On the Notion of Effective Impedance for Finite and Infinite Networks

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Abstract

It is known that electrical networks with resistors are related to the Laplace operator and random walk on weighted graphs. Then the effective resistance is defined for any finite or infinite network. In this thesis we consider a more general electrical network with passive elements (resistors, capacitors and coils) and with an external source of alternating voltage of frequency $\omega > 0$ (AC network). Then the analogue of resistance is called impedance. Although the notion of the effective impedance is widely used in physical and mathematical literature, the problem of justification of this notion in the presence of coils and capacitors was not satisfactorily solved.

Mathematically AC network can be represented by a locally finite connected graph whose edges are endowed with weights depending on parameter λ (by the physical meaning, $\lambda = i\omega$, where i is the imaginary unit). These weights are rational functions of λ with real coefficients and correspond to physical admittances (inverses of impedances). In this thesis, we construct two mathematical models of an AC network.

In the first model we consider admittances of passive elements as complex valued functions of $\lambda \in \mathbb{C}$. The network is considered as a complex-weighted graph. We firstly introduce a mathematically correct definition of the notion of effective admittance (the inverse of the effective impedance) of a finite network. Then we prove some estimates of the effective admittances of finite networks in terms of λ. Using these estimates, we show that, for infinite networks, the sequence of effective admittances of finite network approximations converges in certain regions of the complex plane to a holomorphic function of λ, which allows to define the effective admittance of an infinite network in these regions. As an example of an infinite network, *Feynman's ladder* is considered.

In the second model we consider admittances as elements of an ordered field. The maximum principle holds for the Laplace operator with weights from an ordered field, which allows to uniquely define the effective admittance as an element of the ordered field. We apply these results to the ordered field of rational functions with real coefficients $\mathbb{R}(\lambda)$. In some particular examples, we consider networks over the Levi-Civita field $\mathcal{R}$ and show that the limit of the sequence of effective admittances of finite network approximations does not always exist in non-Archimedean field.

For finite networks, we prove the equivalence of the two aforementioned definitions of the effective admittance in the following sense: the effective admittance from the first model is equal to the effective admittance from the second model (i.e. over $\mathbb{R}(\lambda)$), evaluated at the point λ, for all values of $\lambda \in \mathbb{C}$ except for a finite set lying in $\{\operatorname{Re}\lambda \leq 0\}$.

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Contents

Chapter 1

Introduction

1.1 Motivation

Mathematically, an electrical network can be represented by a locally finite connected graph whose edges are endowed with weights that are determined by the physical properties of the connection between two nodes. Here we deal with the networks consisting of resistors, coils and capacitors. Assuming that an external periodic voltage of frequency $\omega > 0$ is applied to the network, each edge xy between the nodes x and y receives the *impedance*

$$z_{xy} = L_{xy} i\omega + R_{xy} + \frac{1}{C_{xy} i\omega} = L_{xy}\lambda + R_{xy} + \frac{1}{C_{xy}\lambda},$$

where R_{xy} is the resistance of this edge, L_{xy} is the inductance, C_{xy} is the capacitance, and $\lambda = i\omega$, i is the imaginary unit.

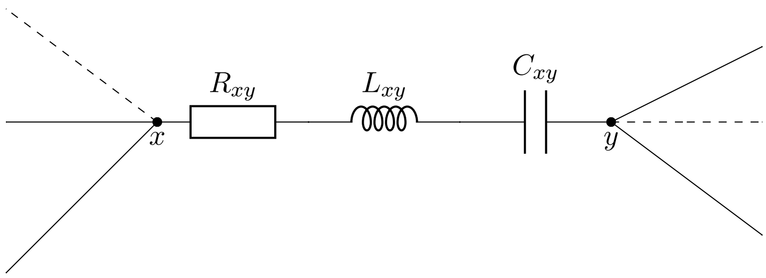


Figure 1.1: One edge of a network with impedances

It was shown in [13] and [23] that there is a tight relation between electrical networks with exclusively resistors, attached to a source of direct current, and weighted graphs. Ohm's and Kirchhoff's laws imply that the voltage in a finite network is a solution of the Dirichlet problem for the discrete Laplace operator on the weighted graph. Due to the maximum principle, the solution of the Dirichlet problem in this case exists and is unique (see, for example, [17]). Hence, this provides a mathematical justification of the notion of effective resistance as the inverse energy of the solution of the Dirichlet problem. Moreover, the network that consists just of resistors determines naturally a reversible Markov chain, and it is related to a random walk on graphs (see e.g. [3], [11], [13], [18], [23], [32], [38]).

For infinite (but locally finite) networks, again in absence of capacitors and coils, one constructs first a sequence $\{Z_n\}$ of *partial effective resistances* that are the effective resistances of an exhaustive sequence of finite networks, and then defines the effective resistance Z of the entire network as the limit $\lim\limits_{n\to\infty} Z_n$. This limit always exists due to the monotonicity of the sequence $\{Z_n\}$ (cf. [13], [18], [23], [32]).

In the case of finite networks with impedances and alternating current, the determinant of the Dirichlet problem, corresponding to the complex Kirchhoff's law (see e.g. [14], [15], [31]), may vanish for some frequencies of the alternating current and the system may have infinitely many solutions or no solution. To the best of our knowledge, there has been no precise mathematical definition of an effective impedance for these cases in the literature, although this notion is widely used in physics.

The case of infinite networks is even more complicated, since the sequence of the partial effective impedances is complex-valued and depends on the frequency of an alternating current. The monotonicity argument in this case is not available in the field $\mathbb{C}$ of complex numbers.

One of the first examples of a computation of an effective impedance for an infinite network was done by Richard Feynman in [15]. As it was observed later (cf. [21], [33], [34], [35], [39]), the sequence $\{Z_n\}$ of partial effective impedances in this network (named *Feynman's ladder*) converges not for all values of the frequency ω, which raises the question about the validity of Feynman's computation as well as the problem about a careful mathematical definition of the effective impedance for infinite networks.

The other example of a calculation of an effective impedance for one particular infinite network (*fractal Feynman-Sierpinski AC circuit*) is given in [1] and [10]. Moreover, in [1], [2] and [10] some relations between electrical networks and Dirichlet forms on graphs and fractals are considered.

1.2 Description of the results

In this thesis we introduce two mathematical models of finite and infinite electrical networks of alternating current (AC) with passive elements and we first give a mathematical definition of an effective impedance for finite networks in each model. For the convenience we will work with the admittance

$$\rho_{xy} = \frac{1}{z_{xy}} = \frac{1}{L_{xy}\lambda + R_{xy} + \dfrac{1}{C_{xy}\lambda}} = \frac{\lambda}{L_{xy}\lambda^2 + R_{xy}\lambda + \dfrac{1}{C_{xy}}} \tag{1.2.1}$$

of an edge, i.e. the inverse of the impedance.

Let (V, E) be a finite connected graph, where V is a set of vertices, E is a set of edges. Let each edge $xy \in E$ is endowed with the admittance ρ_{xy}. Further, let $B \subset V$ is a non-empty set of grounded vertices and $a_0 \notin B$, $a_0 \in V$ is the vertex, where the potential (voltage) with amplitude 1 is maintained. We refer to the structure

$$\Gamma = \{V, \{\rho_{xy}\}, a_0, B\}$$

as a *finite network*.

By Ohm's complex law and Kirchhoff's complex law, the complex voltage $v(x)$ at the vertex x satisfy the following system of linear equations:

$$\begin{cases} \sum_{y:y\sim x}(v(y) - v(x))\rho_{xy} = 0 \text{ on } V \setminus B_0, \\ v(a_0) = 1, \\ v(x) = 0 \text{ on } B, \end{cases} \tag{1.2.2}$$

where $B_0 = B \cup a_0$.

The physical meaning of $v(x)$ is that it is a (complex-valued) amplitude of the voltage at the node x, while the actual alternating voltage at time t is equal to $\mathrm{Re}(v(x)e^{i\omega t})$, where ω is the frequency of an alternating current.

We will consider the system (1.2.2) as a discrete boundary value Dirichlet problem.

In Chapter 2 we consider it over the field $\mathbb{C}$ of complex numbers with parameter λ, i.e. we consider an admittance $\rho_{xy} = \rho_{xy}^{(\lambda)}$ of an edge of a network as a complex-valued function of λ. Although initially $\lambda = i\omega$, where $\omega > 0$ is the frequency of an alternating current, we will consider more general $\lambda \in \mathbb{C}$ (similarly to [7]). One of our main contributions to analysis on graphs and theory of electrical networks is Definition 2.2.3, which gives the mathematical notions of effective impedance $Z(\lambda)$ and effective admittance $\mathcal{P}(\lambda)$ for finite networks. We prove, that the effective impedance and the effective admittance are well-defined, since in the case of multiple solutions of Dirichlet problem for some λ, all these solutions have the same energy. Moreover, we prove that the so defined effective impedance has some expected properties: elementary transforms of an electrical network do not change $Z(\lambda)$ (see Subsection 2.2.2). Using Green's formula (2.2.10), we prove that the effective impedance is a positive real function of λ (see Corollary 2.2.17). The concept of positive real function was introduced in [7] as one of the main properties of physical effective impedance as a function of λ.

In Section 2.4 we give estimates of effective admittances for finite network in some regions of the complex-plane λ. We prove (see Corollary 2.4.4), that for any network the effective admittance $\mathcal{P}(\lambda)$ is holomorphic in the domain $\{\mathrm{Re}\,\lambda > 0\}$ and admits there the estimate:

$$|\mathcal{P}(\lambda)| \leq C_0 \frac{|\lambda|^2 \left(1 + |\lambda|^2\right)}{(\mathrm{Re}\,\lambda)^3},$$

where C_0 does not depend on λ. Some other domains of the complex plane λ, depending on

$$S_D := \sup_{xy} \frac{1}{C_{xy}L_{xy}} \text{ and } S_D^* := \inf_{xy} \frac{1}{C_{xy}L_{xy}}$$

for a given network, are also considered (Theorem 2.4.7 and Theorem 2.4.10).

These estimates allow us to introduce the effective admittance for infinite networks in certain domains using diagonal argument for the sequence $\{\mathcal{P}_n(\lambda)\}$ of the partial effective admittances, i.e. the effective admittances of exhausted finite networks.

The main result of Chapter 2 is Theorem 2.5.2. It says that $\mathcal{P}(\lambda) := \lim_{n\to\infty} \mathcal{P}_n(\lambda)$ exists and is a holomorphic function of λ in the domain $\{\operatorname{Re}\lambda > 0\}$ as well as in some other regions. In the case of a resistance free network, Corollary 2.5.4 says that $\mathcal{P}(\lambda)$ is holomorphic in $\mathbb{C} \setminus \left[-i\sqrt{S_D}, i\sqrt{S_D}\right]$, where again

$$S_D = \sup_{xy} \frac{1}{C_{xy}L_{xy}}.$$

Moreover, as an important example we present the calculation of the effective admittance of the *Feynman's ladder* with zero at infinity (see Subsection 2.5.3).

In Chapter 3 we consider networks over an arbitrary ordered field $(\mathcal{K}, \succ)$. The motivation for this is the fact, that the field $\mathbb{R}(\lambda)$ of rational functions with real coefficients is an ordered field, where admittances ρ_{xy} in the form (1.2.1) are positive elements. The order " $\succ$" in $\mathbb{R}(\lambda)$ is defined as follows: for any rational function

$$f(\lambda) = \frac{b_k\lambda^k + \cdots + b_1\lambda + b_0}{d_m\lambda^m + \cdots + d_1\lambda + d_0} \in \mathbb{R}(\lambda)$$

with $b_k \neq 0, d_m \neq 0$, write

$$f(\lambda) \succ 0, \text{ if } \frac{b_k}{d_m} > 0$$

and say that $f(\lambda)$ is positive (see [6, p. A.VI.21], [37, pp. 231–234] and Appendix B).

Therefore, we can consider the Dirichlet problem (1.2.2) over $\mathbb{R}(\lambda)$. Fortunately, the maximum principle holds for the Laplace operator with weights from an ordered field, which allows to solve uniquely the Dirichlet problem and, hence, to define the effective admittance over an ordered field (see Definition 3.2.4). In particular, for any given physical network, one can uniquely define the effective admittance as a rational function of λ with real coefficients. Then we investigate properties of the effective admittance and prove Dirichlet/Thomson's principle for networks over an ordered field (Theorem 3.2.8). We make a first attempt to introduce an effective admittance of an infinite network over an ordered field. We show, using examples, that for a non-Archimedean ordered field the limit of the partial effective admittances does not always exist. As examples we consider infinite ladder networks over the Levi-Civita field, which is a Cauchy completion of $\mathbb{R}(\lambda)$.

In Section 3.4 we elaborate on connections between the models from Chapter 2 and Chapter 3 for finite networks. Let us denote by $\mathcal{P}_{\mathbb{C}}(\lambda)$ the effective admittance considered in Chapter 2 (see Definition 2.2.3). Let $\mathcal{P}_{\mathbb{R}(\lambda)}(\lambda)$ be the effective admittance from Chapter 3 (see Definition 3.2.4) over the field $\mathcal{K} = \mathbb{R}(\lambda)$. Note that $\mathcal{P}_{\mathbb{R}(\lambda)}(\lambda)$ is a rational function of λ by definition. Therefore, it is a continuous function of $\lambda \in \mathbb{C}$, taking values in $\mathbb{C} \cup \{\infty\}$. The arises question is whether

$$\mathcal{P}_{\mathbb{C}}(\lambda) = \mathcal{P}_{\mathbb{R}(\lambda)}(\lambda) \tag{1.2.3}$$

for all $\lambda \in \mathbb{C}$ and for all networks, which can be considered in both approaches.

The answer is positive for all but finite number of values of λ. Moreover, the equality holds for all λ such that $\operatorname{Re}\lambda > 0$. Therefore, $\mathcal{P}_{\mathbb{C}}(\lambda)$ is a continuous function for any λ in the right half-plane. We prove also that $\mathcal{P}_{\mathbb{C}}(\lambda)$ is a holomorphic function in this region. The question, whether the equality (1.2.3) is true, remains **open** for λ on the imaginary axis. For λ such that $\operatorname{Re}\lambda < 0$ the effective admittances not always coincide (see Examples 2.3.2 and 3.2.21 for $\lambda = -1$).

This thesis contains three appendixes. In Appendix A we collected definitions and auxiliary results from analysis on graphs.

In Appendix B the concept of an ordered field is described. Appendix B also contains the definitions and auxiliary results on the Levi-Civita field $\mathcal{R}$.

In Appendix C some well-known physical laws and statements are presented.

CHAPTER 2

Effective impedance of networks over $\mathbb{C}$

In this chapter we consider an electrical network as a complex-weighted graph, whose each edge xy is endowed with an impedance

$$z_{xy} = z_{xy}^{(\lambda)} = L_{xy}\lambda + R_{xy} + \frac{1}{C_{xy}\lambda},$$

where R_{xy} is the resistance of this edge, L_{xy} is the inductance, C_{xy} is the capacitance, and $\lambda \in \mathbb{C}$ (see [7]). It will be more convenient for us to work with admittances $\rho_{xy}^{(\lambda)} = \frac{1}{z_{xy}^{(\lambda)}}$. The case $\lambda = i\omega$, $\omega > 0$ corresponds to a physical case of an alternating current of the frequency ω.

If the network is finite then the problem of finding voltages amounts to a linear system of Kirchhoff's complex equations. We consider this problem as a boundary value Dirichlet problem (see (2.2.1)). In absence of coils and capacitors this system has always non-zero determinant, which implies that the effective impedance (i.e. resistance) is well-defined and, of course, is independent of λ (see [13], [17], [18], [23]). In the case of networks with passive elements, the determinant of the Dirichlet problem can vanish for some frequencies ω (see Examples 2.2.34 and 2.3.1). Therefore, the definitions of the effective impedance and of the effective admittance in this case require substantial work, which is done in Section 2.2. In the same Section the Green's formula for networks is proved. Moreover, in Subsection 2.2.2 we justify an application of a star-mesh transform, as well as some other physical transforms (including series law and $Y - \Delta$ transform) to our model.

In Section 2.3 we discuss an **unsolved** problem of continuity of the effective impedance as a function of frequency $\omega > 0$.

In Section 2.4 we give estimates for the effective admittance $\mathcal{P}(\lambda)$ of a given finite network in terms of λ.

Then in Section 2.5 we introduce a notion of an effective admittance for an infinite network. We define it exclusively for λ such that the sequence $\{\mathcal{P}_n(\lambda)\}_{n=1}^{\infty}$ of the effective admittances for the exhausted finite networks converges. Therefore, we investigate the problem of convergence of the sequence $\{\mathcal{P}_n(\lambda)\}$ of the partial effective admittances. The main result of this chapter, Theorem 2.5.2, says that $\mathcal{P}(\lambda) := \lim_{n\to\infty} \mathcal{P}_n(\lambda)$ exists and is a holomorphic function of λ in the domain

$\{\operatorname{Re}\lambda > 0\}$ as well as in some other regions. In the case of a resistance free network, Corollary 2.5.4 says that $\mathcal{P}(\lambda)$ is holomorphic in $\mathbb{C} \setminus \left[-i\sqrt{S_D}, i\sqrt{S_D}\right]$, where

$$S_D = \sup_{xy} \frac{1}{C_{xy}L_{xy}}.$$

The proof of the results about infinite networks is based on the estimates of effective admittances for finite networks that are presented in Section 2.4. Moreover, in Section 2.5 we give some examples, including *Feynman's ladder* with zero at infinity. The examples illustrate the domain of convergence of the sequence $\{\mathcal{P}_n(\lambda)\}$.

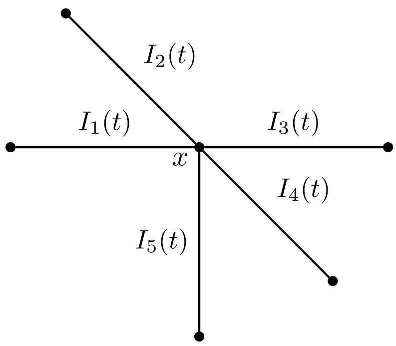
This chapter is based on [25] and [26].

2.1 Description of the model and definition of network

Let us consider an electrical network, containing passive elements: resistors, capacitors and coils (or inductors). We assume that several nodes of a network are grounded and at one node the potential (voltage) with amplitude 1 is maintained. We will refer to nodes, which are neither grounded, nor have maintained potential, *inner* nodes. By Superposition theorem (see Appendix C) the voltages at any physical electrical network with passive elements can be presented as a linear combination of voltages in such networks. By Kirchhoff's law, for any inner node of electrical circuit the sum of currents flowing into the node is equal to the sum of currents flowing out of that node:

$$\sum_j I_j(t) = 0,$$

where t is time and current I is considered as a function of time t.



Differentiating the last equality, one obtains

$$\sum_j I_j'(t) = 0.$$

Using the laws for voltage drop on passive elements ((C.0.1), (C.0.2), and (C.0.3)), the last expression could be rewritten in the following form:

$$\sum_{y:y\overset{C}{\sim}x}(v_t''(x,t)-v_t''(y,t))C_{xy}+\sum_{y:y\overset{R}{\sim}x}\frac{v_t'(x,t)-v_t'(y,t)}{R_{xy}}+\sum_{y:y\overset{L}{\sim}x}\frac{v(x,t)-v(y,t)}{L_{xy}}=0, \tag{2.1.1}$$

where $v(x,t)$ denote a potential (voltage) at the node x at time t, and $y\overset{C}{\sim}x$, $y\overset{R}{\sim}x$, $y\overset{L}{\sim}x$ mean that the edge xy is endowed with capacitor, resistor or coil respectively. Properties of a system of differential equations for electrical networks are discussed in [8] and [9]. In this thesis we will use just Kirchhoff's complex law (C.0.5), which follows from (2.1.1). Indeed, searching for the solution of this equation with a given frequency ω (frequency of the alternating current), i.e. assuming $v(x,t)=v(x)e^{i\omega t}$, one obtains

$$\sum_{y:y\overset{C}{\sim}x}(v(x)-v(y))C_{xy}i\omega+\sum_{y:y\overset{R}{\sim}x}\frac{v(x)-v(y)}{R_{xy}}+\sum_{y:y\overset{L}{\sim}x}\frac{v(x)-v(y)}{L_{xy}i\omega}=0,$$

where $v(x)$ is called complex voltage and does not depend on t. If we refer to the quantities $C_{xy}i\omega$, $\dfrac{1}{L_{xy}i\omega}$, R_{xy} as *impedance (of the segment xy)* and denote it by z_{xy}, we can rewrite Kirchhoff's law in a complex form as

$$\sum_{y:y\sim x}\frac{v(x)-v(y)}{z_{xy}}=0$$

(see e.g. [14] and [15]). In this chapter we make a more general assumption, which can be justified by series law (see Statement C.0.3 and Corollary 2.2.22). Assume that each segment xy is equipped with a resistance R_{xy}, inductance L_{xy}, and capacitance C_{xy}, where $R_{xy}, L_{xy}\in[0,+\infty)$ and $C_{xy}\in(0,+\infty]$.

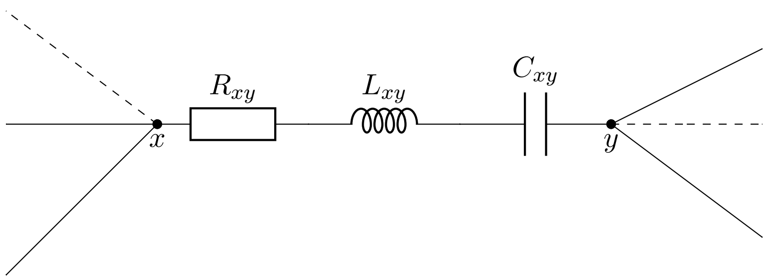


Figure 2.1: One edge of a network

Then the *impedance* of the segment xy is

$$z_{xy}^{(\lambda)}=R_{xy}+L_{xy}i\omega+\frac{1}{C_{xy}i\omega}=R_{xy}+L_{xy}\lambda+\frac{1}{C_{xy}\lambda}. \tag{2.1.2}$$

Although the impedance has physical meaning only for $\lambda = i\omega$, where ω is a positive real number, we will consider more general $\lambda \in \mathbb{C} \setminus \{0\}$ (see [7]).

It will be convenient for us to use the inverse capacitance:

$$D_{xy} = \frac{1}{C_{xy}} \in [0, +\infty),$$

as well as the *admittance* ρ_{xy}:

$$\rho_{xy}^{(\lambda)} = \frac{1}{z_{xy}^{(\lambda)}} = \frac{1}{L_{xy}\lambda + R_{xy} + \dfrac{D_{xy}}{\lambda}} = \frac{\lambda}{L_{xy}\lambda^2 + R_{xy}\lambda + D_{xy}}. \tag{2.1.3}$$

We always assume that for any edge

$$R_{xy} + L_{xy} + D_{xy} > 0.$$

For simplicity of notations we will sometimes omit the superscript in $\rho_{xy}^{(\lambda)}$ when λ is fixed.

Definition 2.1.1. *Let (V, E) be a connected locally finite graph without loops, where V is a set of vertices and E is a set of (unoriented) edges (see Definition A.0.1). Assume that each edge xy is equipped with an admittance $\rho_{xy}^{(\lambda)}$ in the form* (2.1.3). *Let $a_0 \in V$ be a fixed vertex, and $B \subset V$, such that $a_0 \notin B$. We will denote $B_0 = B \cup \{a_0\}$ the set of all* boundary vertices.

We will extend $\rho_{xy}^{(\lambda)}$ to all pairs $x, y \in V$ by setting $\rho_{xy}^{(\lambda)} = 0$, if xy is not an edge.

Then the structure $\Gamma = (V, \rho, a_0, B)$ is called an (electrical) network. *The graph (V, E) is an* underlying graph *of the network Γ.*

We will denote by $|V|$ the number of vertices of a graph (i.e. the cardinality of the set V). The network is called finite, *if $|V| < \infty$. Otherwise, it is called* infinite.

For a finite network the set B should be non-empty. $B = \emptyset$ in an infinite network means the ground at infinity.

Any admittance ρ gives rise to a function on vertices as follows:

$$\rho^{(\lambda)}(x) = \sum_{y:y\sim x} \rho_{xy}^{(\lambda)}.$$

Here and further in notations $\sum\limits_{y}$ means $\sum\limits_{y\in V}$.

We use the term *admittance*, for a function $\rho : V \times V \to \mathbb{C}$, where for any $x, y \in V$ ρ_{xy} has the representation (2.1.3) or is zero. The term *(complex-)weight* will be used more general for any function $\varphi : V \times V \to \mathbb{C}$. Then *(complex-)weighted* graph is a graph endowed with complex weights.

Then the complex voltage $v : V \to \mathbb{C}$ in a finite network satisfies the following conditions:

$$\begin{cases} \sum\limits_{y:y\sim x} (v(y) - v(x))\rho_{xy} = 0 \text{ on } V \setminus B_0, \\ v(a_0) = 1, \\ v(x) = 0 \text{ on } B, \end{cases} \tag{2.1.4}$$

i.e. we consider B as a set of grounded nodes.

2.2 Notion of effective impedance for finite networks

In Subsection 2.2.1 we introduce notions of an effective impedance $Z(\lambda)$ and an effective admittance $\mathcal{P}(\lambda)$ of a finite electrical network. We prove Green's formula on networks (see e.g. [17] for Green's formula on weighted graphs). As a corollary we obtain a conservation of complex power (Theorem 2.2.8). Then we show that $Z(\lambda)$ and $\mathcal{P}(\lambda)$ are positive real functions of λ (see Corollary 2.2.17). In Subsection 2.2.2 we justify an application of some known physical transforms (star-mesh transform, series law, $Y - \Delta$ and $\Delta - Y$ transforms) to a finite network.

2.2.1 Definition of the effective impedance and Green's formula

Let $\Gamma = (V, \rho, a_0, B)$ be a finite *network.*

Definition 2.2.1. *Define the* weighted Laplace operator Δ_ρ *as follows: for any function* $f : V \to \mathbb{C}$

$$\Delta_\rho f(x) = \sum_{y:y\sim x} (f(y) - f(x))\rho_{xy}^{(\lambda)} = \sum_{y:y\sim x} (\nabla_{xy} f)\rho_{xy}^{(\lambda)},$$

where

$$\nabla_{xy} f = f(y) - f(x)$$

is the difference operator.

Therefore, we can rewrite (2.1.4) as follows:

$$\begin{cases} \Delta_\rho v(x) = 0 \text{ on } V \setminus B_0, \\ v(a_0) = 1, \\ v(x) = 0 \text{ on } B, \end{cases} \tag{2.2.1}$$

where $B_0 = B \cup \{a_0\}$. We consider (2.2.1) as a discrete boundary value *Dirichlet problem.*

If $|V| = n$, then (2.2.1) is a $n \times n$ system of linear equations. Let $|B| = k$. Then the system (2.2.1) can be rewritten in a matrix form:

$$\mathbf{A}\hat{v} = \mathbf{b}, \tag{2.2.2}$$

where $\mathbf{A}$ is a symmetric $(n-k-1)$-matrix, $\hat{v}, \mathbf{b}$ are vector-columns of length $(n-k-1)$:

$$\mathbf{A} = \begin{bmatrix} \sum\limits_{x:x\sim x_1} \rho_{xx_1} & -\rho_{x_1x_2} & \cdots & -\rho_{x_1x_{n-k-1}} \\ -\rho_{x_1x_2} & \sum\limits_{x:x\sim x_2} \rho_{xx_2} & \cdots & -\rho_{x_2x_{n-k-1}} \\ \cdots & & & \\ -\rho_{x_1x_{n-k-1}} & -\rho_{x_2x_{n-k-1}} & \cdots & \sum\limits_{x:x\sim x_{n-k-1}} \rho_{xx_{n-k-1}} \end{bmatrix},$$

$$\mathbf{b} = \left[\rho_{a_0x_1}, \rho_{a_0x_2}, \ldots, \rho_{a_0x_{n-k-1}}\right]^T,$$
$$\hat{v} = \left[v(x_1), v(x_2), \ldots, v(x_{n-k-1})\right]^T.$$

Note that we have denoted $x \in V \setminus B_0$ by $x_1, \ldots, x_{n-k-1}$ and we have substituted $v(a_0) = 1$, $v(b) = 0$ for any $b \in B$ in the first $(n-k-1)$ equations. Moreover, we have multipied each line by (-1).

Remark 2.2.2. The existence and uniqueness of the solution of (2.2.1) over $\mathbb{C}$ is not always the case (see Examples 2.2.34, 2.2.36, 2.3.1 and 2.3.2).

Denote by Λ the set of all those values of λ for which $\rho_{xy}^{(\lambda)} \in \mathbb{C} \setminus \{0\}$ for all edges xy. The complement $\mathbb{C} \setminus \Lambda$ consists of $\lambda = 0$ and of all zeros of the equations

$$L_{xy}\lambda^2 + R_{xy}\lambda + D_{xy} = 0. \tag{2.2.3}$$

In particular, $\mathbb{C} \setminus \Lambda$ is a finite set. The roots of the equation (2.2.3) are

$$\lambda = \frac{-R_{xy} \pm \sqrt{R_{xy}^2 - 4L_{xy}D_{xy}}}{2L_{xy}}.$$

Therefore, for every $\lambda \in \mathbb{C} \setminus \Lambda$ we have $\operatorname{Re}\lambda \le 0$ so that

$$\Lambda \supset \{\operatorname{Re}\lambda > 0\}.$$

Observe also that

$$\operatorname{Re}\lambda > 0 \Rightarrow \operatorname{Re} z_{xy}^{(\lambda)} > 0 \Rightarrow \operatorname{Re}\rho_{xy}^{(\lambda)} > 0, \tag{2.2.4}$$

since

$$\operatorname{Re} z_{xy}^{(\lambda)} = R_{xy} + L_{xy}\operatorname{Re}\lambda + \frac{D_{xy}\operatorname{Re}\lambda}{|\lambda|^2} \text{ and } \operatorname{Re}\rho_{xy}^{(\lambda)} = \frac{\operatorname{Re} z_{xy}^{(\lambda)}}{\left|z_{xy}^{(\lambda)}\right|^2}. \tag{2.2.5}$$

Moreover, for $\lambda \in \Lambda$,

$$\operatorname{Re}\lambda \ge 0 \Rightarrow \operatorname{Re} z_{xy}^{(\lambda)} \ge 0 \Rightarrow \operatorname{Re}\rho_{xy}^{(\lambda)} \ge 0 \tag{2.2.6}$$

In what follows we consider the Dirichlet problem (2.2.1) only for $\lambda \in \Lambda$.

If $v\,(x)$ is a solution of the Dirichlet problem (2.2.1) then the total current through a_0 is equal to

$$\sum_{x\in V}(1-v^{(\lambda)}(x))\rho^{(\lambda)}_{xa_0},$$

which motivates the following definition.

Definition 2.2.3. *Let $v(x)$ be a solution of the Dirichlet problem* (2.2.1)*. Define the* effective impedance *of the network Γ by*

$$Z(\lambda)=\frac{1}{\sum\limits_{x:x\sim a_0}(1-v^{(\lambda)}(x))\rho^{(\lambda)}_{xa_0}} \tag{2.2.7}$$

and the effective admittance *by*

$$\mathcal{P}(\lambda)=\frac{1}{Z(\lambda)}=\sum_{x:x\sim a_0}(1-v^{(\lambda)}(x))\rho^{(\lambda)}_{xa_0}, \tag{2.2.8}$$

$\lambda\in\Lambda$. If (2.2.1) *has no solution for some $\lambda\in\Lambda$, then we set $Z(\lambda)=0$ and $\mathcal{P}(\lambda)=\infty$.*

We set $\mathcal{P}(\lambda)=\infty$ in the case of lack of solutions, since it corresponds to a physical phenomenon of resonance (See Example 2.2.34). Note that $Z(\lambda)$ and $\mathcal{P}(\lambda)$ take values in $\mathbb{C}\cup\{\infty\}$. We will prove below (see Theorem 2.2.8) , that in the case when the Dirichlet problem (2.2.1) has multiple solutions for some λ, the values $Z(\lambda)$ and $\mathcal{P}(\lambda)$ are independent of the choice of the solution $v^{(\lambda)}$.

The effective admittance does not necessary possess the representation (2.1.3).

Observe immediately the following symmetry properties that will be used later on.

Lemma 2.2.4. (a) *If $\lambda\in\Lambda$ then also $\overline{\lambda}\in\Lambda$ and*

$$\mathcal{P}(\overline{\lambda})=\overline{\mathcal{P}(\lambda)}. \tag{2.2.9}$$

(b) *Assume in addition that $R_{xy}=0$ for all $xy\in E$. Then $\lambda\in\Lambda$ implies $-\lambda\in\Lambda$ and*

$$\mathcal{P}(-\lambda)=-\mathcal{P}(\lambda).$$

Proof. (a) If λ is a root of the equation $L_{xy}\lambda^2+R_{xy}\lambda+D_{xy}=0$ then $\overline{\lambda}$ is also a root, whence the first claim follows. If v is a solution of the Dirichlet problem (2.2.1) for some λ, then clearly $\overline{v}$ is a solution of (2.2.1) with the parameter $\overline{\lambda}$ instead of λ, since $\rho^{(\overline{\lambda})}=\overline{\rho^{(\lambda)}}$. Substituting into (2.2.8) and using $\rho^{(\overline{\lambda})}=\overline{\rho^{(\lambda)}}$ again, we obtain (2.2.9).

(b) The proof is similar to (a) observing that if λ is a root of $L_{xy}\lambda^2+D_{xy}=0$ then $-\lambda$ is also a root, and $\rho^{(-\lambda)}=-\rho^{(\lambda)}$ in this case. $\square$

Lemma 2.2.5 (Green's formula). *Let $W \subset V$. For any $\lambda \in \Lambda$ and for any two functions $f, g : V \to \mathbb{C}$ the following identity is true:*

$$\sum_{x\in W} \Delta_\rho f(x)g(x) = -\frac{1}{2}\sum_{x,y\in W} (\nabla_{xy} f)(\nabla_{xy} g)\rho_{xy} + \sum_{x\in W}\sum_{y\in V\setminus W} (\nabla_{xy} f)g(x)\rho_{xy}. \tag{2.2.10}$$

Proof.

$$\begin{aligned}\sum_{x\in W} \Delta_\rho f(x)g(x) &= \sum_{x\in W}\left(\sum_{y\in V}(f(y)-f(x))\rho_{xy}\right)g(x)\\ &= \sum_{x\in W}\sum_{y\in V}(f(y)-f(x))g(x)\rho_{xy}\\ &= \sum_{x\in W}\sum_{y\in W}(f(y)-f(x))g(x)\rho_{xy} + \sum_{x\in W}\sum_{y\in V\setminus W}(f(y)-f(x))g(x)\rho_{xy}\\ &= \sum_{y\in W}\sum_{x\in W}(f(x)-f(y))g(y)\rho_{xy} + \sum_{x\in W}\sum_{y\in V\setminus W}(\nabla_{xy} f)g(x)\rho_{xy},\end{aligned}$$

where in the last line we have switched notation of the variables x and y in the first sum. Adding together the last two lines and dividing by 2, we obtain (2.2.10). ☐

If we put $W = V$, then $V \setminus W$ is empty so that the last term in (2.2.10) vanishes, and we obtain

$$\sum_{x\in V} \Delta_\rho f(x)g(x) = -\frac{1}{2}\sum_{x,y\in V} (\nabla_{xy} f)(\nabla_{xy} g)\rho_{xy}. \tag{2.2.11}$$

Corollary 2.2.6. *For any function $f : V \to \mathbb{C}$,*

$$\sum_{x\in V} \Delta_\rho f(x) = 0. \tag{2.2.12}$$

Proof. Apply (2.2.11) for $g \equiv 1$. ☐

Lemma 2.2.7. *For any solution v of the Dirichlet problem* (2.2.1) *we have*

$$\sum_{x:x\sim a_0} (1-v(x))\rho_{xa_0} = -\Delta_\rho v(a_0) = \sum_{b\in B}\Delta_\rho v(b), \tag{2.2.13}$$

Moreover,

$$\sum_{x:x\sim a_0} (1-v(x))\rho_{xa_0} = \frac{1}{2}\sum_{x,y\in V}(\nabla_{xy} v)(\nabla_{xy} u)\rho_{xy}, \tag{2.2.14}$$

where $u : V \to \mathbb{C}$ is any function such that $u(a_0) = 1$ and $u|_B \equiv 0$.

Proof. We have

$$\Delta_\rho v(a_0) = \sum_{x:x\sim a_0} (v(x) - v(a_0))\rho_{xa_0} = \sum_{x:x\sim a_0} (v(x) - 1)\rho_{xa_0}$$

since $v(a_0) = 1$. This proves the first identity in (2.2.13). Since by (2.2.12)

$$\sum_{x\in V} \Delta_\rho f(x) = 0$$

and $\Delta_\rho v(x) = 0$ for all $x \in V \setminus B_0$, we obtain

$$\Delta_\rho v(a_0) + \sum_{b\in B} \Delta_\rho v(b) = 0$$

whence the second identity in (2.2.13) follows.

Finally, to prove (2.2.14), we apply Green's formula (2.2.11) to the right-hand side and obtain

$$\frac{1}{2}\sum_{x,y\in V} (\nabla_{xy} v)(\nabla_{xy} u)\rho_{xy} = -\sum_{x\in V} \Delta_\rho v(x)u(x) = -\Delta_\rho v(a_0),$$

because $\Delta_\rho v(x) = 0$ for all $x \in V \setminus B_0$, while $u(a_0) = 1$ and $u|_B \equiv 0$. □

Theorem 2.2.8. *For any $\lambda \in \Lambda$ the values of the effective admittance $\mathcal{P}(\lambda)$ and the effective impedance $Z(\lambda)$ do not depend on the choice of a solution $v^{(\lambda)}$ of the Dirichlet problem* (2.2.1). *Besides, we have the identity*

$$\frac{1}{2}\sum_{x,y\in V} \left|\nabla_{xy} v^{(\lambda)}\right|^2 \rho_{xy}^{(\lambda)} = \mathcal{P}(\lambda) \tag{2.2.15}$$

(conservation of complex power).

Physically, the left hand side in (2.2.15) means the sum of complex powers, absorbed by passive elements, and the right hand size means the power, delivered by a source, due to the unit voltage (comp. Statement C.0.4).

Proof. Let v_1 and v_2 be two solutions of (2.2.1) for the same λ. By (2.2.14) we have

$$\sum_{x:x\sim a_0} (1 - v_1(x))\rho_{xa_0} = \frac{1}{2}\sum_{x,y\in V} (\nabla_{xy} v_1)(\nabla_{xy} v_2)\rho_{xy}$$

and also

$$\sum_{x:x\sim a_0} (1 - v_2(x))\rho_{xa_0} = \frac{1}{2}\sum_{x,y\in V} (\nabla_{xy} v_2)(\nabla_{xy} v_1)\rho_{xy},$$

whence the identity

$$\sum_{x:x\sim a_0} (1 - v_1(x))\rho_{xa_0} = \sum_{x:x\sim a_0} (1 - v_2(x))\rho_{xa_0}$$

follows. Hence, v_1 and v_2 determine the same admittance and impedance. Applying (2.2.14) with $u = \overline{v_1}$, we obtain

$$\frac{1}{2}\sum_{x,y\in V}|\nabla_{xy}v_1|^2\rho_{xy} = \frac{1}{2}\sum_{x,y\in V}(\nabla_{xy}v_1)(\overline{\nabla_{xy}v_1})\rho_{xy} = \mathcal{P}(\lambda).$$

□

Let us denote by Λ_1 the domain

$$\Lambda_1 = \{\lambda \in \Lambda \text{ such that the Dirichlet problem (2.2.1) has a solution}\},$$

i.e. the domain, where $\mathcal{P}(\lambda) \neq \infty$ is defined.

