathbf{A}, F \rangle$ defines a consequence relation $\models_{\langle \mathbf{A}, F \rangle}$ over the set of formulas by the condition $\Gamma \models_{\langle \mathbf{A}, F \rangle} \varphi$ if, for each $h \in Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{A})$, $h(\Gamma) \subseteq F$ implies $h(\varphi) \in F$.

Let $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ be a deductive system. A matrix $\langle \mathbf{A}, F \rangle$ is a **matrix model** of $\mathcal{S}$ (or an $\mathcal{S}$**-matrix**) if $\Gamma \vdash_{\mathcal{S}} \varphi$ implies $\Gamma \models_{\langle \mathbf{A}, F \rangle} \varphi$. In this case we say that F is an $\mathcal{S}$**-filter**. Let $M = \{\langle \mathbf{A}_i, F_i \rangle : i \in I\}$ be a class of $\mathcal{S}$-matrices. M is a **matrix semantics** of $\mathcal{S}$ if $\Gamma \models_{\langle \mathbf{A}_i, F_i \rangle} \varphi$ for all $i \in I$ implies $\Gamma \vdash_{\mathcal{S}} \varphi$.

Example 1.2 If the set A is finite, then the consequence relation $\models_{\langle \mathbf{A},F\rangle}$ is finitary and $\mathcal{S}_{\langle \mathbf{A},F\rangle} = \langle \mathcal{L}, \models_{\langle \mathbf{A},F\rangle}\rangle$ is a deductive system, called the deductive system determined by the matrix $\langle \mathbf{A}, F\rangle$. In this case $\langle \mathbf{A}, F\rangle$ is an $\mathcal{S}_{\langle \mathbf{A},F\rangle}$-matrix and $\{\langle \mathbf{A}, F\rangle\}$ is a matrix semantics for $\mathcal{S}_{\langle \mathbf{A},F\rangle}$ (cf. [17]). ∎

Definition 1.3 *A finite valued logic over a propositional language $\mathcal{L}$ is any deductive system of the form $\mathcal{S}_{\langle \mathbf{A},F\rangle}$, where $\mathbf{A}$ is a finite $\mathcal{L}$-algebra and $F \subseteq A$.*

The deduction detachment theorem for deductive systems

Let us recall the deduction detachment theorem for deductive systems (see [5, III] and [8]).

Let $\mathcal{S}$ be a deductive system over a propositional language $\mathcal{L}$. A finite set $E(p,q) = \{\eta_0(p,q), \ldots, \eta_{k-1}(p,q)\}$ of formulas in the two variables p and q is called a **deduction detachment set** for $\mathcal{S}$ if, for all $\Gamma \subseteq Fm_{\mathcal{L}}$ and all $\varphi, \psi \in Fm_{\mathcal{L}}$,

$$\Gamma \cup \{\varphi\} \vdash_{\mathcal{S}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathcal{S}} E(\varphi,\psi);$$

where $\Gamma \vdash_{\mathcal{S}} E(\varphi,\psi)$ is an abbreviation for the conjunction of the assertions $\Gamma \vdash_{\mathcal{S}} \eta_i(\varphi,\psi)$, $i < k$.

$\mathcal{S}$ has the **deduction detachment theorem** (DDT for short) if there is some deduction-detachment set for $\mathcal{S}$.

Example 1.4 *As is well known, the set $E(p,q) = \{p \to q\}$ is a deduction detachment set for the Classical Propositional Calculus.* ∎

Note that in the abstract definition of the deduction detachment theorem, the set $E(p,q)$ collectively acts as a kind of implication.

m-sequents and m-sequent calculi

An **m-sequent**, also called m-dimensional sequent or m-sided sequent, is a sequence $(\Gamma_0, \Gamma_1, \ldots, \Gamma_{m-1})$ where each Γ_i is a finite sequence of $\mathcal{L}$-formulas, which is called the i-th **component** (or **place**) of the sequent. Those sequents have been taken into account in [16], [3], [2], [18] and [11]. As in these works we will write $\Gamma_0 \mid \Gamma_1 \mid \ldots \mid \Gamma_{m-1}$ for $(\Gamma_0, \Gamma_1, \ldots, \Gamma_{m-1})$. We denote by m-$Seq_{\mathcal{L}}$ the set of m-sequents.

Thus in the 2-dimensional case we will write $\Gamma \mid \Delta$ instead of the more common notations $\Gamma \vdash \Delta$ or $\Gamma \to \Delta$. The use of the symbol $\mid$ as a separator of the components prevents us from thinking of entailment relations between the components of a sequent. Note that in our notation the symbol $\vdash$ is only used, possibly with a subindex, to denote consequence relations on the sets considered (formulas, k-formulas, sequents or m-sequents).

If we have two or more sequents, we will separate them by the symbol ";". In this way there will be no confusion between, for instance, the 3-sequent $\Gamma, x \mid \Delta, y \mid \Pi$ and the two 2-sequents $\Gamma, x \mid \Delta;\ y \mid \Pi$. The comma will be reserved for the juxtaposition operation on sequences: that is, expressions such as Γ, δ will stand for $(\gamma_0, \ldots, \gamma_{k-1}, \delta)$, where $\Gamma = (\gamma_0, \ldots, \gamma_{k-1})$.

If Γ is a sequence of formulas and φ occurs in Γ, we will write $\varphi \in \Gamma$. Also, we write $\Gamma \subseteq \{\varphi_0, \ldots, \varphi_n\}$ to denote that all the formulas that occur in Γ are in $\{\varphi_0, \ldots, \varphi_n\}$.

To increase the readability of some of the results of this paper in which we use simultaneously formulas, sequences of formulas, sequents and sets of sequents, we will use the following notation: lowercase letters from the end of the alphabet, possibly with subindex and superindex $(p, q, p_i^j, \ldots)$ to denote propositional variables; Greek letters $(\varphi, \psi, \varphi_i^j, \ldots,)$ to denote formulas; uppercase Greek letters $(\Gamma, \Delta, \ldots)$ to denote sequences and sets of formulas; boldface uppercase Greek letters $(\mathbf{\Gamma}, \mathbf{\Delta}, \mathbf{\Delta}_i, \ldots)$ to denote sequents, and *boldmath* uppercase letters of the end of the alphabet $(\boldsymbol{T}, \boldsymbol{S}, \ldots,)$ to denote sets of sequents.

If $\mathbf{\Gamma}$ is an m-sequent and $i < m$, then $\mathbf{\Gamma}(i)$ denotes the i-th component of $\mathbf{\Gamma}$. If Δ is a sequence of formulas and $I = \{i_1, \ldots, i_n\} \subseteq \{0, \ldots, m-1\}$, we denote by $[I : \Delta]$ the m-sequent whose i-th component is Δ if $i \in I$ and is empty otherwise, that is:

$$[I : \Delta](i) = \begin{cases} \Delta & \text{if } i \in I \\ \emptyset & \text{if } i \notin I. \end{cases}$$

We will write $[i_1, \ldots, i_n : \Delta]$ for $[\{i_i, \ldots, i_n\} : \Delta]$.

If $\mathbf{\Gamma}$ and $\mathbf{\Pi}$ are m-sequents then we denote by $[\mathbf{\Gamma}, \mathbf{\Pi}]$ the m-sequent

$$\mathbf{\Gamma}(0), \mathbf{\Pi}(0) \mid \ldots \mid \mathbf{\Gamma}(m-1), \mathbf{\Pi}(m-1),$$

and by $\emptyset_m$ the sequent such that all its components are empty.

Note that while $\mathbf{\Gamma}_1, \mathbf{\Gamma}_2 \ldots$ are sequents, the expression $\mathbf{\Gamma}(i)$ stands for the i-th component of the sequent $\mathbf{\Gamma}$.

If $I \subseteq \{0, \ldots, m-1\}$, then we will write I^C for the set $\{j < m : j \notin I\}$.

Example 1.5 *Let* $m = 3$.

(i) $[\{0,2\} : \varphi] = \varphi \mid \emptyset \mid \varphi$.

(ii) $[\{2\}^C : \varphi] = \varphi \mid \varphi \mid \emptyset$.

(iii) $[\![[\{1\}^C : \varphi], [1 : \varphi]]\!] = [\![[0,2 : \varphi], [1 : \varphi]]\!] = \varphi \mid \varphi \mid \varphi$. ∎

An m-**rule of inference** is a set (r) of ordered pairs of the form $\langle \boldsymbol{T}, \boldsymbol{\Gamma} \rangle$, where $\boldsymbol{T} \cup \{\boldsymbol{\Gamma}\} \subseteq m\text{-}Seq_{\mathcal{L}}$ and $\boldsymbol{T}$ is finite, such that it is closed under substitutions, i.e., for every $h \in Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{Fm}_{\mathcal{L}})$, if $\langle \boldsymbol{T}, \boldsymbol{\Gamma} \rangle \in (r)$ then $\langle h(\boldsymbol{T}), h(\boldsymbol{\Gamma}) \rangle \in (r)$. Rules having all pairs of the form $\langle \emptyset, \boldsymbol{\Gamma} \rangle$ are called **axioms** and, in this case, $\boldsymbol{\Gamma}$ is called an **instance** of the axiom.

Rules are often written in a schematic form; for instance,

$$\frac{\boldsymbol{\Gamma}}{[\![\boldsymbol{\Gamma}, [i : \varphi]]\!]}$$

denotes the rule $\{\langle \{\boldsymbol{\Gamma}\}, [\![\boldsymbol{\Gamma}, [i : \varphi]]\!] \rangle : \boldsymbol{\Gamma}$ is an m-sequent and φ is a formula$\}$.

An m-**sequent calculus** is a set of m-rules of inference.

The following are called **structural rules** (see [2]), where $\boldsymbol{\Gamma}$ and $\boldsymbol{\Delta}$ are arbitrary sequents and φ and ψ are arbitrary $\mathcal{L}$-formulas:

- **Axiom**: $[0, \ldots, m-1 : \varphi]$.

- **Weakening rule** $(w : i)$ for the place $i < m$:

$$\frac{\boldsymbol{\Gamma}}{[\![\boldsymbol{\Gamma}, [i : \varphi]]\!]}\, w : i$$

- **Contraction rule** $(c : i)$ for the place $i < m$:

$$\frac{[\![\boldsymbol{\Gamma}, [i : \varphi, \varphi]]\!]}{[\![\boldsymbol{\Gamma}, [i : \varphi]]\!]}\, c : i$$

- **Exchange rule** $(x : i)$ for the place $i < m$:

$$\frac{[\![\boldsymbol{\Gamma}, [i\colon \varphi, \psi], \boldsymbol{\Delta}]\!]}{[\![\boldsymbol{\Gamma}, [i\colon \psi, \varphi], \boldsymbol{\Delta}]\!]}\, x\colon i$$

- **Cut rule** ($cut : i, j$) for the places $i < m, j < m$, $i \neq j$:

$$\frac{[\![\boldsymbol{\Gamma}, [i\colon \varphi]]\!] \quad [\![\boldsymbol{\Delta}, [j\colon \varphi]]\!]}{[\![\boldsymbol{\Gamma}, \boldsymbol{\Delta}]\!]}\, cut\colon i, j$$

Note that we have a structural rule of each kind for each component of the sequents (or pair of components, in the case of the cut rule). If $m = 2$ there is only one cut rule, which has the usual form:

$$\frac{\Gamma_0, \varphi \mid \Gamma_1 \quad \Delta_0 \mid \Delta_1, \varphi}{\Gamma_0, \Delta_0 \mid \Gamma_1, \Delta_1}\, cut : 0, 1$$

Example 1.6 *Let $m = 3$:*

(i) *Axiom:* $\varphi \mid \varphi \mid \varphi$.