For any $\lambda \in \Lambda_1$ the effective admittance can be written using the Dirichlet problem in matrix form (2.2.2) of the Dirichlet problem:

$$\mathcal{P} = (\mathbf{A}e - \mathbf{b})^T\hat{v} + \sum_{b\in B}\rho_{a_0b}, \tag{2.2.16}$$

where $\mathbf{e} = [1, 1, \ldots, 1]^T$. Indeed,

$$(\mathbf{A}e - \mathbf{b})^T\hat{v} + \sum_{b\in B}\rho_{a_0b} = \left[\sum_{b\in B}\rho_{bx_1}, \sum_{b\in B}\rho_{bx_2}, \ldots, \sum_{b\in B}\rho_{bx_{n-k-1}}\right]\begin{bmatrix} v(x_1)\\ v(x_2)\\ \ldots\\ v(x_3)\end{bmatrix} + \sum_{b\in B}\rho_{a_0b}$$

$$= \sum_{i=1}^{n-k-1}\sum_{b\in B}\rho_{bx_i}v(x_i) + \sum_{b\in B}\rho_{a_0b} = \sum_{b\in B}\sum_{x}\rho_{xb}v(x) = \sum_{b\in B}\Delta_\rho v(b) = \mathcal{P}$$

by (2.2.8) and (2.2.13), since $v|_B \equiv 0$.

The identity (2.2.16) gives us another proof of the fact, that the effective admittance (and, consequently, the effective impedance) does not depend on the choice of a solution of the Dirichlet problem. Indeed, for any two solutions v_1, v_2 of (2.2.2) we have

$$\begin{aligned}(\mathbf{A}\mathbf{e} - \mathbf{b})^T\hat{v}_1 =& \mathbf{e}^T\mathbf{A}^T\hat{v}_1 - \mathbf{b}^T\hat{v}_1 = \mathbf{e}^T\mathbf{A}\hat{v}_1 - (\mathbf{A}\hat{v}_2)^T\hat{v}_1 = \mathbf{e}^T\mathbf{b} - \hat{v}_2^T\mathbf{A}^T\hat{v}_1\\ =& \mathbf{e}^T\mathbf{A}\hat{v}_2 - \hat{v}_2^T\mathbf{A}\hat{v}_1 = \mathbf{e}^T\mathbf{A}\hat{v}_2 - \hat{v}_2^T\mathbf{b} = \mathbf{e}^T\mathbf{A}\hat{v}_2 - \hat{v}_2^T\mathbf{A}\hat{v}_2\\ =& \mathbf{e}^T\mathbf{A}^T\hat{v}_2 - \hat{v}_2^T\mathbf{A}^T\hat{v}_2 = \mathbf{e}^T\mathbf{A}^T\hat{v}_2 - (\mathbf{A}\hat{v}_2)^T\hat{v}_2 = \mathbf{e}^T\mathbf{A}^T\hat{v}_2 - \mathbf{b}^T\hat{v}_2\\ =& (\mathbf{A}\mathbf{e} - \mathbf{b})^T\hat{v}_2,\end{aligned}$$

which, by (2.2.16) gives the required.

Theorem 2.2.9. *(a) If for some $\lambda \in \Lambda_1$ $\operatorname{Re}\rho_{xy}^{(\lambda)} \geq 0$ for all $xy \in E$, then $\operatorname{Re}\mathcal{P}(\lambda) \geq 0$ and $\operatorname{Re}Z(\lambda) \geq 0$. Moreover, if for some $\lambda \in \Lambda_1$ $\operatorname{Re}\rho_{xy}^{(\lambda)} > 0$ for all $xy \in E$, then $\operatorname{Re}\mathcal{P}(\lambda) > 0$ and $\operatorname{Re}Z(\lambda) > 0$.*

(b) If for some $\lambda \in \Lambda_1$ $\operatorname{Im}\rho_{xy}^{(\lambda)} \geq 0$ for all $xy \in E$, then $\operatorname{Im}\mathcal{P}(\lambda) \geq 0$. Moreover, if for some $\lambda \in \Lambda_1$ $\operatorname{Im}\rho_{xy}^{(\lambda)} > 0$ for all $xy \in E$, then $\operatorname{Im}\mathcal{P}(\lambda) > 0$.

(*c*) *If for some* $\lambda \in \Lambda_1$ $\operatorname{Im}\rho_{xy}^{(\lambda)} \leq 0$ *for all* $xy \in E$*, then* $\operatorname{Im}\mathcal{P}(\lambda) \leq 0$*. Moreover, if for some* $\lambda \in \Lambda_1$ $\operatorname{Im}\rho_{xy}^{(\lambda)} < 0$ *for all* $xy \in E$*, then* $\operatorname{Im}\mathcal{P}(\lambda) < 0$*.*

Proof. (*a*) By the conservation of complex power (2.2.15) we have

$$\operatorname{Re}\mathcal{P} = \frac{1}{2}\sum_{x,y\in V} |\nabla_{xy} v|^2 \operatorname{Re}\rho_{xy} = \sum_{e\in E} |\nabla_e v|^2 \operatorname{Re}\rho_e,$$

therefore, $\operatorname{Re}\rho_{xy} \geq 0$ for all xy implies $\operatorname{Re}\mathcal{P} \geq 0$. Further, $\operatorname{Re}\rho_{xy} > 0$ for all xy implies $\operatorname{Re}\mathcal{P} > 0$ by connectedness of a graph and the fact, that $v(a_0) = 1$ and $v|_B \equiv 0$. The statements for $\operatorname{Re} Z$ follows from

$$\operatorname{Re} Z = \operatorname{Re}\frac{1}{\mathcal{P}} = \frac{\operatorname{Re}\mathcal{P}}{|\mathcal{P}|^2}.$$

The statements about $\operatorname{Im}\mathcal{P}$ are handled in the same way. □

Corollary 2.2.10. *For any* $\lambda \in \Lambda_1$*, if* $\operatorname{Re}\lambda > 0$*, then* $\operatorname{Re}\mathcal{P}(\lambda) > 0$ *and* $\operatorname{Re} Z(\lambda) > 0$ *for any finite network. Moreover, if* $\operatorname{Re}\lambda \geq 0$ *then* $\operatorname{Re}\mathcal{P}(\lambda) \geq 0$ *and* $\operatorname{Re} Z(\lambda) \geq 0$*.*

Proof. It follows from (2.2.4) and (2.2.6).

□

Theorem 2.2.11. *The Dirichlet problem (2.2.1) has a unique solution* $v = v^{(\lambda)}$ *for all* $\lambda \in \Lambda_0$ *where* Λ_0 *is a subset of* Λ *such that* $\Lambda \setminus \Lambda_0$ *is finite. Besides,* Λ_0 *contains the domains*

$$\Lambda \cap \{\operatorname{Re}\rho_{xy}^{(\lambda)} > 0 \ \forall xy \in E\}, \tag{2.2.17}$$

$$\Lambda \cap \{\operatorname{Im}\rho_{xy}^{(\lambda)} > 0\, \forall xy \in E\} \quad \textit{and} \quad \Lambda \cap \{\operatorname{Im}\rho_{xy}^{(\lambda)} < 0\, \forall xy \in E\}. \tag{2.2.18}$$

Consequently, $\mathcal{P}(\lambda)$ *is a rational* $\mathbb{C}$*-valued function with real coefficients in* Λ_0 *and, hence, in any of the domains* (2.2.17) *and* (2.2.18)*.*

Proof. Let us consider the Dirichlet problem in the matrix form (2.2.2):

$$\mathbf{A}\hat{v} = \mathbf{b}.$$

Set also

$$\mathcal{D} = \det(\mathbf{A})$$

and let $\mathcal{D}_j$ be the determinant of the matrix obtained by replacing the column j in the matrix $\mathbf{A}$ by the column $\mathbf{b}$. Then, by Cramer's rule,

$$\hat{v}_j = \frac{\mathcal{D}_j}{\mathcal{D}} \tag{2.2.19}$$

provided $\mathcal{D} \neq 0$. Of course, all these quantities are functions of λ. Since all the coefficients $\mathbf{A}_{ij}$ and $\mathbf{b}_i$ are rational functions of λ with real coefficients, also $\mathcal{D} = \mathcal{D}(\lambda)$ and $\mathcal{D}_j = \mathcal{D}_j(\lambda)$ are rational functions of λ with real coefficients. For all

$\lambda \in \Lambda$ but a finite number, all functions $\mathcal{D}_j(\lambda)$ and $\mathcal{D}(\lambda)$ take values in $\mathbb{C}$. The existence and uniqueness of a solution is equivalent to $\mathcal{D}(\lambda) \neq 0$. Hence, define Λ_0 as the subset of Λ where all functions $\mathcal{D}_j(\lambda)$ and $\mathcal{D}(\lambda)$ take values in $\mathbb{C}$ and, besides, $\mathcal{D}(\lambda) \neq 0$. Since $\mathcal{D}(\lambda)$ is a rational function of λ, it may vanish only when λ is a root of the numerator of $\mathcal{D}(\lambda)$ or $\mathcal{D}(\lambda) \equiv 0$. In the former case the number of such values of λ is finite.

Hence, it suffices to exclude the latter case, that is, to show that $\Lambda_0 \neq \emptyset$. For that, let us prove that Λ_0 contains the domain (2.2.17) that in turn, by (2.2.4), contains $\{\operatorname{Re}\lambda > 0\}$ and, hence, is non-empty. In order to show that $\mathcal{D}(\lambda) \neq 0$ for any λ from (2.2.17), it suffices to verify that the homogeneous Dirichlet problem

$$\begin{cases} \Delta_\rho u(x) = 0 \text{ on } V \setminus B_0, \\ u(x) = 0 \text{ on } B_0 \end{cases} \tag{2.2.20}$$

has a unique solution $u \equiv 0$. Indeed, by Green's formula we have

$$\begin{aligned} \sum_{xy \in E} |\nabla_{xy} u|^2 \rho_{xy} = & \frac{1}{2} \sum_{x,y \in V} |\nabla_{xy} u|^2 \rho_{xy} = - \sum_{x,y \in V} \Delta_\rho u(x) \overline{u}(x) \\ = & - \sum_{x \in V \setminus B_0} \Delta_\rho u(x) \overline{u}(x) - \sum_{x \in B_0} \Delta_\rho u(x) \overline{u}(x) = 0, \end{aligned}$$

since u is a solution of (2.2.20). Since $\operatorname{Re}\rho_{xy} > 0$, we conclude that $|\nabla_{xy} u| = 0$ on all the edges. By the connectedness of the graph this implies that $u = \text{const}$. Since $u\big|_{B_0} \equiv 0$, we conclude that $u \equiv 0$.

In the same way the domains (2.2.18) are subsets of Λ_0.

Finally, by the above argument (2.2.19), $v^{(\lambda)}(x)$ is a rational function of λ with real coefficients, so that the last claim follows from (2.2.8). $\square$

Remark 2.2.12. Since $\{\operatorname{Re}\lambda > 0\}$ is contained in Λ_0, we see that $\mathcal{P}(\lambda)$ is a holomorphic function in $\{\operatorname{Re}\lambda > 0\}$. If $R_{xy} > 0$ for all $xy \in E$, then by also $\Lambda \cap \{\operatorname{Re}\lambda \geq 0\}$ is a subset of (2.2.17) by (2.2.6).

Remark 2.2.13. The uniqueness of the solution of the Dirichlet problem for the domain (2.2.17) follows also from [36, Lemma 4.4].

Remark 2.2.14. Due to (2.1.3), (2.2.19) and (2.2.8) the effective admittance and the effective impedance are rational functions of λ with real coefficients for $\lambda \in \Lambda_1$.

Definition 2.2.15. *[7, p. 25] A function $F(\lambda) : S \to \mathbb{C}$, where $S \subset \mathbb{C}$, is called a* positive function, *if it satisfies the following conditions:*

- $\operatorname{Re} F(\lambda) > 0$, *when* $\operatorname{Re}\lambda > 0$,
- $\operatorname{Re} F(\lambda) \geq 0$, *when* $\operatorname{Re}\lambda = 0$.

If in addition $F(\lambda)$ is real for all real λ, then $F(\lambda)$ is called a positive real function.

Remark 2.2.16. By (2.1.3), (2.2.4) and (2.2.6) the admittances $\rho_{xy}^{(\lambda)}$ are positive real functions of $\lambda \in \Lambda$.

Corollary 2.2.17. *An effective admittance $P(\lambda)$ and an effective impedance $Z(\lambda)$ of a finite network are positive real functions of $\lambda \in \Lambda_1$. Consequently, by Remark 2.2.12, they are positive real functions on $\{\operatorname{Re}\lambda > 0\}$.*

Proof. The positivity follows from Corollary 2.2.10. □

2.2.2 Basic properties of the effective impedance

Star-mesh transform, series law, parallel law, $Y - \Delta$ and $\Delta - Y$ transforms are wide used in physics. We give a rigorous justification of their use for our model and describe the values of λ, for which they can be used. Note that the complex weights of the edges, obtained after some of these transforms do not have representation (2.1.3) in general. Therefore, strictly speaking, after transformations, we obtain not networks, but complex-weighted graphs, whose weights are (rational) functions of λ. Moreover, these functions can be not positive real (see Examples 2.2.41 and 2.2.42).

Theorem 2.2.18 (Star-mesh transform)**.** *Let $\Gamma = (V, \rho, a_0, B)$ be a finite network, $|V| = n$, $B_0 = B \cup \{a_0\}$, and $x_1, \ldots, x_m \in V$, $3 \le m \le n$, are such, that*

- $x_1 \notin B_0$,
- $y \not\sim x_1$ *for all* $y \in V \setminus \{x_2, \ldots, x_m\}$,

Let $\lambda \in \Lambda_1$ be such that

$$\rho^{(\lambda)}(x_1) \neq 0.$$

If one removes the vertex x_1, edges $x_1x_i, i = \overline{2,m}$ and change the weights of the edges $x_ix_j, i, j = \overline{2,m}, i \neq j$ as follows

$$\widetilde{\rho}_{x_ix_j} = \rho_{x_ix_j} + \frac{\rho_{x_1x_i}\rho_{x_1x_j}}{\rho(x_1)}, \tag{2.2.21}$$

not changing the other admittances, then for the new complex-weighted graph the solution of the Dirichlet problem (2.2.1) for all the vertices will be the same as the solution for the original network at corresponding vertices (for the same λ).

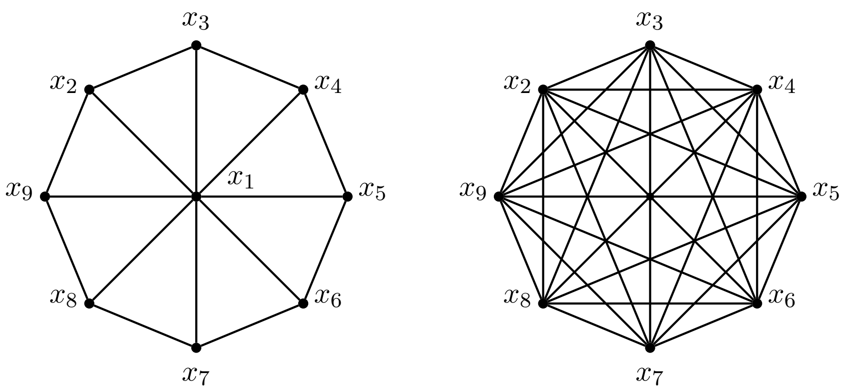


Figure 2.2: Star-mesh transform for $m = 9$

Remark 2.2.19. The condition $\rho^{(\lambda)}(x_1) \neq 0$ is crucial (see Example 2.2.38).

Proof. Let us consider the Dirichlet problem for the network Γ in matrix form (2.2.2), i.e.

$$\mathbf{A}\hat{v} = \mathbf{b}.$$

Without loss of generality we can assume that $x_1, \dots, x_l \notin B_0$, where $l = m - |\{x_1, \dots, x_m\} \cap B_0|$. Writing equations for $x_1, \dots, x_l$ as the first ones and denoting $k = |B|$, we have

$$\mathbf{A} = \begin{bmatrix} \rho(x_1) & -\rho_{x_1x_2} & \dots & -\rho_{x_1x_l} & 0 & \dots & 0 \\ -\rho_{x_1x_2} & \rho(x_2) & \dots & -\rho_{x_2x_l} & -\rho_{x_2x_{m+1}} & \dots & -\rho_{x_2x_{n-k-1}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -\rho_{x_1x_l} & -\rho_{x_2x_l} & \dots & \rho(x_l) & -\rho_{x_lx_{m+1}} & \dots & -\rho_{x_lx_{n-k-1}} \\ 0 & -\rho_{x_2x_{m+1}} & \dots & -\rho_{x_lx_{m+1}} & \rho(x_{m+1}) & \dots & -\rho_{x_{m+1}x_{n-k-1}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & -\rho_{x_2x_{n-k-1}} & \dots & -\rho_{x_lx_{n-k-1}} & -\rho_{x_{m+1}x_{n-k-1}} & \dots & \rho(x_{n-k-1}) \end{bmatrix},$$

since $y \not\sim x_1$ for all $y \in V \setminus \{x_2, \dots, x_m\}$, and

$$\mathbf{b} = (\rho_{a_0x_1}, \rho_{a_0x_2}, \dots, \rho_{a_0x_l}, \rho_{a_0x_{m+1}}, \dots, \rho_{a_0x_{n-k-1}})^T,$$

Now we can verify, that star-mesh transform is an application of the Gaussian elimination method for the first row. Indeed, applying the Gaussian elimination method for the first row of the augmented matrix $\overline{\mathbf{A}} = [\mathbf{A}|\mathbf{b}]$ we obtain the matrix $\widetilde{\mathbf{A}} =$

$$\left[\begin{array}{ccccccc|c} 1 & -\dfrac{\rho_{x_1x_2}}{\rho(x_1)} & \dots & -\dfrac{\rho_{x_1x_l}}{\rho(x_1)} & 0 & \dots & 0 & \dfrac{\rho_{a_0x_1}}{\rho(x_1)} \\ 0 & \rho^*(x_2) & \dots & -\widetilde{\rho}_{x_2x_l} & -\rho_{x_2x_{m+1}} & \dots & -\rho_{x_2x_{n-k-1}} & \rho^*_{a_0x_2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & -\widetilde{\rho}_{x_2x_l} & \dots & \rho^*(x_l) & -\rho_{x_lx_{m+1}} & \dots & -\rho_{x_lx_{n-k-1}} & \rho^*_{a_0x_l} \\ 0 & -\rho_{x_2x_{m+1}} & \dots & -\rho_{x_lx_{m+1}} & \rho(x_{m+1}) & \dots & -\rho_{x_{m+1}x_{n-k-1}} & \rho_{a_0x_{m+1}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & -\rho_{x_2x_{n-k-1}} & \dots & -\rho_{x_lx_{n-k-1}} & -\rho_{x_{m+1}x_{n-k-1}} & \dots & \rho(x_{n-k-1}) & \rho_{a_0x_{n-k-1}} \end{array}\right],$$

since $\rho(x_1) \neq 0$, where

$$\rho^*(x_i) = \rho(x_i) - \frac{\rho^2_{x_1x_i}}{\rho(x_1)} \text{ and } \rho^*_{a_0x_i} = \rho_{a_0x_i} + \frac{\rho_{x_1x_i}\rho_{a_0x_1}}{\rho(x_1)} \text{ for all } i = \overline{2, l},$$

and $\widetilde{\rho}_{x_i x_j}$ are as in (2.2.21). Note that for all $i = \overline{2, l}$

$$
\begin{aligned}
\widetilde{\rho}(x_i) =& \rho(x_i) - \rho_{x_1 x_i} - \sum_{\substack{j=2 \\ j \neq i}}^{m} \rho_{x_i x_j} + \sum_{\substack{j=2 \\ j \neq i}}^{m} \widetilde{\rho}_{x_i x_j} \\
=& \rho(x_i) - \rho_{x_1 x_i} - \sum_{\substack{j=2 \\ j \neq i}}^{m} \rho_{x_i x_j} + \sum_{\substack{j=2 \\ j \neq i}}^{m} \left(\rho_{x_i x_j} + \frac{\rho_{x_1 x_i} \rho_{x_1 x_j}}{\rho(x_1)} \right) \\
=& \rho(x_i) - \rho_{x_1 x_i} + \sum_{\substack{j=2 \\ j \neq i}}^{m} \frac{\rho_{x_1 x_i} \rho_{x_1 x_j}}{\rho(x_1)} \\
=& \rho(x_i) - \rho_{x_1 x_i} + \frac{\rho_{x_1 x_i}}{\rho(x_1)} \sum_{\substack{j=2 \\ j \neq i}}^{m} \rho_{x_1 x_j} \\
=& \rho(x_i) - \rho_{x_1 x_i} + \frac{\rho_{x_1 x_i}}{\rho(x_1)} \sum_{j=2}^{m} \rho_{x_1 x_j} - \frac{\rho_{x_1 x_i}}{\rho(x_1)} \rho_{x_1 x_i} \\
=& \rho(x_i) - \rho_{x_1 x_i} + \frac{\rho_{x_1 x_i}}{\rho(x_1)} \rho(x_1) - \frac{\rho^2_{x_1 x_i}}{\rho(x_1)} \\
=& \rho(x_i) - \frac{\rho^2_{x_1 x_i}}{\rho(x_1)} = \rho^*(x_i)
\end{aligned}
$$

and

$$
\rho^*_{a_0 x_i} = \rho_{a_0 x_i} + \frac{\rho_{x_1 x_i} \rho_{a_0 x_1}}{\rho(x_1)} = \begin{cases} \widetilde{\rho}_{a_0 x_i}, & \text{if } a_0 \in \{x_2, \dots, x_m\} \\ \rho_{a_0 x_i}, & \text{otherwise, since } \rho_{a_0 x_1} = 0. \end{cases}
$$

Hence, $\widetilde{\mathbf{A}}$

$$
\left[
\begin{array}{ccccccc|c}
1 & -\dfrac{\rho_{x_1 x_2}}{\rho(x_1)} & \dots & -\dfrac{\rho_{x_1 x_l}}{\rho(x_1)} & 0 & \dots & 0 & \dfrac{\rho_{a_0 x_1}}{\rho(x_1)} \\
0 & \widetilde{\rho}(x_2) & \dots & -\widetilde{\rho}_{x_2 x_l} & -\rho_{x_2 x_{m+1}} & \dots & -\rho_{x_2 x_{n-k-1}} & \rho^*_{a_0 x_2} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & -\widetilde{\rho}_{x_2 x_l} & \dots & \widetilde{\rho}(x_l) & -\rho_{x_l x_{m+1}} & \dots & -\rho_{x_l x_{n-k-1}} & \rho^*_{a_0 x_l} \\
0 & -\rho_{x_2 x_{m+1}} & \dots & -\rho_{x_l x_{m+1}} & \rho(x_{m+1}) & \dots & -\rho_{x_{m+1} x_{n-k-1}} & \rho_{a_0 x_{m+1}} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & -\rho_{x_2 x_{n-k-1}} & \dots & -\rho_{x_l x_{n-k-1}} & -\rho_{x_{m+1} x_{n-k-1}} & \dots & \rho(x_{n-k-1}) & \rho_{a_0 x_{n-k-1}}
\end{array}
\right],
$$

Therefore, we can eliminate the variable $v(x_1)$ from the Dirichlet problem, changing complex weights as in the statement of the theorem. □

Corollary 2.2.20. *Under the star-mesh transform of a part of network the effective admittance does not change for all $\lambda \in \Lambda$, such that*

$$
\rho^{(\lambda)}(x_1) \neq 0. \tag{2.2.22}
$$

Consequently, the effective impedance also does not change.

Remark 2.2.21. Here we use the term "effective admittance" (resp. "effective impedance") refer to the quantity, calculated by (2.2.8) (resp. (2.2.7)) on a complex-weighted graph with given boundary vertices $a_0 \cup B$.

Proof. In the proof we will use the notations from the proof of the Theorem 2.2.18. If the Dirichlet problem $\mathbf{A}\hat{v} = \mathbf{b}$ has no solutions, it still will not have solutions after the elimination of variable $v(x_1)$, since $\rho^{(\lambda)}(x_1) \neq 0$. Therefore, it is enough to prove the statement for the case $\mathcal{P}(\lambda) < \infty$.

The case $\{x_1, \ldots, x_m\} \cap B_0 = \emptyset$ is trivial. The cases, when $\{x_1, \ldots, x_m\} \cap B = \emptyset$ or $\{x_1, \ldots, x_m\} \cap \{a_0\} = \emptyset$ are obvious, due to (2.2.8) and (2.2.13).

Otherwise, we can assume, without loss of generality, that $x_m = a_0$. Then, if we denote the effective admittance of the new complex-weighted graph by $\widetilde{\mathcal{P}}$, we have

$$\begin{aligned}
\mathcal{P} = &\sum_{x \neq a_0} (1 - v(x))\rho_{xa_0} \\
=&(1 - v(x_1))\rho_{x_1a_0} + \sum_{i=2}^{m-1} (1 - v(x_i))\rho_{x_ia_0} + \sum_{x \notin \{x_1,\ldots,x_m\}} (1 - v(x))\rho_{xa_0} \\
=&\widetilde{\mathcal{P}} - \sum_{i=2}^{m-1} (1 - v(x_i))\widetilde{\rho}_{x_ia_0} + (1 - v(x_1))\rho_{x_1a_0} + \sum_{i=2}^{m-1} (1 - v(x_i))\rho_{x_ia_0} \\
=&\widetilde{\mathcal{P}} - \sum_{i=2}^{m-1} (1 - v(x_i))\frac{\rho_{x_1a_0}\rho_{x_1x_i}}{\rho(x_1)} + (1 - v(x_1))\rho_{x_1a_0} \\
=&\widetilde{\mathcal{P}} - \rho_{x_1a_0} \sum_{i=2}^{m-1} (1 - v(x_i))\frac{\rho_{x_1x_i}}{\rho(x_1)} + \left(1 - \sum_{i=2}^{m-1} v(x_i)\frac{\rho_{x_1x_i}}{\rho(x_1)} - \frac{\rho_{x_1a_0}}{\rho(x_1)}\right)\rho_{x_1a_0} \\
=&\widetilde{\mathcal{P}} - \rho_{x_1a_0} \left(\sum_{i=2}^{m-1} \frac{\rho_{x_1x_i}}{\rho(x_1)} - \sum_{i=2}^{m-1} v(x_i)\frac{\rho_{x_1x_i}}{\rho(x_1)}\right) + \left(1 - \sum_{i=2}^{m-1} v(x_i)\frac{\rho_{x_1x_i}}{\rho(x_1)} - \frac{\rho_{x_1a_0}}{\rho(x_1)}\right)\rho_{x_1a_0} \\
=&\widetilde{\mathcal{P}} - \rho_{x_1a_0} \left(\frac{\rho(x_1)}{\rho(x_1)} - \frac{\rho_{x_1x_m}}{\rho(x_1)} - \sum_{i=2}^{m-1} v(x_i)\frac{\rho_{x_1x_i}}{\rho(x_1)}\right) + \left(1 - \sum_{i=2}^{m-1} v(x_i)\frac{\rho_{x_1x_i}}{\rho(x_1)} - \frac{\rho_{x_1a_0}}{\rho(x_1)}\right)\rho_{x_1a_0} \\
=&\widetilde{\mathcal{P}}
\end{aligned}$$

since

$$v(x_1) = \sum_{i=2}^{l} v(x_i)\frac{\rho_{x_1x_i}}{\rho(x_1)} + \frac{\rho_{x_1a_0}}{\rho(x_1)} = \sum_{i=2}^{m-1} v(x_i)\frac{\rho_{x_1x_i}}{\rho(x_1)} + \frac{\rho_{x_1a_0}}{\rho(x_1)}$$

(see the first line of $\overline{\mathbf{A}}$ and note that $v(x_j) = 0$ for all $j = \overline{j+1, m-1}$ and $a_0 = x_m$). □

Series law and $Y - \Delta$ *transform* are particular cases of star-mesh transform. Since multigraphs are not allowed in this thesis, we will use a modification of parallel law and refer to it as *parallel-series law.*

Corollary 2.2.22 (Series law). *Let* $\Gamma = (V, \rho, a_0, B)$ *be a finite network,* $B_0 = B \cup \{a_0\}$. *Let* $a, b, c \in V$ *are such, that*

- $b \notin B_0$,
- $a \not\sim c$, $a \sim b$, $b \sim c$,
- $b \not\sim x$ *for all* $x \notin \{a, c\}$.

Let $\lambda \in \Lambda_1$ *be such that* $\rho^{(\lambda)}(b) \neq 0$.

If one removes the vertex b*, edges* ab, bc *and add the edge* ac *with the admittance*

$$\widetilde{\rho}_{ac} = \frac{\rho_{ab}\rho_{bc}}{\rho_{ab} + \rho_{bc}},$$

not changing other admittances, then for the new network the solution of the Dirichlet problem (2.2.1) *for all the vertices will be the same as the solution for the original network at corresponding vertices (for the same* λ*). The effective admittance (impedance) of the new network coincide with the effective admittance (impedance) of the original one.*

Remark 2.2.23. After this transform the weights of the edges possess the representation (2.1.3). Indeed,

$$\begin{aligned}\widetilde{\rho}_{ac} =& \frac{\rho_{ab}\rho_{bc}}{\rho_{ab} + \rho_{bc}} = \frac{\dfrac{\lambda}{L_{ab}\lambda^2 + R_{ab}\lambda + D_{ab}} \cdot \dfrac{\lambda}{L_{bc}\lambda^2 + R_{bc}\lambda + D_{bc}}}{\dfrac{\lambda}{L_{ab}\lambda^2 + R_{ab}\lambda + D_{ab}} + \dfrac{\lambda}{L_{bc}\lambda^2 + R_{bc}\lambda + D_{bc}}} \\ =& \frac{\lambda}{(L_{ab} + L_{bc})\lambda^2 + (R_{ab} + R_{bc})\lambda + (D_{ab} + D_{bc})}.\end{aligned}$$

Therefore, we obtain a network.

Note that the corresponding equation for the impedances is then

$$\widetilde{z}_{ac} = z_{ab} + z_{bc},$$

which corresponds to the well-known physical series law.

Figure 2.3: Series law

Remark 2.2.24. $\rho(b) = 0$ means either b is an isolated vertex or $z_{ab} = -z_{bc}$. In the latter case z_{ac} should be calculated as 0 ($\rho_{ac} = \infty$), which is not allowed.

Proof. Apply Theorem 2.2.18 and Corollary 2.2.20 ($x_1 = b$) for the case $m = 3$ and $\rho_{ac} = 0$. □

Corollary 2.2.25 (Parallel-series law). *Let $\Gamma = (V, \rho, a_0, B)$ be a finite network, $B_0 = B \cup \{a_0\}$.*
Let $a, b, c \in V$ are such, that

- $b \notin B_0$,
- $a \sim b, b \sim c, a \sim c$,
- $b \not\sim x$ *for all* $x \notin \{a, c\}$.

Let $\lambda \in \Lambda_1$ be such that $\rho^{(\lambda)}(b) \neq 0$.

If one removes the vertex b, edges ab, bc and add the edge ac with the admittance

$$\widetilde{\rho}_{ac} = \frac{\rho_{ab}\rho_{bc}}{\rho_{ab} + \rho_{bc}} + \rho_{ac},$$

not changing other admittances, then for the new complex-weighted graph the solution of the Dirichlet problem (2.2.1) for all the vertices will be the same as the solution on the original network for corresponding vertices (for the same λ). The effective admittance (impedance) of the new network coincides with the effective admittance (impedance) of the original one.

Remark 2.2.26. Note that the corresponding equation for the impedances is then

$$\frac{1}{\widetilde{z}_{ac}} = \frac{1}{z_{ab} + z_{bc}} + \frac{1}{z_{ac}}.$$

This corresponds to an application of the physical series law and then an application of the physical parallel law.

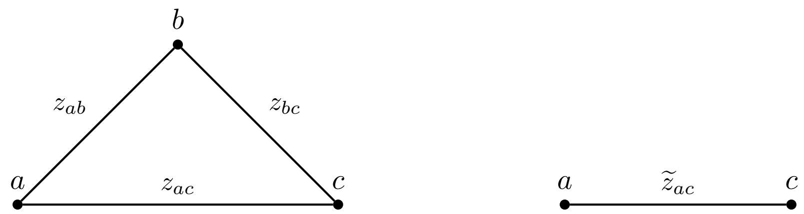


Figure 2.4: Parallel-series law

Proof. Apply Theorem 2.2.18 and Corollary 2.2.20 ($x_1 = b$) for the case $m = 3$. □

Theorem 2.2.27 ($Y - \Delta$ transform). *Let $\Gamma = (V, \rho, a_0, B)$ be a finite network, $B_0 = B \cup \{a_0\}$. Let $a, b, c, d \in V$ are such, that*

- $d \notin B_0$,
- $d \sim a, d \sim b, d \sim c$,

Let $\lambda \in \Lambda_1$ be such that

$$\rho^{(\lambda)}(d) \neq 0. \tag{2.2.23}$$

If one removes the vertex d, edges da, db, dc and set

$$
\begin{aligned}
\widetilde{\rho}_{ab} =& \frac{\rho_{da}\rho_{db}}{\rho_{da}+\rho_{db}+\rho_{dc}} + \rho_{ab},\\
\widetilde{\rho}_{bc} =& \frac{\rho_{db}\rho_{dc}}{\rho_{da}+\rho_{db}+\rho_{dc}} + \rho_{bc},\\
\widetilde{\rho}_{ac} =& \frac{\rho_{da}\rho_{dc}}{\rho_{da}+\rho_{db}+\rho_{dc}} + \rho_{ac},
\end{aligned}
\tag{2.2.24}
$$

not changing other admittances, then for the new complex-weighted graph the solution of the Dirichlet problem (2.2.1) for all the vertices will be the same as the solution for the corresponding vertices at original network (for the same λ). The effective admittance (impedance) of the new network coincides with the effective admittance (impedance) of the original one.

Remark 2.2.28. The corresponding equalities for the impedances are

$$
\begin{aligned}
\widetilde{z}_{ab} =& \frac{1}{\dfrac{z_{dc}}{z_{da}z_{db}+z_{db}z_{dc}+z_{da}z_{dc}} + \dfrac{1}{z_{ab}}},\\
\widetilde{z}_{bc} =& \frac{1}{\dfrac{z_{da}}{z_{da}z_{db}+z_{db}z_{dc}+z_{da}z_{dc}} + \dfrac{1}{z_{bc}}},\\
\widetilde{z}_{ac} =& \frac{1}{\dfrac{z_{db}}{z_{da}z_{db}+z_{db}z_{dc}+z_{da}z_{dc}} + \dfrac{1}{z_{ac}}}.
\end{aligned}
$$

From the physical point of view, if $\rho_{ab}, \rho_{bc}, \rho_{ac}$ are all equal to zero, then it is just $Y-\Delta$ transform, otherwise, it is $Y-\Delta$ transform and the parallel law.

Remark 2.2.29. Example 2.2.38 shows that in the case

$$\rho^{(\lambda)}(d) = \rho^{(\lambda)}_{da} + \rho^{(\lambda)}_{db} + \rho^{(\lambda)}_{dc} = 0$$

one can not use the expressions (2.2.24) formally, i.e. glue the vertices together.

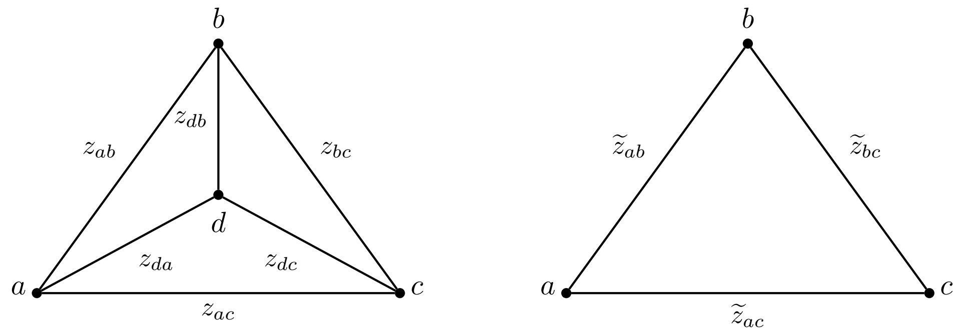


Figure 2.5: $Y-\Delta$ transform

Proof. Theorem 2.2.18 and Corollary 2.2.20 ($x_1 = d$) for the case $m = 4$. □

The $Y-\Delta$ transform is invertible. In general, it is not the case for the star-mesh transform.

Theorem 2.2.30 ($\Delta - Y$ transform). *Let $\widetilde{\Gamma} = (\widetilde{V}, \widetilde{\rho}, a_0, B)$ be a finite network and let $a, b, c \in V$ are such, that $a \sim b, b \sim c, a \sim c$.*

Let $\lambda \in \Lambda_1$ be such that

$$\frac{1}{\widetilde{\rho}_{ab}^{(\lambda)}} + \frac{1}{\widetilde{\rho}_{bc}^{(\lambda)}} + \frac{1}{\widetilde{\rho}_{ac}^{(\lambda)}} \neq 0. \tag{2.2.25}$$

If one add a vertex d and edges da, db, dc setting

$$\begin{aligned} \rho_{da} &= \frac{\widetilde{\rho}_{ac}\widetilde{\rho}_{bc} + \widetilde{\rho}_{ac}\widetilde{\rho}_{ab} + \widetilde{\rho}_{ab}\widetilde{\rho}_{bc}}{\widetilde{\rho}_{bc}}, \\ \rho_{db} &= \frac{\widetilde{\rho}_{ac}\widetilde{\rho}_{bc} + \widetilde{\rho}_{ac}\widetilde{\rho}_{ab} + \widetilde{\rho}_{ab}\widetilde{\rho}_{bc}}{\widetilde{\rho}_{ac}}, \\ \rho_{dc} &= \frac{\widetilde{\rho}_{ac}\widetilde{\rho}_{bc} + \widetilde{\rho}_{ac}\widetilde{\rho}_{ab} + \widetilde{\rho}_{ab}\widetilde{\rho}_{bc}}{\widetilde{\rho}_{ab}}, \end{aligned} \tag{2.2.26}$$

and remove the edges ab, bc, ac not changing other admittances, then for the new complex-weighted graph $(V = \widetilde{V} \cup \{d\}, \rho)$, the solution of the Dirichlet problem (2.2.1) *for all the vertices will be the same as the solution for corresponding vertices at the original network (for the same λ). Moreover, the effective admittance (impedance) does not change under this transform.*

Remark 2.2.31. The condition (2.2.25) can be rewritten as

$$\widetilde{\rho}_{ac}^{(\lambda)}\widetilde{\rho}_{bc}^{(\lambda)} + \widetilde{\rho}_{ac}^{(\lambda)}\widetilde{\rho}_{ab}^{(\lambda)} + \widetilde{\rho}_{ab}^{(\lambda)}\widetilde{\rho}_{bc}^{(\lambda)} \neq 0,$$

since

$$\frac{1}{\widetilde{\rho}_{ab}^{(\lambda)}} + \frac{1}{\widetilde{\rho}_{bc}^{(\lambda)}} + \frac{1}{\widetilde{\rho}_{ac}^{(\lambda)}} = \frac{\widetilde{\rho}_{ac}^{(\lambda)}\widetilde{\rho}_{bc}^{(\lambda)} + \widetilde{\rho}_{ac}^{(\lambda)}\widetilde{\rho}_{ab}^{(\lambda)} + \widetilde{\rho}_{ab}^{(\lambda)}\widetilde{\rho}_{bc}^{(\lambda)}}{\widetilde{\rho}_{bc}^{(\lambda)}\widetilde{\rho}_{ac}^{(\lambda)}\widetilde{\rho}_{ab}^{(\lambda)}}.$$

Remark 2.2.32. The corresponding equalities for the impedances are

$$\begin{aligned} z_{da} &= \frac{\widetilde{z}_{ab}\widetilde{z}_{ac}}{\widetilde{z}_{ab} + \widetilde{z}_{bc} + \widetilde{z}_{ac}}, \\ z_{db} &= \frac{\widetilde{z}_{ab}\widetilde{z}_{bc}}{\widetilde{z}_{ab} + \widetilde{z}_{bc} + \widetilde{z}_{ac}}, \\ z_{dc} &= \frac{\widetilde{z}_{bc}\widetilde{z}_{ac}}{\widetilde{z}_{ab} + \widetilde{z}_{bc} + \widetilde{z}_{ac}}, \end{aligned}$$

and the condition (2.2.25) could be rewritten as $\widetilde{z}_{ab}^{(\lambda)} + \widetilde{z}_{bc}^{(\lambda)} + \widetilde{z}_{ac}^{(\lambda)} \neq 0$.