(ii) *Weakening:1*

$$\frac{\Gamma_0 \mid \Gamma_1 \mid \Gamma_2}{\Gamma_0 \mid \Gamma_1, \varphi \mid \Gamma_2}\, w : 1$$

(iii) *Cut:0,1*

$$\frac{\Gamma_0, \varphi \mid \Gamma_1 \mid \Gamma_2 \quad \Delta_0 \mid \Delta_1, \varphi \mid \Delta_2}{\Gamma_0, \Delta_0 \mid \Gamma_1, \Delta_1 \mid \Gamma_2, \Delta_2}\, cut : 0, 1$$

∎

Example 1.7 The sequent calculus LK for the Classical Propositional Calculus is shown in Table 1. It is a 2-dimensional sequent calculus that contains all the structural rules. In Table 2 we show a 3-dimensional sequent calculus obtained from the three-element MV-algebra following [2]. ∎

If LX is a sequent calculus that contains some of the cut rules, we say that the **cut elimination theorem** holds for LX if every sequent LX-provable can be proved without using any of the cut rules.

The VL-sequent calculi

Each finite $\mathcal{L}$-algebra of cardinal m induces a semantical interpretation on the set of m-sequents, in such a way that several m-sequent calculi are known to be complete with respect to this semantical interpretation. Since in this paper we will extend the definition of the semantical interpretation to a semantical consequence relation on the set of m-sequents we will now recall some of the basic definitions involved.

Definition 1.8 *Let* $\mathbf{L}$ *be a finite* $\mathcal{L}$*-algebra with universe* $L = \{v_0, \ldots, v_{m-1}\}$ *of cardinal* m*.*

(i) *Let* $h \in Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{L})$*.* h **L-satisfies** *an* m*-sequent* $\Gamma(0) \mid \ldots \mid \Gamma(m-1)$*, if there is an* $i < m$ *such that, for some formula* $\gamma \in \Gamma(i)$*,* $h(\gamma) = v_i$*.*

(ii) *If* $\mathbf{\Gamma}$ *is an* m*-sequent,* $s(\mathbf{\Gamma})$ *is the set of homomorphisms that* $\mathbf{L}$*-satisfy the sequent* $\mathbf{\Gamma}$*.*

(iii) $\mathbf{\Gamma} \in m\text{-}Seq_{\mathcal{L}}$ *is* **L-valid** *if for every* $h \in Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{L})$*,* h $\mathbf{L}$*-satisfies* $\mathbf{\Gamma}$*, that is, iff* $s(\mathbf{\Gamma}) = Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{L})$*.*

(iv) $\boldsymbol{T} \subseteq m\text{-}Seq_{\mathcal{L}}$ *is* **simultaneously L-satisfiable** *if* $\bigcap_{\mathbf{\Gamma} \in \boldsymbol{T}} s(\mathbf{\Gamma}) \neq \emptyset$*.*

The above definition of validity is the restriction to the propositional case of [2, Def 3.2] and of the definitions given in [3] and [18].

The elements of L are called truth values and if $\mathbf{\Gamma}$ is an $\mathbf{L}$-valid sequent we will write, following [18], $\models_{\mathbf{L}} \mathbf{\Gamma}$.

A dual semantical interpretation of sequents, which corresponds to analytic tableaux is studied, for instance, in [18].

It is always possible to find sequent calculi complete with respect to this definition of $\mathbf{L}$-validity (see [2] and [18] for historical remarks). The calculi we will deal with were defined by M. Baaz *et al.* in [2] and they play the same role with respect to the algebra $\mathbf{L}$ as the sequent calculus LK does with respect to the two-element Boolean algebra $\mathbf{2}$, in which case, as is well known, a sequent $\Gamma \mid \Delta$ is LK-provable iff for every interpretation of the variables of the sequent in $\mathbf{2}$, some formula in Γ is false or some formula in Δ is true; thus a sequent is LK-provable iff it is $\mathbf{2}$-valid (see [2, p. 336] and [18, p. 31 and 33]).

We will now recall the definition of the introduction rules of these calculi, which are called in this paper **VL**-rules, that is, preceding the name of the algebra with the letter **V**.

Definition 1.9 (cf. [2, Definition 3.3]) and [3]). *A* **VL-introduction rule** $(\Box : i)$ *for a connective* $\Box$ *at place* i *is a schema of the form:*

$$\frac{\{\Gamma_0, \Delta_0^j \mid \ldots \mid \Gamma_{m-1}, \Delta_{m-1}^j\}_{j\in I}}{\Gamma_0 \mid \ldots \mid \Gamma_i, \Box(\varphi_0, \ldots, \varphi_{n-1}) \mid \ldots \mid \Gamma_{m-1}} \; \Box{:}\, i \tag{1}$$

where $\Delta_l^j \subseteq \{\varphi_0, \ldots, \varphi_{n-1}\}$, *for every* $l < m$ *and* $j \in I$, $\Box$ *is a propositional connective of rank* n, I *is a finite set, and, for each* $h \in Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{L})$, *the following properties are equivalent:*

(VL1) h **L**-*satisfies the sequent* $\Delta_0^j \mid \ldots \mid \Delta_{m-1}^j$ *for every* $j \in I$.

(VL2) $h(\Box(\varphi_0, \ldots, \varphi_{n-1})) = v_i$.

The existence of such rules for an arbitrary algebra is proved in [16, Lemma 1]. As pointed out in [2], it should be stressed that for any connective $\Box$ and any $i < m$, there may be different rules that satisfy the definition of a **VL** introduction rule $(\Box : i)$. In [2] there is a description of how to find these rules from the *partial normal forms* in the sense of [15] (see also [18, p. 8-9]). A procedure to find rules that are minimal with respect to the number of premises and the number of formulas per premise has been implemented in the system MULTLOG (see [3] and [4]).

With these introduction rules and the structural rules, a class of sequent calculi associated with the algebra **L**, and denoted by the letter **V** in front of the name of the algebra, is defined as follows:

Definition 1.10 (see [2]). *Let* **L** *be a finite* $\mathcal{L}$-*algebra. A* **VL**-*sequent calculus consists of*

(i) *A* **VL**-*introduction rule* $(\Box : i)$ *for every connective* $\Box$ *and every place* $i < m$,

(ii) *All the structural rules, that is, the axiom and the rules* $(w : i)$, $(c : i)$, $(x : i)$ *for all* $i < m$, *and the rules* $(cut : i, j)$ *for all* $i, j < m$, $i \neq j$.

This definition corresponds to the propositional fragment of the sequent calculi **VL** defined in [3] and [18], and in [2] with the name **LM**. Among the properties of the sequent calculi just defined we are interested in the restriction to the propositional case of the following result:

Theorem 1.11 (Completeness and Cut Elimination) *Let* **L** *be a finite algebra, then the following properties hold:*

(i) *If an m-sequent is provable in a* **VL***-sequent calculus, then it is* **L***-valid.*

(ii) *If an m-sequent is* **L***-valid, then it is provable in any* **VL***-sequent calculus without cuts.* ∎

Proof: (i) See [2, Theorem 3.1]. (ii) See [2, Theorem 3.2]. ∎

Since the **VL** introduction rules for a given connective are not unique, for any finite $\mathcal{L}$-algebra **L** there may be several calculi that satisfy the definition of a **VL**-sequent calculus. However, it follows from Theorem 1.11 that each **VL**-sequent calculus has the same provable sequents. More generally, we will prove in Theorem 3.7 that each **VL**-sequent calculus determines the same consequence relation over the set of m-sequents, that is, the same Gentzen system.

G. Rousseau defines in [16] another class of sequent calculi for each $\mathcal{L}$-algebra **L**, which we refer to as **RL**-calculus, and which have a different axiom and no structural rules. Although for any finite $\mathcal{L}$-algebra **L** the **VL** and **RL**-sequent calculi have the same provable sequents, the absence of the structural rules makes the later less appropriate for our purposes. In [11] some differences between the consequence relations associated with these classes of calculi are considered.

Example 1.12 Let $\mathbf{L} = (\{f, p, t\}, \wedge)$ be an algebra of three elements where the truth table of the connective is given by

$\wedge$	f	p	t
f	f	f	f
p	f	p	p
t	f	p	t

Let us now find a **VL**-rule $(\wedge : 1)$. First note that $h(\varphi \wedge \psi) = p \iff (h(\varphi) = p$ or $h(\psi) = p)$ and $(h(\varphi) = p$ or $h(\varphi) = t)$ and $(h(\psi) = t$ or $h(\psi) = p)$.

This expression corresponds to a *conjunctive normal form*. Now, each of the conjuncts can be expressed by the fact that the homomorphism satisfies a certain sequent; for instance

$$(h(\varphi) = p \text{ or } h(\psi) = p) \iff h \in s(\emptyset \mid \varphi, \psi \mid \emptyset).$$

Indeed $h(\varphi \wedge \psi) = p \iff h \in s(\emptyset \mid \varphi, \psi \mid \emptyset) \cap s(\emptyset \mid \varphi \mid \varphi) \cap s(\emptyset \mid \psi \mid \psi)$.

Now, writing this equivalence in the usual rule-style form we obtain the following introduction schema

$$\frac{\emptyset \mid \varphi, \psi \mid \emptyset \quad \emptyset \mid \varphi \mid \varphi \quad \emptyset \mid \psi \mid \psi}{\emptyset \mid \varphi \wedge \psi \mid \emptyset}$$

from which we obtain the **VL**-introduction rule of the connective $\wedge$ in the place 1:

$$\frac{\Gamma \mid \Delta, \varphi, \psi \mid \Pi \quad \Gamma \mid \Delta, \varphi \mid \Pi, \varphi \quad \Gamma \mid \Delta, \psi \mid \Pi, \psi}{\Gamma \mid \Delta, \varphi \wedge \psi \mid \Pi}$$

∎

The relation between the **VL**-provable sequents and the finite valued logics defined over **L** is given in the following result:

Theorem 1.13 *Let* $F \subseteq L = \{v_0, \dots, v_{m-1}\}$ *and let* $I_F = \{i < m : v_i \in F\}$. *For every set of formulas* Γ *and every formula* φ*:*

(i) $\models_{\mathbf{L}} [I_F : \varphi] \iff \emptyset \models_{\langle \mathbf{L}, F \rangle} \varphi$

(ii) $\models_{\mathbf{L}} [\![[I_F{}^C : \Gamma], [I_F : \varphi]]\!] \iff \Gamma \models_{\langle \mathbf{L}, F \rangle} \varphi$

Proof: (i) Note that for each $h \in Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{L})$, $h \in s([I_F : \varphi])$ iff $h(\varphi) = v_i$ for some i such that $i \in I_F$, that is, iff $h(\varphi) \in F$.

(ii) $\Leftarrow$) Let $h \in Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{L})$. Either $h(\Gamma) \subseteq F$, in which case, by hypothesis $h(\varphi) \in F$ and then h **L**-satisfies the sequent $[\![[I_F{}^C : \Gamma], [I_F : \varphi]]\!]$, or $h(\Gamma) \not\subseteq F$, in which case there exists some $\gamma \in \Gamma$ such that $h(\gamma) = v_j$ for some $j \notin I_F$, and then h also **L**-satisfies the sequent $[\![[I_F{}^C : \Gamma], [I_F : \varphi]]\!]$.

$\Rightarrow$) Let $h \in Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{L})$. If $h(\Gamma) \subseteq F$ and since the sequent $[\![[I_F{}^C : \Gamma], [I_F : \varphi]]\!]$ is **L**-satisfied by h, there must exist some $i \in I_F$ such that $h(\varphi) = v_i$, that is, $h(\varphi) \in F$. ∎

2 Gentzen Systems

In order to study the consequence relations determined by the **VL**-sequent calculi, we first recall the abstract definition of an m-dimensional Gentzen system. These systems were introduced in [11] (with some limitations on the length of the sequent, not considered in this paper) and can be seen as a generalization of the 2-dimensional Gentzen systems introduced in [14] (allowing an arbitrary but fixed number of components in the sequents), and also as a generalization of the m-dimensional deductive systems (considering m-tuples of sequences of formulas, instead of m-tuples of formulas).

An **m-dimensional Gentzen system** is a pair $\mathcal{G} = \langle \mathcal{L}, \vdash_{\mathcal{G}} \rangle$ where $\vdash_{\mathcal{G}}$ is a finitary consequence relation on the set of m-sequents, m-$Seq_{\mathcal{L}}$, which is also **structural** in the following sense: For every $h \in Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{Fm}_{\mathcal{L}})$,$\boldsymbol{T} \vdash_{\mathcal{G}} \boldsymbol{\Gamma}$ implies $h(\boldsymbol{T}) \vdash_{\mathcal{G}} h(\boldsymbol{\Gamma})$ where $h(\boldsymbol{T}) = \{h(\boldsymbol{\Delta}) : \boldsymbol{\Delta} \in \boldsymbol{T}\}$ and

$$h(\gamma_0^0, \ldots, \gamma_0^{t_0-1} \mid \gamma_1^0, \ldots, \gamma_1^{t_1-1} \mid \ldots \mid \gamma_{m-1}^0, \ldots, \gamma_{m-1}^{t_{m-1}-1})$$

stands for the sequent

$$h(\gamma_0^0), \ldots, h(\gamma_0^{t_0-1}) \mid h(\gamma_1^0), \ldots, h(\gamma_1^{t_1-1}) \mid \ldots \mid h(\gamma_{m-1}^0), \ldots, h(\gamma_{m-1}^{t_{m-1}-1}).$$

Let $\mathcal{G}$ be a Gentzen system. If $\boldsymbol{\Gamma} \vdash_{\mathcal{G}} \boldsymbol{\Delta}$ and $\boldsymbol{\Delta} \vdash_{\mathcal{G}} \boldsymbol{\Gamma}$ we will say that $\boldsymbol{\Gamma}$ and $\boldsymbol{\Delta}$ are **$\mathcal{G}$-equivalent**. If $\emptyset \vdash_{\mathcal{G}} \boldsymbol{\Gamma}$ we will say that $\boldsymbol{\Gamma}$ is **$\mathcal{G}$-derivable**. We will sometimes write $\boldsymbol{T}; \boldsymbol{\Gamma} \vdash_{\mathcal{G}} \boldsymbol{\Delta}$ for $\boldsymbol{T} \cup \{\boldsymbol{\Gamma}\} \vdash_{\mathcal{G}} \boldsymbol{\Delta}$.

Every m-sequent calculus, LX, determines a Gentzen system $\mathcal{G}_{LX} = \langle \mathcal{L}, \vdash_{LX} \rangle$ by using the rules of the calculus to derive sequents from sets of sequents, not just from the axiom alone, as stated in the following definition (cf. [14, p.14] and [1, p. 267]):

Definition 2.1 *Given* $\boldsymbol{T} \cup \{\boldsymbol{\Gamma}\} \subseteq m$-$Seq_{\mathcal{L}}$ *, we say that* $\boldsymbol{\Gamma}$ **follows from** $\boldsymbol{T}$ *in* $\mathcal{G}_{LX}$*, in symbols* $\boldsymbol{T} \vdash_{LX} \boldsymbol{\Gamma}$ *iff there is a finite sequence of sequents* $\boldsymbol{\Gamma}_0, \ldots, \boldsymbol{\Gamma}_{n-1}$*, $(n \geq 1)$, called a* **proof** *of* $\boldsymbol{\Gamma}$ *from* $\boldsymbol{T}$*, such that* $\boldsymbol{\Gamma}_{n-1} = \boldsymbol{\Gamma}$ *and for each* $i < n$ *one of the following conditions holds:*

(i) $\boldsymbol{\Gamma}_i$ *is an instance of an axiom;*

(ii) $\boldsymbol{\Gamma}_i \in \boldsymbol{T}$*;*

(iii) $\boldsymbol{\Gamma}_i$ *is obtained from* $\{\boldsymbol{\Gamma}_j : j < i\}$ *by using a rule* (r) *of* LX*, i.e.,* $\langle \boldsymbol{S}, \boldsymbol{\Gamma}_i \rangle \in (r)$ *for some* $\boldsymbol{S} \subseteq \{\boldsymbol{\Gamma}_j : j < i\}$*.*

n is called the **length** *of the proof.*

Example 2.2 The Gentzen system determined by the sequent calculus LK is studied in [14], where it is denoted by $\mathcal{G}_{CPC}$. This Gentzen system is equivalent to the Classical Propositional Calculus and is algebraizable, the variety of Boolean algebras being its equivalent quasivariety semantics (see [14] for the definitions of equivalence and algebraizability of 2-dimensional Gentzen systems). Other Gentzen systems, obtained modifying some of the rules of LK, are studied in [14] and [13]. ∎

Definition 2.3 *Let* $\mathbf{L}$ *be a finite* $\mathcal{L}$*-algebra. A* **VL-Gentzen system** *is a Gentzen system determined by a* $\mathbf{VL}$*-sequent calculus.*

Example 2.4 The Gentzen system determined by the $\mathbf{VS}(3)$-sequent calculus given in Table 2 is studied in [11] and [10]. This Gentzen system is equivalent to the 3-valued Łukasiewicz propositional logic and is algebraizable, the variety generated by the three-element MV-algebra being its equivalent quasivariety semantics (see [11] for the definitions of equivalence and algebraizability of m-dimensional Gentzen systems). The Gentzen systems determined by a $\mathbf{VS}(m)$-sequent calculus (defined in [2]), where $\mathbf{S}(m)$ is the linear MV-algebra of m elements are also studied in [11]. ∎

Definition 2.5 *An m-dimensional Gentzen system satisfies an m-rule (r) if $\boldsymbol{T}\vdash_{\mathcal{G}}\boldsymbol{\Gamma}$ for every $\langle \boldsymbol{T}, \boldsymbol{\Gamma}\rangle \in (r)$.*

The structural rules

The structural rules play an important role in the proof of some basic theorems of this paper. We introduce now some technical lemmas that will help us to shorten some proofs where different structural rules are involved.

Lemma 2.6 *Let $\mathcal{G} = \langle \mathcal{L}, \vdash_{\mathcal{G}}\rangle$ be an m-dimensional Gentzen system that satisfies the contraction and exchange rules for a given place $i < m$. For every $\{\boldsymbol{\Gamma}; \boldsymbol{\Pi}\} \subseteq m\text{-}Seq_{\mathcal{L}}$ and every sequence of formulas Δ*

$$[\![\boldsymbol{\Gamma}, [i : \Delta, \Delta], \boldsymbol{\Pi}]\!]\vdash_{\mathcal{G}}[\![\boldsymbol{\Gamma}, [i : \Delta], \boldsymbol{\Pi}]\!].$$

Proof: Straightforward. ∎

Lemma 2.7 *Let $\mathcal{G} = \langle \mathcal{L}, \vdash_{\mathcal{G}} \rangle$ be an m-dimensional Gentzen system that satisfies the axiom and the weakening rules. For all $\{\mathbf{\Gamma}; \mathbf{\Pi}\} \subseteq m\text{-}Seq_{\mathcal{L}}$ and every $\varphi \in Fm_{\mathcal{L}}$*

$$\emptyset \vdash_{\mathcal{G}} [\![[0, \dots, m-1 : \varphi], \mathbf{\Pi}]\!].$$

If, in addition, $\mathcal{G}$ satisfies the exchange rules, then

$$\emptyset \vdash_{\mathcal{G}} [\![\mathbf{\Gamma}, [0, \dots, m-1 : \varphi], \mathbf{\Pi}]\!].$$

Proof: Straightforward. ∎

As regards the cut rules (*cut* : i, j), although the cut is made on the last formula of the components i, j, if $\mathcal{G}$ satisfies the exchange rule for these components, the cut can be made on some other formulas, that is:

Lemma 2.8 *Let $k < m$ and $j < m$, $k \neq j$. Let $\mathcal{G} = \langle \mathcal{L}, \vdash_{\mathcal{G}} \rangle$ be an m-dimensional Gentzen system that satisfies the rule (cut : k, j) and the exchange rules for the components k and j. For every $\{\mathbf{\Gamma}_0; \mathbf{\Gamma}_1; \mathbf{\Delta}_0; \mathbf{\Delta}_1\} \subseteq m\text{-}Seq_{\mathcal{L}}$*

$$[\![\mathbf{\Gamma}_0, [k\colon \varphi], \mathbf{\Gamma}_1]\!]; [\![\mathbf{\Delta}_0, [j\colon \varphi], \mathbf{\Delta}_1]\!] \vdash_{\mathcal{G}} [\![\mathbf{\Gamma}_0, \mathbf{\Gamma}_1, \mathbf{\Delta}_0, \mathbf{\Delta}_1]\!].$$

Proof:

$$\dfrac{\begin{array}{c} [\![\mathbf{\Gamma}_0, [k\colon \varphi], \mathbf{\Gamma}_1]\!] \\ \vdots\, x : k \\ [\![\mathbf{\Gamma}_0, \mathbf{\Gamma}_1, [k\colon \varphi]]\!] \end{array} \qquad \begin{array}{c} [\![\mathbf{\Delta}_0, [j\colon \varphi], \mathbf{\Delta}_1]\!] \\ \vdots\, x : j \\ [\![\mathbf{\Delta}_0, \mathbf{\Delta}_1, [j\colon \varphi]]\!] \end{array}}{[\![\mathbf{\Gamma}_0, \mathbf{\Gamma}_1, \mathbf{\Delta}_0, \mathbf{\Delta}_1]\!]}\ \mathit{cut}\colon k, j$$

∎

The next lemma allows us to cut a formula not only in a pair of different components, but in a pair of disjoint sets of components.