Remark 2.2.33. The condition (2.2.25) is crucial, see Example 2.2.40.

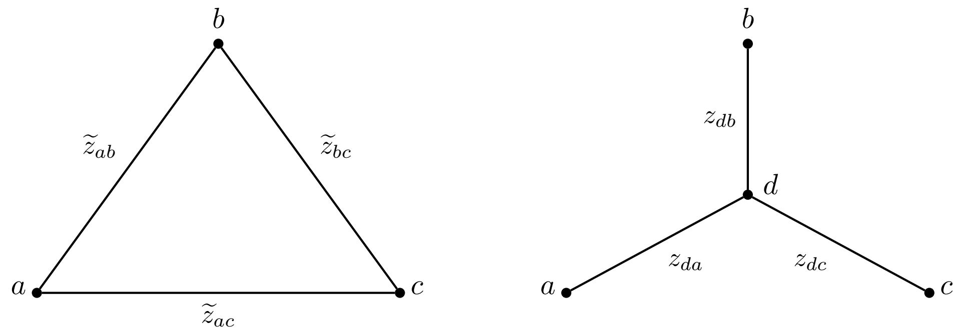


Figure 2.6: $\Delta - Y$ transform

Proof. To prove the theorem it is enough to express ρ_{da}, ρ_{db} and ρ_{dc} from (2.2.24), assuming $\rho_{ab} = 0$, $\rho_{bc} = 0$, and $\rho_{ac} = 0$. Summing up the inverses of all three equations one obtains

$$\frac{1}{\widetilde{\rho}_{ab}} + \frac{1}{\widetilde{\rho}_{bc}} + \frac{1}{\widetilde{\rho}_{ac}} = \frac{(\rho_{da} + \rho_{db} + \rho_{dc})^2}{\rho_{da}\rho_{db}\rho_{dc}}$$

Since left hand side (consequently also right hand side) is not equal to zero by (2.2.25), the last equation is equivalent to

$$\frac{\widetilde{\rho}_{ab}\widetilde{\rho}_{bc}\widetilde{\rho}_{ac}}{\widetilde{\rho}_{ab}\widetilde{\rho}_{bc} + \widetilde{\rho}_{bc}\widetilde{\rho}_{ac} + \widetilde{\rho}_{ab}\widetilde{\rho}_{ac}} = \frac{\rho_{da}\rho_{db}\rho_{dc}}{(\rho_{da} + \rho_{db} + \rho_{dc})^2} \tag{2.2.27}$$

Multiplying the both sides of (2.2.27) by

$$\frac{1}{\widetilde{\rho}_{ab}\widetilde{\rho}_{ac}} = \frac{(\rho_{da} + \rho_{db} + \rho_{dc})^2}{\rho_{da}^2\rho_{db}\rho_{dc}},$$

which follows from (2.2.24), we get

$$\frac{\widetilde{\rho}_{bc}}{\widetilde{\rho}_{ab}\widetilde{\rho}_{bc} + \widetilde{\rho}_{bc}\widetilde{\rho}_{ac} + \widetilde{\rho}_{ab}\widetilde{\rho}_{ac}} = \frac{1}{\rho_{da}}.$$

Then the equation for ρ_{da} follows. To obtain the equations for ρ_{db} and ρ_{dc} one should multiply (2.2.27) by $\dfrac{1}{\widetilde{\rho}_{ab}\widetilde{\rho}_{bcac}}$ and $\dfrac{1}{\widetilde{\rho}_{ac}\widetilde{\rho}_{bc}}$ respectively.

The fact that the effective admittance does not change follows from Theorem 2.2.27 since

$$\begin{aligned}\rho(d) =& \rho_{da} + \rho_{db} + \rho_{dc} = \frac{(\widetilde{\rho}_{ac}\widetilde{\rho}_{bc} + \widetilde{\rho}_{ac}\widetilde{\rho}_{ab} + \widetilde{\rho}_{ab}\widetilde{\rho}_{bc})^2}{\widetilde{\rho}_{bc}\widetilde{\rho}_{ac}\widetilde{\rho}_{ab}}, \\ &= \left(\frac{1}{\widetilde{\rho}_{ab}} + \frac{1}{\widetilde{\rho}_{bc}} + \frac{1}{\widetilde{\rho}_{ac}}\right)^2 \widetilde{\rho}_{ab}\widetilde{\rho}_{bc}\widetilde{\rho}_{ac} \neq 0.\end{aligned}$$

□

2.2.3 Examples of finite networks

Example 2.2.34 (Resonance)**.** Let us consider a network consisting of one coil and one capacitor (Figure 2.7), $a_0 = 1, B = \{0\}$, $L, C > 0$.

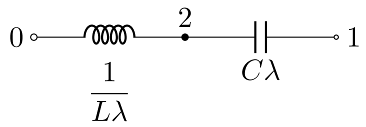


Figure 2.7: LC-network

For this network $\Lambda = \mathbb{C} \setminus \{0\}$. The Dirichlet problem is

$$\begin{cases} \dfrac{(v(0) - v(2))}{L\lambda} + (v(1) - v(2))C\lambda = 0, \\ v(0) = 0, \\ v(1) = 1. \end{cases}$$

The determinant of this linear system is

$$\mathcal{D}(\lambda) = \frac{LC\lambda^2 + 1}{L\lambda}.$$

Therefore, $\Lambda_0 = \mathbb{C} \setminus \left\{0, \pm\dfrac{i}{\sqrt{LC}}\right\}$. The solution of the Dirichlet problem is

$$v = (v(0), v(1), v(2)) = \left(0, 1, \frac{LC\lambda^2}{1 + LC\lambda^2}\right), \lambda \in \Lambda_0.$$

and the effective admittance is

$$\mathcal{P}(\lambda) = \frac{1}{L\lambda} v(2) = \frac{C\lambda}{LC\lambda^2 + 1}, \lambda \in \Lambda_0.$$

In the case $\lambda = \pm\dfrac{i}{\sqrt{LC}}$ the Dirichlet problem has no solution and we have $\mathcal{P} = \infty$ by definition. Therefore, in fact,

$$\mathcal{P}(\lambda) = \frac{C\lambda}{LC\lambda^2 + 1}, \text{ for all } \lambda \in \Lambda$$

and

$$Z(\lambda) = \frac{LC\lambda^2 + 1}{C\lambda}, \text{ for all } \lambda \in \Lambda.$$

One can also calculate the effective impedance using the series law:

$$Z(\lambda) = L\lambda + \frac{1}{C\lambda} = \frac{LC\lambda^2 + 1}{C\lambda}.$$

Note that the case $\lambda = \dfrac{i}{\sqrt{LC}}$ (the frequency $\omega = \dfrac{1}{\sqrt{LC}}$) corresponds to a resonance in the network, which means the lack of a solution $v(t)$ of (2.1.1) in the form $v(x,t) = v(x)e^{i\omega t}$. In this case the frequency ω is called *resonance frequency*. Then $\mathcal{P} = \infty$ matches the physical phenomenon that in this case the current becomes arbitrary large over time.

Example 2.2.35 (Simple network)**.** Let us consider a network as at Figure 2.8, $a_0 = 1, B = \{0\}$, $L, C > 0$.

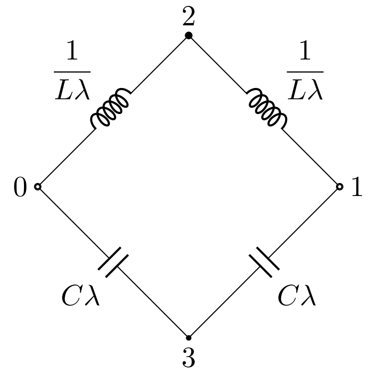


Figure 2.8: Simple network

For this network $\Lambda = \mathbb{C} \setminus \{0\}$. The Dirichlet problem is

$$\begin{cases} \dfrac{(v(0) - v(2))}{L\lambda} + \dfrac{(v(1) - v(2))}{L\lambda} = 0, \\ (v(0) - v(3))C\lambda + (v(1) - v(3))C\lambda = 0, \\ v(0) = 0, \\ v(1) = 1. \end{cases}$$

The determinant of this linear system is

$$\mathcal{D}(\lambda) = \frac{4C}{L}.$$

Therefore, $\Lambda_0 = \mathbb{C} \setminus \{0\} = \Lambda$. The solution of the Dirichlet problem is

$$v = (v(0), v(1), v(2), v(3)) = \left(0, 1, \frac{1}{2}, \frac{1}{2}\right)$$

and the effective admittance is

$$\mathcal{P}(\lambda) = \frac{1}{L\lambda} v(2) + C\lambda v(3) = \frac{LC\lambda^2 + 1}{2L\lambda}, \lambda \in \Lambda.$$

Then the effective impedance is

$$Z(\lambda) = \frac{2L\lambda}{LC\lambda^2 + 1}, \lambda \in \Lambda.$$

Note that the impedances of the edges are

$$z_{02} = z_{12} = L\lambda, z_{03} = z_{13} = \frac{1}{C\lambda},$$

and then the effective impedance can be calculated using series and parallel-series laws (Corollaries 2.2.22 and 2.2.25).

$$Z(\lambda) = \frac{1}{\mathcal{P}(\lambda)} = \frac{1}{\dfrac{1}{z_{02}+z_{12}} + \dfrac{1}{z_{03}+z_{13}}} = \frac{1}{\dfrac{1}{2L\lambda} + \dfrac{C\lambda}{2}} = \frac{2L\lambda}{1+LC\lambda^2}.$$

Example 2.2.36 (Non-uniqueness of the solution of the Dirichlet problem)**.** Let us consider a network as at Figure 2.9, $a_0 = 1, B = \{0\}$, $L, C > 0$.

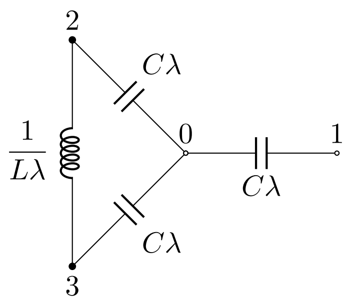


Figure 2.9: Non-uniqueness of the solution of the Dirichlet problem

For this network $\Lambda = \mathbb{C} \setminus \{0\}$. The Dirichlet problem is

$$\begin{cases} \dfrac{(v(3)-v(2))}{L\lambda} + (v(0)-v(2))C\lambda = 0, \\ \dfrac{(v(2)-v(3))}{L\lambda} + (v(0)-v(3))C\lambda = 0, \\ v(0) = 0, \\ v(1) = 1. \end{cases}$$

The determinant of this linear system is

$$\mathcal{D}(\lambda) = \frac{LC^2\lambda^2 + 2C}{L}.$$

Therefore, $\Lambda_0 = \mathbb{C} \setminus \left\{0, \pm i\sqrt{\dfrac{2}{LC}}\right\}$. The solution of the Dirichlet problem is

$$v = (v(0), v(1), v(2), v(3)) = (0,1,0,0), \lambda \in \Lambda_0.$$

and the effective admittance is

$$\mathcal{P}(\lambda) = C\lambda v(1) = C\lambda, \lambda \in \Lambda_0.$$

In the cases $\lambda = \pm i\sqrt{\dfrac{2}{LC}}$ the Dirichlet problem has infinitely many solutions

$$v = (0, 1, c, -c), c \in \mathbb{C}.$$

The effective admittance in this case is, obviously, also

$$\mathcal{P}(\lambda) = C\lambda, \lambda = \pm i\sqrt{\frac{2}{LC}}.$$

Therefore, $\mathcal{P}(\lambda) = C\lambda, \lambda \in \Lambda$.

Example 2.2.37 (Complex-weighted graph, which is not a network)**.** Let us consider the complex weights, which are not admittances (i.e. they do not posses the representation (2.1.3)), to show that Theorem 2.2.11 is true only for networks.

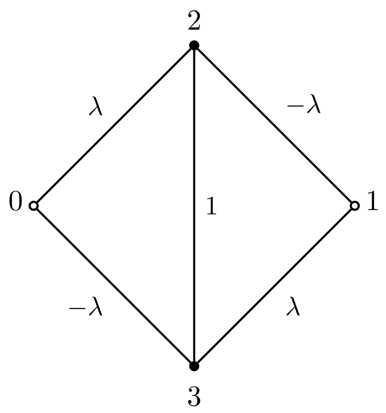


Figure 2.10: Example of a complex-weighted graph, which is not a network

Let us consider the following Dirichlet problem (see Figure 2.10):

$$\begin{cases} (v(0) - v(2))\lambda + (v(1) - v(2))(-\lambda) + (v(3) - v(2)) = 0, \\ (v(0) - v(3))(-\lambda) + (v(1) - v(3))\lambda + (v(2) - v(3)) = 0 \\ v(0) = 0, \\ v(1) = 1. \end{cases}$$

The determinant of this system is

$$\mathcal{D} = \mathcal{D}(\lambda) \equiv 0$$

and the Dirichlet problem has infinitely many solutions

$$v(2) = c, v(3) = c + \lambda, c \in \mathbb{C}$$

for any $\lambda \in \mathbb{C}$.

Example 2.2.38 ($Y-\Delta$ transform)**.** Let us consider a network as at Figure 2.11 ($a_0 = 1, B = \{0\}$, $L, C > 0$). Then $\Lambda = \mathbb{C} \setminus \{0\}$.

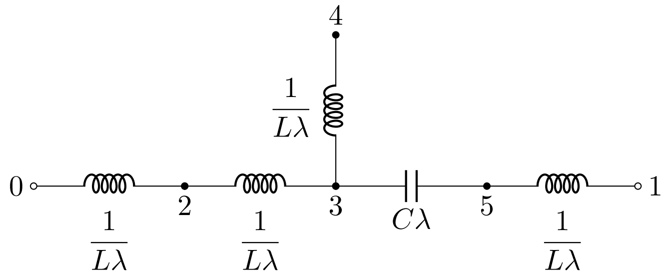


Figure 2.11: An example of $Y - \Delta$ transform

The Dirichlet problem for this network is

$$\begin{cases} \dfrac{(v(0)-v(2))}{L\lambda} + \dfrac{(v(3)-v(2))}{L\lambda} = 0, \\ \dfrac{(v(2)-v(3))}{L\lambda} + \dfrac{(v(4)-v(3))}{L\lambda} + (v(5)-v(3))C\lambda = 0, \\ \dfrac{(v(3)-v(4))}{L\lambda} = 0, \\ (v(3)-v(5))C\lambda + \dfrac{(v(1)-v(5))}{L\lambda} = 0, \\ v(0) = 0, \\ v(1) = 1, \end{cases}$$

and its determinant is

$$\mathcal{D}(\lambda) = \frac{3LC\lambda^2 + 1}{L^4\lambda^4}.$$

Therefore, $\Lambda_0 = \mathbb{C} \setminus \left\{0, \pm\dfrac{i}{\sqrt{3LC}}\right\}$. The solution of the Dirichlet problem is

$$\begin{cases} v(0) = 0, \\ v(1) = 1, \\ v(2) = \dfrac{LC\lambda^2}{3LC\lambda^2+1}, \\ v(3) = \dfrac{2LC\lambda^2}{3LC\lambda^2+1}, \\ v(4) = \dfrac{2LC\lambda^2}{3LC\lambda^2+1}, \\ v(5) = \dfrac{2LC\lambda^2+1}{3LC\lambda^2+1}. \end{cases}$$

The effective admittance is

$$\mathcal{P}(\lambda) = \frac{C\lambda}{3LC\lambda^2+1}, \lambda \in \Lambda_0.$$

In the cases $\lambda = \pm\frac{i}{\sqrt{3LC}}$ the Dirichlet problem has no solutions and $\mathcal{P}(\lambda) = \infty$ by the definition. Therefore, in fact

$$\mathcal{P}(\lambda) = \frac{C\lambda}{3LC\lambda^2 + 1}, \lambda \in \Lambda. \tag{2.2.28}$$

The domain of holomorphicity of $\mathcal{P}(\lambda)$ is $\mathbb{C} \setminus \left\{0, \pm\frac{i}{\sqrt{3LC}}\right\} = \Lambda_0$.

Let us apply the particular case of star-mesh transform, i.e. $Y - \Delta$ transform, to this network. In the case

$$\rho(3) = \frac{2}{L\lambda} + C\lambda \neq 0,$$

we have by (2.2.24)

$$\begin{aligned}
\widetilde{\rho}_{24} =& \frac{\rho_{32}\rho_{34}}{\rho_{32} + \rho_{34} + \rho_{35}} = \frac{1}{L\lambda(CL\lambda^2 + 2)} \\
\widetilde{\rho}_{45} =& \frac{\rho_{34}\rho_{35}}{\rho_{32} + \rho_{34} + \rho_{35}} = \frac{C\lambda}{CL\lambda^2 + 2}, \\
\widetilde{\rho}_{25} =& \frac{\rho_{32}\rho_{35}}{\rho_{32} + \rho_{34} + \rho_{35}} = \frac{C\lambda}{CL\lambda^2 + 2}.
\end{aligned} \tag{2.2.29}$$

The equations (2.2.29) suggest us to glue together vertices $2, 3, 4$ and 5 in the case $\frac{2}{L\lambda} + C\lambda = 0$ (i.e. $\lambda = \pm i\sqrt{\frac{2}{LC}}$), but the effective admittance of the obtained network will not be the same. Indeed, by (2.2.28).

$$\mathcal{P}\left(\pm i\sqrt{\frac{2}{LC}}\right) = \mp\frac{i}{5}\sqrt{\frac{2C}{L}}.$$

But the network, obtaining by gluing $2, 3, 4$ and 5, is shown at Figure 2.12 and its effective admittance is

$$\widetilde{\mathcal{P}} = \frac{1}{2L\lambda} = \mp\frac{i}{2}\sqrt{\frac{C}{2L}} \neq \mp\frac{i}{5}\sqrt{\frac{2C}{L}}.$$

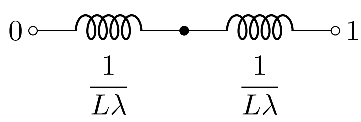


Figure 2.12: Suggested $Y - \Delta$ transform for the case $\lambda = \pm i\sqrt{\frac{2}{LC}}$

Therefore, this example shows, that the condition (2.2.22) in Corollary 2.2.20 (and the condition (2.2.23) in $Y - \Delta$ transform) is crucial.

Example 2.2.39 ($\Delta - Y$ transform)**.** Let us consider a network as at the Figure 2.13 ($a_0 = 1, B = \{0\}$, $L, C > 0$). Then $\Lambda = \mathbb{C} \setminus \{0\}$.

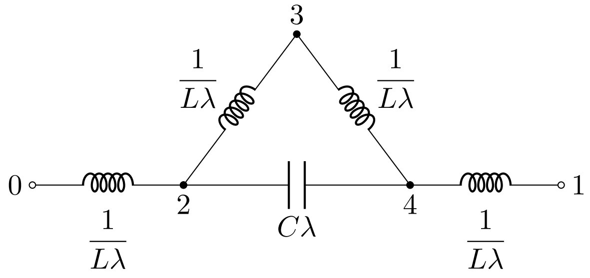


Figure 2.13: An example of $\Delta - Y$ transform

The Dirichlet problem for this network is

$$\begin{cases} \dfrac{(v(0)-v(2))}{L\lambda} + \dfrac{(v(3)-v(2))}{L\lambda} + (v(4)-v(2))C\lambda = 0, \\ \dfrac{(v(2)-v(3))}{L\lambda} + \dfrac{(v(4)-v(3))}{L\lambda} = 0, \\ (v(2)-v(4))C\lambda + \dfrac{(v(3)-v(4))}{L\lambda} + \dfrac{(v(1)-v(4))}{L\lambda} = 0, \\ v(0) = 0, \\ v(1) = 1, \end{cases}$$

and its determinant is

$$\mathcal{D} = \mathcal{D}(\lambda) = -\frac{4CL\lambda^2 + 4}{L^3\lambda^3}.$$

Therefore, $\Lambda_0 = \mathbb{C} \setminus \left\{0, \pm i\dfrac{1}{\sqrt{LC}}\right\}$. The solution of the Dirichlet problem is

$$\begin{cases} v(0) = 0, \\ v(1) = 1, \\ v(2) = \dfrac{2LC\lambda^2 + 1}{4LC\lambda^2 + 4}, \\ v(3) = \dfrac{1}{2}, \\ v(4) = \dfrac{2LC\lambda^2 + 3}{4LC\lambda^2 + 4}, \end{cases}$$

and the effective admittance is

$$\mathcal{P}(\lambda) = \frac{1}{L\lambda} v(2) = \frac{2LC\lambda^2 + 1}{4L^2C\lambda^3 + 4L\lambda}, \lambda \in \Lambda_0.$$

Since in the case $\lambda = \pm i\dfrac{1}{\sqrt{LC}}$ the Dirichlet problem has no solutions, we have, in fact

$$\mathcal{P}(\lambda) = \frac{2LC\lambda^2 + 1}{4L^2C\lambda^3 + 4L\lambda}, \lambda \in \Lambda.$$

The domain of holomorphicity of $\mathcal{P}(\lambda)$ is $\mathbb{C} \setminus \left\{0, \pm i\frac{1}{\sqrt{LC}}\right\} = \Lambda_0$.

After $\Delta - Y$ transform, adding the vertex 5, one gets by (2.2.26)

$$\begin{aligned}
\rho_{52} &= \frac{\widetilde{\rho}_{24}\widetilde{\rho}_{34} + \widetilde{\rho}_{24}\widetilde{\rho}_{23} + \widetilde{\rho}_{23}\widetilde{\rho}_{34}}{\widetilde{\rho}_{34}} = \frac{2LC\lambda^2 + 1}{L\lambda}, \\
\rho_{53} &= \frac{\widetilde{\rho}_{24}\widetilde{\rho}_{34} + \widetilde{\rho}_{24}\widetilde{\rho}_{23} + \widetilde{\rho}_{23}\widetilde{\rho}_{34}}{\widetilde{\rho}_{24}} = \frac{2LC\lambda^2 + 1}{CL^2\lambda^3}, \\
\rho_{54} &= \frac{\widetilde{\rho}_{24}\widetilde{\rho}_{34} + \widetilde{\rho}_{24}\widetilde{\rho}_{23} + \widetilde{\rho}_{23}\widetilde{\rho}_{34}}{\widetilde{\rho}_{23}} = \frac{2LC\lambda^2 + 1}{L\lambda},
\end{aligned} \tag{2.2.30}$$

if

$$\frac{1}{\widetilde{\rho}_{ab}} + \frac{1}{\widetilde{\rho}_{bc}} + \frac{1}{\widetilde{\rho}_{ac}} = 2L\lambda + \frac{1}{C\lambda} \neq 0 \text{ i.e. } \lambda \neq \pm i\frac{1}{\sqrt{2LC}}.$$

In the case $\lambda = \pm i\frac{1}{\sqrt{2LC}}$ the formulas (2.2.30) suggest us to make the graph disconnected, which matches the fact that $\mathcal{P}\left(\pm i\frac{1}{\sqrt{2LC}}\right) = 0$. But the next example shows, that this is not a general case.

Example 2.2.40 (The condition (2.2.25) in $\Delta - Y$ transform)**.** This example shows, that the condition (2.2.25) in $\Delta - Y$ transform is crucial. Let us consider a network as at Figure 2.14 ($a_0 = 1, B = \{0\}$, $L, C > 0$). Then $\Lambda = \mathbb{C} \setminus \{0\}$.

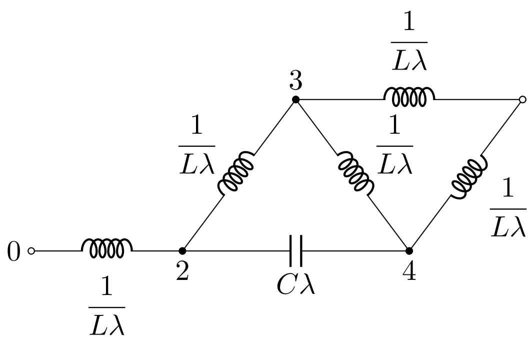


Figure 2.14: Condition (2.2.25) in $\Delta - Y$ transform

The Dirichlet problem for this network is

$$\begin{cases}
\dfrac{(v(0)-v(2))}{L\lambda} + \dfrac{(v(3)-v(2))}{L\lambda} + (v(4)-v(2))C\lambda = 0, \\
\dfrac{(v(2)-v(3))}{L\lambda} + \dfrac{(v(4)-v(3))}{L\lambda} + \dfrac{(v(1)-v(3))}{L\lambda} = 0, \\
(v(2)-v(4))C\lambda + \dfrac{(v(3)-v(4))}{L\lambda} + \dfrac{(v(1)-v(4))}{L\lambda} = 0, \\
v(0) = 0, \\
v(1) = 1,
\end{cases}$$

and its determinant is

$$\mathcal{D} = \mathcal{D}(\lambda) = -\frac{8CL\lambda^2 + 8}{L^3\lambda^3}.$$

Therefore, $\Lambda_0 = \mathbb{C} \setminus \left\{0, \pm i\frac{1}{\sqrt{LC}}\right\}$. The solution of the Dirichlet problem is

$$\begin{cases} v(0) = 0, \\ v(1) = 1, \\ v(2) = \dfrac{5LC\lambda^2 + 3}{8LC\lambda^2 + 8}, \\ v(3) = \dfrac{3}{4}, \\ v(4) = \dfrac{5LC\lambda^2 + 7}{8LC\lambda^2 + 8}, \end{cases}$$

and the effective admittance is

$$\mathcal{P}(\lambda) = \frac{1}{L\lambda}v(2) = \frac{5LC\lambda^2 + 3}{8L^2C\lambda^3 + 8L\lambda}, \lambda \in \Lambda_0.$$

Since in the case $\lambda = \pm i\frac{1}{\sqrt{LC}}$ the Dirichlet problem has no solutions ($\mathcal{P} = \infty$), we have, in fact

$$\mathcal{P}(\lambda) = \frac{5LC\lambda^2 + 3}{8L^2C\lambda^3 + 8L\lambda}, \lambda \in \Lambda.$$

The domain of holomorphicity of $\mathcal{P}(\lambda)$ is $\mathbb{C} \setminus \left\{0, \pm i\frac{1}{\sqrt{LC}}\right\} = \Lambda_0$.

After $\Delta - Y$ transform, applied to the vertices $2, 3$ and 4, adding the vertex 5, one gets by (2.2.26)

$$\begin{aligned} \rho_{52} &= \frac{\widetilde{\rho}_{24}\widetilde{\rho}_{34} + \widetilde{\rho}_{24}\widetilde{\rho}_{23} + \widetilde{\rho}_{23}\widetilde{\rho}_{34}}{\widetilde{\rho}_{34}} = \frac{2LC\lambda^2 + 1}{L\lambda}, \\ \rho_{53} &= \frac{\widetilde{\rho}_{24}\widetilde{\rho}_{34} + \widetilde{\rho}_{24}\widetilde{\rho}_{23} + \widetilde{\rho}_{23}\widetilde{\rho}_{34}}{\widetilde{\rho}_{24}} = \frac{2LC\lambda^2 + 1}{L^2C\lambda^3}, \\ \rho_{54} &= \frac{\widetilde{\rho}_{24}\widetilde{\rho}_{34} + \widetilde{\rho}_{24}\widetilde{\rho}_{23} + \widetilde{\rho}_{23}\widetilde{\rho}_{34}}{\widetilde{\rho}_{23}} = \frac{2LC\lambda^2 + 1}{L\lambda}, \end{aligned} \tag{2.2.31}$$

if

$$\frac{1}{\widetilde{\rho}_{ab}} + \frac{1}{\widetilde{\rho}_{bc}} + \frac{1}{\widetilde{\rho}_{ac}} = 2L\lambda + \frac{1}{C\lambda} \neq 0 \text{ i.e. } \lambda \neq \pm i\frac{1}{\sqrt{2LC}}.$$

In the case $\lambda = \pm i\frac{1}{\sqrt{2LC}}$ the formulas (2.2.31) suggest us to make the graph disconnected, but for the original network

$$\mathcal{P}\left(\pm\frac{i}{\sqrt{2LC}}\right) = \mp\frac{i}{8}\sqrt{\frac{2C}{L}} \neq 0.$$

Example 2.2.41 (The weights after $Y-\Delta$ transform are not positive real functions)**.** Let us consider $Y-\Delta$ transform on the part of a network as at Figure 2.15.

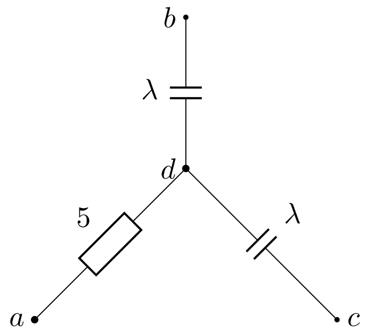


Figure 2.15: The weight after $Y-\Delta$ transform is not positive real function

After $Y-\Delta$ transform one get by (2.2.24)

$$\begin{aligned}\widetilde{\rho}_{ab} =& \frac{\rho_{da}\rho_{db}}{\rho_{da}+\rho_{db}+\rho_{dc}} = \frac{5\lambda}{2\lambda+5}\\ \widetilde{\rho}_{bc} =& \frac{\rho_{db}\rho_{dc}}{\rho_{da}+\rho_{db}+\rho_{dc}} = \frac{\lambda^2}{2\lambda+5},\\ \widetilde{\rho}_{ac} =& \frac{\rho_{da}\rho_{dc}}{\rho_{da}+\rho_{db}+\rho_{dc}} = \frac{5\lambda}{2\lambda+5},\end{aligned}$$

and $\widetilde{\rho}_{bc}$ is not a positive real function. Indeed, taking $\lambda = 1+2i$, we get

$$\widetilde{\rho}_{bc}^{(1+2i)} = -\frac{1}{13}+\frac{8}{13}i.$$

Moreover, for $\lambda = i$ we have $\operatorname{Re}\widetilde{\rho}_{bc}^{(i)} = -\dfrac{5}{29} < 0$.

Example 2.2.42 (The weights after $\Delta-Y$ transform are not positive real functions)**.** Let us consider $\Delta-Y$ transform on the part of a network as at Figure 2.16.

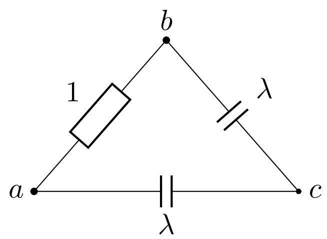


Figure 2.16: The weight after $\Delta-Y$ transform is not positive real function

After $\Delta - Y$ transform one get by (2.2.26)

$$\rho_{da} = \frac{\widetilde{\rho}_{ac}\widetilde{\rho}_{bc} + \widetilde{\rho}_{ac}\widetilde{\rho}_{ab} + \widetilde{\rho}_{ab}\widetilde{\rho}_{bc}}{\widetilde{\rho}_{bc}} = 2 + \lambda,$$
$$\rho_{db} = \frac{\widetilde{\rho}_{ac}\widetilde{\rho}_{bc} + \widetilde{\rho}_{ac}\widetilde{\rho}_{ab} + \widetilde{\rho}_{ab}\widetilde{\rho}_{bc}}{\widetilde{\rho}_{ac}} = 2 + \lambda,$$
$$\rho_{dc} = \frac{\widetilde{\rho}_{ac}\widetilde{\rho}_{bc} + \widetilde{\rho}_{ac}\widetilde{\rho}_{ab} + \widetilde{\rho}_{ab}\widetilde{\rho}_{bc}}{\widetilde{\rho}_{ab}} = 2\lambda + \lambda^2,$$

and ρ_{dc} is not a positive real function. Indeed, taking $\lambda = 1 + 2i$, we get

$$\rho_{dc}^{(1+2i)} = -1 + 8i.$$

Moreover, for $\lambda = i$ we have $\operatorname{Re} \rho_{dc}^{(i)} = -1 < 0$.

2.2.4 Calculation of the effective admittance of a finite $\alpha\beta$-network

Consider the finite graph (V, E), where

$$V = \{0, 1, 2, 3, 4, \ldots (2n-1), (2n-2)\} \cup \{2n\}$$

and E is given by $(2k-2) \sim 2k$, $k = \overline{1, n}$ and $(2k-1) \sim 2k$ for $k = \overline{1, (n-1)}$. Let us consider a network as on Figure 2.17.

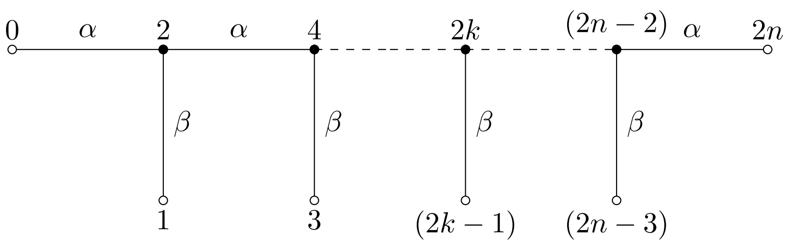


Figure 2.17: Finite ladder network

That is, let admittances of the edges $(2k-2) \sim 2k$ be $\alpha = \alpha^{(\lambda)}$ and admittances of the edges $2k - 1 \sim 2k$ be $\beta^{(\lambda)}$. Set also $a_0 = 0$, while

$$B = \{1, 3, \ldots, 2n-3\} \cup \{2n\}.$$

We will refer to such a network as a *finite $\alpha\beta$-network* and denote it by $\Gamma_n^{\alpha\beta}$. The Dirichlet problem (2.2.1) for this network is as follows:

$$\begin{cases} v(2k-2) + \mu v(2k-1) + v(2k+2) - (2+\mu)v(2k) = 0, \quad k = \overline{1, n-1}, \\ v(0) = 1, \\ v(2k-1) = 0, \quad k = \overline{1, n-1}, \\ v(2n) = 0, \end{cases} \tag{2.2.32}$$

where $\mu = \dfrac{\beta}{\alpha}$.

Substituting the equations from the third line of (2.2.32) to the first line and denoting $v_k = v(2k)$, we obtain the following recurrence relation for v_k:

$$v_{k+1} - (2+\mu)\, v_k + v_{k-1} = 0. \tag{2.2.33}$$

The characteristic polynomial of (2.2.33) is

$$\psi^2 - (2+\mu)\, \psi + 1 = 0. \tag{2.2.34}$$

By the definition of a network $\mu \neq 0$ (i.e. $\mu^{(\lambda)} \neq 0$ for $\lambda \in \Lambda$). If $\mu \neq -4$, then the equation (2.2.34) has two different complex roots ψ_1, ψ_2 and its solution is

$$v_k = c_1 \psi_1^k + c_2 \psi_2^k, \tag{2.2.35}$$

where $c_1, c_2 \in \mathbb{C}$ are arbitrary constants.

We use the second and fourth equations of (2.2.32) as boundary conditions for this recurrence equation. Substituting (2.2.35) in the boundary conditions we obtain the following equations for the constants:

$$\begin{cases} c_1 + c_2 = 1, \\ c_1 \psi_1^n + c_2 \psi_2^n = 0. \end{cases}$$

Therefore,

$$\begin{cases} c_1 = \dfrac{1}{1-\psi_1^{2n}} = \dfrac{-\psi_2^{2n}}{1-\psi_2^{2n}}, \\ c_2 = \dfrac{1}{1-\psi_2^{2n}} = \dfrac{-\psi_1^{2n}}{1-\psi_1^{2n}}, \end{cases}$$

since $\psi_1 \psi_2 = 1$ by (2.2.34).

Now we can calculate the effective admittance of $\Gamma_n^{\alpha\beta}$ in the case $\mu \neq -4$:

$$\begin{aligned} \mathcal{P}_n^{\alpha\beta} =& \alpha\,(1 - v(2)) = \alpha\,(1 - v_1) = \alpha\,(1 - c_1\psi_1 - c_2\psi_2) \\ =& \frac{\alpha\left(\psi_1^{2n-1}+1\right)(\psi_1 - 1)}{\left(\psi_1^{2n}-1\right)} = \frac{\alpha\left(\psi_2^{2n-1}+1\right)(\psi_2 - 1)}{\left(\psi_2^{2n}-1\right)}. \end{aligned} \tag{2.2.36}$$

By Theorem 2.2.11 $\mathcal{P}_n^{\alpha\beta}$ is a rational function of α and β. Indeed, using binomial expansion, it can be written as a rational function of α and β, without usage of ψ:

$$\begin{aligned} \mathcal{P}_n^{\alpha\beta} =& \alpha\,(1 - c_1\psi_1 - c_2\psi_2) = \alpha\left(1 - \frac{\psi_1}{1-\psi_1^{2n}} - \frac{\psi_2}{1-\psi_2^{2n}}\right) \\ =& \alpha\left(1 - \frac{\psi_1\left(1-\psi_2^{2n}\right) + \psi_2\left(1-\psi_1^{2n}\right)}{\left(1-\psi_1^{2n}\right)\left(1-\psi_2^{2n}\right)}\right) = \alpha\left(1 - \frac{\psi_1 + \psi_2 - \left(\psi_2^{2n-1} + \psi_1^{2n-1}\right)}{2 - \left(\psi_1^{2n} + \psi_2^{2n}\right)}\right) \end{aligned}$$

$$=\alpha\left(1-\frac{2+\dfrac{\beta}{\alpha}-2\sum\limits_{k=0}^{n-1}\binom{2n-1}{2k}\left(1+\dfrac{\beta}{2\alpha}\right)^{2n-2k-1}\left(\dfrac{\beta}{\alpha}+\left(\dfrac{\beta}{2\alpha}\right)^2\right)^k}{2-2\sum\limits_{k=0}^{n}\binom{2n}{2k}\left(1+\dfrac{\beta}{2\alpha}\right)^{2n-2k}\left(\dfrac{\beta}{\alpha}+\left(\dfrac{\beta}{2\alpha}\right)^2\right)^k}\right),$$

since $\psi_1+\psi_2=2+\mu=2+\dfrac{\beta}{\alpha}$ by (2.2.34).

Let us now consider the case $\mu=-4$. Then the solution of the recurrence relation (2.2.33) is

$$v_k=c_1(-1)^k+c_2k(-1)^k,$$

where $c_1,c_2\in\mathbb{C}$ are arbitrary constants. And using boundary conditions for the recurrence relation, we obtain

$$\begin{cases} c_1=1 \\ c_2=-\dfrac{1}{n}. \end{cases}$$

Then the effective admittance is

$$\mathcal{P}_n=\alpha\left(1-v(2)\right)=\alpha\left(1-v_1\right)=\alpha\left(1-\left(-1+\frac{1}{n}\right)\right)=\frac{\alpha(2n-1)}{n}.$$

Therefore, for a finite $\alpha\beta$-network we have for $\lambda\in\Lambda$

$$\mathcal{P}_n=\begin{cases}\dfrac{\alpha(2n-1)}{n}, \dfrac{\beta}{\alpha}=-4, \\ \mathcal{P}_n^{\alpha\beta}, \text{otherwise.}\end{cases} \tag{2.2.37}$$

We will use this result later for calculations of effective admittances of infinite ladder networks (for example, Feynman's ladder with zero at infinity, see Subsection 2.5.3).