Lemma 2.9 *Let $\mathcal{G}$ be an m-dimensional Gentzen system ($m > 2$), such that it satisfies the exchange, cut and contraction rules. Let I, J be a pair of disjoint and nonempty subsets of the set $\{0, \dots, m-1\}$. For any sequents $\mathbf{\Gamma}$ and $\mathbf{\Delta}$ and for any formula φ,*

$$\{[\![\mathbf{\Gamma}, [J : \varphi]]\!]; [\![\mathbf{\Delta}, [I : \varphi]]\!]\} \vdash_{\mathcal{G}} [\![\mathbf{\Gamma}, \mathbf{\Delta}]\!]. \tag{2}$$

Proof: First we prove the Lemma for the case $I = \{i\}$. We proceed by induction on $Card(J)$. If $Card(J) = 1$, $J = \{j\}$, $(i \neq j)$, and we can use $(cut\colon j, i)$. If $J = \{j_1, \ldots, j_n\}, n > 1$, then by $(cut : j_n, i)$

$$\{[\![\Gamma, [j_1, \ldots, j_{n-1}, j_n : \varphi]]\!]; [\![\Delta, [i : \varphi]]\!]\}\vdash_{\mathcal{G}}[\![\Gamma, \Delta, [j_1, \ldots, j_{n-1} : \varphi]]\!].$$

By inductive hypothesis and by using Lemma 2.6 several times we have

$$[\![\Gamma, \Delta, [j_1, \ldots, j_{n-1} : \varphi]]\!]; [\![\Delta, [i : \varphi]]\!]\vdash_{\mathcal{G}}[\![\Gamma, \Delta]\!]$$

so

$$[\![\Gamma, [J : \varphi]]\!]; [\![\Delta, [i : \varphi]]\!]\vdash_{\mathcal{G}}[\![\Gamma, \Delta]\!]. \tag{3}$$

In the general case we proceed by induction on $Card(I)$.
If $Card(I) = 1$ it is done. If $I = \{i_1, \ldots, i_n\}, n > 1$, then $[\![\Delta, [I : \varphi]]\!] = [\![\Delta, [i_1, \ldots, i_{n-1} : \varphi], [i_n : \varphi]]\!]$ and then by using (3),

$$\frac{[\![\Gamma, [J : \varphi]]\!] \qquad [\![\Delta, [i_1, \ldots, i_{n-1} : \varphi], [i_n : \varphi]]\!]}{[\![\Gamma, \Delta, [i_1, \ldots, i_{n-1} : \varphi]]\!]}$$

By inductive hypothesis, as in the previous case, we obtain

$$[\![\Gamma, [J : \varphi]]\!]; \quad [\![\Gamma, \Delta, [i_1, \ldots, i_{n-1} : \varphi]]\!]\vdash_{\mathcal{G}}[\![\Gamma, \Delta]\!]. \qquad \blacksquare$$

Note that if $m = 2$, then $I = \{0\}$ and $J = \{1\}$ or viceversa, so (2) is the cut rule.

Lemma 2.10 *Let $\mathcal{G} = \langle \mathcal{L}, \vdash_{\mathcal{G}} \rangle$ be an m-dimensional Gentzen system that satisfies the exchange, contraction and cut rules. For any sequents Γ and Δ,*

(i) *If $k < m$ then*

$$\Gamma; \{[\![\Delta, [\{k\}^C : \gamma]]\!] : \gamma \in \Gamma(k)\}\vdash_{\mathcal{G}}\Pi,$$

where

$$\Pi(i) = \begin{cases} \Gamma(i), \Delta(i) & \text{if } i \neq k \\ \Delta(k) & \text{if } i = k \end{cases}$$

(ii)

$$\Gamma; \bigcup_{k<m} \{[\![\Delta, [\{k\}^C : \gamma]]\!] : \gamma \in \Gamma(k)\}\vdash_{\mathcal{G}}\Delta.$$

Proof: (i) Let $\mathbf{\Gamma}(k) = (\gamma_0, \ldots, \gamma_{r-1})$. For every $j < r-1$ let $\mathbf{\Pi}_j$ be the sequent defined by

$$\mathbf{\Pi}_j(i) = \begin{cases} \mathbf{\Gamma}(i), \mathbf{\Delta}(i) & \text{if } i \neq k \\ \gamma_{j+1}, \ldots, \gamma_{r-1}, \mathbf{\Delta}(k) & \text{if } i = k. \end{cases}$$

The result follows from the following chain of inferences, which are obtained by Lemma 2.9 and Lemma 2.6 (thus using only the exchange, contraction and cut rules)

$$\begin{aligned} \mathbf{\Gamma}; [\![\mathbf{\Delta}, [\{k\}^C : \gamma_0]]\!] &\vdash_{\mathcal{G}} \mathbf{\Pi}_0 \\ \mathbf{\Pi}_0; [\![\mathbf{\Delta}, [\{k\}^C : \gamma_1]]\!] &\vdash_{\mathcal{G}} \mathbf{\Pi}_1 \\ &\vdots \\ \mathbf{\Pi}_{r-2}; [\![\mathbf{\Delta}, [\{k\}^C : \gamma_{r-1}]]\!] &\vdash_{\mathcal{G}} \mathbf{\Pi} \end{aligned}$$

Notice that $\mathbf{\Pi}_0 = \mathbf{\Pi}$ in the case $r = 1$.

(ii) For every $k < m-1$, let $\mathbf{\Delta}_k$ be the sequent defined by

$$\mathbf{\Delta}_k(i) = \begin{cases} \mathbf{\Gamma}(i), \mathbf{\Delta}(i) & \text{if } i > k \\ \mathbf{\Delta}(i) & \text{if } i \leq k. \end{cases}$$

The result follows from the following chain of inferences, which are obtained by Lemma 2.6 and (i) (thus using only the exchange, contraction and cut rules)

$$\begin{aligned} \mathbf{\Gamma}; \{[\![\mathbf{\Delta}, [\{0\}^C : \gamma]]\!] : \gamma \in \mathbf{\Gamma}(0)\} &\vdash_{\mathcal{G}} \mathbf{\Delta}_0 \\ \mathbf{\Delta}_0; \{[\![\mathbf{\Delta}, [\{1\}^C : \gamma]]\!] : \gamma \in \mathbf{\Gamma}(1)\} &\vdash_{\mathcal{G}} \mathbf{\Delta}_1 \\ &\vdots \\ \mathbf{\Delta}_{m-2}; \{[\![\mathbf{\Delta}, [\{m-1\}^C : \gamma]]\!] : \gamma \in \mathbf{\Gamma}(m-1)\} &\vdash_{\mathcal{G}} \mathbf{\Delta}. \end{aligned}$$ ∎

Accumulative Gentzen systems

An m-dimensional Gentzen system $\mathcal{G} = \langle \mathcal{L}, \vdash_{\mathcal{G}} \rangle$ is called **i-accumulative** ($i < m$) if it satisfies the following properties:

(i) For every $\varphi \in Fm_{\mathcal{L}}$, $\emptyset \vdash_{\mathcal{G}} [0, \ldots, m-1 : \varphi]$;

(ii) For every $\boldsymbol{T} \cup \{\boldsymbol{\Gamma}; \boldsymbol{\Delta}\} \subseteq m\text{-}Seq_{\mathcal{L}}$ and every formula φ,

$$\boldsymbol{T}; \boldsymbol{\Gamma} \vdash_{\mathcal{G}} \boldsymbol{\Delta} \quad \text{implies} \quad \boldsymbol{T}; [\![\boldsymbol{\Gamma}, [i : \varphi]]\!] \vdash_{\mathcal{G}} [\![\boldsymbol{\Delta}, [i : \varphi]]\!].$$

We will say that a Gentzen system is **accumulative** if it is i-accumulative for every $i < m$. Accumulative (2-)dimensional Gentzen systems have been defined and studied in [14]. (2-)dimensional systems satisfying a similar property have been studied by A. Avron in [1] with the name of "pure". Now we will give sufficient conditions for a Gentzen system to be accumulative.

Proposition 2.11 *Let $\mathcal{G} = \langle \mathcal{L}, \vdash_{\mathcal{L}} \rangle$ be an m-dimensional Gentzen system and $i < m$. If $\mathcal{G}$ satisfies the following two properties:*

(i) *$\mathcal{G}$ satisfies the axiom and the weakening rule for the place i,*

(ii) *$\mathcal{G}$ can be defined by a set of rules such that an arbitrary sequence of formulas Λ appears at the end of the i-th component of all the sequents that appear in the rule. That is, $\mathcal{G}$ can be defined by a set of rules of the following form:*

$$\frac{\{[\![\boldsymbol{\Pi}_j, [i : \Lambda]]\!] : j < n\}}{[\![\boldsymbol{\Pi}, [i : \Lambda]]\!]}$$

where $\{\boldsymbol{\Pi}_j : j < n\} \cup \{\boldsymbol{\Pi}\} \subseteq m\text{-}Seq_{\mathcal{L}}$ and Λ is an arbitrary sequence of $\mathcal{L}$-formulas,

then $\mathcal{G}$ is i-accumulative.

Proof: Let $\boldsymbol{T} \cup \{\boldsymbol{\Gamma}; \boldsymbol{\Delta}\}$ be a set of sequents and φ a formula. Suppose $\boldsymbol{T}; \boldsymbol{\Gamma} \vdash_{\mathcal{G}} \boldsymbol{\Delta}$ and let us see that $\boldsymbol{T}; [\![\boldsymbol{\Gamma}, [i : \varphi]]\!] \vdash_{\mathcal{G}} [\![\boldsymbol{\Delta}, [i : \varphi]]\!]$. We apply induction on the length l of a proof of $\boldsymbol{\Delta}$ from $\boldsymbol{T} \cup \{\boldsymbol{\Gamma}\}$:

If $l = 1$ then we have two cases:

(i) If $\boldsymbol{\Delta}$ is an axiom, by $(w : i)$, $\emptyset \vdash_{\mathcal{G}} [\![\boldsymbol{\Delta}, [i : \varphi]]\!]$;

(ii) If $\boldsymbol{\Delta} \in \boldsymbol{T}$, by using $(w : i)$ we obtain $\boldsymbol{\Delta} \vdash_{\mathcal{G}} [\![\boldsymbol{\Delta}, [i : \varphi]]\!]$;

and in both cases $\boldsymbol{T}; [\![\boldsymbol{\Gamma}, [i : \varphi]]\!] \vdash_{\mathcal{G}} [\![\boldsymbol{\Delta}, [i : \varphi]]\!]$.

Let $l > 1$. If the last rule applied in the proof of $\boldsymbol{\Delta}$ from $\boldsymbol{T}$ is (r), then $\boldsymbol{\Delta} = [\![\boldsymbol{\Pi}, [i : \Lambda]]\!]$, thus we have the following instance of (r):

$$\frac{\{[\![\boldsymbol{\Pi}_j, [i : \Lambda]]\!] : j < n\}}{[\![\boldsymbol{\Pi}, [i : \Lambda]]\!]} r.$$

By induction hypothesis

$$T; [\![\Gamma, [i : \varphi]]\!] \vdash_{\mathcal{G}} [\![\Pi_j, [i : \Lambda, \varphi]]\!] \tag{4}$$

for all $j < n$. Now, by the same rule (r), we obtain

$$\{[\![\Pi_j, [i : \Lambda, \varphi]]\!] : j < n\} \vdash_{\mathcal{G}} [\![\Pi, [i : \Lambda, \varphi]]\!]. \tag{5}$$

Thus, by using (4) and (5) $T; [\![\Gamma, [i : \varphi]]\!] \vdash_{\mathcal{G}} [\![\Delta, [i : \varphi]]\!]$. ∎

Corollary 2.12 *Let $\mathcal{G} = \langle \mathcal{L}, \vdash_{\mathcal{L}} \rangle$ be an m-dimensional Gentzen system. If $\mathcal{G}$ satisfies the axiom and the weakening rules and can be defined by a set of rules of the form*

$$\frac{\{[\![\Pi_j, \Gamma]\!] : j < n\}}{[\![\Pi, \Gamma]\!]}$$

where $\{\Pi_j : j < n\} \cup \{\Pi; \Gamma\} \subseteq m\text{-}Seq_{\mathcal{L}}$, then $\mathcal{G}$ is accumulative. ∎

Theorem 2.13 *Let* **L** *be a finite $\mathcal{L}$-algebra. The* **VL**-*Gentzen systems are accumulative.*

Proof: Since every **VL**-sequent calculus contains the axiom and the weakening and exchange rules, the Gentzen system determined satisfies these rules. The weakening, the contraction rules and all the **VL**-introduction rules can be written as in the hypothesis of Corollary 2.12, by using the exchange rules. The exchange rules are already in the desired form and finally, by using the contraction and the exchange rules, the cut rule can be replaced by

$$\frac{[\![\Gamma_0, [k \colon \varphi], \Gamma]\!] \quad [\![\Delta_0, [j \colon \varphi], \Gamma]\!]}{[\![\Gamma_0, \Delta_0, \Gamma]\!]}.$$

So each **VL**-Gentzen system satisfies the hypothesis of Corollary 2.12. Thus they are accumulative. ∎

Matrices for Gentzen systems

Our definition of the notion of a matrix model of a Gentzen system is similar to the corresponding definition for a deductive system.

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Let $\mathbf{A}$ be an $\mathcal{L}$–algebra. An m-relation on $\mathbf{A}$ is a set $R \subseteq \bigcup\{A^{n_0} \times \ldots \times A^{n_{m-1}} : n_i \in \omega, i < m\}$, that is, a set of m-tuples formed by finite sequences of elements of A; $\mathcal{R}_m(\mathbf{A})$ will be the set of all m-relations on $\mathbf{A}$. If there is no risk of confusion we write $\mathcal{R}_m$ instead of $\mathcal{R}_m(\mathbf{A})$.

An m**–matrix**, or just a **matrix**, is a pair $\langle \mathbf{A}, R\rangle$ where R is an m-relation on $\mathbf{A}$. Notice that instead of considering a set $F \subseteq A$, we consider an m-relation.

Let $h \in Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{A})$. If $\mathbf{\Gamma}$ is the sequent

$$\gamma_0^0, \ldots, \gamma_0^{t_0-1} \mid \ldots \mid \gamma_{m-1}^0, \ldots, \gamma_{m-1}^{t_{m-1}-1},$$

then $h(\mathbf{\Gamma})$ stands for

$$h(\gamma_0^0), \ldots, h(\gamma_0^{t_0-1}) \mid \ldots \mid h(\gamma_{m-1}^0), \ldots, h(\gamma_{m-1}^{t_{m-1}-1}) \in A^{t_0} \times \ldots \times A^{t_{m-1}}.$$

Let $\mathcal{G}$ be an m-dimensional Gentzen system and let (r) be an m–rule of inference. An element $R \in \mathcal{R}_m$ is **closed under the rule** (r) if for every pair $\langle \boldsymbol{T}, \mathbf{\Gamma}\rangle \in (r)$, and every $h \in Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{A})$, $h(\boldsymbol{T}) \subseteq R$ implies $h(\mathbf{\Gamma}) \in R$. A $\mathcal{G}$**-filter** is a set $R \in \mathcal{R}_m$ such that for every set of sequents $\boldsymbol{T} \cup \{\mathbf{\Gamma}\}$ and for every $h \in Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{A})$, $\boldsymbol{T} \vdash_{\mathcal{G}} \mathbf{\Gamma}$ and $h(\boldsymbol{T}) \subseteq R$ imply $h(\mathbf{\Gamma}) \in R$. When $\mathcal{G}$ is defined by means of some axioms and inference rules, R is a $\mathcal{G}$-filter iff R contains all the interpretations of these axioms and is closed under each of these rules. A matrix $\langle \mathbf{A}, R\rangle$ is called a **matrix model** of $\mathcal{G}$ (or $\mathcal{G}$**-matrix**) if R is a $\mathcal{G}$-filter.

If $\langle \mathbf{A}, R\rangle$ is an m-matrix, let $\models_{\langle \mathbf{A}, R\rangle}$ be the structural consequence relation on the set m-$Seq_{\mathcal{L}}$ defined by the following condition: $\boldsymbol{T} \models_{\langle \mathbf{A}, R\rangle} \mathbf{\Gamma}$ iff for every $h \in Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{A})$, $h(\boldsymbol{T}) \subseteq R$ implies $h(\mathbf{\Gamma}) \in R$.

Now we are going to define a semantical consequence relation over the set of m-sequents based on the definition of $\mathbf{L}$-satisfaction. This consequence relation is defined from an m–matrix on the algebra $\mathbf{L}$. So we start by defining the following m–relation, which contains the interpretation of the valid sequents:

Definition 2.14 *Let* $\mathbf{L}$ *be a finite algebra with universe* $L = \{v_0, \ldots, v_{m-1}\}$ *of cardinal* m*, then*

$$D_{\mathbf{L}} = \left\{ \begin{array}{l} (X(0), \ldots, X(m-1)) \in L^{n_0} \times \ldots \times L^{n_{m-1}} : n_i \in \omega \textit{ for } i < m, \\ \textit{and exists } i < m \textit{ such that } v_i \in X(i) \end{array} \right\}$$

The connection between the m-matrix $\langle \mathbf{L}, D_{\mathbf{L}} \rangle$ and the definition of $\mathbf{L}$-validity and $\mathbf{L}$-satisfaction is shown in the following

Theorem 2.15 *Let $\boldsymbol{T} \cup \{\boldsymbol{\Gamma}\} \subseteq m\text{-}Seq_{\mathcal{L}}$. The following properties hold:*

(i) *If $h \in Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{L})$, then*

$$h \in s(\boldsymbol{\Gamma}) \iff h(\boldsymbol{\Gamma}) \in D_{\mathbf{L}}.$$

(ii) $\emptyset \models_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle} \boldsymbol{\Gamma} \iff \boldsymbol{\Gamma}$ *is an* $\mathbf{L}$*-valid sequent.*

(iii) $\boldsymbol{T} \models_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle} \boldsymbol{\Gamma} \iff \bigcap_{\boldsymbol{\Pi} \in \boldsymbol{T}} s(\boldsymbol{\Pi}) \subseteq s(\boldsymbol{\Gamma})$.

Proof: Straightforward. ∎

Definition 2.16 *Let $\boldsymbol{T} \cup \{\boldsymbol{\Gamma}\} \subseteq m\text{-}Seq_{\mathcal{L}}$, if $\boldsymbol{T} \models_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle} \boldsymbol{\Gamma}$ we say that $\boldsymbol{\Gamma}$* **follows semantically** *from $\boldsymbol{T}$.*

In order to prove that the consequence relation associated to any **VL**-sequent calculus and the semantical consequence relation $\models_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle}$ are equal, and since the first one is, by definition, finitary, we will show first that the second is also finitary. We will follow the topological proof given by Łos and Suszko (and sketched by Wojcicki in [17, p. 262]) of the fact that the consequence relation determined by a finite class of finite matrices is finitary. The topological basis of the proof can be found in [16].

Let us assign to L the discrete topology and to L^{Var} the product topology, where Var is the set of propositional variables. Since L is a compact Hausdorff space, L^{Var} is also compact Hausdorff, by Tychonoff's Theorem. Now, by identifying each homomorphism $h : \mathbf{Fm}_{\mathcal{L}} \mapsto \mathbf{L}$ with its restriction $\overline{a} : Var \mapsto L$, we can identify the sets $Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{L})$ and L^{Var}, so $Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{L})$ is also a compact Hausdorff space.