2.3 On continuity of the effective admittance for finite networks

In this section we discuss a continuity of the effective admittance $\mathcal{P}(\lambda)$ in different domains of a complex plane λ. By Theorem 2.2.11 $\mathcal{P}(\lambda)$ has not more than finitely many discontinuities. From the same theorem and (2.2.4) it follows that the effective admittance $\mathcal{P}(\lambda)$ is a continuous function of λ in the right-half plane $\{\operatorname{Re}\lambda>0\}$. Example 2.3.2 shows, that $\mathcal{P}(\lambda)$ can be discontinuous for λ in the left-half plane $\{\operatorname{Re}\lambda<0\}$. The continuity of the effective admittance $\mathcal{P}(\lambda)$ for $\lambda=i\omega$, $\omega>0$ is an open question. This question is very interesting, since it corresponds to the physical case. Example 2.3.1 shows, how the continuity can be attained in the case of zero determinant of the Dirichlet problem. In Subsection 2.3.2 we present the proof of the continuity of an effective admittance for $\lambda=i\omega$ for the cycle graph with four vertices. We stress that the proof does not apply to general rational functions, but hinges on the representation (2.1.3). Example 2.3.3 shows, that this representation, as a condition, can not be omitted. Note that if for some $\lambda_0 \in \Lambda$ the solution of the Dirichlet problem (2.2.2) has a limit $v_0 = \lim\limits_{\lambda \to \lambda_0} v^{(\lambda)}$, then v_0 is a solution of the Dirichlet problem

$$\mathbf{A}^{(\lambda_0)} \hat{v} = \mathbf{b}^{(\lambda_0)}.$$

This follows from taking limits of sums and products. Therefore, $\lambda_0 \in \Lambda_1$ in this case and $\mathcal{P}(\lambda)$ is continuous at λ_0.

2.3.1 Particular examples

Example 2.3.1 (Continuity of the effective admittance)**.** Let us consider the finite network as at Figure 2.18, where $R, L, C > 0$, with $a_0 = 1$, $B = \{0\}$. Then $\Lambda = \mathbb{C} \setminus \left\{0, -\dfrac{R}{L}\right\}$.

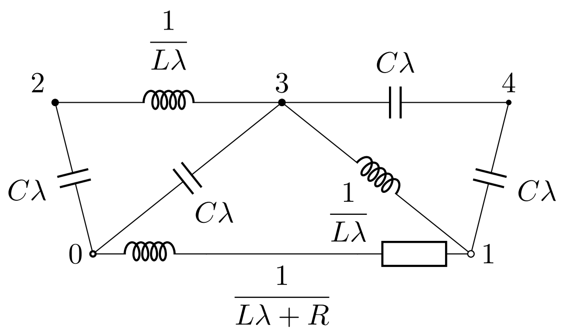


Figure 2.18: Finite network with continuous effective admittance for any $\lambda \in \Lambda$

The Dirichlet problem for this network is

$$\begin{cases} (v(0) - v(2))C\lambda + \dfrac{(v(3) - v(2))}{L\lambda} = 0, \\ (v(0) - v(3))C\lambda + \dfrac{(v(2) - v(3))}{L\lambda} + \dfrac{(v(1) - v(3))}{L\lambda} + (v(4) - v(3))C\lambda = 0, \\ (v(3) - v(4))C\lambda + (v(1) - v(4))C\lambda = 0, \\ v(0) = 0, \\ v(1) = 1. \end{cases} \tag{2.3.1}$$

The determinant of this linear system is

$$\mathcal{D} = \mathcal{D}(\lambda) = \frac{-C(3LC\lambda^2 + 1)(LC\lambda^2 + 2)}{L^2\lambda}$$

and it has four zeros: $\pm i\sqrt{\dfrac{2}{LC}}$ and $\pm i\sqrt{\dfrac{1}{3LC}}$.

Therefore, $\Lambda_0 = \mathbb{C} \setminus \left\{0, -\frac{R}{L}, \pm i\sqrt{\frac{2}{LC}}, \pm i\sqrt{\frac{1}{3LC}}\right\}$

In the case $\mathcal{D}(\lambda) \neq 0$ the solution of the Dirichlet problem (2.3.1) is

$$\begin{cases} v(0) = 0, \\ v(1) = 1, \\ v(2) = \dfrac{1}{3LC\lambda^2 + 1}, \\ v(3) = \dfrac{LC\lambda^2 + 1}{3LC\lambda^2 + 1}, \\ v(4) = \dfrac{2LC\lambda^2 + 1}{3LC\lambda^2 + 1}. \end{cases}$$

In this case

$$\mathcal{P}(\lambda) = C\lambda v(2) + C\lambda v(3) + \frac{1}{L\lambda + R} = \frac{L^2C^2\lambda^4 + RLC^2\lambda^3 + 5LC\lambda^2 + 2RC\lambda + 1}{3L^2C\lambda^3 + 3RLC\lambda^2 + L\lambda + R}.$$

In the cases $\lambda = \pm i\sqrt{\frac{2}{LC}}$ the Dirichlet problem (2.3.1) has infinitely many solutions

$$v = (0, 1, -2c + 1, 2c - 1, c), c \in \mathbb{C}.$$

Then

$$\mathcal{P}\left(\pm i\sqrt{\frac{2}{LC}}\right) = \frac{1}{R \pm i\sqrt{\frac{2L}{C}}}.$$

In the case $\pm i\sqrt{\frac{1}{3LC}}$ the Dirichlet problem (2.3.1) has no solution.

Therefore, the effective admittance is

$$\mathcal{P}(\lambda) = \begin{cases} \dfrac{1}{R \pm i\sqrt{\frac{2L}{C}}}, \lambda = \pm i\sqrt{\dfrac{2}{LC}}; \\ \infty, \lambda = \pm i\sqrt{\dfrac{1}{3LC}}; \\ \dfrac{L^2C^2\lambda^4 + RLC^2\lambda^3 + 5LC\lambda^2 + 2RC\lambda + 1}{3L^2C\lambda^3 + 3RLC\lambda^2 + L\lambda + R}, \text{ for the other } \lambda \in \Lambda. \end{cases}$$

One can verify, that in fact

$$\mathcal{P}(\lambda) = \frac{L^2C^2\lambda^4 + RLC^2\lambda^3 + 5LC\lambda^2 + 2RC\lambda + 1}{3L^2C\lambda^3 + 3RLC\lambda^2 + L\lambda + R} \text{ for any } \lambda \in \Lambda.$$

Therefore, the domain of continuity of the effective admittance is $\mathbb{C} \setminus \left\{0, -\frac{R}{L}\right\} = \Lambda$ and the domain of holomorphicity is $\mathbb{C} \setminus \left\{0, -\frac{R}{L}, \pm i\sqrt{\frac{1}{3LC}}\right\}$.

Example 2.3.2 (Non-continuity of the effective admittance)**.** Let us consider the finite network as at Figure 2.19, where all inductances, capacitances and resistance are equal to 1, with $a_0 = 1$, $B = \{0\}$. Then $\Lambda = \mathbb{C} \setminus \{0\}$.

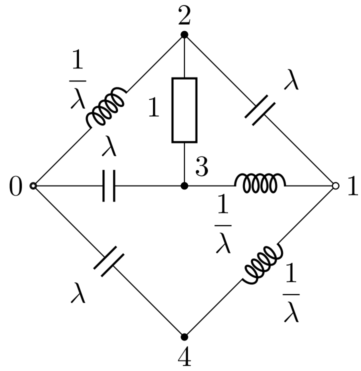


Figure 2.19: Example of a finite network with non-continuous admittance

The Dirichlet problem for this network is

$$\begin{cases} \dfrac{(v(0) - v(2))}{\lambda} + (v(1) - v(2))\lambda + (v(3) - v(2)) = 0, \\ (v(0) - v(3))\lambda + \dfrac{(v(1) - v(3))}{\lambda} + (v(2) - v(3)) = 0, \\ (v(0) - v(4))\lambda + \dfrac{(v(1) - v(4))}{\lambda} = 0, \\ v(0) = 0, \\ v(1) = 1. \end{cases}$$

The determinant of this system is

$$\begin{aligned} \mathcal{D} = \mathcal{D}(\lambda) &= -\left(\frac{1}{\lambda^2} + \frac{2}{\lambda} + 2 + 2\lambda + \lambda^2\right)\left(\lambda + \frac{1}{\lambda}\right) \\ &= -\frac{1}{\lambda^3}(\lambda + 1)^2(\lambda^2 + 1)^2, \end{aligned}$$

and it is easy to see, that $\lambda = \pm i$ and $\lambda = -1$ are its zeros. Therefore, $\Lambda_0 = \mathbb{C} \setminus \{0, \pm i, -1\}$.

In case $\mathcal{D}(\lambda) \neq 0$ the solution of the Dirichlet problem is

$$\begin{cases} v(0) = 0, \\ v(1) = 1, \\ v(2) = \dfrac{\lambda}{1+\lambda}, \\ v(3) = \dfrac{1}{1+\lambda}, \\ v(4) = \dfrac{1}{1+\lambda^2}. \end{cases}$$

and the effective admittance is

$$\mathcal{P}(\lambda) = \frac{1}{\lambda} v(2) + \lambda v(3) + \lambda v(4) = \frac{\lambda^2+\lambda+1}{\lambda^2+1}.$$

Note that the finite limit of $v^{(\lambda)}$ does not exist when λ goes to i or λ goes to -1.

The Dirichlet problem in the case $\lambda = i$ is

$$\begin{cases} -(v(0)-v(2))i + (v(1)-v(2))i + (v(3)-v(2)) = 0, \\ (v(0)-v(3))i - (v(1)-v(3))i + (v(2)-v(3)) = 0, \\ (v(0)-v(4))i - (v(1)-v(4))i = 0, \\ v(0) = 0, \\ v(1) = 1. \end{cases}$$

and has no solution, which by definition of an effective impedance implies $\mathcal{P}(i) = \infty$. The case $\lambda = -i$ similarly has no solution.

The Dirichlet problem in the case $\lambda = -1$ is

$$\begin{cases} -(v(0)-v(2)) - (v(1)-v(2)) + (v(3)-v(2)) = 0, \\ -(v(0)-v(3)) - (v(1)-v(3)) + (v(2)-v(3)) = 0, \\ -(v(0)-v(4)) - (v(1)-v(4)) = 0, \\ v(0) = 0, \\ v(1) = 1. \end{cases}$$

and it has multiple solutions

$$v = (v(0), v(1), v(2), v(3), v(4)) = \left(0, 1, c, 1-c, \frac{1}{2}\right), c \in \mathbb{C}.$$

Then

$$\mathcal{P}(-1) = (-1)c + (-1)(1-c) + (-1)\frac{1}{2} = -\frac{3}{2}.$$

Therefore, we have

$$\mathcal{P}(\lambda) = \begin{cases} \infty, & \lambda = \pm i, \\ -\dfrac{3}{2}, & \lambda = -1, \\ \dfrac{\lambda^2+\lambda+1}{\lambda^2+1}, & \text{otherwise } (\lambda \neq 0), \end{cases}$$

and the domain of continuity of the effective admittance is $\mathbb{C} \setminus \{0, -1\}$, since

$$\left(\frac{\lambda^2+\lambda+1}{\lambda^2+1}\right)\bigg|_{\lambda=-1} = \frac{1}{2} \neq -\frac{3}{2}.$$

The domain of holomorphicity is $\mathbb{C} \setminus \{0, -1, \pm i\} = \Lambda_0$.

Example 2.3.3 (Non-continuity of the admittance in the case of complex-weighted graph, which is not a network)**.** Let us consider graph with complex-weights of the edges as at Figure 2.20. Note that here the weight of the edge $2 \sim 3$ is *not* in the form (2.1.3) (and can not be rewritten in this form), so it is not a network.

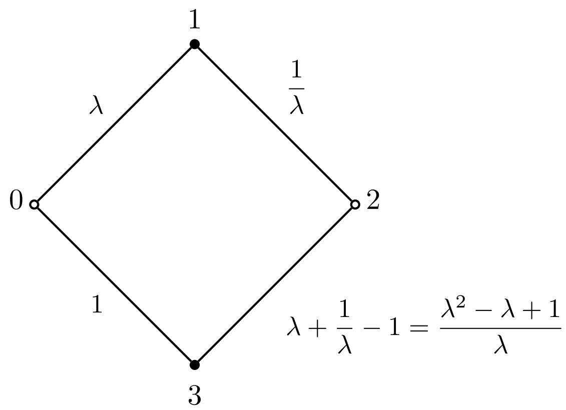


Figure 2.20: Complex-weighted graph C_4, which is not a network

Let us consider the following Dirichlet problem:

$$\begin{cases} (v(0)-v(1))\lambda + \dfrac{v(2)-v(1)}{\lambda} = 0, \\ (v(0)-v(3)) + (v(2)-v(3))(\lambda + \dfrac{1}{\lambda} - 1) = 0, \\ v(0) = 0, \\ v(2) = 1. \end{cases}$$

The determinant of this system is

$$\mathcal{D} = \mathcal{D}(\lambda) = \frac{1}{\lambda^2}(\lambda^2+1)^2$$

and its zeros are i and $-i$.

In case $\mathcal{D}(\lambda) \neq 0$ the solution of the Dirichlet problem is

$$v(1) = \frac{1}{\lambda^2+1}, v(3) = \frac{\lambda^2-\lambda+1}{\lambda^2+1}.$$

and it has no finite limit as $\lambda \to i$.

Let us calculate an analogue of the effective admittance:

$$\mathcal{P}(\lambda) = \lambda v(1) + v(3) = 1, \lambda \neq \pm i, 0.$$

The Dirichlet problem in the cases $\lambda = \pm i$ is

$$\begin{cases} (v(0)-v(1))i - (v(2)-v(1))i = 0, \\ (v(0)-v(3)) - (v(2)-v(3)) = 0, \\ v(0) = 0, \\ v(2) = 1. \end{cases}$$

and, obviously, has no solutions.

Therefore,

$$\mathcal{P}(\lambda) = \begin{cases} \lambda v(1) + v(3) = 1, \lambda \neq \pm i, 0 \\ \infty, \lambda = \pm i, \end{cases}$$

is not a continuous function of $\lambda = i\omega$, $\omega > 0$ at the point $\omega = 1$.

2.3.2 Continuity of the effective admittance for the network on C_4 (cycle graph with four vertices)

Proposition 2.3.4. *For any network Γ, whose underlying graph is C_4, the effective admittance $\mathcal{P}(\lambda)$, $\lambda \in \Lambda$ is a continuous function of $\lambda = i\omega$, where $\omega > 0$.*

Proof. Consider a network whose underlying graph is C_4. More precisely, $V = \{0,1,2,3\}$, $a_0 = 2$, $B = \{0\}$ and

$$0 \sim 1 \sim 2 \sim 3 \sim 0.$$

Let the admittances of the edges be

$$\rho_{01}^{(\lambda)} = \alpha(\lambda) = \alpha, \rho_{12}^{(\lambda)} = \beta(\lambda) = \beta, \rho_{23}^{(\lambda)} = \nu(\lambda) = \nu, \rho_{30}^{(\lambda)} = \gamma(\lambda) = \gamma.$$

That is, all the functions $\alpha(\lambda), \beta(\lambda), \nu(\lambda), \gamma(\lambda)$ are in the form (2.1.3), i.e.

$$\frac{\lambda}{L\lambda^2 + R\lambda + D}, L, R, D \geq 0. \tag{2.3.2}$$

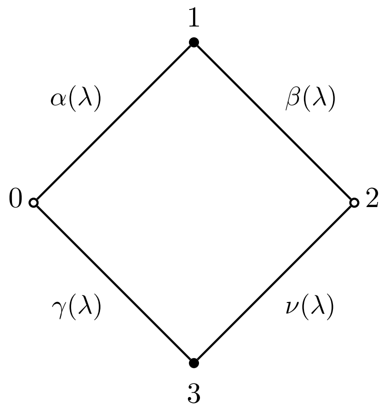


Figure 2.21: A network on C_4

Then the Dirichlet problem for a given network is the following:

$$\begin{cases} v(0)\alpha + v(2)\beta - v(1)(\alpha+\beta) = 0, \\ v(0)\gamma + v(2)\nu - v(3)(\gamma+\nu) = 0, \\ v(0) = 0, \\ v(2) = 1. \end{cases} \tag{2.3.3}$$

Let us denote $v_1 = v(1)$ and $v_3 = v(3)$. The solution of the Dirichlet problem (2.3.3) is

$$v_1 = \frac{\beta}{\alpha+\beta} \text{ and } v_3 = \frac{\nu}{\gamma+\nu}.$$

Therefore,

$$\mathcal{P}(\lambda) = \alpha v_1 + \gamma v_3 = \frac{\alpha\beta}{\alpha+\beta} + \frac{\gamma\nu}{\gamma+\nu}. \tag{2.3.4}$$

Let us prove that $\mathcal{P}(\lambda)$ is continuous at any $\lambda_0 = i\omega_0$. Without loss of generality take $\lambda_0 = i$ (otherwise, change L, R, D). If $\alpha(i) + \beta(i) \neq 0$ and $\gamma(i) + \nu(i) \neq 0$ then all is trivial. Assume that

$$(\alpha+\beta)(i) = 0. \tag{2.3.5}$$

Then $-i$ is also a root of $\alpha+\beta$, and we obtain,

$$(\alpha+\beta)(\lambda) = (\lambda^2+1)Q_1(\lambda), \tag{2.3.6}$$

where $Q_1(\lambda)$ is a rational function with real coefficients such that $Q_1(i) \in \mathbb{C}$.

(Note that in fact $Q_1(i) \neq 0$ because otherwise we have

$$(\alpha+\beta)(\lambda) = (\lambda^2+1)^2 Q(\lambda),$$

which is not possible for functions α and β of type (2.3.2), since the degree of the nominator of $\alpha+\beta$ is at most 3.)

Under the assumption (2.3.5) we have

$$v_1(\lambda) \to \infty \text{ as } \lambda \to i. \tag{2.3.7}$$

Our purpose is to show that

$$\mathcal{P}(\lambda) \to \infty \text{ as } \lambda \to i, \tag{2.3.8}$$

since $\mathcal{P}(i) = \infty$ because at $\lambda = i$ the first equation of (2.3.3) in the view of (2.3.5) has no solutions.

Let us now concentrate on (2.3.8). We have by conservation of complex power (2.2.15) that

$$\mathcal{P} = \alpha|v_1|^2 + \beta|1 - v_1|^2 + \nu|1 - v_3|^2 + \gamma|v_3|^2 \text{ for } \lambda \in \Lambda_1. \tag{2.3.9}$$

Hence, if $\operatorname{Re}\alpha(i) > 0$ we have

$$\operatorname{Re}\mathcal{P}(\lambda) \geq \operatorname{Re}\alpha(\lambda)|v_1|^2 \to \infty \text{ as } \lambda \to \infty,$$

and the same holds if $\operatorname{Re}\beta(i) > 0$. Hence, we assume in what follows

$$\operatorname{Re}\alpha(\lambda) = \operatorname{Re}\beta(\lambda) = 0.$$

(by (2.2.5) this is true not only for $\lambda = i$ but for any $\lambda = i\omega$, $\omega > 0$).

It follows from (2.3.4) and (2.3.7) that if $|v_3|$ remains bounded as $\lambda \to i$, then (2.3.8) holds. Hence, we may assume in what follows that also

$$v_3(\lambda) \to \infty \text{ as } \lambda \to i,$$

that is $(\gamma + \nu)(i) = 0$ and, hence

$$(\gamma + \nu)(\lambda) = (\lambda^2 + 1)Q_2(\lambda), \tag{2.3.10}$$

where $Q_2(\lambda)$ is a rational function with real coefficients such that $Q_2(i) \in \mathbb{C}$.

(Similarly to $Q_1(i)$, we have $Q_2(i) \neq 0$.)

Using (2.3.9) we obtain as above that

$$\operatorname{Re}\gamma(\lambda) = \operatorname{Re}\nu(\lambda) = 0$$

for any $\lambda = i\omega$, $\omega > 0$, since otherwise (2.3.8) is trivially satisfied.

Using (2.3.6) and (2.3.10), we obtain

$$\mathcal{P}(\lambda) = \left(\frac{\alpha\beta}{\alpha + \beta} + \frac{\gamma\nu}{\gamma + \nu}\right)(\lambda) = \frac{1}{\lambda^2 + 1}\left(\frac{\alpha\beta Q_2 + \gamma\nu Q_1}{Q_1 Q_2}\right)(\lambda).$$

In order to prove that $\mathcal{P}(\lambda) \to \infty$ as $\lambda \to \infty$, it suffices to prove that

$$(\alpha\beta Q_2 + \gamma\nu Q_1)(i) \neq 0.$$

We have from (2.3.6)

$$\beta(\lambda) = (\lambda^2 + 1)Q_1(\lambda) - \alpha(\lambda)$$

and from (2.3.10)

$$\nu(\lambda) = (\lambda^2 + 1)Q_2(\lambda) - \gamma(\lambda).$$

It follows that

$$\begin{aligned}\alpha\beta Q_2 + \gamma\nu Q_1 =& \alpha Q_2((\lambda^2 + 1)Q_1 - \alpha) + \gamma Q_1((\lambda^2 + 1)Q_2 - \gamma)\\ =&(\lambda_2 + 1)Q_1 Q_2(\alpha + \gamma) - (\alpha^2 Q_2 + \gamma^2 Q_1),\end{aligned}$$

where we omit λ as a parameter of the functions for the brievity.

Hence, it remains to verify that

$$(\alpha^2 Q_2 + \gamma^2 Q_1)(i) \neq 0.$$

Now we use an explicit form of admittances of edges. Let

$$\alpha(\lambda) = \frac{\lambda}{L_1\lambda^2 + D_1} \text{ and } \beta(\lambda) = \frac{\lambda}{L_2\lambda^2 + D_2},$$

where L_1, D_1, L_2, D_2 are non-negative reals. We have

$$(\alpha + \beta)(\lambda) = \frac{((L_1 + L_2)\lambda^2 + D_1 + D_2)\lambda}{(L_1\lambda^2 + D_1)(L_2\lambda^2 + D_2)}$$

so that (2.3.6) is only possible if

$$L_1 + L_2 = D_1 + D_2$$

and, hence,

$$(\alpha + \beta)(\lambda) = \left(\lambda^2 + 1\right) \frac{(L_1 + L_2)\lambda}{(L_1\lambda^2 + D_1)(L_2\lambda^2 + D_2)}$$

so that

$$Q_1(\lambda) = \frac{(L_1 + L_2)\lambda}{(L_1\lambda^2 + D_1)(L_2\lambda^2 + D_2)}.$$

In the same way setting

$$\gamma(\lambda) = \frac{\lambda}{L_3\lambda^2 + D_3} \text{ and } \nu(\lambda) = \frac{\lambda}{L_4\lambda^2 + D_4},$$

where L_3, D_3, L_4, D_4 are non-negative reals, we obtain from (2.3.10) that

$$L_3 + L_4 = D_3 + D_4$$

and

$$Q_2(\lambda) = \frac{(L_3 + L_4)\lambda}{(L_3\lambda^2 + D_3)(L_4\lambda^2 + D_4)}.$$

Therefore,

$$\begin{aligned}
&\left(\alpha^2 Q_2+\gamma^2 Q_1\right)(i)\\
&\quad=\left(\frac{\lambda^2}{(L_1\lambda^2+D_1)^2}\frac{(L_3+L_4)\lambda}{(L_3\lambda^2+D_3)(L_4\lambda^2+D_4)}+\frac{\lambda^2}{(L_3\lambda^2+D_3)^2}\frac{(L_1+L_2)\lambda}{(L_1\lambda^2+D_1)(L_2\lambda^2+D_2)}\right)\Bigg|_{\lambda=i}\\
&\quad=\left(\frac{-1}{(-L_1+D_1)^2}\frac{(L_3+L_4)i}{(-L_3+D_3)(-L_4+D_4)}+\frac{-1}{(-L_3+D_3)^2}\frac{(L_1+L_2)i}{(-L_1+D_1)(-L_2+D_2)}\right).
\end{aligned}$$

Since

$$D_1-L_1=L_2-D_2 \text{ and } D_3-L_3=L_4-D_4$$

it follows that

$$\left(\alpha^2 Q_2+\gamma^2 Q_1\right)(i)=\frac{i(L_1+L_2+L_3+L_4)}{(D_1-L_1)^2(D_3-L_3)^2}.$$

We see that this expression cannot vanish since otherwise we must have

$$L_1=L_2=L_3=L_4=0$$

and, hence, also

$$D_1=D_2=D_3=D_4=0$$

which is impossible.

Hence, all the above proves that

$$\mathcal{P}(\lambda)\to\infty \text{ as } \lambda\to i,$$

which proves the continuity of $\mathcal{P}(\lambda)$ of $\lambda=i\omega$, $\omega>0$. □

Remark 2.3.5. Example 2.3.3 shows that the representation of admittances (2.1.3) is crucial for the proof.

2.4 Estimates of the effective admittance for finite networks

In this section we present estimates of the effective admittance for a given finite network in terms of λ in different domains of a complex plane λ (see Corollaries 2.4.4 and 2.4.11, Theorems 2.4.7 and 2.4.10).

Let $\Gamma=(V,\rho,a_0,B)$ be a finite network.

2.4.1 An upper bound of the admittance using $\operatorname{Re}\lambda$

Theorem 2.4.1. *Let $\lambda\in\Lambda$ be fixed and assume that, for some $\epsilon>0$,*

$$\inf_{xy\in E}\frac{\operatorname{Re}\rho_{xy}}{|\rho_{xy}|}\geq\epsilon. \tag{2.4.1}$$

Then

$$|\mathcal{P}| \leq \frac{1}{\epsilon^2} \sum_{x \sim a_0} |\rho_{xa_0}| \,. \tag{2.4.2}$$

The same result is true if one assumes instead of (2.4.1) that

$$\inf_{xy \in E} \frac{\operatorname{Im} \rho_{xy}}{|\rho_{xy}|} \geq \epsilon$$

or

$$\inf_{xy \in E} \frac{-\operatorname{Im} \rho_{xy}}{|\rho_{xy}|} \geq \epsilon.$$

Proof. Under the hypothesis (2.4.1) the Dirichlet problem (2.2.1) has a unique solution v by Theorem 2.2.11. We have by a conservation of a complex power (2.2.15)

$$|\mathcal{P}| \geq \operatorname{Re} \mathcal{P} = \sum_{xy \in E} |\nabla_{xy} v|^2 \operatorname{Re} \rho_{xy} \geq \epsilon \sum_{xy \in E} |\nabla_{xy} v|^2 |\rho_{xy}| \,. \tag{2.4.3}$$

Applying (2.2.14) with the function

$$u = \mathbf{1}_{\{a_0\}}$$

and using the inequality

$$2|z_1 z_2| \leq \epsilon |z_1|^2 + \frac{1}{\epsilon} |z_2|^2 \text{ for any } z_1, z_2 \in \mathbb{C}$$

we obtain

$$|\mathcal{P}| \leq \sum_{xy \in E} |\nabla_{xy} v| \, |\nabla_{xy} u| \, |\rho_{xy}| \leq \frac{\epsilon}{2} \sum_{xy \in E} |\nabla_{xy} v|^2 |\rho_{xy}| + \frac{1}{2\epsilon} \sum_{xy \in E} |\nabla_{xy} u|^2 |\rho_{xy}| \,.$$

Setting

$$U := \sum_{xy \in E} |\nabla_{xy} u|^2 |\rho_{xy}| = \sum_{x \sim a_0} |\rho_{xa_0}| \,,$$

we obtain

$$|\mathcal{P}| \leq \frac{\epsilon}{2} \sum_{xy \in E} |\nabla_{xy} v|^2 |\rho_{xy}| + \frac{1}{2\epsilon} U.$$

Combing this with (2.4.3) yields

$$\frac{\epsilon}{2} \sum_{xy \in E} |\nabla_{xy} v|^2 |\rho_{xy}| \leq \frac{1}{2\epsilon} U$$

whence by (2.2.15)

$$|\mathcal{P}| = \left| \sum_{xy \in E} |\nabla_{xy} v|^2 \rho_{xy} \right| \leq \sum_{xy \in E} |\nabla_{xy} v|^2 |\rho_{xy}| \leq \frac{1}{\epsilon^2} U.$$

The conditions with $\operatorname{Im} \rho_{xy}$ are handled in the same way. $\square$

In order to be able to verify (2.4.1), we need the following lemma.

Lemma 2.4.2. *Let L, R, D be non-negative real numbers and $\lambda \in \mathbb{C} \setminus \{0\}$. If*

$$z := R + L\lambda + \frac{D}{\lambda} \neq 0$$

then

$$\frac{\operatorname{Re} z}{|z|} \geq \frac{\operatorname{Re} \lambda}{|\lambda|}$$

and, for $\rho = \dfrac{1}{z}$,

$$\frac{\operatorname{Re} \rho}{|\rho|} \geq \frac{\operatorname{Re} \lambda}{|\lambda|}.$$

Proof. We have

$$\operatorname{Re} z = R + L \operatorname{Re} \lambda + \frac{D \operatorname{Re} \lambda}{|\lambda|^2} \geq \left(R + L|\lambda| + \frac{D}{|\lambda|} \right) \frac{\operatorname{Re} \lambda}{|\lambda|}$$

and

$$|z| \leq R + L|\lambda| + \frac{D}{|\lambda|},$$

whence

$$\frac{\operatorname{Re} z}{|z|} \geq \frac{\operatorname{Re} \lambda}{|\lambda|}.$$

Finally, we have

$$\rho = \frac{1}{z} = \frac{\operatorname{Re} z - i \operatorname{Im} z}{|z|^2}$$

and, hence,

$$\frac{\operatorname{Re} \rho}{|\rho|} = \frac{\operatorname{Re} z}{|z|^2} |z| = \frac{\operatorname{Re} z}{|z|} \geq \frac{\operatorname{Re} \lambda}{|\lambda|}.$$

□

Corollary 2.4.3. *If $\operatorname{Re} \lambda > 0$ then*

$$|\mathcal{P}(\lambda)| \leq \frac{|\lambda|^2}{(\operatorname{Re} \lambda)^2} \sum_{x \sim a_0} \left| \rho_{x a_0}^{(\lambda)} \right|. \tag{2.4.4}$$

Proof. Indeed, we have for all $xy \in E$

$$\frac{\operatorname{Re} \rho_{xy}^{(\lambda)}}{|\rho_{xy}^{(\lambda)}|} \geq \frac{\operatorname{Re} \lambda}{|\lambda|} =: \epsilon.$$

Substituting into (2.4.2) we obtain (2.4.4). □

Corollary 2.4.4. *If* $\operatorname{Re}\lambda > 0$ *then*

$$|\mathcal{P}(\lambda)| \le C_0 \frac{|\lambda|^2 \left(1+|\lambda|^2\right)}{(\operatorname{Re}\lambda)^3}, \tag{2.4.5}$$

where

$$C_0 = \sum_{x\sim a_0} \frac{1}{R_{xa_0} + L_{xa_0} + D_{xa_0}}.$$

Proof. We have for $z = R + L\lambda + \dfrac{D}{\lambda}$

$$\begin{aligned}
|z| \ge \operatorname{Re} z \ge & R + L\operatorname{Re}\lambda + \frac{D\operatorname{Re}\lambda}{|\lambda|^2} \\
\ge & (R+L+D)\min\left(1, \operatorname{Re}\lambda, \frac{\operatorname{Re}\lambda}{|\lambda|^2}\right) \\
\ge & (R+L+D)\min\left(\operatorname{Re}\lambda, \frac{\operatorname{Re}\lambda}{|\lambda|^2}\right) \\
= & (R+L+D)(\operatorname{Re}\lambda)\min\left(1, |\lambda|^{-2}\right) \\
= & (R+L+D)\frac{\operatorname{Re}\lambda}{\max\left(1,|\lambda|^2\right)} \\
\ge & (R+L+D)\frac{\operatorname{Re}\lambda}{1+|\lambda|^2},
\end{aligned}$$

since $\dfrac{(\operatorname{Re}\lambda)^2}{|\lambda|^2} \le 1$ and, therefore, either $\operatorname{Re}\lambda \le 1$ or $\dfrac{\operatorname{Re}\lambda}{|\lambda|^2} \le 1$. It follows that, for $\rho = \dfrac{1}{z}$,

$$|\rho| \le \frac{1}{R+L+D}\frac{1+|\lambda|^2}{\operatorname{Re}\lambda}.$$

Hence,

$$\sum_{x\sim a_0} \left|\rho^{(\lambda)}_{xa_0}\right| \le \frac{1+|\lambda|^2}{\operatorname{Re}\lambda} \sum_{x\sim a_0} \frac{1}{R_{xa_0} + L_{xa_0} + D_{xa_0}}.$$

Substituting into (2.4.4) we obtain (2.4.5). □

Remark 2.4.5. In the case $L > 0$ we can use in the domain $\{\operatorname{Re}\lambda > 0\}$ the estimate $|z| \ge L\operatorname{Re}\lambda$ which implies

$$|\rho| \le \frac{1}{L\operatorname{Re}\lambda}.$$

Hence, if $L_{xa_0} > 0$ for all $x \sim a_0$ then

$$\sum_{x\sim a_0} \left|\rho^{(\lambda)}_{xa_0}\right| \le C' \frac{1}{\operatorname{Re}\lambda},$$

where

$$C' = \sum_{x\sim a_0} \frac{1}{L_{xa_0}}.$$

Therefore, by (2.4.4) in this case, in the domain $\{\operatorname{Re}\lambda > 0\}$ we have

$$|\mathcal{P}(\lambda)| \le \frac{C'|\lambda|^2}{(\operatorname{Re}\lambda)^3}.$$

2.4.2 An upper bound of the admittance using large $\operatorname{Im}\lambda$

Lemma 2.4.6. *Let R, L, D be non-negative numbers. Let $L > 0$ and $\lambda \in \mathbb{C}$ be such that*

$$\operatorname{Im}\lambda > 0, \quad |\lambda|^2 > \frac{D}{L}.$$

Then

$$z := R + L\lambda + \frac{D}{\lambda} \neq 0$$

and for $\rho = \frac{1}{z}$ we have

$$-\frac{\operatorname{Im}\rho}{|\rho|} \ge \frac{1 - \dfrac{D}{L|\lambda|^2}}{|\lambda| + \dfrac{D}{L|\lambda|} + \dfrac{R}{L}} \operatorname{Im}\lambda \tag{2.4.6}$$

and

$$|\rho| \le \frac{1}{\left(L - \dfrac{D}{|\lambda|^2}\right)\operatorname{Im}\lambda}.$$

Proof. We have

$$\operatorname{Im} z = L\operatorname{Im}\lambda - \frac{D\operatorname{Im}\lambda}{|\lambda|^2} = \left(L - \frac{D}{|\lambda|^2}\right)\operatorname{Im}\lambda > 0.$$

In particular, $z \neq 0$. We have also

$$|z| \le R + L|\lambda| + \frac{D}{|\lambda|}.$$

It follows that

$$-\frac{\operatorname{Im}\rho}{|\rho|} = \frac{\operatorname{Im} z}{|z|^2}|z| = \frac{\operatorname{Im} z}{|z|} \ge \frac{\left(L - \dfrac{D}{|\lambda|^2}\right)\operatorname{Im}\lambda}{R + L|\lambda| + \dfrac{D}{|\lambda|}} \ge \frac{1 - \dfrac{D}{L|\lambda|^2}}{|\lambda| + \dfrac{D}{L|\lambda|} + \dfrac{R}{L}} \operatorname{Im}\lambda$$

which proves (2.4.6). Finally, we have

$$|\rho| = \frac{1}{|z|} \le \frac{1}{\operatorname{Im} z} = \frac{1}{\left(L - \dfrac{D}{|\lambda|^2}\right)\operatorname{Im}\lambda}.$$

□

Theorem 2.4.7. *Assume that $L_{xy} > 0$ for all $xy \in E$. Set*

$$S_D = \sup_{xy \in E} \frac{D_{xy}}{L_{xy}}, \quad S_R = \sup_{xy \in E} \frac{R_{xy}}{L_{xy}},$$

and

$$C' = \sum_{x \sim a_0} \frac{1}{L_{xa_0}}.$$

Then, in the domain,

$$\Omega = \{\lambda \in \mathbb{C} \mid \operatorname{Im} \lambda \neq 0 \quad \textit{and} \quad |\lambda|^2 > S_D\}$$

the function $\mathcal{P}(\lambda)$ is holomorphic and

$$|\mathcal{P}(\lambda)| \leq \frac{C'(2|\lambda| + S_R)^2 |\lambda|^6}{(|\lambda|^2 - S_D)^3 |\operatorname{Im} \lambda|^3}.$$

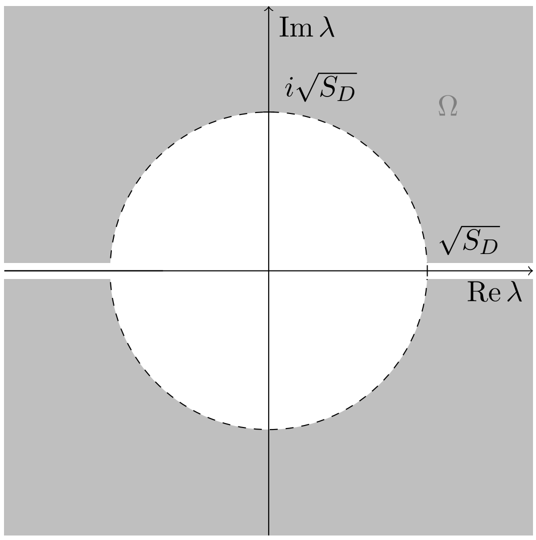


Figure 2.22: The domain $\Omega = \{\lambda \in \mathbb{C} \mid \operatorname{Im} \lambda \neq 0 \text{ and } |\lambda|^2 > S_D\}$.

Proof. By the symmetry $\lambda \to \overline{\lambda}$ (see Lemma 2.2.4 (a)), it suffices to prove the both claims in the domain

$$\Omega_+ = \{\lambda \in \mathbb{C} \mid \operatorname{Im} \lambda > 0 \text{ and } |\lambda|^2 > S_D\}.$$

For $\lambda \in \Omega_+$ we have by Lemma 2.4.6 that $z_{xy} \neq 0$, whence $\lambda \in \Lambda$. By Lemma 2.4.6 we have for all $xy \in E$ and $\lambda \in \Omega_+$

$$-\frac{\operatorname{Im} \rho_{xy}}{|\rho_{xy}|} \geq \frac{1 - \dfrac{D_{xy}}{L_{xy}|\lambda|^2}}{|\lambda| + \dfrac{D_{xy}}{L_{xy}|\lambda|} + \dfrac{R_{xy}}{L_{xy}}} \operatorname{Im} \lambda \geq \frac{1 - \dfrac{S_D}{|\lambda|^2}}{|\lambda| + \dfrac{S_D}{|\lambda|} + S_R} \operatorname{Im} \lambda \geq \frac{1 - \dfrac{S_D}{|\lambda|^2}}{2|\lambda| + S_R} \operatorname{Im} \lambda > 0, \tag{2.4.7}$$

since $\dfrac{S_D}{|\lambda|} < |\lambda|$. By Theorem 2.2.11 we conclude that $\mathcal{P}(\lambda)$ is a holomorphic function in Ω_+.