Theorem 2.17 ([16, p. 25-26] *Let $\mathbf{L}$ be a finite $\mathcal{L}$-algebra. Let $\boldsymbol{\Gamma} \in m\text{-}Seq_{\mathcal{L}}$ and let $s(\boldsymbol{\Gamma})$ be the set of homomorphisms that $\mathbf{L}$-satisfy the sequent $\boldsymbol{\Gamma}$. Then the set $s(\boldsymbol{\Gamma})$ is open and closed in the space $Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{L})$*

Proof: Let $Var(\mathbf{\Gamma}) = \{p_0, \ldots, p_{f-1}\}$ be the set of propositional variables occurring in $\mathbf{\Gamma}$. Since $Var(\mathbf{\Gamma})$ is finite, the restrictions to $Var(\mathbf{\Gamma})$ of the elements of $s(\mathbf{\Gamma})$ form a finite set $\{w^0, \ldots, w^{s-1}\}$. Now we have the following equivalences:

$$\begin{aligned} h \in s(\mathbf{\Gamma}) &\iff \text{there exists} \quad j < s : h/Var(\mathbf{\Gamma}) = w^j \\ &\iff \text{there exists} \quad j < s : h/Var(\mathbf{\Gamma})(p_i) = w^j(p_i) \quad (i < f) \\ &\iff \text{there exists} \quad j < s : \\ &\qquad h \in pr_{p_1}^{-1}(w^j(p_1)) \cap \ldots \cap pr_{p_{f-1}}^{-1}(w^j(p_{f-1})), \end{aligned}$$

(where $pr_{p_i} : L^{Var} \to L$ is the projection over the propositional variable $p_i \in Var$), thus

$$s(\mathbf{\Gamma}) = \bigcup_{j \in \{1, \ldots, s-1\}} \{pr_{p_1}^{-1}(w^j(p_1)) \cap \ldots \cap pr_{p_{f-1}}^{-1}(w^j(p_{f-1}))\},$$

and $s(\mathbf{\Gamma})$ is open and closed in $Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{L})$. ∎

With this result, an immediate proof of the compactness theorem can be given:

Theorem 2.18 [16, Theorem 2] *Let* $\mathbf{L}$ *be a finite* $\mathcal{L}$*-algebra. A set of sequents is simultaneously* $\mathbf{L}$*-satisfiable iff every finite subset is simultaneously* $\mathbf{L}$*-satisfiable.*

Proof: If $\boldsymbol{T}$ is not simultaneously $\mathbf{L}$-satisfiable, by definition, $\cap_{\mathbf{\Gamma} \in \boldsymbol{T}} s(\mathbf{\Gamma}) = \emptyset$. Now, since $s(\mathbf{\Gamma})$ is closed for every $\mathbf{\Gamma}$, and $Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{L})$ is compact Hausdorff, there exists a finite subset $\boldsymbol{T}' \subseteq \boldsymbol{T}$, such that $\cap_{\mathbf{\Gamma} \in \boldsymbol{T}'} s(\mathbf{\Gamma}) = \emptyset$, and then $\boldsymbol{T}'$, being finite, is not simultaneously $\mathbf{L}$-satisfiable. The other implication is obvious. ∎

Theorem 2.19 (Finitariety) *Let* $\mathbf{L}$ *be a finite* $\mathcal{L}$*-algebra. Let* $\boldsymbol{T} \cup \{\mathbf{\Gamma}\}$ *be a set of m-sequents. The following properties are equivalent:*

(i) $\boldsymbol{T} \models_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle} \mathbf{\Gamma}$.

(ii) *There exists a finite subset* $\boldsymbol{T}' \subseteq \boldsymbol{T}$*, such that* $\boldsymbol{T}' \models_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle} \mathbf{\Gamma}$.

Proof: Assume $\boldsymbol{T} \models_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle} \boldsymbol{\Gamma}$, by Theorem 2.15, $\cap_{\boldsymbol{\Delta} \in \boldsymbol{T}} s(\boldsymbol{\Delta}) \subseteq s(\boldsymbol{\Gamma})$, and then

$$\left(\bigcap_{\boldsymbol{\Delta} \in \boldsymbol{T}} s(\boldsymbol{\Delta}) \right)^C \cup s(\boldsymbol{\Gamma}) = \left(\bigcup_{\boldsymbol{\Delta} \in \boldsymbol{T}} (s(\boldsymbol{\Delta}))^C \right) \cup s(\boldsymbol{\Gamma}) = Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{L}), \qquad (6)$$

where, if $H \subseteq Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{L})$, H^C denotes $Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{L}) \backslash H$.

Since $Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{L})$ is a compact Hausdorff space, and $s(\boldsymbol{\Delta})$ is always closed, there is a finite subset of $\boldsymbol{T}$ that satisfies (6), that is, there exists a finite subset $\boldsymbol{T}' \subseteq \boldsymbol{T}$ satisfying

$$\left(\bigcap_{\boldsymbol{\Delta} \in \boldsymbol{T}'} s(\boldsymbol{\Delta}) \right)^C \cup s(\boldsymbol{\Gamma}) = \left(\bigcup_{\boldsymbol{\Delta} \in \boldsymbol{T}'} (s(\boldsymbol{\Delta}))^C \right) \cup s(\boldsymbol{\Gamma}) = Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{Fm}_{\mathcal{L}}),$$

thus $\cap_{\boldsymbol{\Delta} \in \boldsymbol{T}'} s(\boldsymbol{\Delta}) \subseteq s(\boldsymbol{\Gamma})$ and then $\boldsymbol{T}' \models_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle} \boldsymbol{\Gamma}$. The other implication is straightforward. ∎

3 The deduction detachment theorem for Gentzen systems

The definition we give of the deduction detachment theorem (DDT for short) is a generalization of the one given in [14] for (2-)dimensional Gentzen systems, which is, in turn, a generalization of the deduction detachment theorem for deductive systems. This DDT is formulated by means of a deduction detachment set (DD-set for short) which acts as a kind of implication between sequents. If a Gentzen system has the DDT, then the DD-set makes it possible to carry one sequent from left to right of the consequence relation by using a finite number of sequents. These sequents will be in as many variables as there are formulas in the two sequents involved. Thus we need different sets of sequents, according to the lengths of the components of the sequents.

After giving the definition, we obtain sufficient conditions for a Gentzen system to have the deduction detachment theorem.

Let $\mathcal{G} = \langle \mathcal{L}, \vdash_{\mathcal{G}} \rangle$ be an m-dimensional Gentzen system.
If $\vec{k} = (k_0, \ldots, k_{m-1}) \in \omega^m$ and $\vec{l} = (l_0, \ldots, l_{m-1}) \in \omega^m$, $\mathbf{p}_{\vec{k}}$ and $\mathbf{q}_{\vec{l}}$ will denote disjoint sequences of distinct variables

$$\mathbf{p}_{\vec{k}} = p_0^0, p_0^1, \ldots, p_0^{k_0-1}, p_1^0, p_1^1, \ldots, p_1^{k_1-1}, \ldots, p_{m-1}^0, p_{m-1}^1, \ldots, p_{m-1}^{k_{m-1}-1}$$

$$\mathbf{q}_{\vec{l}} = q_0^0, q_0^1, \ldots, q_0^{l_0-1}, q_1^0, q_1^1, \ldots, q_1^{l_1-1}, \ldots, q_{m-1}^0, q_{m-1}^1, \ldots, q_{m-1}^{l_{m-1}-1}$$

Let $E = \{\boldsymbol{E}_{\vec{k},\vec{l}} : \vec{k}, \vec{l} \in \omega^m\}$, where each $\boldsymbol{E}_{\vec{k},\vec{l}}$ is a finite set of m-sequents in the variables $\mathbf{p}_{\vec{k}}$ and $\mathbf{q}_{\vec{l}}$ (at most). In the case $k_i = l_i = 0$ for all $i < m$, $\boldsymbol{E}_{(0,\ldots,0),(0,\ldots,0)}$ is a finite set of m-sequents in one variable, (at most). Then

$$\boldsymbol{E}_{\vec{k},\vec{l}}\,(\mathbf{p}_{\vec{k}}, \mathbf{q}_{\vec{l}}) = \{\boldsymbol{\Delta}_i(\mathbf{p}_{\vec{k}}, \mathbf{q}_{\vec{l}}) : i < \omega\} \subseteq m\text{-}Seq_{\mathcal{L}}$$

and in the case $k_i = l_i = 0$ for all $i < m$,

$$\boldsymbol{E}_{(0,\ldots,0),(0,\ldots,0)}(p_0) = \{\boldsymbol{\Pi}_i(p_0) : i < l\} \subseteq m\text{-}Seq_{\mathcal{L}}\ .$$

If $\boldsymbol{\Gamma}$ and $\boldsymbol{\Pi}$ are the sequents
$\boldsymbol{\Gamma} = \gamma_0^0, \gamma_0^1, \ldots, \gamma_0^{k_0-1} \mid \gamma_1^0, \gamma_1^1, \ldots, \gamma_1^{k_1-1} \mid \ldots \mid \gamma_{m-1}^0, \gamma_{m-1}^1, \ldots, \gamma_{m-1}^{k_{m-1}-1}$ and
$\boldsymbol{\Pi} = \pi_0^0, \pi_0^1, \ldots, \pi_0^{l_0-1} \mid \pi_1^0, \pi_1^1, \ldots, \pi_1^{l_1-1} \mid \ldots \mid \pi_{m-1}^0, \pi_{m-1}^1, \ldots, \pi_{m-1}^{l_{m-1}-1}$
we will write

$$E(\boldsymbol{\Gamma}, \boldsymbol{\Pi}) = \boldsymbol{E}_{(k_0,\ldots,k_{m-1}),(l_0,\ldots,l_{m-1})}(\boldsymbol{\Gamma}, \boldsymbol{\Pi}),$$

the result of replacing the variable p_i^j by γ_i^j $(i < m, j < k_i)$, and the variable q_i^j by π_i^j $(i < m, j < l_i)$ in every sequent of $\boldsymbol{E}_{\vec{k},\vec{l}}$.
In the case $k_i = l_i = 0$ for all $i < m$, we will write

$$E(\emptyset_m, \emptyset_m) = \boldsymbol{E}_{(0,\ldots,0),(0,\ldots,0)}(p_0).$$

The set E is called **deduction-detachment set** (DD-set, for short) for $\mathcal{G}$ if for all $\boldsymbol{T} \cup \{\boldsymbol{\Gamma}; \boldsymbol{\Pi}\} \subseteq m\text{-}Seq_{\mathcal{L}}$,

$$\boldsymbol{T}; \boldsymbol{\Gamma} \vdash_{\mathcal{G}} \boldsymbol{\Pi} \quad \text{iff} \quad \boldsymbol{T} \vdash_{\mathcal{G}} E(\boldsymbol{\Gamma}, \boldsymbol{\Pi}).$$

Definition 3.1 *A Gentzen system $\mathcal{G}$ has the* **deduction-detachment theorem** *(DDT, for short) if it possesses some deduction-detachment set.*

If a Gentzen system $\mathcal{G}$ has the deduction detachment theorem, the consequence relation $\vdash_{\mathcal{G}}$ is determined by the $\mathcal{G}$-derivable sequents and the DD-set, as shown in the following

Theorem 3.2 *Let $\mathcal{G}_1 = \langle \mathcal{L}, \vdash_{\mathcal{G}_1} \rangle$ and $\mathcal{G}_2 = \langle \mathcal{L}, \vdash_{\mathcal{G}_2} \rangle$ be two m-dimensional Gentzen systems. If $\mathcal{G}_1$ and $\mathcal{G}_2$ have the deduction detachment theorem with respect to the same deduction-detachment set and they have the same derivable sequents, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are equal.*

Proof: We prove that for any finite set of sequents $\boldsymbol{T}$ and any sequent $\boldsymbol{\Gamma}$

$$\boldsymbol{T} \vdash_{\mathcal{G}_1} \boldsymbol{\Gamma} \Longleftrightarrow \boldsymbol{T} \vdash_{\mathcal{G}_2} \boldsymbol{\Gamma}$$

by induction on the cardinal of the set $\boldsymbol{T}$.
If $\emptyset \vdash_{\mathcal{G}_1} \boldsymbol{\Gamma}$, then $\boldsymbol{\Gamma}$ is $\mathcal{G}_1$-derivable and, by hypothesis, $\emptyset \vdash_{\mathcal{G}_2} \boldsymbol{\Gamma}$.
Let E be a DD-set for both Gentzen systems $\mathcal{G}_1$ and $\mathcal{G}_2$. If $\boldsymbol{T}'; \boldsymbol{\Pi} \vdash_{\mathcal{G}_1} \boldsymbol{\Gamma}$, then, by the deduction theorem, $\boldsymbol{T}' \vdash_{\mathcal{G}_1} E(\boldsymbol{\Pi}, \boldsymbol{\Gamma})$. By inductive hypothesis $\boldsymbol{T}' \vdash_{\mathcal{G}_2} E(\boldsymbol{\Pi}, \boldsymbol{\Gamma})$, and since $\mathcal{G}_2$ has the same DD-set,

$$\boldsymbol{T}'; \boldsymbol{\Pi} \vdash_{\mathcal{G}_2} \boldsymbol{\Gamma},$$

and this finishes the proof. ∎

Now we will give sufficient conditions for a Gentzen system to have the deduction detachment theorem. First we show that, in a Gentzen system that satisfies the axiom, weakening and exchange rules, we can associate to each sequent $\boldsymbol{\Gamma}$ a set of derivable sequents: those obtained adding a formula that occurs in any component of $\boldsymbol{\Gamma}$ to all the other components, that is:

Lemma 3.3 *Let $\mathcal{G} = \langle \mathcal{L}, \vdash_{\mathcal{G}} \rangle$ be an m-dimensional Gentzen system that satisfies the axiom, and the weakening and exchange rules. If $\boldsymbol{\Gamma} \in m\text{-}Seq_{\mathcal{L}}$, then, for every $k < m$ and every $\gamma \in \boldsymbol{\Gamma}(k)$, the sequent $[\![\boldsymbol{\Gamma}, [\{k\}^C : \gamma]]\!]$ is $\mathcal{G}$-derivable.*

Proof: Note that, if $\gamma \in \boldsymbol{\Gamma}(k)$ and $i < m$, then

$$[\![\boldsymbol{\Gamma}, [\{k\}^C : \gamma]]\!](i) = \begin{cases} \boldsymbol{\Gamma}(i), \gamma & \text{if } i \neq k \\ \boldsymbol{\Gamma}(k) & \text{if } i = k \end{cases}$$

and then $\gamma \in [\![\boldsymbol{\Gamma}, [\{k\}^C : \gamma]]\!](i)$ for all $i < m$, and the result follows from Lemma 2.7. ∎

Theorem 3.4 *Let $\mathcal{G} = \langle \mathcal{L}, \vdash_{\mathcal{G}} \rangle$ be a Gentzen system that satisfies all the structural rules and is accumulative. Then $\mathcal{G}$ satisfies the deduction detachment theorem with respect to the DD-set given by the following equivalences, where $\boldsymbol{T} \cup \{\boldsymbol{\Gamma}; \boldsymbol{\Delta}\} \subseteq m\text{-}Seq_{\mathcal{L}}$*

(i) *If $\boldsymbol{\Gamma} \neq \emptyset_m$,*

$$\boldsymbol{T}; \boldsymbol{\Gamma} \vdash_{\mathcal{G}} \boldsymbol{\Delta} \Longleftrightarrow \boldsymbol{T} \vdash_{\mathcal{G}} [\![\boldsymbol{\Delta}, [\{k\}^C : \gamma]]\!], \textit{ for all } k < m \textit{ and all } \gamma \in \boldsymbol{\Gamma}(k).$$

(ii) *If* $\boldsymbol{\Delta} \neq \emptyset_m$,

$$\boldsymbol{T}; \emptyset_m \vdash_{\mathcal{G}} \boldsymbol{\Delta} \quad \textit{iff} \quad \boldsymbol{T} \vdash_{\mathcal{G}} \boldsymbol{\Delta}(0), \ldots, \boldsymbol{\Delta}(m-1) \mid \ldots \mid \boldsymbol{\Delta}(0), \ldots, \boldsymbol{\Delta}(m-1).$$

(iii) $\boldsymbol{T}; \emptyset_m \vdash_{\mathcal{G}} \emptyset_m \quad \textit{iff} \quad \boldsymbol{T} \vdash_{\mathcal{G}} p_0 \mid \ldots \mid p_0.$

Proof: Cases (ii) and (iii) are obvious, because both sides of the equivalence are always true. Let us prove (i): Assume $\boldsymbol{T}; \boldsymbol{\Gamma} \vdash_{\mathcal{G}} \boldsymbol{\Delta}$ and let $k < m$ and $\gamma \in \boldsymbol{\Gamma}(k)$. Since $\mathcal{G}$ is accumulative

$$\boldsymbol{T}; [\![\boldsymbol{\Gamma}, [\{k\}^C : \gamma]]\!] \vdash_{\mathcal{G}} [\![\boldsymbol{\Delta}, [\{k\}^C : \gamma]]\!]. \tag{7}$$

By Lemma 3.3, $\emptyset \vdash_{\mathcal{G}} [\![\boldsymbol{\Gamma}, [\{k\}^C : \gamma]]\!]$ and then, by using (7)

$$\boldsymbol{T} \vdash_{\mathcal{G}} [\![\boldsymbol{\Delta}, [\{k\}^C : \gamma]]\!].$$

To prove the other implication, by using Lemma 2.10

$$\boldsymbol{\Gamma}; \bigcup_{k<m} \{[\![\boldsymbol{\Delta}, [\{k\}^C : \gamma]]\!] : \gamma \in \boldsymbol{\Gamma}(k)\} \vdash_{\mathcal{G}} \boldsymbol{\Delta}. \tag{8}$$

But by hypothesis $\boldsymbol{T} \vdash_{\mathcal{G}} [\![\boldsymbol{\Delta}, [\{k\}^C : \gamma]]\!]$ for any $k < m$ and $\gamma \in \boldsymbol{\Gamma}(k)$. So, by using (8), $\boldsymbol{T}; \boldsymbol{\Gamma} \vdash_{\mathcal{G}} \boldsymbol{\Delta}$. ∎

Note that only variables occur in the sequents of the DD-set of this theorem.

Example 3.5 Let $\mathcal{G}$ be a 3-dimensional Gentzen system that satisfies the hypotheses of Theorem 3.4. Then, by applying the deduction detachment theorem

$$\begin{array}{rll} \boldsymbol{T}; \varphi \mid \psi \mid \eta \vdash_{\mathcal{G}} \varphi_1 \mid \psi_1 \mid \eta_1 \iff & \boldsymbol{T} \vdash_{\mathcal{G}} & \varphi_1 \mid \psi_1, \varphi \mid \eta_1, \varphi \\ & \boldsymbol{T} \vdash_{\mathcal{G}} & \varphi_1, \psi \mid \psi_1 \mid \eta_1, \psi \\ & \boldsymbol{T} \vdash_{\mathcal{G}} & \varphi_1, \eta \mid \psi_1, \eta \mid \eta_1. \end{array}$$

∎

Since the **VL**-Gentzen systems satisfy all the structural rules, and are accumulative, they satisfy the deduction-detachment theorem.

Corollary 3.6 *Let* $\mathbf{L}$ *be a finite* $\mathcal{L}$*-algebra. Let* $\mathcal{G} = \langle \mathcal{L}, \vdash_{\mathcal{G}} \rangle$ *be a* $\mathbf{VL}$*-Gentzen system. Then* $\mathcal{G}$ *satisfies the deduction detachment theorem given in Theorem 3.4.* ∎

Although all the $\mathbf{VL}$-Gentzen systems satisfy this DDT, by using special properties of the algebras involved, it may be possible to prove other DDT. For instance, in [11, Theorem 45 and 48] we give two different DDT for the $\mathbf{VS}(m)$-Gentzen systems, where $\mathbf{S}(m)$ is the linear MV-algebra of m-elements, in which the sets $\boldsymbol{E}_{\vec{k},\vec{l}}$ consists of a single sequent, for all $\vec{k}, \vec{l} \in \omega^m$.

Theorem 3.7 *Let* $\mathbf{L}$ *be a finite* $\mathcal{L}$*-algebra. If* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *are* $\mathbf{VL}$*-Gentzen systems, then* $\mathcal{G}_1 = \mathcal{G}_2$*, that is, each* $\mathbf{VL}$*-sequent calculus determines the same Gentzen system.*

Proof: All the $\mathbf{VL}$-Gentzen systems satisfy the deduction detachment theorem with respect to the DD-set given in Theorems 3.4 and Corollary 3.6, and they all have the same set of derivable sequents (see Theorem 1.11). Thus, by Theorem 3.2, all these Gentzen systems are equal. ∎

This theorem proves that the consequence relation determined by any $\mathbf{VL}$-sequent calculus is independent of the set of introduction rules we choose for each connective and place. The consequence relation associated with any $\mathbf{VL}$-sequent calculus will be denoted by $\vdash_{\mathbf{VL}}$, and the Gentzen system determined by any $\mathbf{VL}$-sequent calculus will be denoted by the expression $\mathcal{G}_{\mathbf{VL}} = \langle \mathcal{L}, \vdash_{\mathbf{VL}} \rangle$. This Gentzen system will be called the **Gentzen system associated with the algebra L**.

The Strong cut elimination Theorem

In [1, pag. 270] there is a proof of the *Strong Cut Elimination Theorem* for the "Gentzen-type system for the Classical Propositional Logic". By using Theorem 3.4 we now extend this theorem to a wide class of Gentzen systems:

Theorem 3.8 *Let* LX *be a sequent calculus such that the cut elimination theorem holds for* LX *and such that the Gentzen system* $\mathcal{G}_{LX}$ *is accumulative and satisfies all the structural rules. For all* $\boldsymbol{T} \cup \{\boldsymbol{\Delta}\} \subseteq m\text{-}Seq_{\mathcal{L}}$ *, if* $\boldsymbol{T} \vdash_{LX} \boldsymbol{\Delta}$*, then there is a proof of* $\boldsymbol{\Delta}$ *from* $\boldsymbol{T}$ *in which every cut is made on a formula that occurs in some sequent of* $\boldsymbol{T}$*.*

Proof: By induction on the number of sequents n in $\boldsymbol{T}$, which can be assumed to be finite. The case $n = 0$ is just the cut elimination theorem for LX. If $n > 0$, let $\boldsymbol{T} = \boldsymbol{T}' \cup \{\boldsymbol{\Gamma}\}$. If $\boldsymbol{\Gamma} \neq \emptyset$, since $\mathcal{G}_{LX}$ satisfies the deduction detachment theorem given in Theorem 3.4, for all $k < m$ and all $\gamma \in \boldsymbol{\Gamma}(k)$ we have

$$\boldsymbol{T}' \vdash_{LX} [\![\boldsymbol{\Delta}, [\{k\}^C : \gamma]]\!].$$

By inductive hypothesis, there is a proof from $\boldsymbol{T}'$ of every sequent

$$[\![\boldsymbol{\Delta}, [\{k\}^C : \gamma]]\!]$$

where each cut is made on a formula that occurs in some sequent of $\boldsymbol{T}'$. With all these proofs, and since in the proof of

$$\boldsymbol{\Gamma}; \bigcup_{k<m} \{[\![\boldsymbol{\Delta}, [\{k\}^C : \gamma]]\!] : \gamma \in \boldsymbol{\Gamma}(k)\} \vdash_{LX} \boldsymbol{\Delta}$$

(see Lemma 2.10) the cuts are made on formulas that occur in $\boldsymbol{\Gamma}$ (thus in a sequent of $\boldsymbol{T}$), we obtain a proof of $\boldsymbol{\Delta}$ from $\boldsymbol{T}$ as desired.
If $\boldsymbol{\Gamma} = \emptyset_m$, then $\boldsymbol{\Delta}$ can be proved from $\boldsymbol{\Gamma}$ using only the weakening rules. ∎

4 Strong Completeness Theorem

Next we prove a Strong Completeness theorem for the Gentzen system associated with a finite algebra $\mathbf{L}$ with respect to the semantical consequence relation introduced in Definition 2.1. This theorem provides, from a semantical point of view, a characterization of this Gentzen system and, from the syntactical point of view, an axiomatization of the relation $\models_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle}$.