Using (2.4.7), we obtain by Theorem 2.4.1 that for $\lambda \in \Omega_+$

$$|\mathcal{P}(\lambda)| \le \left(\frac{2|\lambda| + S_R}{\left(1 - \frac{S_D}{|\lambda|^2}\right) \operatorname{Im}\lambda} \right)^2 \sum_{x \sim a_0} \left| \rho_{xa_0}^{(\lambda)} \right|. \tag{2.4.8}$$

Next, we have by Lemma 2.4.6

$$|\rho_{xy}| \le \frac{1}{\left(L_{xy} - \frac{D_{xy}}{|\lambda|^2}\right) \operatorname{Im}\lambda} \le \frac{1}{L_{xy}\left(1 - \frac{S_D}{|\lambda|^2}\right) \operatorname{Im}\lambda},$$

whence

$$\sum_{x \sim a_0} |\rho_{xa_0}| \le \frac{\sum_{x \sim a_0} (L_{xa_0})^{-1}}{\left(1 - \frac{S_D}{|\lambda|^2}\right) \operatorname{Im}\lambda} = \frac{C'}{\left(1 - \frac{S_D}{|\lambda|^2}\right) \operatorname{Im}\lambda}.$$

It follows from (2.4.8) that

$$|\mathcal{P}(\lambda)| \le \left(\frac{2|\lambda| + S_R}{\left(1 - \frac{S_D}{|\lambda|^2}\right) \operatorname{Im}\lambda} \right)^2 \frac{C'}{\left(1 - \frac{S_D}{|\lambda|^2}\right) \operatorname{Im}\lambda} = \frac{C'(2|\lambda| + S_R)^2}{\left(1 - \frac{S_D}{|\lambda|^2}\right)^3 (\operatorname{Im}\lambda)^3}$$

which was to be proved. □

Corollary 2.4.8. *Under the hypothesis of Theorem 2.4.7, assume in addition that $R_{xy} = 0$ for all $xy \in E$. Then $\mathcal{P}(\lambda)$ is holomorphic in $\mathbb{C} \setminus J$ where*

$$J = \left[-i\sqrt{S_D}, i\sqrt{S_D} \right].$$

Proof. In this case we have the symmetry $\mathcal{P}(-\lambda) = -\mathcal{P}(\lambda)$ (see Lemma 2.2.4 (b)). By Theorem 2.2.11 (see Remark 2.2.12) and Theorem 2.4.7 $\mathcal{P}(\lambda)$ is holomorphic in the union

$$\{\operatorname{Re}\lambda \ne 0\} \cup \{(\operatorname{Im}\lambda)^2 > S_D\},$$

that coincides with $\mathbb{C} \setminus J$. □

2.4.3 An upper bound of the admittance using small $\operatorname{Im}\lambda$

Lemma 2.4.9. *Let R, L, D be non-negative numbers. Let $L > 0$ and $\lambda \in \mathbb{C}$ be such that*

$$\operatorname{Im}\lambda > 0 \quad \text{and} \quad |\lambda|^2 < \frac{D}{L}.$$

Then

$$z := R + L\lambda + \frac{D}{\lambda} \neq 0$$

and, for $\rho = \dfrac{1}{z}$, *we have*

$$\frac{\operatorname{Im}\rho}{|\rho|} \geq \frac{\dfrac{D}{L|\lambda|^2} - 1}{|\lambda| + \dfrac{D}{L|\lambda|} + \dfrac{R}{L}} \operatorname{Im}\lambda \tag{2.4.9}$$

and

$$|\rho| \leq \frac{1}{\left(\dfrac{D}{|\lambda|^2} - L\right) \operatorname{Im}\lambda}.$$

Proof. We have

$$-\operatorname{Im} z = -L \operatorname{Im}\lambda + \frac{D \operatorname{Im}\lambda}{|\lambda|^2} = \left(\frac{D}{|\lambda|^2} - L\right) \operatorname{Im}\lambda > 0.$$

In particular, $z \neq 0$. We have also

$$|z| \leq R + L|\lambda| + \frac{D}{|\lambda|}.$$

It follows that

$$\frac{\operatorname{Im}\rho}{|\rho|} = -\frac{\operatorname{Im} z}{|z|} \geq \frac{\left(\dfrac{D}{|\lambda|^2} - L\right) \operatorname{Im}\lambda}{R + L|\lambda| + \dfrac{D}{|\lambda|}} \geq \frac{\dfrac{D}{L|\lambda|^2} - 1}{|\lambda| + \dfrac{D}{L|\lambda|} + \dfrac{R}{L}} \operatorname{Im}\lambda$$

which proves (2.4.9). Finally, we have

$$|\rho| = \frac{1}{|z|} \leq \frac{1}{-\operatorname{Im} z} = \frac{1}{\left(\dfrac{D}{|\lambda|^2} - L\right) \operatorname{Im}\lambda}.$$

□

Theorem 2.4.10. *Assume that* $L_{xy} > 0$ *for all* $xy \in E$. *Set*

$$S_D = \sup_{xy \in E} \frac{D_{xy}}{L_{xy}}, \quad S_D^* = \inf_{xy \in E} \frac{D_{xy}}{L_{xy}}, \quad S_R = \sup_{xy \in E} \frac{R_{xy}}{L_{xy}},$$

and

$$C' = \sum_{x \sim a_0} \frac{1}{L_{x a_0}}.$$

Then in the domain

$$\Omega^* = \{\lambda \in \mathbb{C} \mid \operatorname{Im}\lambda \neq 0 \quad and \quad |\lambda|^2 < S_D^*\}$$

the function $\mathcal{P}(\lambda)$ *is holomorphic and*

$$|\mathcal{P}(\lambda)| \le \frac{C'(|\lambda|^2 + S_R|\lambda| + S_D)^2|\lambda|^4}{(S_D^* - |\lambda|^2)^3|\operatorname{Im}\lambda|^3}. \tag{2.4.10}$$

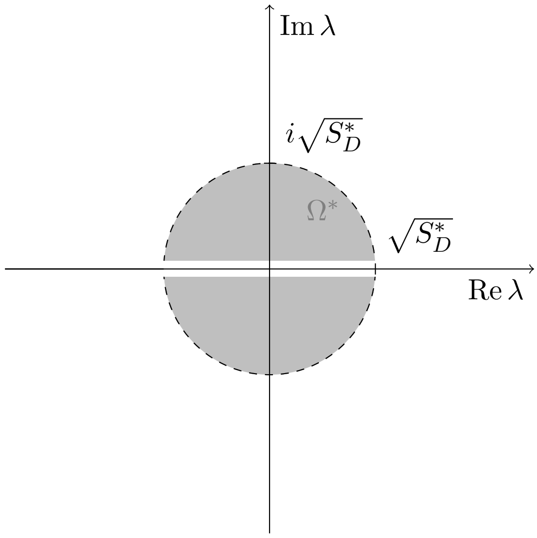


Figure 2.23: The domain $\Omega^* = \{\lambda \in \mathbb{C} \mid \operatorname{Im}\lambda \neq 0 \text{ and } |\lambda|^2 < S_D^*\}$

Proof. If $S_D^* = 0$ then $\Omega^* = \emptyset$ and there is nothing to prove. Let $S_D^* > 0$. By the symmetry $\lambda \to \overline{\lambda}$, it suffices to prove the both claims in the domain

$$\Omega_+^* = \{\lambda \in \mathbb{C} \mid \operatorname{Im}\lambda > 0 \text{ and } |\lambda|^2 < S_D^*\}.$$

For $\lambda \in \Omega_+^*$ we have by Lemma 2.4.9 that $z_{xy} \neq 0$, whence $\lambda \in \Lambda$. By Lemma 2.4.9 we have for all $xy \in E$ and $\lambda \in \Omega_+^*$

$$\frac{\operatorname{Im}\rho_{xy}}{|\rho_{xy}|} \ge \frac{\dfrac{D_{xy}}{L_{xy}|\lambda|^2} - 1}{|\lambda| + \dfrac{D_{xy}}{L_{xy}|\lambda|} + \dfrac{R_{xy}}{L_{xy}}} \operatorname{Im}\lambda \ge \frac{\dfrac{S_D^*}{|\lambda|^2} - 1}{|\lambda| + \dfrac{S_D}{|\lambda|} + S_R} \operatorname{Im}\lambda > 0, \tag{2.4.11}$$

By Theorem 2.2.11 we conclude that $\mathcal{P}(\lambda)$ is a holomorphic function in Ω_+^*.

Using (2.4.11), we obtain by Theorem 2.4.1 that for all $\lambda \in \Omega_+^*$

$$|\mathcal{P}(\lambda)| \le \left(\frac{|\lambda| + \dfrac{S_D}{|\lambda|} + S_R}{\left(\dfrac{S_D^*}{|\lambda|^2} - 1\right)\operatorname{Im}\lambda} \right)^2 \sum_{x \sim a_0} \left|\rho_{xa_0}^{(\lambda)}\right|.$$

Next, we have by Lemma 2.4.9

$$|\rho_{xy}| \le \frac{1}{\left(\dfrac{D_{xy}}{|\lambda|^2} - L_{xy}\right)\operatorname{Im}\lambda} = \frac{1}{L_{xy}\left(\dfrac{D_{xy}}{L_{xy}|\lambda|^2} - 1\right)\operatorname{Im}\lambda} \le \frac{1}{L_{xy}\left(\dfrac{S_D^*}{|\lambda|^2} - 1\right)\operatorname{Im}\lambda},$$

whence

$$\sum_{x\sim a_0} |\rho_{xa_0}| \leq \frac{\sum_{x\sim a_0}(L_{xa_0})^{-1}}{\left(\frac{S_D^*}{|\lambda|^2}-1\right)\operatorname{Im}\lambda} = \frac{C'}{\left(\frac{S_D^*}{|\lambda|^2}-1\right)\operatorname{Im}\lambda}.$$

It follows that

$$|\mathcal{P}(\lambda)| \leq \frac{C'\left(|\lambda| + \frac{S_D}{|\lambda|} + S_R\right)^2}{\left(\frac{S_D^*}{|\lambda|^2}-1\right)^3 (\operatorname{Im}\lambda)^3}$$

whence (2.4.10) follows. □

Corollary 2.4.11. *Under the hypothesis of Theorem 2.4.10, assume in addition that* $R_{xy} = 0$ *for all* $xy \in E$. *Then* $\mathcal{P}(\lambda)$ *is holomorphic in the domain* $\mathbb{C}\setminus J^*$ *where*

$$J^* = \left[i\sqrt{S_D^*}, i\sqrt{S_D}\right] \cup \left[-i\sqrt{S_D}, -i\sqrt{S_D^*}\right] \cup \{0\}.$$

Proof. In this case we have the symmetry $\mathcal{P}(-\lambda) = -\mathcal{P}(\lambda)$. By Theorem 2.2.11 (see Remark 2.2.12), Theorems 2.4.7 and 2.4.10 $\mathcal{P}(\lambda)$ is holomorphic in the union

$$\{\operatorname{Re}\lambda \neq 0\} \cup \{(\operatorname{Im}\lambda)^2 > S_D\} \cup \{0 < (\operatorname{Im}\lambda)^2 < S_D^*\}$$

that coincides with $\mathbb{C}\setminus J^*$. □

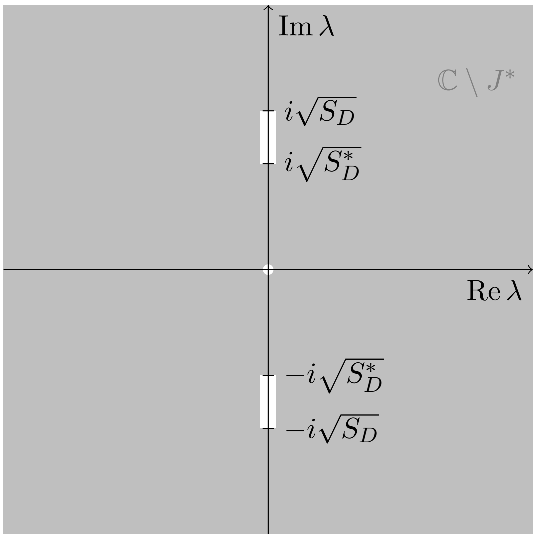


Figure 2.24: The domain $\mathbb{C}\setminus J^*$

2.5 Effective admittance of infinite networks

In the case of an infinite network with just resistors the effective resistance can be defined as the limit of monotonically increasing sequence of the effective resistances of the exhausted finite networks (see [13], [18], [32]). In Subsection 2.5.1 we consider a sequence of exhausted finite networks for an infinite network with passive elements. We present a definition of an effective admittance $\mathcal{P}(\lambda)$ of an infinite network (Definition 2.5.1) and consider the domains of the complex plane λ, where the effective admittance is well-defined (see Theorem 2.5.2, which is the main theorem of this chapter). The Corollaries 2.5.3 and 2.5.4 states the results for resistor-free network.

In Subsection 2.5.2 we consider several examples of infinite networks, including two different ladder networks.

2.5.1 Main result

Let $\Gamma = (V, \rho, a_0, B)$ be an **infinite network**.

Let dist (x, y) be the graph distance on V, that is, the minimal value of n such that there exists a path $\{x_k\}_{k=0}^{n}$ connecting x and y, that is,

$$x = x_0 \sim x_1 \sim ... \sim x_n = y.$$

Let us consider a sequence of finite graphs (V_n, E_n), $n \in \mathbb{N}$, where

$$V_n = \{x \in V \mid \operatorname{dist}(a_0, x) \leq n\}$$

and E_n consists of all the edges of E with the endpoints in V_n. We endow the finite graph (V_n, E_n) with the admittance $\rho_n = \rho|_{E_n}$.

Consider the set

$$\partial V_n = \{x \in V \mid \operatorname{dist}(a_0, x) = n\},$$

that will be regarded as the *boundary* of the graph (V_n, E_n). Note that $V_n = \partial V_n \cup V_{n-1}$. Let us set

$$B_n = (B \cap V_n) \cup \partial V_n$$

and consider the following sequence of finite networks

$$\Gamma_n = (V_n, \rho_n, a_0, B_n), \quad n \in \mathbb{N}.$$

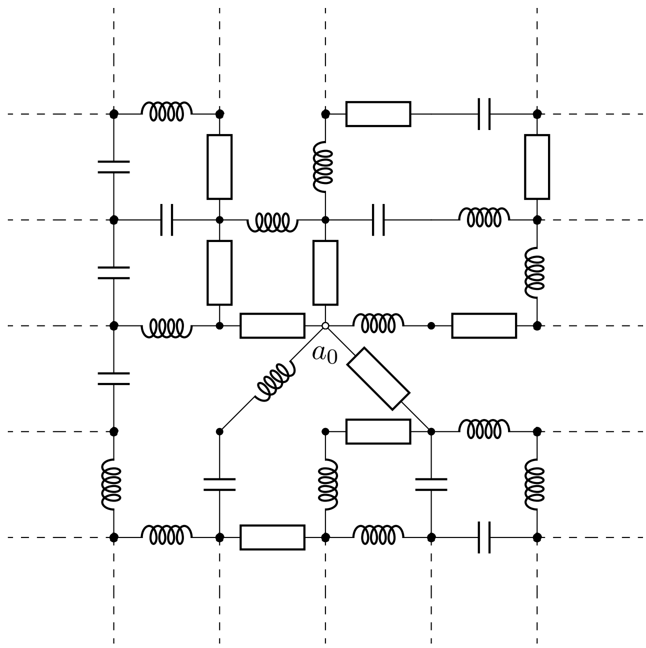


Figure 2.25: Infinite network Γ

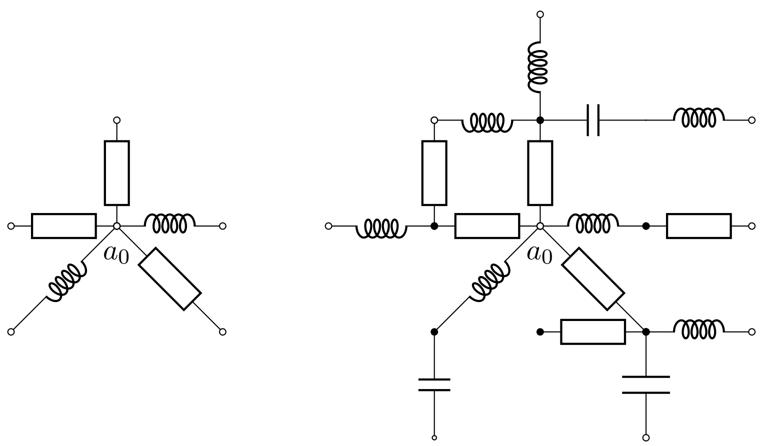


Figure 2.26: Networks Γ_1 and Γ_2

Let $\mathcal{P}_n(\lambda)$ be the effective admittance of Γ_n.

Definition 2.5.1. *Define the* effective admittance *of* Γ *as*

$$\mathcal{P}(\lambda) = \lim_{n\to\infty} \mathcal{P}_n(\lambda)$$

for those $\lambda \in \mathbb{C} \setminus \{0\}$ *where the limit exists.*

Theorem 2.5.2. (*a*) *The sequence* $\{\mathcal{P}_n(\lambda)\}$ *converges as* $n \to \infty$ *locally uniformly in the domain* $\{\operatorname{Re}\lambda > 0\}$.

(*b*) *If* $L_{xy} > 0$ *for all* $xy \in E$ *and*

$$S_D := \sup_{xy \in E} \frac{D_{xy}}{L_{xy}} < \infty \text{ and } S_R := \sup_{xy \in E} \frac{R_{xy}}{L_{xy}} < \infty \tag{2.5.1}$$

then $\{\mathcal{P}_n(\lambda)\}$ *converges as* $n \to \infty$ *locally uniformly in the domain*

$$\Omega = \{\lambda \in \mathbb{C} \mid \operatorname{Im} \lambda \neq 0 \text{ and } |\lambda|^2 > S_D\}.$$

(*c*) *If in addition to (2.5.1) also*

$$S_D^* := \inf_{xy \in E} \frac{D_{xy}}{L_{xy}} > 0$$

then $\{\mathcal{P}_n(\lambda)\}$ *converges as* $n \to \infty$ *locally uniformly in the domain*

$$\Omega^* = \{\lambda \in \mathbb{C} \mid \operatorname{Im} \lambda \neq 0 \text{ and } |\lambda|^2 < S_D^*\}.$$

In all the cases, the limit

$$\mathcal{P}(\lambda) = \lim_{n \to \infty} \mathcal{P}_n(\lambda)$$

is a holomorphic function in the domains in question.

Proof. (*a*) By Corollary 2.4.4, the sequence $\{\mathcal{P}_n(\lambda)\}$ is uniformly bounded in any domain

$$\{\operatorname{Re} \lambda \geq \epsilon, \ |\lambda| \leq c\}$$

with $0 < \epsilon < c < \infty$. Hence, the sequence $\{\mathcal{P}_n(\lambda)\}$ is precompact in such a domain and, hence, has a convergent subsequence. By a diagonal process, we obtain a convergent subsequence $\{\mathcal{P}_{n_k}(\lambda)\}$ in the entire domain $\{\operatorname{Re} \lambda > 0\}$, and the limit is a holomorphic function in this domain. On the other hand, for positive real values of λ also all $\rho_{xy}^{(\lambda)}$ are real and positive on the edges, and in this case the sequence $\{\mathcal{P}_n(\lambda)\}$ is known to be positive and decreasing (from the theory of random walks on graphs, see e.g. [18], [32]). Hence, this sequence has a limit for all positive real λ. Since every holomorphic function in $\{\operatorname{Re} \lambda > 0\}$ is uniquely determined by its values on positive reals, we obtain that $\lim \mathcal{P}_{n_k}(\lambda)$ is independent of the choice of a subsequence. Hence, the entire sequence $\{\mathcal{P}_n(\lambda)\}$ converges as $n \to \infty$ in the domain $\{\operatorname{Re} \lambda > 0\}$, and the limit is a holomorphic function in this domain.

(*b*) By Theorem 2.4.7 all functions $\{\mathcal{P}_n(\lambda)\}$ are holomorphic in the domain

$$\Omega = \{\lambda \in \mathbb{C} \mid \operatorname{Im} \lambda \neq 0 \text{ and } |\lambda|^2 > S_D\}$$

and admit the estimate

$$|\mathcal{P}_n(\lambda)| \leq \frac{C'(2|\lambda| + S_R)^2 |\lambda|^6}{(|\lambda|^2 - S_D)^3 |\operatorname{Im} \lambda|^3}.$$

Hence, the sequence $\{\mathcal{P}_n(\lambda)\}$ is locally uniformly bounded in Ω and, hence, is precompact. All the limits of convergent subsequences of $\{\mathcal{P}_n(\lambda)\}$ coincide by (a) in the domain

$$\Omega \cap \{\operatorname{Re}\lambda > 0\}$$

which implies that they coincide also in Ω. Hence, $\{\mathcal{P}_n(\lambda)\}$ converges in Ω to a holomorphic function.

(c) By Theorem 2.4.10 all functions $\{\mathcal{P}_n(\lambda)\}$ are holomorphic

$$\Omega^* = \{\lambda \in \mathbb{C} \mid \operatorname{Im}\lambda \neq 0 \text{ and } |\lambda|^2 < S_D^*\}$$

and admit the estimate

$$|\mathcal{P}_n(\lambda)| \leq \frac{C'(|\lambda|^2 + S_R|\lambda| + S_D)^2|\lambda|^4}{(S_D^* - |\lambda|^2)^3|\operatorname{Im}\lambda|^3}.$$

Hence, the sequence $\{\mathcal{P}_n(\lambda)\}$ is locally uniformly bounded in Ω^* and, hence, is precompact. All the limits of convergent subsequences of $\{\mathcal{P}_n(\lambda)\}$ coincide in the domain

$$\Omega^* \cap \{\operatorname{Re}\lambda > 0\}$$

which implies that they coincide also in Ω^*. Hence, $\{\mathcal{P}_n(\lambda)\}$ converges in Ω^* to a holomorphic function. $\square$

Corollary 2.5.3. *Assume that* $R_{xy} = 0$ *for all* $xy \in E$. *Then* $\mathcal{P}(\lambda) = \lim_{n\to\infty} \mathcal{P}_n(\lambda)$ *is well-defined and holomorphic in the domain* $\mathbb{C} \setminus J^*$, *where*

$$J^* = \left[i\sqrt{S_D^*}, i\sqrt{S_D}\right] \cup \left[-i\sqrt{S_D}, -i\sqrt{S_D^*}\right] \cup \{0\}.$$

Proof. Note that we assume neither $S_D < \infty$ nor $S_D^* > 0$. By the symmetry $\mathcal{P}(-\lambda) = -\mathcal{P}(\lambda)$ an by Theorem 2.5.2, the sequence $\{\mathcal{P}_n(\lambda)\}$ converges locally uniformly in the union

$$\{\operatorname{Re}\lambda \neq 0\} \cup \{(\operatorname{Im}\lambda)^2 > S_D\} \cup \{0 < (\operatorname{Im}\lambda)^2 < S_D^*\}$$

that coincides with $\mathbb{C} \setminus J^*$. $\square$

The next statement is a simplified version of Corollary 2.5.3.

Corollary 2.5.4. *Assume that* $R_{xy} = 0$ *for all* $xy \in E$ *and set*

$$S_D := \sup_{xy\in E} \frac{1}{C_{xy}L_{xy}}.$$

Then $\mathcal{P}(\lambda) = \lim_{n\to\infty} \mathcal{P}_n(\lambda)$ *is well-defined and holomorphic in the domain*

$$\mathbb{C} \setminus \left[-i\sqrt{S_D}, i\sqrt{S_D}\right].$$

2.5.2 Examples

Example 2.5.5 (Chain network)**.** Consider the infinite graph (V, E), where

$$V = \{0, 1, 2, \dots\}$$

and E is given by

$$0 \sim 1 \sim 2 \sim \cdots \sim n \sim (n+1) \sim \dots$$

Define the impedance of the edge $k \sim (k+1)$ by

$$z_{k(k+1)} = L_k \lambda + \frac{D_k}{\lambda},$$

where $L_k \geq D_k > 0$ (and $R_k = 0$) (see Figure 2.27), i.e. $\rho_{k(k+1)} = \dfrac{\lambda}{L_k \lambda^2 + D_k}$.

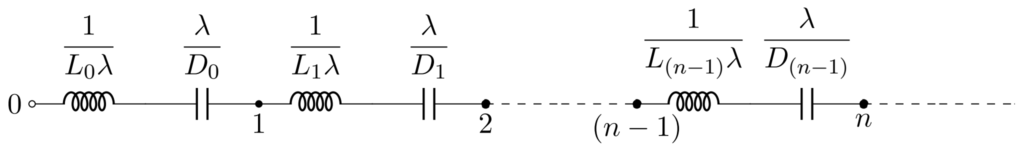


Figure 2.27: Chain network

Set $a_0 = 0$ and $B = \emptyset$. Then we have $V_n = \{0, ..., n\}$ and $B_n = \{n\}$. It follows by series law (Theorem 2.2.22), that

$$\mathcal{P}_n(\lambda) = \frac{1}{\sum\limits_{k=0}^{n-1} \left(L_k \lambda + \dfrac{D_k}{\lambda} \right)} = \frac{1}{l_n \lambda + \dfrac{d_n}{\lambda}},$$

where

$$l_n = \sum_{k=0}^{n-1} L_k \ \text{ and } d_n = \sum_{k=0}^{n-1} D_k.$$

Assume further that

$$\sum_{k=0}^{\infty} L_k = \sum_{k=0}^{\infty} D_k = \infty$$

that is

$$\lim_{n\to\infty} l_n = \lim_{n\to\infty} d_n = +\infty.$$

Then for any λ with $\operatorname{Re} \lambda \neq 0$ we obtain

$$\operatorname{Re} \left(l_n \lambda + \frac{d_n}{\lambda} \right) \to \infty$$

whence

$$\mathcal{P}(\lambda) = \lim_{n\to\infty} \mathcal{P}_n(\lambda) = 0.$$

For $\lambda = i\omega$ with real ω we have

$$\mathcal{P}_n(i\omega) = -\frac{i}{l_n\omega - \dfrac{d_n}{\omega}}.$$

Assume in addition that

$$\sum_{k=0}^{\infty}(L_k - D_k) =: c \in (0,\infty)$$

that is

$$\lim_{n\to\infty}(l_n - d_n) = c \in (0,\infty).$$

Then for $\omega = 1$ we have

$$\mathcal{P}_n(i) = -\frac{i}{l_n - d_n},$$

whence

$$\mathcal{P}(i) = \lim_{n\to\infty}\mathcal{P}_n(i) = -\frac{i}{c}.$$

It follows also that

$$\mathcal{P}(-i) = \frac{i}{c}.$$

For $\omega \neq \pm 1$ we have

$$l_n\omega - \frac{d_n}{\omega} = (l_n - d_n)\omega + d_n\left(\omega - \frac{1}{\omega}\right) \to \pm\infty$$

whence it follows that

$$\mathcal{P}(i\omega) = \lim_{n\to\infty}\mathcal{P}_n(i\omega) = 0.$$

Hence, for any $\lambda \in \mathbb{C}\setminus\{0\}$ we have

$$\mathcal{P}(\lambda) = \begin{cases} -\dfrac{\lambda}{c}, & \text{if } \lambda = \pm i \\ 0, & \text{otherwise.}\end{cases}$$

In particular, $\mathcal{P}(\lambda)$ is holomorphic in $\mathbb{C}\setminus\{\pm i, 0\}$ but is discontinuous at $\lambda = \pm i$.

On the other hand we have

$$S_D = \sup_n \frac{D_n}{L_n} \quad \text{and} \quad S_D^* = \inf_n \frac{D_n}{L_n}.$$

Since $c < \infty$ then necessarily $S_D = 1$. Since $c > 0$ then $S_D^* < 1$.

Wee see that the points $\lambda = \pm i$ (where $\mathcal{P}$ looses continuity) lie in the set

$$J^* = \left[-i\sqrt{S_D}, -i\sqrt{S_D^*}\right] \cup \left[i\sqrt{S_D^*}, i\sqrt{S_D}\right] \cup \{0\}$$

that matches Corollary 2.5.3. For example, choose

$$L_n = 1 \text{ and } D_n = 1 - \epsilon 2^{-n},$$

where $\epsilon > 0$. Then $S_D = 1$ and $S_D^* = 1 - \epsilon$, so that the interval

$$\left[i\sqrt{S_D^*}, i\sqrt{S_D}\right] = [i\sqrt{1-\varepsilon}, i],$$

containing $\lambda = i$, can have an arbitrary small length.

Example 2.5.6 (Modified ladder)**.** Consider the infinite graph (V, E), where

$$V = \{0, 1, 2, 3, 4, \dots\}$$

and E is given by $(2k-2) \sim 2k$ and $(2k-1) \sim 2k$ for $k = \overline{1, \infty}$. Let us consider a network as at Figure 2.28.

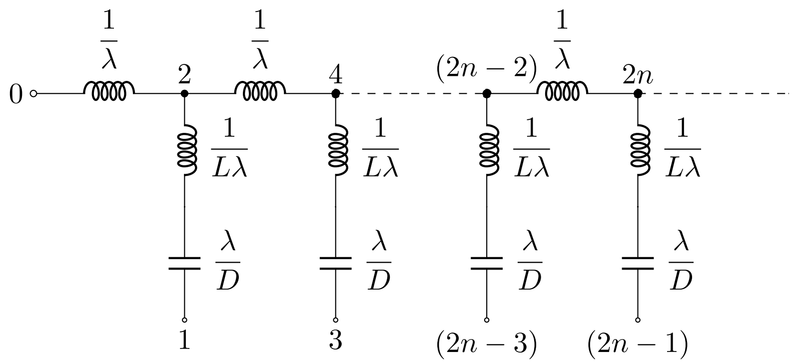


Figure 2.28: Modified ladder network

That is, let the admittances of the edges $(2k-2) \sim 2k$ be $\dfrac{1}{\lambda}$ and the admittances of the edges $2k-1 \sim 2k$ be $\dfrac{\lambda}{L\lambda^2 + D}$, where $L > 0$ and $D > 0$. Set also $a_0 = 0$, while

$$B = \{1, 3, \dots\}.$$

This network is similar to Feynman's ladder network (see [15]), but we add coils to the "vertical" edges and ground at infinity. Clearly, we have

$$V_n = \{0, 1, \dots, 2n-2\} \cup \{2n\} \ \text{ and } \ B_n = \{1, 3, \dots, 2n-3\} \cup \{2n\}.$$

Therefore, this network can be exhausted by finite $\alpha\beta$-networks (see Subsection 2.2.4, $\alpha = \dfrac{1}{\lambda}$, $\beta = \dfrac{\lambda}{L\lambda^2 + D}$), whose effective admittances by (2.2.36) and (2.2.37) are

$$\mathcal{P}_n(\lambda) = \begin{cases} \dfrac{\alpha(2n-1)}{n}, \text{ if } \mu = -4, \\ \dfrac{\alpha\left(\psi_1^{2n-1}+1\right)(\psi_1 - 1)}{\left(\psi_1^{2n} - 1\right)}, \text{otherwise}, \\ \text{not defined}, \lambda \in \left\{0, \pm i\sqrt{\dfrac{D}{L}}\right\}, \end{cases}$$

where $\mu = \dfrac{\lambda^2}{L\lambda^2 + D}$.

Since $\psi_{1,2}$ are the square roots of the equation

$$\psi^2 - (2+\mu)\,\psi + 1 = 0, \tag{2.5.2}$$

we can assume, without loss of generality, that $|\psi_1| \le |\psi_2|$. Then, since $\psi_1\psi_2 = 1$ by (2.5.2), we have either $|\psi_1| < 1 < |\psi_2|$ or $|\psi_1| = |\psi_2| = 1$.

In the case $|\psi_1| < 1 < |\psi_2|$ we obtain

$$\mathcal{P} = \mathcal{P}(\lambda) = \lim_{n\to\infty} \mathcal{P}_n(\lambda) = \frac{1-\psi_1}{\lambda}.$$

In the case $|\psi_1| = |\psi_2| = 1$, $\psi_1 \ne \psi_2 \ne -1$ the sequence $\{\mathcal{P}_n(\lambda)\}$ has no limit.

In the case $|\psi_1| = |\psi_2| = -1$ (i.e. $\mu = -4$)

$$\mathcal{P} = \mathcal{P}(\lambda) = \lim_{n\to\infty} \mathcal{P}_n(\lambda) = \frac{2}{\lambda}.$$

Therefore, for the infinite network we have

$$\mathcal{P}(\lambda) = \begin{cases} \dfrac{1-\psi_1}{\lambda}, & \text{if } |\psi_1| < 1 < |\psi_2|, \\ \dfrac{2}{\lambda}, & \text{if } \psi_1 = \psi_2 = -1, \\ \text{not defined, otherwise.} \end{cases} \tag{2.5.3}$$

We will reformulate the identity (2.5.3) in terms of λ, but firstly we prove an auxiliary claim.

Claim 2.5.7. *Let $\mu \ne -4$ and $\mu \ne 0$. Then the condition $|\psi_1| = |\psi_2| = 1$ occurs if and only if $\mu \in (-4, 0)$.*

Proof. "$\Rightarrow$" Let $|\psi_1| = 1$. Then $|\psi_2| = 1$. Since $\mu \ne 0$ and $\mu \ne -4$, it follows from (2.5.2) and $|\psi_1| = |\psi_2| = 1$ that $\psi_1, \psi_2 \notin \mathbb{R}$. Therefore, $\psi_2 = \overline{\psi_1}$, since $\psi_1\overline{\psi_1} = |\psi_1|^2 = 1$. Moreover, by (2.5.2) we have $2 + \mu = \psi_1 + \psi_2 \in \mathbb{R}$, i.e. $\mu \in \mathbb{R}$. Also $\psi_1, \psi_2 \notin \mathbb{R}$ means that the determinant of (2.5.2)

$$(2+\mu)^2 - 4 = 4\mu + \mu^2$$

is not positive, i.e. $\mu \notin (-\infty, -4) \cup (0, \infty)$. Therefore,

$$\mu \in \mathbb{R} \setminus ((-\infty, -4) \cup (0, \infty))$$

which was to be proved.

"$\Leftarrow$" Let $\mu \in (-4, 0)$. Then the determinant of (2.5.2) is negative and

$$|\psi_{1,2}|^2 = \left| 1 + \frac{\mu}{2} \pm i\sqrt{-\mu - \left(\frac{\mu}{2}\right)^2} \right|^2 = \left(1 + \frac{\mu}{2}\right)^2 - \mu - \left(\frac{\mu}{2}\right)^2 = 1.$$

$\square$

Since $\mu = -4$ at the points $\lambda = \pm i\sqrt{\dfrac{D}{L+1/4}}$, we have

$$\mathcal{P}(\lambda) = \begin{cases} \dfrac{1-\psi_1}{\lambda}, \text{ if } \lambda \in \mathbb{C} \setminus \left(\left[-i\sqrt{\dfrac{D}{L+1/4}}, i\sqrt{\dfrac{D}{L+1/4}} \right] \bigcup \left\{ \pm i\sqrt{\dfrac{D}{L}} \right\} \right), \\ \dfrac{2}{\lambda}, \qquad \text{if } \lambda = \pm i\sqrt{\dfrac{D}{L+1/4}}, \\ \text{not defined, if } \lambda \in \left(-i\sqrt{\dfrac{D}{L+1/4}}, i\sqrt{\dfrac{D}{L+1/4}} \right) \bigcup \left\{ \pm i\sqrt{\dfrac{D}{L}} \right\}, \end{cases}$$

where ψ_1 is the root of the equation (2.5.2) with $|\psi| < 1$. Note that this function is continuous at the points $\lambda = \pm i\sqrt{\dfrac{D}{L+1/4}}$. Indeed,

$$\psi \to -1,$$

when $\lambda \to \pm i\sqrt{\dfrac{D}{L+1/4}}$, since roots of the quadratic equation are continuous functions of coefficients. Hence, the continuity of $\mathcal{P}(\lambda)$ at given points follows. Therefore, $\mathcal{P}(\lambda)$ is well-defined and continuous in the domain

$$\mathbb{C} \setminus \left(\left(-i\sqrt{\dfrac{D}{L+1/4}}, i\sqrt{\dfrac{D}{L+1/4}} \right) \bigcup \left\{ \pm i\sqrt{\dfrac{D}{L}} \right\} \right).$$

In particular, the domain of holomorphicity of $\mathcal{P}(\lambda)$ is

$$\mathbb{C} \setminus \left(\left[-i\sqrt{\dfrac{D}{L+1/4}}, i\sqrt{\dfrac{D}{L+1/4}} \right] \bigcup \left\{ \pm i\sqrt{\dfrac{D}{L}} \right\} \right). \tag{2.5.4}$$

On the other hand we have $S_D^* = 0$, $S_D = \dfrac{D}{L}$, therefore, the Corollary 2.5.3 states, that $\mathcal{P}(\lambda)$ is holomorphic in the domain

$$\mathbb{C} \setminus \left[-i\sqrt{\dfrac{D}{L}}, i\sqrt{\dfrac{D}{L}} \right]. \tag{2.5.5}$$

Comparison of (2.5.4) and (2.5.5) shows the sharpness of Corollary 2.5.3.

2.5.3 Feynman's ladder with zero at infinity

Consider the infinite ladder network (V, E), where

$$V = \{0, 1, 2, 3, 4, \dots\}$$

and E is given by $(2k-2) \sim 2k$ and $(2k-1) \sim 2k$ for $k = \overline{1, \infty}$ (see Figure 2.29)

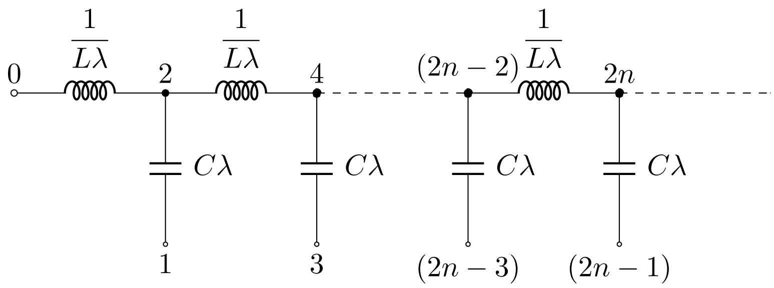


Figure 2.29: Feynman's ladder with zero at infinity

Let the admittance of the edges $(2k-2) \sim 2k$ be $\dfrac{1}{L\lambda}$ and admittance of the edges $2k-1 \sim 2k$ be $C\lambda$, where $L > 0$ and $C > 0$. Set also $a_0 = 0$, while $B = \{1, 3, \dots\}$.

This network is very similar to Feynman's ladder network (see [15]), but has ground at infinity.

Theorem 2.5.2 is applicable in this case only in the right half-plane $\{\operatorname{Re}\lambda > 0\}$. In this subsection we will analyze the behavior of the sequence of the effective admittances for the exhausted networks in the whole plane and compare our calculations with the result stated by Richard Feynman in [15].

This network can be exhausted by finite $\alpha\beta$-networks (see Subsection 2.2.4, $\alpha = \dfrac{1}{L\lambda}$, $\beta = C\lambda$), whose effective admittances by (2.2.36) and (2.2.37) are

$$\mathcal{P}_n(\lambda) = \begin{cases} \dfrac{2n-1}{L\lambda n}, \text{ if } \mu = -4, \\ \text{not defined, if } \lambda = 0, \\ \mathcal{P}_n^{\alpha\beta}(\lambda) = \dfrac{\left(\psi_1^{2n-1}+1\right)(\psi_1 - 1)}{L\lambda\left(\psi_1^{2n}-1\right)}, \text{otherwise,} \end{cases}$$

where $\mu = LC\lambda^2$ and ψ_1 is any root of the equation

$$\psi^2 - \left(2 + LC\lambda^2\right)\psi + 1 = 0. \tag{2.5.6}$$

Let us analyze the sequence $\{\mathcal{P}_n^{\alpha\beta}(\lambda)\}$, $n \to \infty$.