Theorem 4.1 (Strong Completeness) *Let* $\mathbf{L}$ *be a finite* $\mathcal{L}$*-algebra with universe* $L = \{v_0, \ldots, v_{m-1}\}$. *If* $\boldsymbol{T} \cup \{\boldsymbol{\Gamma}\} \subseteq m\text{-}Seq_{\mathcal{L}}$ *, then*

$$\boldsymbol{T} \models_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle} \boldsymbol{\Gamma} \iff \boldsymbol{T} \vdash_{\mathbf{VL}} \boldsymbol{\Gamma}.$$

Proof: Since the consequence relation $\models_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle}$ is finitary (Theorem 2.19), $\mathcal{G}_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle} = \langle \mathcal{L}, \models_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle} \rangle$ is a Gentzen system. It is easy to see that $\mathcal{G}_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle}$ satisfies all the structural rules; we show here that it satisfies the cut rules, i.e.,

$$[\![\boldsymbol{\Gamma}, [i : \varphi]]\!]; [\![\boldsymbol{\Delta}, [j : \varphi]]\!] \models_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle} [\![\boldsymbol{\Gamma}, \boldsymbol{\Delta}]\!]$$

for $i, j < m$, $i \neq j$.

Let $h \in Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{L})$ such that $h \in s([\![\mathbf{\Gamma}, [i : \varphi]]\!]) \cap s([\![\mathbf{\Delta}, [j : \varphi]]\!])$. If $h \in s(\mathbf{\Gamma}) \cup s(\mathbf{\Delta})$ then $h \in s([\![\mathbf{\Gamma}, \mathbf{\Delta}]\!])$. Otherwise we have $h(\varphi) = v_i$ and $h(\psi) = v_j$, a contradiction.

Now let us see that $\mathcal{G}_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle}$ is accumulative. Assume $\boldsymbol{T}; \mathbf{\Gamma} \models_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle} \mathbf{\Delta}$ and let us show that

$$\boldsymbol{T}; [\![\mathbf{\Gamma}, [i : \varphi]]\!] \models_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle} [\![\mathbf{\Delta}, [i : \varphi]]\!]$$

for $i < m$ and $\varphi \in Fm_{\mathcal{L}}$. Let $h \in Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{L})$ such that

$$h \in \left(\bigcap_{\mathbf{\Pi} \in \boldsymbol{T}} s(\mathbf{\Pi}) \right) \cap s([\![\mathbf{\Gamma}, [i : \varphi]]\!]).$$

If $h \in s(\mathbf{\Gamma})$, then $h \in s(\mathbf{\Delta})$ and so $h \in s([\![\mathbf{\Delta}, [i : \varphi]]\!])$. Otherwise we have $h(\varphi) = i$ and hence $h \in s([\![\mathbf{\Delta}, [i : \varphi]]\!])$.

Thus $\mathcal{G}_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle}$, as well as $\mathcal{G}_{\mathbf{VL}}$, satisfies the hypotheses of Theorem 3.4. Therefore $\mathcal{G}_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle}$ and $\mathcal{G}_{\mathbf{VL}}$ have the DDT with respect to the same DD-set. As, in addition, they have the same provable sequents (see Theorems 1.11 and 2.15), by Theorem 3.2, $\mathcal{G}_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle}$ and $\mathcal{G}_{\mathbf{VL}}$ coincide. ∎

The special case of the strong completeness theorem for the $\mathbf{VS}(m)$-Gentzen systems, where $\mathbf{VS}(m)$ is the linear MV-algebra of m-elements, was obtained in [11, Theorem 6.6] by using the fact that these Gentzen systems are equivalent to the m-valued Łukasiewicz propositional logic. This equivalence is proved by using the fact that every m-sequent is $\mathbf{VS}(m)$-equivalent to a sequent of the form $[m-1 : \varphi]$ (see [11, Theorem 5.12]), that is, a sequent such that all its components are empty, except the last one, which consists of a single formula. This formula is obtained by means of McNaughton's Theorem (see [12]). The strong completeness theorem proved in this paper holds for any finite algebra.

5 Tables

Table 1: LK

$$\varphi \mid \varphi \text{ (axiom)} \qquad \frac{\Gamma \mid \Delta, \varphi \quad \varphi, \Pi \mid \Lambda}{\Gamma, \Pi \mid \Delta, \Lambda}\,(cut)$$

$$\frac{\Gamma \mid \Delta}{\Gamma, \varphi \mid \Delta}\,(w:0) \qquad \frac{\Gamma \mid \Delta}{\Gamma \mid \Delta, \varphi}\,(w:1)$$

$$\frac{\Gamma, \varphi, \varphi \mid \Delta}{\Gamma, \varphi \mid \Delta}\,(c:0) \qquad \frac{\Gamma \mid \Delta, \varphi, \varphi}{\Gamma \mid \Delta, \varphi}\,(c:1)$$

$$\frac{\Gamma, \varphi, \psi, \Pi \mid \Delta}{\Gamma, \psi, \varphi, \Pi \mid \Delta}\,(e:0) \qquad \frac{\Gamma \mid \Delta, \varphi, \psi, \Lambda}{\Gamma \mid \Delta, \psi, \varphi, \Lambda}\,(e:1)$$

$$\frac{\Gamma \mid \Delta, \varphi}{\neg\varphi, \Gamma \mid \Delta}\,(\neg:0) \qquad \frac{\varphi, \Gamma \mid \Delta}{\Gamma \mid \Delta, \neg\varphi}\,(\neg:1)$$

$$\frac{\varphi, \Gamma \mid \Delta}{\varphi \wedge \psi, \Gamma \mid \Delta} \qquad \frac{\psi, \Gamma \mid \Delta}{\varphi \wedge \psi, \Gamma \mid \Delta}\,(\wedge:0) \qquad \frac{\Gamma \mid \Delta, \varphi \quad \Gamma \mid \Delta, \psi}{\Gamma \mid \Delta, \varphi \wedge \psi}\,(\wedge:1)$$

$$\frac{\varphi, \Gamma \mid \Delta \quad \psi, \Gamma \mid \Delta}{\varphi \vee \psi, \Gamma \mid \Delta}\,(|:0) \qquad \frac{\Gamma \mid \Delta, \varphi}{\Gamma \mid \Delta, \varphi \vee \psi} \qquad \frac{\Gamma \mid \Delta, \psi}{\Gamma \mid \Delta, \varphi \vee \psi}\,(\vee:1)$$

$$\frac{\Gamma \mid \Delta, \varphi \quad \psi, \Pi \mid \Lambda}{\varphi \to \psi, \Gamma, \Pi \mid \Delta, \Lambda}\,(\to:0) \qquad \frac{\varphi, \Gamma \mid \Delta, \psi}{\Gamma \mid \Delta, \varphi \to \psi}\,(\to:1)$$

Table 2: A **VS**(3)-sequent calculus

Introduction rules

$$\frac{\Gamma \mid \Delta \mid \Pi, \varphi}{\Gamma, \neg\varphi \mid \Delta \mid \Pi}\neg : 0 \qquad \frac{\Gamma \mid \Delta, \varphi \mid \Pi}{\Gamma \mid \Delta, \neg\varphi \mid \Pi}\neg : 1 \qquad \frac{\Gamma, \varphi \mid \Delta \mid \Pi}{\Gamma \mid \Delta \mid \Pi, \neg\varphi}\neg : 2$$

$$\frac{\Gamma, \varphi \mid \Delta \mid \Pi \quad \Gamma, \psi \mid \Delta \mid \Pi}{\Gamma, \varphi \vee \psi \mid \Delta \mid \Pi}\vee : 0$$

$$\frac{\Gamma \mid \Delta, \varphi, \psi \mid \Pi \quad \Gamma, \varphi \mid \Delta, \varphi \mid \Pi \quad \Gamma, \psi \mid \Delta, \psi \mid \Pi}{\Gamma \mid \Delta, \varphi \vee \psi \mid \Pi}\vee : 1$$

$$\frac{\Gamma \mid \Delta \mid \Pi, \varphi, \psi}{\Gamma \mid \Delta \mid \Pi, \varphi \vee \psi}\vee : 2$$

$$\frac{\Gamma, \varphi, \psi \mid \Delta \mid \Pi}{\Gamma, \varphi \wedge \psi \mid \Delta \mid \Pi}\wedge : 0$$

$$\frac{\Gamma \mid \Delta, \varphi, \psi \mid \Pi \quad \Gamma \mid \Delta, \varphi \mid \Pi, \varphi \quad \Gamma \mid \Delta, \psi \mid \Pi, \psi}{\Gamma \mid \Delta, \varphi \wedge \psi \mid \Pi}\wedge : 1$$

$$\frac{\Gamma \mid \Delta \mid \Pi, \varphi \quad \Gamma \mid \Delta \mid \Pi, \psi}{\Gamma \mid \Delta \mid \Pi, \varphi \wedge \psi}\wedge : 2$$

$$\frac{\Gamma \mid \Delta \mid \Pi, \varphi \quad \Gamma, \psi \mid \Delta \mid \Pi}{\Gamma, \varphi \to \psi \mid \Delta \mid \Pi}\to : 0$$

$$\frac{\Gamma \mid \Delta, \varphi \mid \Pi, \varphi \quad \Gamma \mid \Delta, \varphi, \psi \mid \Pi \quad \Gamma, \psi \mid \Delta \mid \Pi, \varphi}{\Gamma \mid \Delta, \varphi \to \psi \mid \Pi}\to : 1$$

$$\frac{\Gamma, \varphi \mid \Delta, \varphi \mid \Pi, \psi \quad \Gamma, \varphi \mid \Delta, \psi \mid \Pi, \psi}{\Gamma \mid \Delta \mid \Pi, \varphi \to \psi}\to : 2$$

Structural rules: All of them.

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