Since ψ_1 is any root of the quadratic equation (2.5.6), we can assume, without loss of generality, that $|\psi_1| \le 1 \le |\psi_2|$. Note that then $\{\mathcal{P}_n^{\alpha\beta}(\lambda)\}$ has limit if and only if $|\psi_1| < 1$. This limit is equal to $\dfrac{1-\psi_1}{L\lambda}$.

Claim 2.5.8. *Let $\lambda \in \mathbb{C} \setminus \{0\}$, $\lambda^2 \neq -\dfrac{4}{LC}$. Then the condition $|\psi_1| = |\psi_2| = 1$ occurs if and only if $\lambda^2 \in \left(-\dfrac{4}{LC}, 0\right)$.*

Proof. The proof follows the same outline as the proof of Claim 2.5.7. □

Therefore, for any $\lambda \in \mathbb{C} \setminus \left[-i\sqrt{\frac{4}{LC}}, i\sqrt{\frac{4}{LC}}\right]$ we can write

$$\psi_1(\lambda) = 1 + \frac{LC\lambda^2}{2} + \lambda\xi(\lambda), \tag{2.5.7}$$

where $\xi(\lambda)$ is the square root of $LC + \frac{L^2C^2\lambda^2}{4}$, such that $|\psi_1(\lambda)| < 1$.

Let us denote $\gamma = LC + \frac{L^2C^2\lambda^2}{4}$. Let us consider $\gamma \in \mathbb{C} \setminus (-i\infty, 0)$, i.e. $\gamma = re^{i\phi}$, $\phi \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ in a polar coordinates. Then there are two continuous functions ξ_1, ξ_2, which give square roots of γ:

$$\xi_1(\gamma) = \sqrt{r}e^{i\frac{\phi}{2}}, \xi_2(\gamma) = -\xi_1(\gamma)$$

(see Figure 2.30).

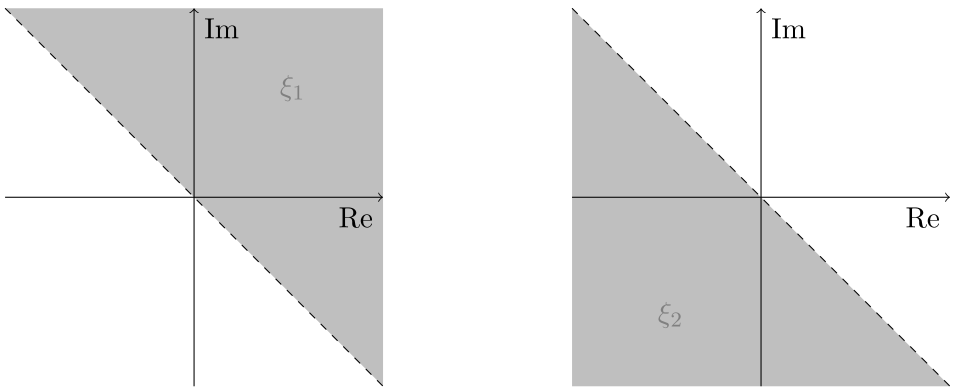


Figure 2.30: The images of ξ_1 and ξ_2

Since

$$\begin{aligned}\gamma =& LC + \frac{L^2C^2\lambda^2}{4} = LC + \frac{L^2C^2\left(\operatorname{Re}\lambda + i\operatorname{Im}\lambda\right)^2}{4}\\ =& LC + \frac{L^2C^2\left(\operatorname{Re}\lambda\right)^2 - L^2C^2\left(\operatorname{Im}\lambda\right)^2}{4} + i\frac{L^2C^2(\operatorname{Re}\lambda)(\operatorname{Im}\lambda)}{2},\end{aligned}$$

$\xi_1(\gamma(\lambda))$ and $\xi_2(\gamma(\lambda))$ are defined for all $\lambda \in \mathbb{C} \setminus \overline{\Lambda}$, where

$$\overline{\Lambda} = \left\{1 + \frac{LC\left(\operatorname{Re}\lambda\right)^2 - LC\left(\operatorname{Im}\lambda\right)^2}{4} = 0 \text{ and } \left(\operatorname{Re}\lambda\right)\left(\operatorname{Im}\lambda\right) < 0\right\},$$

see Figure 2.31.

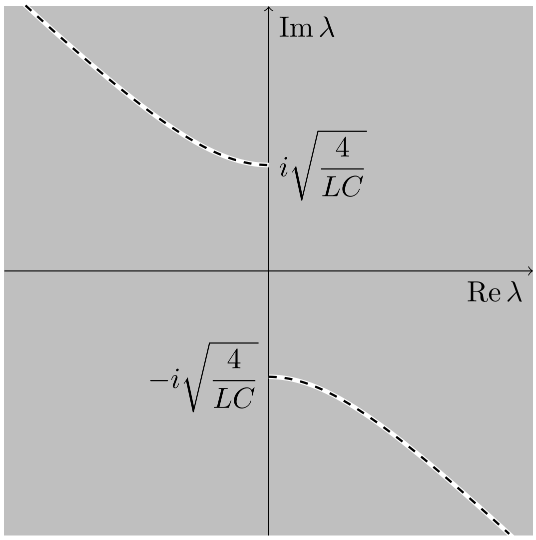


Figure 2.31: The domain of $\xi_{1,2}(\gamma(\lambda))$

Since the functions

$$\left|1+\frac{LC\lambda^2}{2}+\lambda\xi_1(\lambda)\right| \text{ and } \left|1+\frac{LC\lambda^2}{2}+\lambda\xi_2(\lambda)\right|$$

are continuous on λ, the choice of the function $\xi_1(\lambda)$ or $\xi_2(\lambda)$ in (2.5.7), can not change inside the domains

$$\Omega_1=\left\{\operatorname{Re}^2\lambda-\operatorname{Im}^2\lambda>-\sqrt{\frac{4}{LC}},\operatorname{Re}\lambda>0\right\}\cup\left\{\operatorname{Re}^2\lambda-\operatorname{Im}^2\lambda<-\sqrt{\frac{4}{LC}},\operatorname{Im}\lambda>0\right\}$$

and

$$\Omega_2=\left\{\operatorname{Re}^2\lambda-\operatorname{Im}^2\lambda>-\sqrt{\frac{4}{LC}},\operatorname{Re}\lambda<0\right\}\cup\left\{\operatorname{Re}^2\lambda-\operatorname{Im}^2\lambda<-\sqrt{\frac{4}{LC}},\operatorname{Im}\lambda<0\right\}$$

(see Figure 2.32).

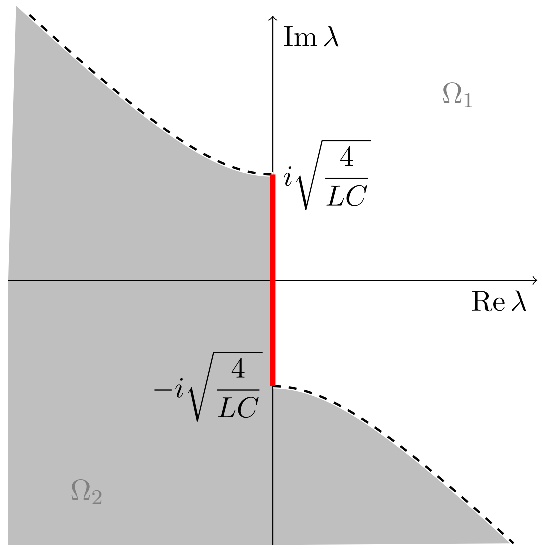


Figure 2.32: The domains Ω_1 and Ω_2

Taking $\lambda = 2 \in \Omega_1$ we have

$$\left|1 + \frac{LC\lambda^2}{2} + \lambda\xi_1(\gamma(\lambda))\right| = \left|1 + 2LC + 2\xi_1(LC + L^2C^2)\right| = \left|1 + 2LC + 2\sqrt{LC + L^2C^2}\right| > 1.$$

Therefore,

$$\psi_1(\lambda) = 1 + \frac{LC\lambda^2}{2} + \lambda\xi_2(\gamma(\lambda)), \lambda \in \Omega_1.$$

In the same way, taking $\lambda = -2 \in \Omega_2$ we obtain

$$\left|1 + \frac{LC\lambda^2}{2} + \lambda\xi_1(\gamma(\lambda))\right| = \left|1 + 2LC - 2\xi_1(LC + L^2C^2)\right| = \left|1 + 2LC - 2\sqrt{LC + L^2C^2}\right| < 1.$$

Therefore,

$$\psi_1(\lambda) = 1 + \frac{LC\lambda^2}{2} + \lambda\xi_1(\gamma(\lambda)), \lambda \in \Omega_2.$$

Therefore, we can calculate the effective admittance of the infinite network for any $\lambda \in \mathbb{C} \setminus \left(\overline{\Lambda} \cup \left[-i\sqrt{\frac{4}{LC}}, i\sqrt{\frac{4}{LC}}\right]\right)$ as

$$\mathcal{P}(\lambda) = \frac{1-\psi_1}{L\lambda} = \begin{cases} -\frac{C\lambda}{2} - \frac{\xi_2(\gamma(\lambda))}{L}, \lambda \in \Omega_1, \\ -\frac{C\lambda}{2} - \frac{\xi_1(\gamma(\lambda))}{L}, \lambda \in \Omega_2. \end{cases}$$

The effective admittance for the $\lambda \in \overline{\Lambda}$ we can calculate, considering another cut of the plane γ. Moreover, since the cut $(-i\infty, 0)$ has been chosen arbitrary, the limits of the effective admittances from the both sides of the curves $\overline{\Lambda}$ will coincide and give the required quantity.

Therefore, we can calculate the effective admittance of the infinite network for any $\lambda \in \mathbb{C} \setminus \left[-i\sqrt{\frac{4}{LC}}, i\sqrt{\frac{4}{LC}}\right]$.

A natural question is whether the right and left limits of the effective admittance (or of the $\xi_1(\lambda)$, $\xi_2(\lambda)$) at the segment $\left[-i\sqrt{\frac{4}{LC}}, i\sqrt{\frac{4}{LC}}\right]$ coincide (see Figure 2.33). The answer is negative.

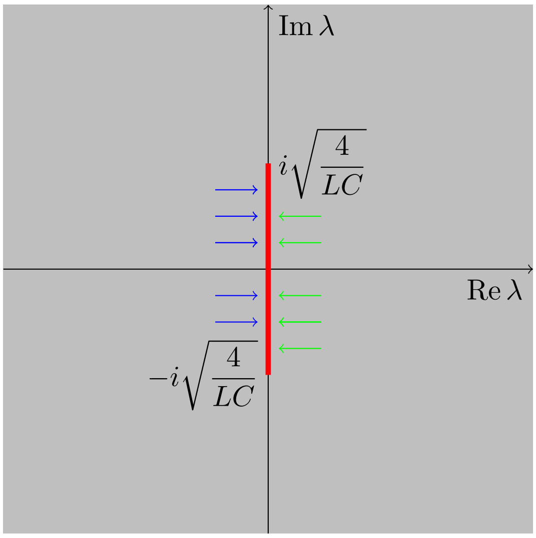


Figure 2.33: The limits of $\xi_1(\lambda)$ and $\xi_2(\lambda)$

Indeed, let $\lambda = \epsilon + i\omega$, $1 \gg \epsilon > 0$, $\omega \in \left(-\sqrt{\frac{4}{LC}}, \sqrt{\frac{4}{LC}}\right)$, $\lambda \in \Omega_1$. Then

$$\gamma(\epsilon + i\omega) = LC + \frac{L^2C^2}{4}\left(\epsilon^2 - \omega^2\right) + i\frac{L^2C^2}{2}\epsilon\omega$$

and

$$\begin{aligned} \lim_{\lambda \to +i\omega} \mathcal{P}(\lambda) &= \lim_{\epsilon \to 0}\left(-\frac{C(\epsilon + i\omega)}{2} - \frac{\xi_2(\gamma(\epsilon + i\omega))}{L}\right) \\ &= -\frac{Ci\omega}{2} + \sqrt{\frac{C}{L} - \frac{C^2\omega^2}{4}}, \end{aligned} \tag{2.5.8}$$

since $\operatorname{Re}\gamma > 0$, $|\operatorname{Im}\gamma| \ll 1$ provides $\operatorname{Re}\xi_2 < 0$.

Let $\lambda = -\epsilon + i\omega$, $1 \gg \epsilon > 0$, $\omega \in \left(-\sqrt{\frac{4}{LC}}, \sqrt{\frac{4}{LC}}\right)$, $\lambda \in \Omega_2$. Then

$$\gamma(-\epsilon + i\omega) = LC + \frac{L^2C^2}{4}\left(\epsilon^2 - \omega^2\right) - i\frac{L^2C^2}{2}\epsilon\omega$$

and

$$\lim_{\lambda\to -i\omega}\mathcal{P}(\lambda) = \lim_{\epsilon\to 0}\left(-\frac{C(-\epsilon+i\omega)}{2} - \frac{\xi_1(\gamma(-\epsilon+i\omega))}{L}\right)$$
$$= -\frac{Ci\omega}{2} - \sqrt{\frac{C}{L} - \frac{C^2\omega^2}{4}},$$

since $\operatorname{Re}\gamma > 0$, $|\operatorname{Im}\gamma| \ll 1$ provides $\operatorname{Re}\xi_1 > 0$.

The limit (2.5.8) coincides with the one, stated by R. Feynman in [15]. Indeed, by [15, p. 22-13] we have the effective admittance

$$\mathcal{P} = \frac{1}{Z} = \frac{1}{i\omega L/2 + \sqrt{(L/C) - (\omega^2 L^2/4)}} = \frac{i\omega L/2 - \sqrt{(L/C) - (\omega^2 L^2/4)}}{(i\omega L/2)^2 - ((L/C) - (\omega^2 L^2/4))}$$
$$= -\frac{C}{L}\left(\frac{i\omega L}{2} - \sqrt{(L/C) - (\omega^2 L^2/4)}\right) = -\frac{Ci\omega}{2} + \sqrt{\frac{C}{L} - \frac{C^2\omega^2}{4}}.$$

This corresponds to the ideas in [1] and in [39] to calculate the effective impedance as right half-plane limit. Physically it makes sense, since the real resistance in any part of a physical network is always greater than zero. Unfortunately, as one can see in Example 2.5.5, the right half-plane limit does not necessarily coincide with the effective admittance at the point, provided by Definition 2.5.1 (look at the points $\pm i$ in Example 2.5.5).

Chapter 3

Effective impedance of networks over an ordered field

A physical electrical network with impedances can contain three types of passive elements:

- resistor, whose impedance is $R \in [0, \infty)$,
- inductor (coil), whose impedance is $Li\omega$, $L \in [0, \infty)$,
- capacitor, whose impedance is $\dfrac{D}{i\omega}$, $D \in [0, \infty)$,

where $\omega > 0$ is the frequency of an alternating current, i is the imaginary unit (see [14] and Appendix C).

If we denote $\lambda = i\omega$ (see [2], [7]), then the impedance of the edge xy of a network is

$$z_{xy} = R_{xy} + L_{xy}i\omega + \frac{D_{xy}}{i\omega} = R_{xy} + L_{xy}\lambda + \frac{D_{xy}}{\lambda},$$

where $R_{xy}, L_{xy}, D_{xy} \geq 0$ and $R_{xy} + L_{xy} + D_{xy} \neq 0$.

Therefore, the admittance of an edge

$$\rho_{xy} = \frac{1}{z_{xy}} = \frac{\lambda}{L_{xy}\lambda^2 + R_{xy}\lambda + D_{xy}}, L_{xy}, R_{xy}, D_{xy} \geq 0, L_{xy}+R_{xy}+D_{xy} \neq 0. \quad (3.0.1)$$

can be considered as an element of the ordered field $(\mathbb{R}(\lambda), \succ)$ of rational functions with real coefficients (see [37]). This ordered field is non-Archimedean. Its smallest extension, which is both real-closed and complete in the order topology is the Levi-Civita field $\mathcal{R}$ (see e.g. [30]). See also Appendix B.

In this chapter we introduce and consider a more general concept of a network over an arbitrary ordered field.

In Section 3.1 we introduce a notion of a network over an ordered field $\mathcal{K}$ (see Appendix B for more details about ordered fields). In Section 3.2 we prove that the maximum principle holds for the weighted Laplace operator with weights from $\mathcal{K}$, which allows us to solve uniquely the discrete boundary value Dirichlet problem over $\mathcal{K}$. Therefore, we define an effective admittance of a network as an element of $\mathcal{K}$. Then we investigate properties of the effective admittance and give examples.

For the Archimedean field $(\mathbb{R}, >)$ the effective admittance of infinite network is defined as the limit of the decreasing sequence $\{\mathcal{P}_n\}_{n=1}^{\infty}$ of partial effective admittances (i.e the effective admittances of the exhausted finite networks), since the admittance is this case is just capacitance (see [13], [18], [32]). In Section 3.3 we prove the monotonicity of the sequence $\{\mathcal{P}_n\}_{n=1}^{\infty}$ for any ordered field (Theorem 3.3.1). Then we consider several examples of infinite networks over the Levi-Civita field $\mathcal{R}$ to show, that the monotonicity does not imply convergence in the case of a non-Archimedean field.

In Section 3.4 we show some relations between networks over $\mathbb{C}$, introduced in Chapter 2, and networks over an ordered field, discussed in this chapter.

This chapter is based on [24] and [26].

3.1 Settings, assumptions and definitions

In this section we remind some definitions from theory of ordered fields (see also Appendix B), introduce notations and define a network over an ordered field.

Let $(\mathcal{K}, \succ)$ be an arbitrary ordered field (see Definition B.1.1). We say that $k \in \mathcal{K}$ is *positive* if $k \succ 0$. For $k_1, k_2 \in \mathcal{K}$ we will write

$$k_1 \succeq k_2, \text{ if } k_1 \succ k_2 \text{ or } k_1 = k_2.$$

Moreover, we will write

$$k_1 \prec k_2, \text{ if } k_2 \succ k_1$$

and

$$k_1 \preceq k_2, \text{ if } k_1 \prec k_2 \text{ or } k_1 = k_2.$$

Definition 3.1.1. *[6, p. A.VI.21] Define in* $\mathbb{R}(\lambda)$ *an* order "$\succ$" *as follows: for any rational function*

$$f(\lambda) = \frac{b_k\lambda^k + \cdots + b_1\lambda + b_0}{d_m\lambda^m + \cdots + d_1\lambda + d_0} \in \mathbb{R}(\lambda)$$

with $b_k \neq 0, d_m \neq 0$, *write*

$$f(\lambda) \succ 0, \text{ if } \frac{b_k}{d_m} > 0$$

and

$$f(\lambda) \succ g(\lambda), \text{ if } f(\lambda) - g(\lambda) \succ 0.$$

Statement 3.1.2. *(see [6, p. A.VI.21], [37, pp. 231–234])* $(\mathbb{R}(\lambda), \succ)$ *is an ordered field.*

Remark 3.1.3. The field $(\mathbb{R}(\lambda), \succ)$ is non-Archimedean. Indeed, $\lambda \succ n$ for any $n \in \mathbb{N}$, i.e. $n = \underbrace{1 + \cdots + 1}_{n}$.

The admittance ρ_{xy} in the form (3.0.1) is, obviously, a positive element of $\mathbb{R}(\lambda)$.

Further in this section the definitions will be given for an arbitrary ordered field $(\mathcal{K}, \succ)$.

Definition 3.1.4. *Let (V, E) be a connected locally finite graph, where V is a set of vertices and E is a set of (unoriented) edges (see Definition A.0.1). Let $\rho : E \to \mathcal{K}$ be a positive function (i.e. for every $xy \in E$, $\rho_{xy} \succ 0$ in $\mathcal{K}$). Let us extend ρ_{xy} to all pairs $x, y \in V$ by setting $\rho_{xy} = 0$ if xy is not an edge (also $\rho_{xx} = 0$).*

Then a network over the ordered field $\mathcal{K}$ *is a structure*

$$\Gamma = (V, \rho, a_0, B),$$

where $a_0 \in V$, $B \subset V$ is not empty, and $a_0 \notin B$. We will denote by $B_0 = B \cup \{a_0\}$ the set of boundary vertices.

The function ρ_{xy} is called admittance *or* weight. *The function $z_{xy} = \dfrac{1}{\rho_{xy}}$ is called* impedance.

The weight ρ_{xy} gives rise to a function on vertices as follows:

$$\rho(x) = \sum_y \rho_{xy}, \tag{3.1.1}$$

where the notation $\sum\limits_y$ means $\sum\limits_{y \in V}$. Then $\rho(x)$ is called the *weight of a vertex x*. By properties of an ordered field, we have $\rho(x) \succ 0$ for any $x \in V$.

By $|V|$ we will denote the cardinality of a set V. If $|V| < \infty$ the network is called *finite*. Otherwise, it is called *infinite*.

Definition 3.1.5. *For any function $f : V \to \mathcal{K}$ the* weighted Laplace operator Δ_ρ *is defined as*

$$\Delta_\rho f(x) = \sum_y (f(y) - f(x))\rho_{xy} = \sum_y (\nabla_{xy} f)\rho_{xy},$$

where

$$\nabla_{xy} f = f(y) - f(x)$$

is the difference operator.

In this chapter we will write *network* instead *network over an ordered field* for brievity.

3.2 Finite networks over an ordered field

In this section we assume that $\Gamma = (V, \rho, a_0, B)$ is a finite network over an ordered field $(\mathcal{K}, \succ)$. By Kirchhoff's complex law and Ohm's complex law (see Appendix C) the complex voltages in the network, where all the nodes $b \in B$ are grounded and at the node a_0 an external periodic voltage with the amplitude 1 and frequency $\omega = -i\lambda \in \mathbb{R}$ is maintained, satisfy the following linear system:

$$\begin{cases} \Delta_\rho v(x) = 0 \text{ on } V \setminus B_0, \\ v(a_0) = 1, \\ v(x) = 0 \text{ on } B. \end{cases}$$

Therefore, this system, as a boundary value Dirichlet problem, is also satisfied over the field $\mathbb{R}(\lambda)$. We consider more general the Dirichlet problem (3.2.1) over an ordered field $\mathcal{K}$. We prove the maximum/minimum principle (Lemma 3.2.2) and uniqueness of the solution of the Dirichlet problem (Theorem 3.2.1). Then we introduce notions of effective admittance and effective impedance for a network over an ordered field (Definition 3.2.4). Moreover, we prove Green's formula (Theorem 3.2.5) and Dirichlet/Thomson's principle (Theorem 3.2.8). The proofs are analogous to the case $\mathcal{K} = \mathbb{R}$ (see [17] and [23]). In Subsection 3.2.3 we give several examples of finite networks over the ordered field $\mathbb{R}(\lambda)$.

3.2.1 Definition of the effective impedance and main results

Let $\Gamma = (V, \rho, a_0, B)$ be a finite network over an ordered filed $\mathcal{K}$, $B_0 = B \cup \{a_0\}$.
Theorem 3.2.1. *The following* Dirichlet problem*:*

$$\begin{cases} \Delta_\rho v(x) = 0 \ \textit{on } V \setminus B_0, \\ v(a_0) = 1, \\ v(x) = 0 \ \textit{on } B. \end{cases} \tag{3.2.1}$$

where $v : V \to \mathcal{K}$ is an unknown function, has always a unique solution over $\mathcal{K}$.

The key point for the proof of Theorem 3.2.1 is the following lemma.
Lemma 3.2.2 (A maximum/minimum principle)**.** *Let W be a non-empty subset of V, such that $V \setminus W$ is also non-empty. Then, for any function $u : V \to \mathcal{K}$, that satisfies $\Delta_\rho u(x) \succeq 0$ (i.e. u is subharmonic) on $V \setminus W$, we have*

$$\max_{V \setminus W} u \preceq \max_W u \tag{3.2.2}$$

and for any function $u : V \to \mathcal{K}$, that satisfies $\Delta_\rho u(x) \preceq 0$ (i.e. u is superharmonic) on $V \setminus W$, we have

$$\min_{V \setminus W} u \succeq \min_W u. \tag{3.2.3}$$

Proof. It is enough to proof the first claim (then the second claim follows by changing u to $-u$). Set

$$M = \max_{V \setminus W} u,$$

and assume, from the contrary, that $M \succ \max_W u$. Let us consider the set

$$S = \{x \in V : u(x) = M\}.$$

Claim 1. If $x \in S$, then all neighbors of x also belong to S.

Clearly, $S \subset V \setminus W$ and S is non-empty. Hence, for any $x \in S$ we have $\Delta_\rho u(x) \succeq 0$ which can be rewritten in the form

$$u(x) \preceq \sum_{y:y\sim x} \frac{\rho_{xy}}{\rho(x)} u(y). \tag{3.2.4}$$

By properties of positive elements, we have

$$\frac{\rho_{xy}}{\rho(x)} \succ 0 \text{ for any } y \sim x.$$

Also, for any $y \in V$ we have $u(y) \preceq M$ by the definition of maximum. Therefore,

$$\frac{\rho_{xy}}{\rho(x)} u(y) = \frac{\rho_{xy}}{\rho(x)} M, \text{ if } u(y) = M, \tag{3.2.5}$$

$$\text{and } \frac{\rho_{xy}}{\rho(x)} u(y) \prec \frac{\rho_{xy}}{\rho(x)} M, \text{ if } u(y) \prec M, \tag{3.2.6}$$

where the last line is true by properties of positive elements. If there exist $y_0 \sim x$ such that $u(y_0) \prec M$, then, summing up all the equalities (3.2.5) and inequalities (3.2.6), we obtain

$$\sum_{y\sim x} \frac{\rho_{xy}}{\rho(x)} u(y) \prec \sum_{y\sim x} \frac{\rho_{xy}}{\rho(x)} M. \tag{3.2.7}$$

But

$$\sum_{y\sim x} \frac{\rho_{xy}}{\rho(x)} M = M = u(x),$$

therefore, (3.2.7) is a contradiction with (3.2.4).

Claim 2. Let S be a non-empty set of vertices of a connected graph (V,E) such that $x \in S$ implies that all neighbours of x belong to S. Then $S = V$.

Indeed, let $x \in S$ and y be any other vertex. Then by the definition of connected graph, there is a path $\{x_k\}_{k=0}^n$ between x and y, that is,

$$x = x_0 \sim x_1 \sim x_2 \sim \cdots \sim x_n = y.$$

Since $x_0 \in S$ and $x_1 \sim x_0$, we obtain $x_1 \in S$. Since $x_2 \sim x_1$, we obtain $x_2 \in S$. By induction, we conclude that all $x_k \in S$, whence $y \in S$.

It follows from two claims that set S must coincide with V, which is not possible since $u(x) \prec M$ for any $x \in W$. This contradiction shows that $M \preceq \max_W u$. □

Proof of the Theorem 3.2.1. Let us first proof the uniqueness. If we have two solutions v_1 and v_2 of (3.2.1), then the difference $u = v_1 - v_2$ satisfies the conditions

$$\begin{cases} \Delta_\rho u(x) = 0 \text{ on } V \setminus B_0, \\ u(x) = 0 \text{ on } B_0, \end{cases} \tag{3.2.8}$$

and, by Lemma 3.2.2

$$0 = \max_{B_0} u \succeq \max_{V \setminus B_0} u \succeq \min_{V \setminus B_0} u \succeq \min_{B_0} u = 0,$$

whence, $u \equiv 0$ since $u|_{B_0} = 0$.

Let us now prove the existence of a solution of (3.2.1). For any $x \in V \setminus B_0$, rewrite the equation $\Delta_\rho v(x) = 0$ in the form

$$\sum_{y \in V \setminus B_0} \frac{\rho_{xy}}{\rho(x)} v(y) - v(x) = - \sum_{b \in B_0} \frac{\rho_{xb}}{\rho(x)} v(b). \tag{3.2.9}$$

Let us denote by $\mathcal{F}$ the set of all functions v on $V \setminus B_0$ with values in $\mathcal{K}^m$, where $m = |V| - |B_0|$. Then the left hand side of (3.2.9) can be regarded as an operator in this space; let us denote it by Lv, that is

$$Lv(x) = \sum_{y \in V \setminus B_0} \frac{\rho_{xy}}{\rho(x)} v(y) - v(x), \tag{3.2.10}$$

for all $x \in V \setminus B_0$. Rewrite the equation (3.2.9) in the form $Lv = h$, where h is the right hand side of (3.2.9), which is a given function on $V \setminus B_0$. Note that $\mathcal{F}$ is a linear space over the field $\mathcal{K}$. Since the family $\{\mathbf{1}_{\{x\}}\}_{x \in V \setminus B_0}$ of indicator functions form a basis in $\mathcal{F}$, we obtain that $\dim \mathcal{F} = m < \infty$. Hence, the operator $L : \mathcal{F} \to \mathcal{F}$ is a linear operator in a finitely dimensional space, and the first part of the proof shows that $Lv = 0$ implies $v = 0$ (indeed, just set $v(b) = 0$ for any $b \in B_0$ in (3.2.9)), that is, the operator L is injective. By Linear Algebra, any injective operator acting in the spaces of equal dimensions, must be bijective. Hence, for any $h \in \mathcal{F}$ (in particular, for $h(x) = - \sum_{b \in B_0} \frac{\rho_{xb}}{\rho(x)} v(b) = - \frac{\rho_{xa_0}}{\rho(x)}$), there is a solution, which finishes the proof. □

Corollary 3.2.3. *For the solution $v : V \to \mathcal{K}$ of* (3.2.1) *the following inequality*

$$0_{\mathcal{K}} \preceq v(x) \preceq 1_{\mathcal{K}} \tag{3.2.11}$$

is true for any $x \in V$.

Proof. Apply Lemma 3.2.2 for $W = B_0$. □

Now we can define the effective admittance of a network over an ordered field.

Definition 3.2.4. *Let v be the solution of the Dirichlet problem* (3.2.1) *for the network. Then define the* effective admittance *by*

$$\mathcal{P} = \sum_x (1 - v(x)) \rho_{xa_0} \tag{3.2.12}$$

If $\mathcal{P} \neq 0$, then we define the effective impedance *by*

$$Z = \frac{1}{\mathcal{P}} = \frac{1}{\sum_x (1 - v(x)) \rho_{xa_0}},$$

Since by Theorem 3.2.1 the Dirichlet problem (3.2.1) has exactly one solution over the field $\mathcal{K}$, the effective admittance is always well-defined as an element of $\mathcal{K}$.

Moreover, by (3.2.11) we have $\mathcal{P} \succeq 0$ and, hence, if $\mathcal{P} \neq 0$, then $Z \succ 0$.

Theorem 3.2.5 (Green's formula over an ordered field)**.** *Let Γ be a finite network over an ordered field $\mathcal{K}$ with the vertex set V, and let W be a non-empty subset of V. Then, for any two functions $f, g : V \to \mathcal{K}$,*

$$\sum_{x\in W} \Delta_\rho f(x)g(x) = -\frac{1}{2}\sum_{x,y\in W} (\nabla_{xy} f)(\nabla_{xy} g)\rho_{xy} + \sum_{x\in W}\sum_{y\in V\setminus W} (\nabla_{xy} f)g(x)\rho_{xy}. \tag{3.2.13}$$

If $W = V$, then the last term in (3.2.13) *vanishes, and we obtain*

$$\sum_{x\in V} \Delta_\rho f(x)g(x) = -\frac{1}{2}\sum_{x,y\in V} (\nabla_{xy} f)(\nabla_{xy} g)\rho_{xy} \tag{3.2.14}$$

Proof.

$$\begin{aligned}
\sum_{x\in W} \Delta_\rho f(x)g(x) &= \sum_{x\in W}\left(\sum_{y\in V}(f(y)-f(x))\rho_{xy}\right) g(x) \\
&= \sum_{x\in W}\sum_{y\in V}(f(y)-f(x))g(x)\rho_{xy} \\
&= \sum_{x\in W}\sum_{y\in W}(f(y)-f(x))g(x)\rho_{xy} + \sum_{x\in W}\sum_{y\in V\setminus W}(f(y)-f(x))g(x)\rho_{xy} \\
&= \sum_{y\in W}\sum_{x\in W}(f(x)-f(y))g(y)\rho_{xy} + \sum_{x\in W}\sum_{y\in V\setminus W}(\nabla_{xy} f)g(x)\rho_{xy},
\end{aligned}$$

where in the last line we have switched notation of the variables x and y in the first sum. Adding together the last two lines and dividing by 2 (it is possible, since any ordered field has characteristic 0, see e.g. [37]), we obtain

$$\sum_{x\in W} \Delta_\rho f(x)g(x) = -\frac{1}{2}\sum_{x\in W}\sum_{y\in W}(\nabla_{xy} f)(\nabla_{xy} g)\rho_{xy} + \sum_{x\in W}\sum_{y\in V\setminus W}(\nabla_{xy} f)g(x)\rho_{xy},$$

which was to be proved. ☐

Corollary 3.2.6. *For any function $f : V \to \mathcal{K}$,*

$$\sum_{x\in V} \Delta_\rho f(x) = 0. \tag{3.2.15}$$

Proof. Apply (3.2.14) for $g \equiv 1$. ☐

Lemma 3.2.7. *For any network we have*

$$\mathcal{P} = -\Delta_\rho v(a_0) = \sum_{b\in B} \Delta_\rho v(b), \tag{3.2.16}$$

where v is the solution of the Dirichlet problem (3.2.1).

Proof. Using $v(a_0) = 1$, we obtain

$$\Delta_\rho v(a_0) = \sum_{x:x\sim a_0} (v(x) - v(a_0))\rho_{xa_0} = \sum_{x:x\sim a_0} (v(x) - 1)\rho_{a_0 x} = -\mathcal{P}. \tag{3.2.17}$$

The second equality in (3.2.16) follows from (3.2.15), since v is the solution of the Dirichlet problem (3.2.1). □

Theorem 3.2.8 (Conservation of energy, Dirichlet/Thomson's principle)**.** *Let v be the solution of the Dirichlet problem* (3.2.1) *for network Γ over the ordered field $\mathcal{K}$. Then*

$$\frac{1}{2}\sum_{x,y\in V} (\nabla_{xy} v)^2 \rho_{xy} = \mathcal{P}. \tag{3.2.18}$$

Morover, for any other function $f : V \to \mathcal{K}$ such that $f(a_0) = 1$ and $f|_B \equiv 0$, the following inequality holds:

$$\frac{1}{2}\sum_{x,y\in V} (\nabla_{xy} v)^2 \rho_{xy} \preceq \frac{1}{2}\sum_{x,y\in V} (\nabla_{xy} f)^2 \rho_{xy} \tag{3.2.19}$$

We will call the equality (3.2.18) *conservation of energy* (similarly to the real case, see [18, p. 8]).

The inequality (3.2.19) states the *Dirichlet/Thomson's principle* (see [18, p. 9], [23, p. 121], [32, p. 29]), that the solution of the Dirichlet problem minimizes the energy.

Proof. Applying (3.2.14) to the left hand side of (3.2.18) we obtain

$$\begin{aligned}
\frac{1}{2}\sum_{x,y\in V} (\nabla_{xy} v)^2 \rho_{xy} &= -\sum_{x\in V} \Delta_\rho v(x) v(x) \\
&= -\sum_{x\in V\setminus B_0} \Delta_\rho v(x) v(x) - \Delta_\rho v(a_0) v(a_0) - \sum_{b\in B} \Delta_\rho v(b) v(b) \\
&= -\Delta_\rho v(a_0),
\end{aligned}$$

since v is the solution of (3.2.1). The statement (3.2.18) is proved due to Lemma 3.2.7.

Let $g = f - v$. Then $g(b) = 0$ for any $b \in B_0$. Therefore,

$$\begin{aligned}\frac{1}{2}\sum_{x,y\in V}(\nabla_{xy}f)^2\rho_{xy} =&\frac{1}{2}\sum_{x,y\in V}(\nabla_{xy}(g+v))^2\rho_{xy} = \frac{1}{2}\sum_{x,y\in V}(\nabla_{xy}g+\nabla_{xy}v)^2\rho_{xy}\\ =&\frac{1}{2}\sum_{x,y\in V}((\nabla_{xy}g)^2+2(\nabla_{xy}g)(\nabla_{xy}v)+(\nabla_{xy}v)^2)\rho_{xy}\\ =&\frac{1}{2}\sum_{x,y\in V}(\nabla_{xy}v)^2\rho_{xy}+\frac{1}{2}\sum_{x,y\in V}(\nabla_{xy}g)^2\rho_{xy}+\sum_{x,y\in V}(\nabla_{xy}v)(\nabla_{xy}g)\rho_{xy},\end{aligned}$$

where the last term vanishes by Green's formula (3.2.14), since $g(b) = 0$ for any $b \in B_0$ and v is the solution of the Dirichlet problem (3.2.1), and the second term is greater then zero whenever $g \not\equiv 0$. Therefore, (3.2.19) is proved and an equality is attained if and only if $f \equiv v$. $\square$

3.2.2 Basic properties

In this subsection we present some transforms of a finite network, which do not change the solution of the Dirichlet problem (3.2.1) nor the effective admittance of the network. Particularly we prove that some known physical transforms (i.e. star-mesh transform, $Y-\Delta$ transform, series law) follow from Gaussian elimination method for the Dirichlet problem. Moreover, edges of networks have positive weights after these transforms.

Theorem 3.2.9 (Star-mesh transform over an ordered field). *Let $\Gamma = (V, \rho, a_0, B)$ be a finite network over an ordered field $\mathcal{K}$, $|V| = n$, $B_0 = B\cup\{a_0\}$, and $x_1, \ldots, x_m \in V$, $3 \le m \le n$, are such that*

- $x_1 \notin B_0$,
- $y \not\sim x_1$ *for all* $y \in V \setminus \{x_2, \ldots, x_m\}$,

If one removes the vertex x_1, edges $x_1x_i, i = \overline{2,m}$ and change the admittances of the edges $x_ix_j, i,j = \overline{2,m}, i \neq j$ as follows:

$$\widetilde{\rho}_{x_ix_j} = \rho_{x_ix_j} + \frac{\rho_{x_1x_i}\rho_{x_1x_j}}{\rho(x_1)}, \tag{3.2.20}$$

not changing the other admittances, then for the new network the solution of the Dirichlet problem (3.2.1) *for all the vertices will be the same as the solution of the Dirichlet problem on the original network at corresponding vertices.*

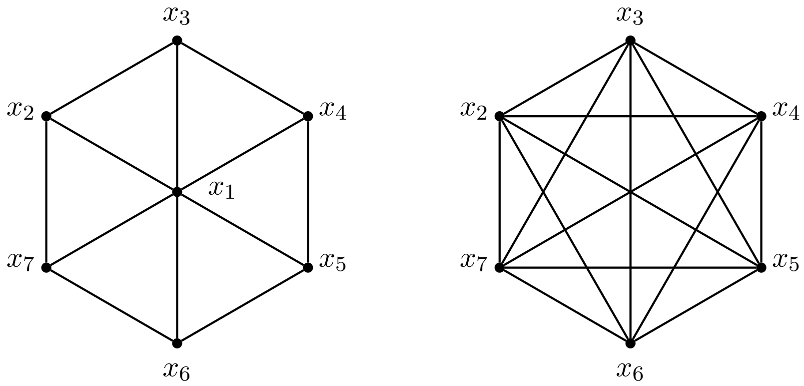


Figure 3.1: Star-mesh transform over an ordered field for $m = 7$

Proof. Without loss of generality we can assume that $x_1, \ldots, x_l \notin B_0$, where $l = m - |\{x_1, \ldots, x_m\} \cap B_0|$. Let $k = |B|$. Then the Dirichlet problem (3.2.1) for the network Γ can be rewritten in a matrix form

$$\mathbf{A}\hat{v} = \mathbf{b},$$

where

$$\mathbf{A} = \begin{bmatrix} \rho(x_1) & -\rho_{x_1x_2} & \cdots & -\rho_{x_1x_l} & 0 & \cdots & 0 \\ -\rho_{x_1x_2} & \rho(x_2) & \cdots & -\rho_{x_2x_l} & -\rho_{x_2x_{m+1}} & \cdots & -\rho_{x_2x_{n-k-1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\rho_{x_1x_l} & -\rho_{x_2x_l} & \cdots & \rho(x_l) & -\rho_{x_lx_{m+1}} & \cdots & -\rho_{x_lx_{n-k-1}} \\ 0 & -\rho_{x_2x_{m+1}} & \cdots & -\rho_{x_lx_{m+1}} & \rho(x_{m+1}) & \cdots & -\rho_{x_{m+1}x_{n-k-1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -\rho_{x_2x_{n-k-1}} & \cdots & -\rho_{x_lx_{n-k-1}} & -\rho_{x_{m+1}x_{n-k-1}} & \cdots & \rho(x_{n-k-1}) \end{bmatrix},$$

$$\hat{v} = [v(x_1), v(x_2), \ldots, v(x_l), v(x_{m+1}), \ldots, v(x_{n-k-1})]^T,$$

and

$$\mathbf{b} = [\rho_{a_0x_1}, \rho_{a_0x_2}, \ldots, \rho_{a_0x_l}, \rho_{a_0x_{m+1}}, \ldots, \rho_{a_0x_{n-k-1}}]^T,$$

since $y \not\sim x_1$ for all $y \in V \setminus \{x_2, \ldots, x_m\}$, $v(a_0) = 1$, $v|_B \equiv 0$.

We will show, that the star-mesh transform is an application of Gaussian elimination method for the first row. Indeed, applying Gaussian elimination method for the first row of the augmented matrix $\bar{\mathbf{A}} = [\mathbf{A}|\mathbf{b}]$ we obtain the matrix $\widetilde{\mathbf{A}} =$

$$\left[\begin{array}{ccccccc|c} 1 & -\dfrac{\rho_{x_1x_2}}{\rho(x_1)} & \cdots & -\dfrac{\rho_{x_1x_l}}{\rho(x_1)} & 0 & \cdots & 0 & \dfrac{\rho_{a_0x_1}}{\rho(x_1)} \\ 0 & \rho^*(x_2) & \cdots & -\widetilde{\rho}_{x_2x_l} & -\rho_{x_2x_{m+1}} & \cdots & -\rho_{x_2x_{n-k-1}} & \rho^*_{a_0x_2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -\widetilde{\rho}_{x_2x_l} & \cdots & \rho^*(x_l) & -\rho_{x_lx_{m+1}} & \cdots & -\rho_{x_lx_{n-k-1}} & \rho^*_{a_0x_l} \\ 0 & -\rho_{x_2x_{m+1}} & \cdots & -\rho_{x_lx_{m+1}} & \rho(x_{m+1}) & \cdots & -\rho_{x_{m+1}x_{n-k-1}} & \rho_{a_0x_{m+1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -\rho_{x_2x_{n-k-1}} & \cdots & -\rho_{x_lx_{n-k-1}} & -\rho_{x_{m+1}x_{n-k-1}} & \cdots & \rho(x_{n-k-1}) & \rho_{a_0x_{n-k-1}} \end{array}\right],$$

since $\rho(x_1) \neq 0$, where

$$\rho^*(x_i) = \rho(x_i) - \frac{\rho^2_{x_1x_i}}{\rho(x_1)} \text{ and } \rho^*_{a_0x_i} = \rho_{a_0x_i} + \frac{\rho_{x_1x_i}\rho_{a_0x_1}}{\rho(x_1)} \text{ for all } i = \overline{2,l}$$

and $\widetilde{\rho}_{x_ix_j}$ as in (3.2.20).

Note that for all $i = \overline{2,l}$

$$\begin{aligned}
\widetilde{\rho}(x_i) =& \rho(x_i) - \rho_{x_1x_i} - \sum_{\substack{j=2\\ j\neq i}}^{m} \rho_{x_ix_j} + \sum_{\substack{j=2\\ j\neq i}}^{m} \widetilde{\rho}_{x_ix_j} \\
=& \rho(x_i) - \rho_{x_1x_i} - \sum_{\substack{j=2\\ j\neq i}}^{m} \rho_{x_ix_j} + \sum_{\substack{j=2\\ j\neq i}}^{m} \left(\rho_{x_ix_j} + \frac{\rho_{x_1x_i}\rho_{x_1x_j}}{\rho(x_1)} \right) \\
=& \rho(x_i) - \rho_{x_1x_i} + \sum_{\substack{j=2\\ j\neq i}}^{m} \frac{\rho_{x_1x_i}\rho_{x_1x_j}}{\rho(x_1)} \\
=& \rho(x_i) - \rho_{x_1x_i} + \frac{\rho_{x_1x_i}}{\rho(x_1)} \sum_{\substack{j=2\\ j\neq i}}^{m} \rho_{x_1x_j} \\
=& \rho(x_i) - \rho_{x_1x_i} + \frac{\rho_{x_1x_i}}{\rho(x_1)} \sum_{j=2}^{m} \rho_{x_1x_j} - \frac{\rho_{x_1x_i}}{\rho(x_1)}\rho_{x_1x_i} \\
=& \rho(x_i) - \rho_{x_1x_i} + \frac{\rho_{x_1x_i}}{\rho(x_1)}\rho(x_1) - \frac{\rho^2_{x_1x_i}}{\rho(x_1)} \\
=& \rho(x_i) - \frac{\rho^2_{x_1x_i}}{\rho(x_1)} = \rho^*(x_i)
\end{aligned}$$

$$\rho^*_{a_0x_i} = \rho_{a_0x_i} + \frac{\rho_{x_1x_i}\rho_{a_0x_1}}{\rho(x_1)} = \begin{cases} \widetilde{\rho}_{a_0x_i}, & \text{if } a_0 \in \{x_2, \dots, x_m\} \\ \rho_{a_0x_i}, & \text{otherwise, since then } \rho_{a_0x_1} = 0. \end{cases}$$

Hence, $\widetilde{\mathbf{A}} =$

$$\left[\begin{array}{ccccccc|c}
1 & -\frac{\rho_{x_1x_2}}{\rho(x_1)} & \dots & -\frac{\rho_{x_1x_l}}{\rho(x_1)} & 0 & \dots & 0 & \frac{\rho_{a_0x_1}}{\rho(x_1)} \\
0 & \widetilde{\rho}(x_2) & \dots & -\widetilde{\rho}_{x_2x_l} & -\rho_{x_2x_{m+1}} & \dots & -\rho_{x_2x_{n-k-1}} & \rho^*_{a_0x_2} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & -\widetilde{\rho}_{x_2x_l} & \dots & \widetilde{\rho}(x_l) & -\rho_{x_lx_{m+1}} & \dots & -\rho_{x_lx_{n-k-1}} & \rho^*_{a_0x_l} \\
0 & -\rho_{x_2x_{m+1}} & \dots & -\rho_{x_lx_{m+1}} & \rho(x_{m+1}) & \dots & -\rho_{x_{m+1}x_{n-k-1}} & \rho_{a_0x_{m+1}} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & -\rho_{x_2x_{n-k-1}} & \dots & -\rho_{x_lx_{n-k-1}} & -\rho_{x_{m+1}x_{n-k-1}} & \dots & \rho(x_{n-k-1}) & \rho_{a_0x_{n-k-1}}
\end{array}\right],$$

Therefore, we can eliminate the variable $v(x_1)$ from the Dirichlet problem, changing admittances as in the statement of the theorem. □

Corollary 3.2.10. *Under the star-mesh transform of the network the effective admittance does not change.*

Proof. In the proof we will use the notations from the proof of the Theorem 3.2.9.

The case $\{x_1, \dots, x_m\} \cap B_0 = \emptyset$ is trivial. The cases, when $\{x_1, \dots, x_m\} \cap B = \emptyset$ or $\{x_1, \dots, x_m\} \cap \{a_0\} = \emptyset$ are obvious, due to (3.2.16).

Otherwise, we can assume, without loss of generality, that $x_m = a_0$. Then, denoting the effective admittance of the new network by $\widetilde{\mathcal{P}}$, we have

$$\begin{aligned}
\mathcal{P} = &\sum_{x \neq a_0} (1 - v(x))\rho_{xa_0} \\
=&(1 - v(x_1))\rho_{x_1a_0} + \sum_{i=2}^{m-1}(1 - v(x_i))\rho_{x_ia_0} + \sum_{x \notin \{x_1,\dots,x_m\}} (1 - v(x))\rho_{xa_0} \\
=&\widetilde{\mathcal{P}} - \sum_{i=2}^{m-1}(1 - v(x_i))\widetilde{\rho}_{x_ia_0} + (1 - v(x_1))\rho_{x_1a_0} + \sum_{i=2}^{m-1}(1 - v(x_i))\rho_{x_ia_0} \\
=&\widetilde{\mathcal{P}} - \sum_{i=2}^{m-1}(1 - v(x_i))\frac{\rho_{x_1a_0}\rho_{x_1x_i}}{\rho(x_1)} + (1 - v(x_1))\rho_{x_1a_0} \\
=&\widetilde{\mathcal{P}} - \rho_{x_1a_0}\sum_{i=2}^{m-1}(1 - v(x_i))\frac{\rho_{x_1x_i}}{\rho(x_1)} + \left(1 - \sum_{i=2}^{m-1} v(x_i)\frac{\rho_{x_1x_i}}{\rho(x_1)} - \frac{\rho_{x_1a_0}}{\rho(x_1)}\right)\rho_{x_1a_0} \\
=&\widetilde{\mathcal{P}} - \rho_{x_1a_0}\left(\sum_{i=2}^{m-1}\frac{\rho_{x_1x_i}}{\rho(x_1)} - \sum_{i=2}^{m-1} v(x_i)\frac{\rho_{x_1x_i}}{\rho(x_1)}\right) + \left(1 - \sum_{i=2}^{m-1} v(x_i)\frac{\rho_{x_1x_i}}{\rho(x_1)} - \frac{\rho_{x_1a_0}}{\rho(x_1)}\right)\rho_{x_1a_0} \\
=&\widetilde{\mathcal{P}} - \rho_{x_1a_0}\left(\frac{\rho(x_1)}{\rho(x_1)} - \frac{\rho_{x_1x_m}}{\rho(x_1)} - \sum_{i=2}^{m-1} v(x_i)\frac{\rho_{x_1x_i}}{\rho(x_1)}\right) + \left(1 - \sum_{i=2}^{m-1} v(x_i)\frac{\rho_{x_1x_i}}{\rho(x_1)} - \frac{\rho_{x_1a_0}}{\rho(x_1)}\right)\rho_{x_1a_0} \\
=&\widetilde{\mathcal{P}},
\end{aligned}$$

since

$$\rho(x_1) = \sum_{i=2}^{m} \rho_{x_1x_i} \text{ and } v(x_1) = \sum_{i=2}^{l} v(x_i)\frac{\rho_{x_1x_i}}{\rho(x_1)} + \frac{\rho_{x_1a_0}}{\rho(x_1)} = \sum_{i=2}^{m-1} v(x_i)\frac{\rho_{x_1x_i}}{\rho(x_1)} + \frac{\rho_{x_1a_0}}{\rho(x_1)}$$

(see the first line of $\widetilde{\mathbf{A}}$ and note that $v(x_j) = 0$ for all $j = \overline{j+1, m-1}$ and $a_0 = x_m$). □

Series law and $Y - \Delta$ *transform* are particular cases of a star-mesh transform. Since multigraphs are not allowed in this thesis, we will use a modification of parallel law and refer to it as *parallel-series law.*

Corollary 3.2.11 (Series law over an ordered field). *Let* $\Gamma = (V, \rho, a_0, B)$ *be a finite network over an ordered field,* $B_0 = B \cup \{a_0\}$. *Let* $a, b, c \in V$ *are such, that*

- $b \notin B_0$,
- $a \not\sim c$, $a \sim b$, $b \sim c$,
- $b \not\sim x$ *for all* $x \in V \setminus \{a, c\}$.

If one removes the vertex b*, edges* ab, bc *and add the edge* ac *with the admittance*

$$\widetilde{\rho}_{ac} = \frac{\rho_{ab}\rho_{bc}}{\rho_{ab} + \rho_{bc}},$$

not changing other admittances, then for the new network the solution of the Dirichlet problem (3.2.1) *for all the vertices will be the same as the solution of the Dirichlet problem on the original network at corresponding vertices. The effective admittance of the new network coincides with the effective admittance of the original one.*

Remark 3.2.12. The corresponding equation for impedances is then

$$\widetilde{z}_{ac} = z_{ab} + z_{bc},$$

which corresponds to the well-known physical series law.

Figure 3.2: Series law over an ordered field

Proof. Apply Theorem 3.2.9 and Corollary 3.2.10 ($x_1 = b$) for the case $m = 3$ and $\rho_{ac} = 0$. □

Corollary 3.2.13 (Parallel-series law over an ordered field)**.** *Let* $\Gamma = (V, \rho, a_0, B)$ *be a finite network over an ordered field,* $B_0 = B \cup \{a_0\}$.
Let $a, b, c \in V$ *are such, that*

- $b \notin B_0$,
- $a \sim b, b \sim c, a \sim c$,
- $b \not\sim x$ *for all* $x \in V \setminus \{a, c\}$.

If one removes the vertex b*, edges* ab, bc *and add the edge* ac *with the admittance*

$$\widetilde{\rho}_{ac} = \frac{\rho_{ab}\rho_{bc}}{\rho_{ab} + \rho_{bc}} + \rho_{ac},$$

not changing other admittances, then for the new network the solution of the Dirichlet problem (3.2.1) *for all the vertices will be the same as the solution of the Dirichlet problem on the original network for corresponding vertices. The effective admittance of the new network coincides with the effective admittance of the original one.*

Remark 3.2.14. The corresponding equation for the impedances is then

$$\frac{1}{\widetilde{z}_{ac}} = \frac{1}{z_{ab} + z_{bc}} + \frac{1}{z_{ac}}$$

which corresponds to an application of the physical series law and then an application of the physical parallel law.

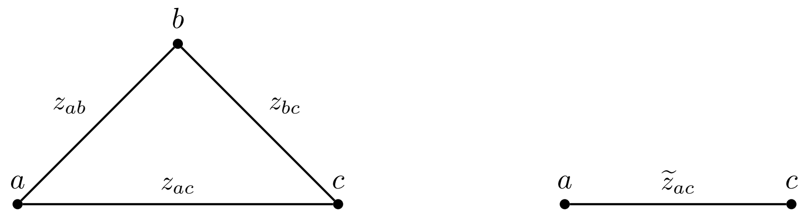


Figure 3.3: Parallel-series law over an ordered field

Proof. Apply Theorem 3.2.9 and Corollary 3.2.10 ($x_1 = b$) for the case $m = 3$. $\square$

Theorem 3.2.15 ($Y - \Delta$ transform over an ordered field)**.** *Let $\Gamma = (V, \rho, a_0, B)$ be a finite network, $B_0 = B \cup \{a_0\}$. Let $a, b, c, d \in V$ are such, that*

- $d \notin B_0$,
- $d \sim a, d \sim b, d \sim c$,
- $d \not\sim x$ *for all* $x \in V \setminus \{a, b, c\}$.

If one removes the vertex d, edges da, db, dc and set

$$\begin{aligned}\widetilde{\rho}_{ab} =& \frac{\rho_{da}\rho_{db}}{\rho_{da}+\rho_{db}+\rho_{dc}} + \rho_{ab},\\ \widetilde{\rho}_{bc} =& \frac{\rho_{db}\rho_{dc}}{\rho_{da}+\rho_{db}+\rho_{dc}} + \rho_{bc},\\ \widetilde{\rho}_{ac} =& \frac{\rho_{da}\rho_{dc}}{\rho_{da}+\rho_{db}+\rho_{dc}} + \rho_{ac},\end{aligned} \tag{3.2.21}$$

not changing other admittances, then for the new network the solution of the Dirichlet problem (3.2.1) *for all the vertices will be the same as the solution of the Dirichlet problem on the original network for the corresponding vertices. The effective admittance of the new network coincides with the effective admittance of the original one.*

Remark 3.2.16. The corresponding equalities for the impedances are

$$\begin{aligned}\frac{1}{\widetilde{z}_{ab}} =& \frac{z_{dc}}{z_{da}z_{db}+z_{db}z_{dc}+z_{da}z_{dc}} + \frac{1}{z_{ab}},\\ \frac{1}{\widetilde{z}_{bc}} =& \frac{z_{da}}{z_{da}z_{db}+z_{db}z_{dc}+z_{da}z_{dc}} + \frac{1}{z_{bc}},\\ \frac{1}{\widetilde{z}_{ac}} =& \frac{z_{db}}{z_{da}z_{db}+z_{db}z_{dc}+z_{da}z_{dc}} + \frac{1}{z_{ac}}.\end{aligned} \tag{3.2.22}$$

From the physical point of view, if $\rho_{ab}, \rho_{bc}, \rho_{ac}$ are all equal to zero, then it is just $Y - \Delta$ transform, otherwise, it is $Y - \Delta$ transform and the parallel law.

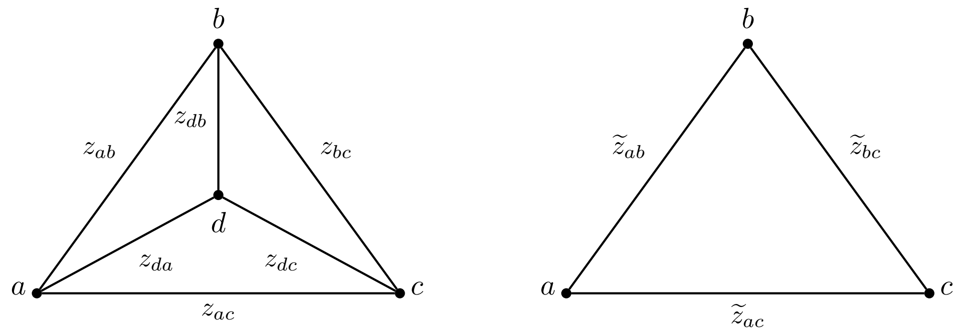


Figure 3.4: $Y - \Delta$ transform over an ordered field

Proof. Theorem 3.2.9 and Corollary 3.2.10 ($x_1 = d$) for the case $m = 4$. □

The $Y - \Delta$ transform is invertible. In general, it is not the case for a star-mesh transform.

Theorem 3.2.17 ($\Delta - Y$ transform over an ordered field)**.** *Let $\widetilde{\Gamma} = (V, \widetilde{\rho}, a_0, B)$ be a finite network and let $a, b, c \in V$ are such, that $a \sim b, b \sim c$, and $a \sim c$. If one add a vertex d and edges da, db, dc setting*

$$\begin{aligned}
\rho_{da} &= \frac{\widetilde{\rho}_{ac}\widetilde{\rho}_{bc} + \widetilde{\rho}_{ac}\widetilde{\rho}_{ab} + \widetilde{\rho}_{ab}\widetilde{\rho}_{bc}}{\widetilde{\rho}_{bc}},\\
\rho_{db} &= \frac{\widetilde{\rho}_{ac}\widetilde{\rho}_{bc} + \widetilde{\rho}_{ac}\widetilde{\rho}_{ab} + \widetilde{\rho}_{ab}\widetilde{\rho}_{bc}}{\widetilde{\rho}_{ac}},\\
\rho_{dc} &= \frac{\widetilde{\rho}_{ac}\widetilde{\rho}_{bc} + \widetilde{\rho}_{ac}\widetilde{\rho}_{ab} + \widetilde{\rho}_{ab}\widetilde{\rho}_{bc}}{\widetilde{\rho}_{ab}},
\end{aligned}$$

and remove the edges ab, bc, ac not changing other admittances, then for the new network

$$\Gamma = (V \cup \{d\}, \rho, a_0, B),$$

the solution of the Dirichlet problem (3.2.1) *for any vertex in V will be the same as the solution of the Dirichlet problem on the original network for the corresponding vertex. Moreover, the effective admittance does not change under this transform.*

Remark 3.2.18. The corresponding equalities for the impedances are

$$\begin{aligned}
z_{da} &= \frac{\widetilde{z}_{ab}\widetilde{z}_{ac}}{\widetilde{z}_{ab} + \widetilde{z}_{bc} + \widetilde{z}_{ac}},\\
z_{db} &= \frac{\widetilde{z}_{ab}\widetilde{z}_{bc}}{\widetilde{z}_{ab} + \widetilde{z}_{bc} + \widetilde{z}_{ac}},\\
z_{dc} &= \frac{\widetilde{z}_{bc}\widetilde{z}_{ac}}{\widetilde{z}_{ab} + \widetilde{z}_{bc} + \widetilde{z}_{ac}}.
\end{aligned}$$

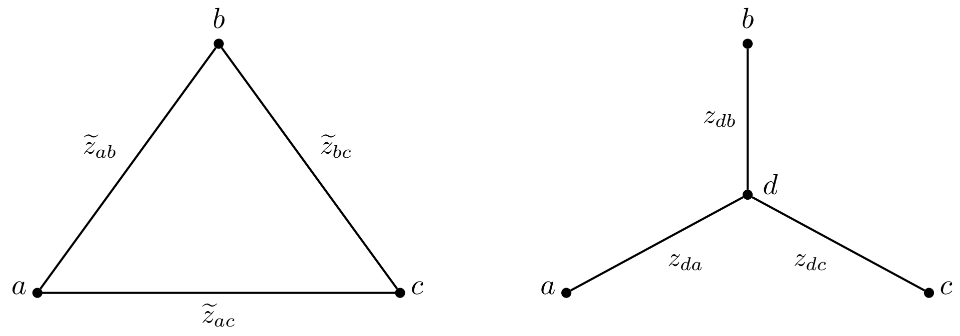


Figure 3.5: $\Delta - Y$ transform over an ordered field

Proof. To prove the theorem it is enough to express ρ_{da}, ρ_{db} and ρ_{dc} from (3.2.21), assuming $\rho_{ab} = 0$, $\rho_{bc} = 0$, and $\rho_{ac} = 0$. Summing up the inverses of all three equations one obtains

$$\frac{1}{\widetilde{\rho}_{ab}} + \frac{1}{\widetilde{\rho}_{bc}} + \frac{1}{\widetilde{\rho}_{ac}} = \frac{(\rho_{ad} + \rho_{bd} + \rho_{cd})^2}{\rho_{da}\rho_{db}\rho_{dc}}$$

Since both sides are strictly positive, the last equation is equivalent to

$$\frac{\widetilde{\rho}_{ab}\widetilde{\rho}_{bc}\widetilde{\rho}_{ac}}{\widetilde{\rho}_{ab}\widetilde{\rho}_{bc} + \widetilde{\rho}_{bc}\widetilde{\rho}_{ac} + \widetilde{\rho}_{ab}\widetilde{\rho}_{ac}} = \frac{\rho_{da}\rho_{db}\rho_{dc}}{(\rho_{ad} + \rho_{bd} + \rho_{cd})^2} \tag{3.2.23}$$

Multiplying the both sides of (3.2.23) by

$$\frac{1}{\widetilde{\rho}_{ab}\widetilde{\rho}_{ac}} = \frac{(\rho_{ad} + \rho_{bd} + \rho_{cd})^2}{\rho_{da}^2\rho_{db}\rho_{dc}},$$

which follows from (3.2.21), we get

$$\frac{\widetilde{\rho}_{bc}}{\widetilde{\rho}_{ab}\widetilde{\rho}_{bc} + \widetilde{\rho}_{bc}\widetilde{\rho}_{ac} + \widetilde{\rho}_{ab}\widetilde{\rho}_{ac}} = \frac{1}{\rho_{da}}.$$

Then the equation for ρ_{da} follows. To obtain the equations for ρ_{db} and ρ_{dc} one should multiply (3.2.23) by $\dfrac{1}{\widetilde{\rho}_{ab}\widetilde{\rho}_{bc}}$ and $\dfrac{1}{\widetilde{\rho}_{ac}\widetilde{\rho}_{bc}}$ respectively.

The fact that effective impedance and effective admittance do not change follows from Theorem 3.2.15. □

Remark 3.2.19. All described in this section transforms preserve the positivity of admittances and impedances on the edges.

3.2.3 Examples of finite networks over the ordered field $\mathbb{R}(\lambda)$

Example 3.2.20 (Simple network over $\mathbb{R}(\lambda)$)**.** Let us consider a network as at Figure 3.6, $a_0 = 1, B = \{0\}$, where the weights are considered as elements of $\mathbb{R}(\lambda)$.

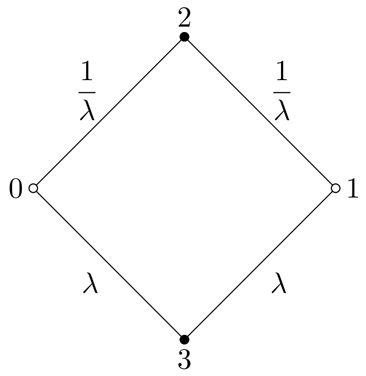


Figure 3.6: Simple network over $\mathbb{R}(\lambda)$

The Dirichlet problem for this network is

$$\begin{cases} \dfrac{(v(0)-v(2))}{\lambda} + \dfrac{(v(1)-v(2))}{\lambda} = 0, \\ (v(0)-v(3))\lambda + (v(1)-v(3))\lambda = 0, \\ v(0) = 0, \\ v(1) = 1. \end{cases}$$

The determinant of this linear system is

$$\mathcal{D}(\lambda) = 4.$$

The solution of the Dirichlet problem is

$$v = (v(0), v(1), v(2), v(3)) = \left(0, 1, \frac{1}{2}, \frac{1}{2}\right)$$

and the effective admittance is

$$\mathcal{P}(\lambda) = \frac{1}{\lambda}v(2) + \lambda v(3) = \frac{\lambda^2+1}{2\lambda}.$$

Note that the effective admittance can be also calculated using series and parallel-series laws (Corollaries 3.2.11 and 3.2.13).

Example 3.2.21 (Network over $\mathbb{R}(\lambda)$)**.** Let us consider the finite network over $\mathbb{R}(\lambda)$ as at Figure 3.7 with $a_0 = 1$, $B = \{0\}$.

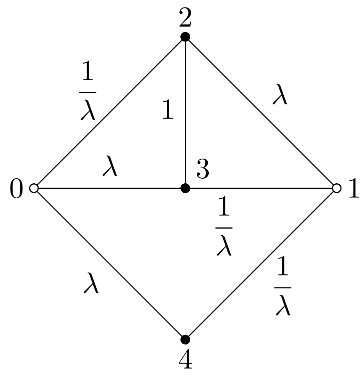


Figure 3.7: Network over $\mathbb{R}(\lambda)$

The Dirichlet problem for this network is

$$\begin{cases} \dfrac{(v(0)-v(2))}{\lambda}+(v(1)-v(2))\lambda+(v(3)-v(2))=0, \\ (v(0)-v(3))\lambda+\dfrac{(v(1)-v(3))}{\lambda}+(v(2)-v(3))=0, \\ (v(0)-v(4))\lambda+\dfrac{(v(1)-v(4))}{\lambda}=0, \\ v(0)=0, \\ v(1)=1. \end{cases}$$

The determinant of this system is

$$\begin{aligned} \mathcal{D}=\mathcal{D}(\lambda) &= -\left(\frac{1}{\lambda^2}+\frac{2}{\lambda}+2+2\lambda+\lambda^2\right)\left(\lambda+\frac{1}{\lambda}\right) \\ &= -\frac{1}{\lambda^3}(\lambda+1)^2(\lambda^2+1)^2, \end{aligned}$$

and the solution of the Dirichlet problem is

$$\begin{cases} v(0)=0, \\ v(1)=1, \\ v(2)=\dfrac{\lambda}{1+\lambda}, \\ v(3)=\dfrac{1}{1+\lambda}, \\ v(4)=\dfrac{1}{1+\lambda^2}. \end{cases}$$

Then the effective admittance is

$$\mathcal{P}(\lambda)=\frac{1}{\lambda}v(2)+\lambda v(3)+\lambda v(4)=\frac{\lambda^2+\lambda+1}{\lambda^2+1}. \tag{3.2.24}$$

Example 3.2.22 (One more network over $\mathbb{R}(\lambda)$)**.** Let us consider the finite network over $\mathbb{R}(\lambda)$ as at Figure 3.8 with $a_0 = 2$, $B = \{0\}$.

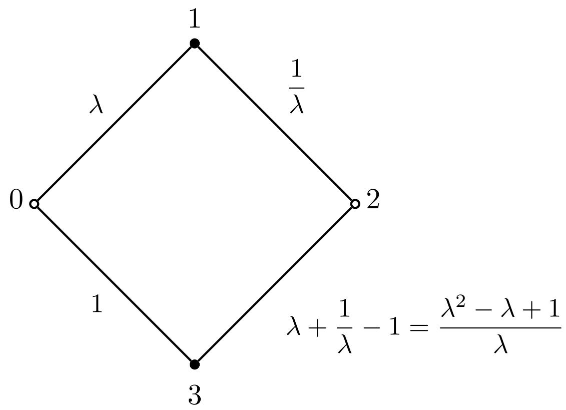


Figure 3.8: One more network over $\mathbb{R}(\lambda)$

The Dirichlet problem is as follows:

$$\begin{cases} (v(0) - v(1))\lambda + \dfrac{v(2) - v(1)}{\lambda} = 0, \\ (v(0) - v(3)) + (v(2) - v(3))(\lambda + \dfrac{1}{\lambda} - 1) = 0, \\ v(0) = 0, \\ v(2) = 1. \end{cases}$$

The determinant of this system is

$$D = D(\lambda) = \frac{1}{\lambda^2}(\lambda^2 + 1)^2.$$

The solution of the Dirichlet problem is

$$v(1) = \frac{1}{\lambda^2 + 1}, v(3) = \frac{\lambda^2 - \lambda + 1}{\lambda^2 + 1}.$$

and the effective admittance is

$$\mathcal{P}(\lambda) = \lambda v(1) + v(3) = 1.$$

3.3 Effective admittance of infinite networks over an ordered field

In this section we consider infinite networks over an ordered field. In Subsection 3.3.1 we generalize the concept of effective admittance for infinite networks (see e.g. [18], [32]) to the case of an ordered field. Unfortunately, in the case of non-Archimedean field the effective admittance can not be defined as a limit of partial effective admittances for any infinite network even in the case of Cauchy complete field (see Subsection 3.3.2 for the examples).

3.3.1 Infinite networks with zero potential at infinity

Let $\Gamma = (V, \rho, a_0, B)$ be an infinite network over an ordered field $(\mathcal{K}, \succ)$.

Let $\mathrm{dist}\,(x, y)$ be the graph distance on V, that is, the minimal value of n such that there exists a path $\{x_k\}_{k=0}^{n}$ connecting x and y, that is,

$$x = x_0 \sim x_1 \sim ... \sim x_n = y.$$

Let us consider the sequence of finite graphs $(V_n,\, \rho|_{V_n \times V_n})$, where

$$V_n = \{x \in V \mid \mathrm{dist}(a_0, x) \leq n\}, n \in \mathbb{N}.$$

We denote by

$$\partial V_n = \{x \in V \mid \mathrm{dist}(a_0, x) = n\}$$

the *boundary* of the graph $(V_n,\, \rho|_{V_n \times V_n})$. Note that $V_{n+1} = \partial V_{n+1} \cup V_n$.

Let us denote $B_n = B \cap V_n$. Let

$$\Gamma_n = (V_n,\, \rho|_{V_n \times V_n}, a_0, B_n \cup \partial V_n), n \in \mathbb{N}$$

Then $\{\Gamma_n\}_{n=1}^{\infty}$ is the *sequence of finite networks exhausted the infinite network* Γ.

This is an analogue of the approach to infinite networks with real weights in [18] and [32].

Let us consider the Dirichlet problem (3.2.1) on each Γ_n:

$$\begin{cases} \sum\limits_{y: y\sim x} (v^{(n)}(y) - v^{(n)}(x))\rho_{xy} = 0 \text{ on } V_n \setminus (\partial V_n \cup B_n \cup \{a_0\}), \\ v^{(n)}(x) = 0 \text{ on } \partial V_n \cup B_n, \\ v^{(n)}(a_0) = 1. \end{cases} \tag{3.3.1}$$

Let us denote by $\mathcal{P}_n$ the effective admittance of Γ_n.

Theorem 3.3.1. *For any infinite network over* $(\mathcal{K}, \succ)$,

$$\mathcal{P}_{n+1} \preceq \mathcal{P}_n. \tag{3.3.2}$$

Proof. By Dirichlet/Thomson's principle we have

$$\mathcal{P}_{n+1} \preceq \frac{1}{2} \sum_{x,y \in V_{n+1}} (\nabla_{xy} f)^2 \rho_{xy} \tag{3.3.3}$$

for any $f: V_{n+1} \to \mathcal{K}$ such that $f(a_0) = 1$, $f|_{\partial V_{n+1} \cup B_{n+1}} \equiv 0$.
Since $(V_{n+1} \setminus \partial V_{n+1}) = V_n$ and $B_{n+1} \cap V_n = B_n$, the inequality (3.3.3) is true for

$$f(x) = \begin{cases} v^{(n)}(x), & \text{if } x \in V_n, \\ 0, & \text{if } x \in \partial V_{n+1}, \end{cases}$$

where $v^{(n)}$ is the solution of (3.3.1) for Γ_n. Then

$$\begin{aligned} \frac{1}{2} \sum_{x,y \in V_{n+1}} (\nabla_{xy} f)^2 \rho_{xy} &= \frac{1}{2} \sum_{x,y \in V_n} (\nabla_{xy} f)^2 \rho_{xy} + \frac{1}{2} \sum_{x,y \in \partial V_{n+1}} (\nabla_{xy} f)^2 \rho_{xy} \\ &= \frac{1}{2} \sum_{x,y \in V_n} (\nabla_{xy} v^{(n)})^2 \rho_{xy} + 0 = \mathcal{P}_n + 0, \end{aligned}$$

where the last equality is true by conservation of energy (3.2.18). Together with (3.3.3), it gives us (3.3.2). □

Remark 3.3.2. Even in a Cauchy complete non-Archimedean ordered field inequalities (3.3.2) for all $n \in \mathbb{N}$ do not imply, that the sequence $\{\mathcal{P}_n\}_{n=1}^{\infty}$ converges.
Definition 3.3.3. *If for a given infinite network Γ the limit of effective admittances of the exhausted finite networks exists in $\mathcal{K}$, we refer to this limit as the* effective admittance of the network Γ (with zero potential at infinity) *and denote it by $\mathcal{P}$.*

3.3.2 Examples: ladder networks over the Levi-Civita field

In this subsection we investigate the behaviour of the sequence $\{\mathcal{P}_n^{\alpha\beta}\}_{n=1}^{\infty}$ of effective admittances of finite networks exhausted the ladder network ($\alpha\beta$-network) as at Figure 3.9 ($\alpha, \beta \in \mathcal{K}$, $\alpha, \beta \succ 0$).

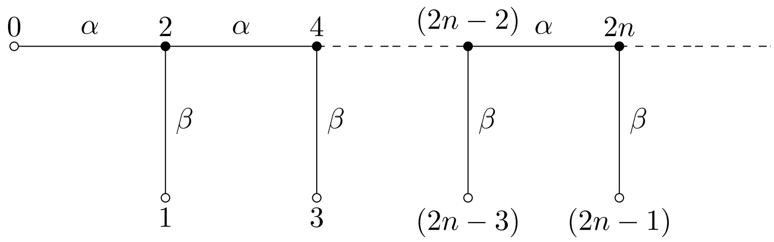


Figure 3.9: Infinite $\alpha\beta$-network over an ordered field

Namely,

$$V = \{0, 1, 2, \dots\}, B = \{1, 3, \dots\}, a_0 = 0.$$

The admittances of the edges $(2k - 2) \sim 2k$ are α and the admittances of the edges $(2k - 1) \sim 2k$ are β.

We will refer to such a network as an *infinite $\alpha\beta$-network over an ordered field $\mathcal{K}$* and denote it by $\Gamma_{\mathcal{K}}^{\alpha\beta}$.

This network is similar to Feynman's ladder network and CL-network (see [15], [39]), but has zero potential at infinity. Therefore, for any ordered field $\mathcal{K}$ the Theorem 3.3.1 is true for this network. We will show (Example 3.3.6 and Example 3.3.10) that whether $\{\mathcal{P}_n\}_{n=1}^{\infty}$ converges in Cauchy completion of $\mathcal{K}$ depends on α and β.

3.3.2.1 Finite ladder network over ordered field

Let $\{\Gamma_n\}_{n\in\mathbb{N}}$ be a sequence of finite networks exhausted an $\alpha\beta$-network (see Figure 3.10).

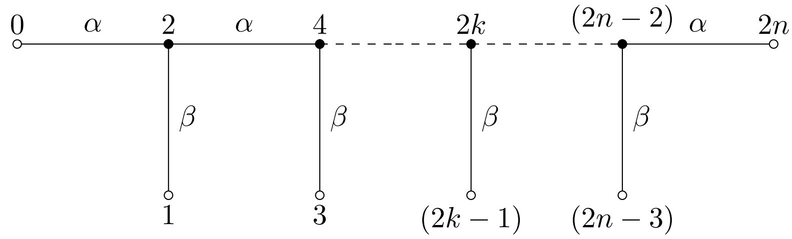


Figure 3.10: Finite ladder network

The Dirichlet problem (3.2.1) for the network Γ_n is the following

$$\begin{cases} \alpha v(2k-2)+\beta v(2k-1)+\alpha v(2k+2)-(2\alpha+\beta)v(2k)=0 \text{ for } k=\overline{1,n-1},\\ v(2n)=0,\\ v(2k-1)=0 \text{ for } k=\overline{1,n-1},\\ v(0)=1. \end{cases} \tag{3.3.4}$$

Denoting $v_k=v(2k)$ we obtain the following recurrence relation for $v_k, k=\overline{1,n-1}$

$$v_{k+1}-\left(2+\frac{\beta}{\alpha}\right)v_k+v_{k-1}=0 \tag{3.3.5}$$

since $v(2k-1)=0$ for $k=\overline{1,n-1}$. The characteristic polynomial of (3.3.5) is

$$\psi^2-\left(2+\frac{\beta}{\alpha}\right)\psi+1=0. \tag{3.3.6}$$

Its roots are

$$\psi_{1,2}=1+\frac{\beta}{2\alpha}\pm\xi,$$

where

$$\xi=\sqrt{\frac{\beta}{\alpha}+\left(\frac{\beta}{2\alpha}\right)^2}. \tag{3.3.7}$$

Note that ξ should not necessary belong to $\mathcal{K}$. It is known, that any ordered field posses a real-closed extension $\overline{\mathcal{K}}$. Then in $\overline{\mathcal{K}}$ exists exactly one positive square root of $\dfrac{\beta}{\alpha}+\left(\dfrac{\beta}{2\alpha}\right)^2$ (see [6, pp. A.VI.23–A.VI.28] and Appendix B). Therefore, we fix the extension $\overline{\mathcal{K}}$, denote the positive square root by ξ, and make all the further calculations in $\overline{\mathcal{K}}$.

The solution of the recurrence relation (3.3.5) is

$$v_k = c_1\psi_1^k + c_2\psi_2^k, \tag{3.3.8}$$

where $c_1, c_2 \in \overline{\mathcal{K}}$ are arbitrary constants.

We use the second and the forth equations in (3.3.4) as boundary conditions for this recurrence equation. Substituting (3.3.8) in the boundary conditions we obtain the following equations for the constants:

$$\begin{cases} c_1 + c_2 = 1, \\ c_1\psi_1^n + c_2\psi_2^n = 0. \end{cases}$$

Therefore,

$$\begin{cases} c_1 = \dfrac{1}{1-\psi_1^{2n}}, \\ c_2 = \dfrac{1}{1-\psi_2^{2n}} = \dfrac{-\psi_1^{2n}}{1-\psi_1^{2n}}, \end{cases}$$

since $\psi_1\psi_2 = 1$ by (3.3.6).

Now we can calculate the effective admittance of Γ_n:

$$\begin{aligned} \mathcal{P}_n =& \alpha\left(1-v(2)\right) = \alpha\left(1-v_1\right) = \alpha\left(1-c_1\psi_1-c_2\psi_2\right) \\ =& \frac{\alpha\left(\left(1+\dfrac{\beta}{2\alpha}+\xi\right)^{2n-1}+1\right)\left(\dfrac{\beta}{2\alpha}+\xi\right)}{\left(1+\dfrac{\beta}{2\alpha}+\xi\right)^{2n}-1}. \end{aligned} \tag{3.3.9}$$

Since $\mathcal{P}_n$ is an element of $\mathcal{K}$ as a rational function of the solution of the Dirichlet problem (3.2.1) over $\mathcal{K}$, it can be written without ξ:

$$\begin{aligned} \mathcal{P}_n =& \alpha\left(1-c_1\psi_1-c_2\psi_2\right) = \alpha\left(1-\frac{\psi_1}{1-\psi_1^{2n}}-\frac{\psi_2}{1-\psi_2^{2n}}\right) \\ =& \alpha\left(1-\frac{\psi_1\left(1-\psi_2^{2n}\right)+\psi_2\left(1-\psi_1^{2n}\right)}{\left(1-\psi_1^{2n}\right)\left(1-\psi_2^{2n}\right)}\right) = \alpha\left(1-\frac{\psi_1+\psi_2-\left(\psi_2^{2n-1}+\psi_1^{2n-1}\right)}{2-\left(\psi_1^{2n}+\psi_2^{2n}\right)}\right) \\ =& \alpha\left(1-\frac{2+\dfrac{\beta}{\alpha}-2\sum\limits_{k=0}^{n-1}\binom{2n-1}{2k}\left(1+\dfrac{\beta}{2\alpha}\right)^{2n-2k-1}\left(\dfrac{\beta}{\alpha}+\left(\dfrac{\beta}{2\alpha}\right)^2\right)^k}{2-2\sum\limits_{k=0}^{n}\binom{2n}{2k}\left(1+\dfrac{\beta}{2\alpha}\right)^{2n-2k}\left(\dfrac{\beta}{\alpha}+\left(\dfrac{\beta}{2\alpha}\right)^2\right)^k}\right), \end{aligned}$$

where in the last line we have used a binomial expansion.

3.3.2.2 Infinite ladder networks over the Levi-Civita field $\mathcal{R}$

We will consider two examples of $\alpha\beta$-network over the Levi-Civita field. Firstly, let us describe the Levi-Civita field $\mathcal{R}$. We take the definition of $\mathcal{R}$ and theorems about its properties from [4], [28], [29] and [30]. For more details see Appendix B.

Definition 3.3.4. *A subset M of the rational numbers $\mathbb{Q}$ is called left-finite if for every $r \in \mathbb{Q}$ there are only finitely elements of M that are smaller than r.*

Then the Levi-Civita field $\mathcal{R}$ is the set of all real valued functions on $\mathbb{Q}$ with left-finite support with the following operations:

for any $\alpha, \beta \in \mathcal{R}$

- addition *is defined component-wise*

$$(\alpha + \beta)(q) = \alpha(q) + \beta(q),$$

- multiplication *is defined as follows*

$$(\alpha \cdot \beta)(q) = \sum_{\substack{q_\alpha, q_\beta \in \mathbb{Q}, \\ q_\alpha + q_\beta = q}} \alpha(q_\alpha) \cdot \beta(q_\beta).$$

It is proved in [4], that $\mathcal{R}$ is an ordered field with a set of positive elements

$$\mathcal{R}^+ = \{\alpha \in \mathcal{R} \mid \alpha(\min\{q \in \mathbb{Q} \mid \alpha(q) \neq 0\}) > 0\}. \tag{3.3.10}$$

We denote by τ the following element in $\mathcal{R}$:

$$\tau(q) = \begin{cases} 1, & \text{if } q = 1, \\ 0, & \text{otherwise,} \end{cases} \tag{3.3.11}$$

which plays role of infinitesimal in the Levi-Civita field. Therefore, the Levi-Civita field is non-Archimedean.

By [4] we can write any $\alpha \in \mathcal{R}$ as

$$\alpha = \sum_{i=1}^{\infty} \alpha(q_i)\tau^{q_i}, \tag{3.3.12}$$

since $\alpha_n = \sum\limits_{i=1}^{n} \alpha(q_i)\tau^{q_i}$ converges strongly to the limit α in the order topology (see Subsection B.2.2).

The set of all polynomials over real numbers

$$\mathbb{R}[\tau] = \{b_0 + b_1\tau + \cdots + b_n\tau^n \mid b_i \in \mathbb{R}, n \in \mathbb{N}_0\}$$

is a subring of the Levi-Civita field due to (3.3.12). Therefore, since $\mathcal{R}$ is a field, the field of rational functions with real coefficients

$$\mathbb{R}(\tau) = \left\{ \frac{b_k\tau^k + b_{k+1}\tau^{k+1} + \cdots + b_n\tau^n}{d_l\tau^l + d_{l+1}\tau^{l+1} + \cdots + d_m\tau^m} \mid b_i, d_i \in \mathbb{R}, n, m, k, l \in \mathbb{N}_0, n \geq k, m \geq l \right\}$$

is isomorphic to a subfield of the Levi-Civita field.

Note that the corresponding order in the field $\mathbb{R}(\tau)$ is the following:

$$\frac{b_k\tau^k + b_{k+1}\tau^{k+1} + \cdots + b_n\tau^n}{d_l\tau^l + d_{l+1}\tau^{l+1} + \cdots + d_m\tau^m} \succ 0 \text{ if } \frac{b_k}{d_l} > 0, \tag{3.3.13}$$

assuming $k \leq n$, $l \leq m$, $b_k \neq 0$, $d_l \neq 0$. Therefore, we can consider the Levi-Civita field $\mathcal{R}$ as an ordered extension of the ordered field $\mathbb{R}(\tau)$ with the positiveness defined as (3.3.13).

Example 3.3.5. Let us find the element in $\mathcal{R}$, which corresponds to the rational function $\dfrac{1}{3 - 4\tau + \tau^2}$, i.e. we should find the sequences $\{q_i\} \in \mathbb{Q}$ and $\{\alpha(q_i)\} \in \mathbb{R}$ such that

$$\left(3 - 4\tau + \tau^2\right)\left(\sum_{i=1}^{\infty} \alpha(q_i)\tau^{q_i}\right) = 1.$$

Comparing the coefficients at powers of τ at right hand side and left hand side, starting from the lowest power, one obtains

$$q_1 = 0, \alpha(q_1) = \frac{1}{3},$$

$$q_2 = 1, \alpha(q_2) = \frac{4}{9},$$

and the recurrence relation

$$q_i = q_{i-1} + 1, 3\alpha(q_i) - 4\alpha(q_{i-1}) + \alpha(q_{i-2}) = 0 \text{ for } i > 2.$$

Therefore, solving the recurrence relation for $\alpha(q_i)$, we obtain

$$\alpha(q_i) = -\frac{1}{2 \cdot 3^i} + \frac{1}{2}$$

and

$$\frac{1}{3 - 4\tau + \tau^2} = \sum_{i=1}^{\infty} \left(-\frac{1}{2 \cdot 3^i} + \frac{1}{2}\right)\tau^{i-1}.$$

To consider $\mathcal{R}$ as an ordered extension of the ordered field $\mathbb{R}(\lambda)$ with the positiveness defined in Definition 3.1.1, we make a substitution

$$\tau = \frac{1}{\lambda}. \tag{3.3.14}$$

Consequently, we can consider admittances in the from (3.0.1), i.e.

$$\rho_{xy} = \frac{\lambda}{L_{xy}\lambda^2 + R_{xy}\lambda + D_{xy}} = \frac{\tau^{-1}}{L_{xy}\tau^{-2} + R_{xy}\tau^{-1} + D_{xy}} = \frac{\tau}{L_{xy} + R_{xy}\tau + D_{xy}\tau^2},$$

$L_{xy}, R_{xy}, D_{xy} \geq 0$, $L_{xy} + R_{xy} + D_{xy} \neq 0$, as elements of the Levi-Civita field and investigate the behavior of the sequence of effective admittances of finite networks. By [4] the Levi-Civita field is Cauchy complete in order topology and real-closed.

Example 3.3.6 (Feynman's ladder)**.** Let us consider the Feynman's infinite ladder LC-network, assuming that it has zero potential at infinity. It is an $\alpha\beta$-network with $\alpha = \dfrac{1}{L\lambda} = L^{-1}\tau$, $\beta = C\lambda = C\tau^{-1}$, where $L, C > 0$, $\alpha, \beta \in \mathcal{R}$.

Statement 3.3.7. *For the Feynman's ladder LC-network ($\alpha = L^{-1}\tau$, $\beta = C\tau^{-1}$, where $L, C > 0$) with zero potential at infinity*

$$\mathcal{P}_n \to \frac{\beta}{\frac{\beta}{2\alpha} + \xi} \quad \text{as } n \to \infty \tag{3.3.15}$$

in the order topology of the Levi-Civita field, where $\mathcal{P}_n$ is the sequence of the effective admittances of the exhausted finite networks, ξ is as in (3.3.7)*.*

Remark 3.3.8. For the Feynman's ladder LC-network

$$\frac{\beta}{\frac{\beta}{2\alpha} + \xi} = -\frac{C}{2\tau} + \frac{\tau}{L}\sqrt{\frac{LC}{\tau^2} + \left(\frac{LC}{2\tau^2}\right)^2}$$

and the motivation for this quantity was Feynman's calculations for infinite ladder LC-network (see [15, p. 22-13]).

Proof. Firstly, we should write ξ as an element of the Levi-Civita field, i.e. as a power series (3.3.12).

$$\begin{aligned}
\xi =& \sqrt{\frac{LC}{\tau^2} + \left(\frac{LC}{2\tau^2}\right)^2} = \frac{LC}{2\tau^2}\sqrt{\frac{4\tau^2}{LC} + 1} = \frac{LC}{2\tau^2}\sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\left(\frac{4\tau^2}{LC}\right)^k \\
=& \frac{LC}{2\tau^2}\cdot 1 + \frac{LC}{2\tau^2}\cdot\frac{1}{2}\cdot\frac{4\tau^2}{LC} - \frac{1}{8}\cdot\frac{LC}{2\tau^2}\cdot\left(\frac{4\tau^2}{LC}\right)^2 + o\left(\tau^2\right) \\
=& \frac{LC}{2}\tau^{-2} + 1 - \tau^2 + o\left(\tau^2\right).
\end{aligned}$$

Note that here and further $o\left(\tau^m\right)$, where $m \in \mathbb{Z}$, means $\sum\limits_{k=m+1}^{\infty} a_k\tau^k, a_k \in \mathbb{R}$.

Let us calculate the difference $\mathcal{P}_n - \dfrac{\beta}{\dfrac{\beta}{2\alpha}+\xi}$ (see (3.3.9)).

$$\begin{aligned}\mathcal{P}_n - \frac{\beta}{\frac{\beta}{2\alpha}+\xi} &= \frac{\alpha\left(\frac{\beta}{2\alpha}+\xi\right)\left(\left(1+\frac{\beta}{2\alpha}+\xi\right)^{2n-1}+1\right)}{\left(1+\frac{\beta}{2\alpha}+\xi\right)^{2n}-1} - \frac{\beta}{\frac{\beta}{2\alpha}+\xi}\\ &= \frac{\alpha\left(\frac{\beta}{2\alpha}+\xi\right)^2\left(\left(1+\frac{\beta}{2\alpha}+\xi\right)^{2n-1}+1\right)-\beta\left(\left(1+\frac{\beta}{2\alpha}+\xi\right)^{2n}-1\right)}{\left(\left(1+\frac{\beta}{2\alpha}+\xi\right)^{2n}-1\right)\left(\frac{\beta}{2\alpha}+\xi\right)}\end{aligned}$$

The nominator A of the last expression is

$$\begin{aligned}A =&\alpha\left(\left(1+\frac{\beta}{2\alpha}+\xi\right)^{2n-1}+1\right)\left(\frac{\beta}{2\alpha}+\xi\right)^2-\beta\left(\left(1+\frac{\beta}{2\alpha}+\xi\right)^{2n}-1\right)\\ =&\alpha\left(A_1+1\right)\left(\frac{\beta}{2\alpha}+\xi\right)^2-\beta\left(\left(1+\frac{\beta}{2\alpha}+\xi\right)A_1-1\right),\end{aligned}$$

where $A_1 = \left(1+\dfrac{\beta}{2\alpha}+\xi\right)^{2n-1}$.

Since

$$\left(\frac{\beta}{2\alpha}+\xi\right)^2 = \frac{\beta^2}{4\alpha^2}+\frac{\beta}{\alpha}\xi+\xi^2 = \frac{\beta^2}{2\alpha^2}+\frac{\beta}{\alpha}+\frac{\beta}{\alpha}\xi = \frac{\beta}{\alpha}\left(\frac{\beta}{2\alpha}+1+\xi\right)$$

and

$$\xi^2 = \frac{\beta}{\alpha}+\frac{\beta^2}{4\alpha^2},$$

we have

$$\begin{aligned}A =&\alpha\left(A_1+1\right)\frac{\beta}{\alpha}\left(\frac{\beta}{2\alpha}+1+\xi\right)-\beta\left(\left(1+\frac{\beta}{2\alpha}+\xi\right)A_1-1\right)\\ =&A_1\cdot 0+2\beta+\frac{\beta^2}{2\alpha}+\beta\xi = 2\alpha\left(\frac{\beta}{\alpha}+\frac{\beta^2}{4\alpha^2}+\frac{\beta}{2\alpha}\xi\right)\\ =&2\alpha\left(\xi^2+\frac{\beta}{2\alpha}\xi\right) = 2\alpha\xi\left(\xi+\frac{\beta}{2\alpha}\right).\end{aligned}$$

Therefore,

$$\mathcal{P}_n - \frac{\beta}{\frac{\beta}{2\alpha}+\xi} = \frac{2\alpha\xi}{\left(1+\frac{\beta}{2\alpha}+\xi\right)^{2n}-1}. \tag{3.3.16}$$

The right hand side of the last expression is positive in $\mathcal{R}$ (since $\alpha, \beta, \xi \succ 0$), therefore

$$\begin{aligned}\left|\mathcal{P}_n - \frac{\beta}{\frac{\beta}{2\alpha}+\xi}\right| =& \frac{2\alpha\xi}{\left(1+\frac{\beta}{2\alpha}+\xi\right)^{2n}-1}\\ =& \frac{2\tau\xi}{L\left(\left(1+\frac{LC}{2\tau^2}+\xi\right)^{2n}-1\right)}\\ =& \frac{LC\tau^{-1}+2\tau-2\tau^3+o\left(\tau^3\right)}{L\left(\left(LC\tau^{-2}+2-\tau^2+o\left(\tau^2\right)\right)^{2n}-1\right)}\\ =& \frac{LC\tau^{-1}+2\tau-2\tau^3+o\left(\tau^3\right)}{L\left(LC\tau^{-2}\right)^{2n}+o\left(\tau^{-4n}\right)}\\ =& \left(C\tau^{-1}+o\left(\tau^{-1}\right)\right)\left(\frac{1}{(LC)^{2n}}\tau^{4n}+o\left(\tau^{4n}\right)\right)\\ =& \frac{C}{(LC)^{2n}}\tau^{4n-1}+o\left(\tau^{4n-1}\right)\to 0,\end{aligned}$$

when $n \to \infty$. Indeed, for any $\epsilon \succ 0$, $\epsilon = \sum_{i=1}^{\infty} \epsilon(q_i)\tau^{q_i}$ we have

$$\frac{C}{(LC)^{2m}}\tau^{4m-1}+o\left(\tau^{4m-1}\right)\prec\epsilon$$

for any $m > \left\lfloor \frac{q_1+1}{4} \right\rfloor + 1$. $\square$

Remark 3.3.9. From the proof one can see that (3.3.15) is true for $\alpha\beta$-network whenever for any $\gamma \in \mathcal{R}$ exists $N_0 \in \mathbb{N}$ such that $n > N_0$ implies $\left(\frac{\beta}{\alpha}\right)^n \succ \gamma$ (see denominator in (3.3.16)).

Example 3.3.10 (CL-network)**.** Let us consider ladder CL-network ([39]), assuming that it has zero potential at infinity. Namely, $\alpha\beta$-network with $\alpha = C\tau^{-1}$, $\beta = L^{-1}\tau$.

Statement 3.3.11. *For the CL-network ($\alpha = C\tau^{-1}$, $\beta = L^{-1}\tau$, $L, C > 0$) the effective admittances of the exhausted finite networks do not converge in $\mathcal{R}$.*

Proof. In this case

$$\xi = \sqrt{\frac{\alpha}{\beta}+\left(\frac{\alpha}{2\beta}\right)^2} = \frac{\tau}{\sqrt{CL}}+\left(\frac{\tau}{2\sqrt{CL}}\right)^3+o(\tau^3).$$

Let us prove, that $\{\mathcal{P}_n\}_{n=1}^{\infty}$ is not a Cauchy sequence in $\mathcal{R}$. Indeed, by (3.3.9)

$$\mathcal{P}_{n+1}-\mathcal{P}_n = \frac{\alpha\left(\frac{\beta}{2\alpha}+\xi\right)\left(\left(1+\frac{\beta}{2\alpha}+\xi\right)^{2n+1}+1\right)}{\left(1+\frac{\beta}{2\alpha}+\xi\right)^{2n+2}-1} - \frac{\alpha\left(\frac{\beta}{2\alpha}+\xi\right)\left(\left(1+\frac{\beta}{2\alpha}+\xi\right)^{2n-1}+1\right)}{\left(1+\frac{\beta}{2\alpha}+\xi\right)^{2n}-1}$$
$$=\alpha\left(\frac{\beta}{2\alpha}+\xi\right)\left(\frac{\left(1+\frac{\beta}{2\alpha}+\xi\right)^{2n+1}+1}{\left(1+\frac{\beta}{2\alpha}+\xi\right)^{2n+2}-1} - \frac{\left(1+\frac{\beta}{2\alpha}+\xi\right)^{2n-1}+1}{\left(1+\frac{\beta}{2\alpha}+\xi\right)^{2n}-1}\right)$$

Since

$$\psi_1 = 1+\frac{\beta}{2\alpha}+\xi = 1+\frac{\tau}{\sqrt{CL}}+o\left(\tau^1\right),$$

we can rewrite

$$\mathcal{P}_{n+1}-\mathcal{P}_n = \alpha\left(\frac{\beta}{2\alpha}+\xi\right)\left(\frac{\psi_1^{2n+1}+1}{\psi_1^{2n+2}-1} - \frac{\psi_1^{2n-1}+1}{\psi_1^{2n}-1}\right)$$
$$=\alpha\left(\frac{\beta}{2\alpha}+\xi\right)\frac{\left(\psi_1^{2n+1}+1\right)\left(\psi_1^{2n}-1\right)-\left(\psi_1^{2n-1}+1\right)\left(\psi_1^{2n+2}-1\right)}{\left(\psi_1^{2n+2}-1\right)\left(\psi_1^{2n}-1\right)}$$
$$=C\tau^{-1}\left(\frac{\tau}{\sqrt{CL}}+o\left(\tau^1\right)\right)\frac{\left(\psi_1^{2n+1}+1\right)\left(\psi_1^{2n}-1\right)-\left(\psi_1^{2n-1}+1\right)\left(\psi_1^{2n+2}-1\right)}{\left(\psi_1^{2n+2}-1\right)\left(\psi_1^{2n}-1\right)}$$

Substituting

$$\left(\psi_1^{2n+1}+1\right)\left(\psi_1^{2n}-1\right)-\left(\psi_1^{2n-1}+1\right)\left(\psi_1^{2n+2}-1\right)$$
$$=\left(2+\frac{\tau}{\sqrt{CL}}(2n+1)+o\left(\tau^1\right)\right)\left(\frac{\tau}{\sqrt{CL}}(2n)+o\left(\tau^1\right)\right)$$
$$-\left(2+\frac{\tau}{\sqrt{CL}}(2n-1)+o\left(\tau^1\right)\right)\left(\frac{\tau}{\sqrt{CL}}(2n+2)+o\left(\tau^1\right)\right)$$
$$=-4\frac{\tau}{\sqrt{CL}}+o\left(\tau^1\right)$$

in the nominator and

$$\left(\psi_1^{2n+2}-1\right)\left(\psi_1^{2n}-1\right)=\left(\frac{\tau}{\sqrt{CL}}(2n+2)+o\left(\tau^1\right)\right)\left(\frac{\tau}{\sqrt{CL}}(2n)+o\left(\tau^1\right)\right)$$
$$=\frac{\tau^2}{CL}(4n^2+4n)+o\left(\tau^2\right)$$

in the denominator, we obtain

$$\mathcal{P}_{n+1}-\mathcal{P}_n=\left(\frac{C}{\sqrt{CL}}+o\left(\tau^0\right)\right)\left(-4\frac{\tau}{\sqrt{CL}}+o\left(\tau^1\right)\right)\left(\frac{CL}{4n^2+4n}\tau^{-2}+o\left(\tau^{-2}\right)\right)$$
$$=\frac{-4C}{4n^2+n}\tau^{-1}+o\left(\tau^{-1}\right)\succ\tau \text{ for any } n\in\mathbb{N} \text{ and for any } L,C>0.$$

Therefore, $\{\mathcal{P}_n\}_{n=1}^{\infty}$ is not a Cauchy sequence in $\mathcal{R}$. □

Therefore, the following question:

> Under what conditions the effective admittance of infinite network over non-Archimedean field could be defined?

remains open. Note that Remark 3.3.9 gives a sufficient condition for $\alpha\beta$-network.

3.4 Relation to the networks over $\mathbb{C}$

In this section we consider some relations between finite networks, introduced in Chapter 2 (Definition 2.1.1) and ones over an ordered field $\mathbb{R}(\lambda)$ (Definition 3.1.4, $\mathcal{K} = \mathbb{R}(\lambda)$).

Denote by $\mathcal{P}_{\mathbb{C}}(\lambda)$ the effective admittance considered in Chapter 2 for a finite network (see Definition 3.2.4). The effective admittance from present chapter (see Definition 3.2.4) for the field $\mathcal{K} = \mathbb{R}(\lambda)$ will be denoted by $\mathcal{P}_{\mathbb{R}(\lambda)}(\lambda)$. Note that it was already defined as a rational function on λ. Any rational function of $\lambda \in \mathbb{C}$ takes values in $\mathbb{C} \cup \{\infty\}$ (for any $\lambda \in \mathbb{C}$). [1] Consequently, any rational function is continuous with the values in $\mathbb{C} \cup \{\infty\}$. In particular, $\mathcal{P}_{\mathbb{R}(\lambda)}(\lambda)$ is a continuos function of $\lambda \in \mathbb{C}$ with values in $\mathbb{C} \cup \{\infty\}$ (including the value ∞).

Firstly, let us point out, that the admittance of an edge in form (2.1.3), i.e.

$$\rho_{xy}^{(\lambda)} = \frac{1}{L_{xy}\lambda + R_{xy} + \frac{D_{xy}}{\lambda}} = \frac{\lambda}{L_{xy}\lambda^2 + R_{xy}\lambda + D_{xy}}, R_{xy} + L_{xy} + D_{xy} > 0$$

is always a positive function in $\mathbb{R}(\lambda)$ (see Definition 3.1.1). But the opposite is not true. For example, the rational function

$$\frac{\lambda^2 - \lambda + 1}{\lambda}$$

is positive in $\mathbb{R}(\lambda)$, but it is not an admittance. Therefore, the weighted graph from Example 2.3.3 and Example 3.2.22 is not a network by definition from Chapter 2, but it is a network over an ordered field $\mathbb{R}(\lambda)$.

For finite networks, which can be considered in both approaches (i.e. for all networks by Definition 2.1.1), the question

> for which $\lambda \in \mathbb{C} \setminus \{0\}$ one have $\mathcal{P}_{\mathbb{C}}(\lambda) = \mathcal{P}_{\mathbb{R}(\lambda)}(\lambda)$?

is the same as the question of continuity of $\mathcal{P}_{\mathbb{C}}(\lambda)$, discussed in Subsection 2.3. Indeed, By Cramer's rule, applied in $\mathbb{C}$, the function $\mathcal{P}_{\mathbb{C}}(\lambda)$ is also a rational function of λ at all λ, where the determinant $\mathcal{D}(\lambda)$ of the Dirichlet problem does not vanish

[1] For any rational function $\frac{P(\lambda)}{Q(\lambda)}$ with polynomials P, Q, one can always assume that P and Q have no common zeros, since otherwise their common linear factor can be cancelled. Hence, one avoids indeterminate form $\frac{0}{0}$.

(see Theorem 2.2.11). Moreover, at those λ, where $\mathcal{D}(\lambda) \neq 0$ the equality $\mathcal{P}_{\mathbb{C}}(\lambda) = \mathcal{P}_{\mathbb{R}(\lambda)}(\lambda)$ is true by Cramer's rule.

By Theorem 2.2.11, $\mathcal{D}(\lambda)$ vanishes only at finitely many values of λ, therefore, the identity will be true at $\lambda = \lambda_0$ if we know that $\mathcal{P}_{\mathbb{C}}(\lambda)$ is continuous in λ_0. Hence, by the same theorem, the identity is true for all but finite number of values λ, and the identity is true for all λ such that $\operatorname{Re}\lambda > 0$. The question remains **open** for λ on the imaginary axis. For λ such that $\operatorname{Re}\lambda < 0$ the effective admittances not always coincide (see Examples 2.3.2 and 3.2.21 for $\lambda = -1$).

An interesing observation is that Theorem 3.2.1 provides another proof of Theorem 2.2.11. Namely, the fact that the determinant of the Dirichlet problem (2.2.1) is not constantly zero follows from Theorem 3.2.1, applied in the ordered field $\mathcal{K} = \mathbb{R}(\lambda)$.

Appendix A

Analysis on Graphs

Here we recall some definitions from analysis on graphs (see, e.g. [17]).

Definition A.0.1. *A* graph *G is a couple (V, E) where V is a set of vertices, that is, an arbitrary set, whose elements are called* vertices*, and E is a set of* edges*, that is E consists of some unordered couples (x, y) where $x, y \in V, x \neq y$. We write $x \sim y$ (x is adjoint to y) if $(x, y) \in E$. We denote the edge (x, y) by $x \sim y$ or xy.*

For each point x, define its degree

$$\deg(x) = |\{y \in V : x \sim y\}|. \tag{A.0.1}$$

that is $\deg(x)$ *is the number of edges with endpoint x.*

Definition A.0.2. *A graph is called* finite*, if it has finite number of vertices. Otherwise, a graph is called* infinite.

A graph *G is called* locally finite *if* $\deg(x) < \infty$ *for all vertices x of the graph).*

Definition A.0.3. *A finite sequence $\{x_k\}_{k=0}^n$ of vertices on a graph is called a* path *if $x_k \sim x_{k+1}$ for all $k = 0, 1, \ldots, n-1$. The number n of edges in the path is referred to as the* length *of the path.*

Definition A.0.4. *A graph (V, E) is called* connected *if, for any two vertices $x, y \in V$, there is a path connecting x and y, that is a path $\{x_k\}_{k=0}^n$ such that $x_0 = x$ and $x_n = y$.*

If a graph (V, E) is connected, then define the graph distance $\mathrm{dist}(x, y)$ between any two vertices x, y as follows: if $x \neq y$ then $\mathrm{dist}(x, y)$ is the minimal length of a path that connects x and y, and if $x = y$, then $\mathrm{dist}(x, y) = 0$.

Lemma A.0.5. *On any connected graph, the graph distance is a metric.*

Theorem A.0.6. *If (V, E) is a connected locally finite graph, then the set of vertices V is either finite or countable. Moreover, $V = \bigcup\limits_{n=1}^{\infty} B_n$, where*

$$B_n = \{y \in V : \mathrm{dist}(x, y) \leq n\}$$

that is a ball with respect to the graph distance, for any fixed reference vertex x.

Definition A.0.7. *A* weighted graph *is a couple (G, μ) where $G = (V, E)$ is a graph and $\mu : V \times V \to \mathbb{R}$ is a non-negative function such that*

- *$\mu_{xy} = \mu_{yx}$,*
- *$\mu_{xy} \neq 0$ if and only if $x \sim y$.*

Any weight μ_{xy} gives rise to a function on vertices as follows:

$$\mu(x) = \sum_{y:y\sim x} \mu_{xy}.$$

Then $\mu(x)$ is called the weight *of a vertex x.*

Since the weight μ contains full information about the set of edges E, the weighted graph can also be denoted by (V, μ).

Definition A.0.8. *Let (V, μ) be a locally finite connected weighted graph. For any function $f : V \to \mathbb{R}$, define the function $\Delta_\mu f$ by*

$$\Delta_\mu f(x) = \frac{1}{\mu(x)} \sum_{y \in V} f(y)\mu_{xy} - f(x). \tag{A.0.2}$$

The operator Δ_μ acting on functions on V, is called the (discrete weighted) Laplace operator *on (V, μ).*

Appendix B

Theory of Ordered Fields

B.1 Definitions and results on ordered fields

In this section we recall some definitions on ordered fields from algebra (see, e.g. [5], [6], [22], [37]).

Definition B.1.1. *A field* $(\mathcal{K}, +, \cdot)$ *is called* ordered *if the property of positiveness* $(\succ 0)$ *is defined for its elements, and if it satisfies the following postulates:*

- *for every element* k *in* $\mathcal{K}$*, just one of the relations*

$$k = 0, k \succ 0, -k \succ 0$$

  is valid.

- *if* $k_1, k_2 \in \mathcal{K}$*,* $k_1 \succ 0$ *and* $k_2 \succ 0$*, then* $k_1 + k_2 \succ 0$ *and* $k_1 \cdot k_2 \succ 0$*.*

If $k \in \mathcal{K}, k \succ 0$*, then we will call* k positive. *We will denote the set of all positive elements by* $\mathcal{K}^+$*.*

If $-k \succ 0$, we say: k is negative $(k \prec 0)$.

The ordering relation $k_1 \succ k_2$ in an ordered field is now defined by

$$\begin{gathered} k_1 \succ k_2, \text{ in words: } k_1 \text{ is greater than } k_2 \\ (\text{or } k_2 \prec k_1, \text{ in words: } k_2 \text{ is less than } k_1) \\ \text{if } k_1 - k_2 \succ 0. \end{gathered}$$

For $k_1, k_2 \in \mathcal{K}$ we will write

$$k_1 \succeq k_2, \text{ if } k_1 \succ k_2 \text{ or } k_1 = k_2.$$

Moreover, we will write

$$k_1 \prec k_2, \text{ if } k_2 \succ k_1$$

and

$$k_1 \preceq k_2, \text{ if } k_1 \prec k_2 \text{ or } k_1 = k_2.$$

Lemma B.1.2. *The ordering relation satisfies the following properties:*

- $1 \succ 0$,

- *for any* $k \in \mathcal{K}$ *from* $k \succ 1$ *follows* $k^{-1} \prec 1$,
- *for any* $k \in \mathcal{K}, k \neq 0$ *from* $k \prec 1$ *follows* $k^{-1} \succ 1$,
- *for any* $k \in \mathcal{K}$ *from* $k \succ 0$ *follows* $k^{-1} \succ 0$,
- *for any* $k_1, k_2 \in \mathcal{K}$ *either* $k_1 \succ k_2$, $k_1 = k_2$ *or* $k_2 \succ k_1$,
- *for any* $k_1, k_2 \in \mathcal{K}$ *from* $k_1 \succ k_2$ *follows* $-k_1 \prec -k_2$,
- *for any* $k_1, k_2 \in \mathcal{K}$ *from* $k_1 \succ 0$, $k_2 \succ 0$ *and* $k_1 \succ k_2$ *follows* $k_1^{-1} \prec k_2^{-1}$,
- *for any* $k_1, k_2, k_3 \in \mathcal{K}$ *from* $k_1 \succ k_2$ *and* $k_2 \succ k_3$ *follows* $k_1 \succ k_3$,
- *for any* $k_1, k_2, k_3 \in \mathcal{K}$ *from* $k_1 \succ k_2$ *follows* $k_1 + k_3 \succ k_2 + k_3$,
- *for any* $k_1, k_2, k_3, k_4 \in \mathcal{K}$ *from* $k_1 \succ k_2$ *and* $k_3 \succ k_4$ *follows* $k_1 + k_3 \succ k_2 + k_4$,
- *for any* $k_1, k_2, k_3 \in \mathcal{K}$ *from* $k_1 \succ k_2$ *and* $k_3 \succ 0$ *follows* $k_1 \cdot k_3 \succ k_2 \cdot k_3$,
- *for any* $k_1, k_2, k_3, k_4 \in \mathcal{K}$ *from* $k_1 \succ k_2$, $k_3 \succ k_4 \succ 0$ *follows* $k_1 \cdot k_3 \succ k_2 \cdot k_4$.

Moreover,

Theorem B.1.3. "$\succeq$" *defines a* total order *on the set* $\mathcal{K}$, *i.e.*

- *for any* $k \in \mathcal{K}$, $k \succeq k$,
- *for any* $k_1, k_2, k_3 \in \mathcal{K}$, *if* $k_1 \succeq k_2$ *and* $k_2 \succeq k_3$, *then* $k_1 \succeq k_3$,
- *for any* $k_1, k_2 \in \mathcal{K}$, *if* $k_1 \succeq k_2$ *and* $k_2 \succeq k_1$, *then* $k_1 = k_2$,
- *for any* $k_1, k_2 \in \mathcal{K}$ *either* $k_1 \succeq k_2$ *or* $k_2 \succeq k_1$.

Lemma B.1.4. *Any finite non-empty subset* F *of an ordered field has maximum and minimum.*

Proof. Induction by $|F|$ (i.e. number of elements in F), using the fact that "$\succeq$" is a total order. ☐

Definition B.1.5. *An ordered field* $\mathcal{K}$ *is called* Archimedean *if for any* $k \in \mathcal{K}$ *there exist* $n \in \mathbb{N}$, *i. e.*

$$n = \underbrace{1 + \cdots + 1}_{n} \in \mathcal{K},$$

such that $n \succ k$.

Definition B.1.6. *The* absolute value *of* $k \in \mathcal{K}$ *is defined as follows:*

$$|k| = \begin{cases} k \text{ if } k \succeq 0, \\ -k \text{ otherwise (if } k \prec 0). \end{cases}$$

Lemma B.1.7. *The mapping* "$|\cdot|$" $: \mathcal{K} \to \mathcal{K}$ *has the following properties:*

for any $k, k_1, k_2 \in \mathcal{K}$

- $|k| = 0$ *if and only if* $k = 0$,

- $k \preceq |k|$,
- $|k_1 \cdot k_2| = |k_1| \cdot |k_2|$,
- $|k_1 + k_2| \preceq |k_1| + |k_2|$,
- $||k_1| - |k_2|| \preceq |k_1 - k_2|$,
- $|k_1| \prec k_2 \Leftrightarrow -k_2 \prec k_1 \prec k_2$,
- *if* $k_1 \neq 0$ *then* $|k_1^{-1}| = |k_1|^{-1}$.

Definition B.1.8. *The characteristic of a field is the smallest number* $n \in \mathbb{N}$ *such that*

$$n = \underbrace{1 + \cdots + 1}_{n} = 0 \in \mathcal{K}.$$

Lemma B.1.9. *The characteristic of an ordered field is zero.*

B.2 Levi-Civita field $\mathcal{R}$

In this section we recall some definitions and results on Levi-Civita field $\mathcal{R}$ (see, e.g. [4], [19], [28], [29], [30]). See also [6, pp. A.VI.20-A.VI.28] for the general properties of ordered fields.

B.2.1 Basic definitions

Definition B.2.1. *A subset M of the rational numbers $\mathbb{Q}$ is called* left-finite *if for every number $r \in \mathbb{Q}$ there are only finitely many elements of M that are smaller than r. The set of all left-finite subsets of $\mathbb{Q}$ will be denoted by $\mathbb{S}$.*

Definition B.2.2. *We define the Levi-Civita field $\mathcal{R}$ as*

$$\mathcal{R} = \{\alpha : \mathbb{Q} \to \mathbb{R} \mid \operatorname{supp} \alpha(q) \subset \mathbb{S}\}, \tag{B.2.1}$$

i. e. the field consists of real valued functions on $\mathbb{Q}$ with left-finite support.

Definition B.2.3. Addition *in $\mathcal{R}$ is defined componentwise:*

$$(\alpha + \beta)(q) = \alpha(q) + \beta(q) \tag{B.2.2}$$

for any $\alpha, \beta \in \mathcal{R}$.

Multiplication *in $\mathcal{R}$ is defined as follows*

$$(\alpha \cdot \beta)(q) = \sum_{\substack{q_\alpha, q_\beta \in \mathbb{Q}, \\ q_\alpha + q_\beta = q}} \alpha(q_\alpha) \cdot \beta(q_\beta). \tag{B.2.3}$$

for any $\alpha, \beta \in \mathcal{R}$.

Theorem B.2.4. $(\mathcal{R}, +, \cdot)$ *is a field.*

Theorem B.2.5. *If we define in $\mathcal{R}$ the set of positive elements as*

$$\mathcal{R}^+ = \{\alpha \in \mathcal{R} \mid \alpha(\min\{\mathrm{supp}(\alpha)\}) > 0\}, \tag{B.2.4}$$

then $(\mathcal{R}, +, \cdot)$ becomes an ordered field.
Theorem B.2.6. *The field of real numbers $\mathbb{R}$ can be embedded in $\mathcal{R}$ under preservation of its arithmetic and order structure. The embedding Π is as follows: for any $h \in \mathbb{R}$*

$$\Pi_h(q) = \begin{cases} h, & \textit{if } q = 0, \\ 0, & \textit{otherwise}. \end{cases} \tag{B.2.5}$$

Theorem B.2.7. *Let $\alpha \in \mathcal{R}$ be nonzero.*

- *If $n \in \mathbb{N}$ is even and $\alpha \succ 0$, then α has two nth roots in $\mathcal{R}$.*
- *If $n \in \mathbb{N}$ is even and $\alpha \prec 0$, then α has no nth roots in $\mathcal{R}$.*
- *If $n \in \mathbb{N}$ is odd, then α has a unique root in $\mathcal{R}$.*

In other words, $\mathcal{R}$ is a real-closed field.

Therefore, since for any $\beta \in \mathcal{R}$ from $\beta^2 = \alpha$ follows $(-\beta)^2 = \alpha$, there is exactly one positive square root of any $\alpha \succ 0$. We will denote it by $\sqrt{\alpha}$.
Definition B.2.8. *We define* the element $\tau \in \mathcal{R}$ *as follows:*

$$\tau(q) = \begin{cases} 1, & \textit{if } q = 1, \\ 0, & \textit{otherwise}. \end{cases} \tag{B.2.6}$$

The element τ is *infinitesimal* in $\mathcal{R}$, i.e. it is less, than every positive real number (see (B.2.5)).
Remark B.2.9. The field $\mathcal{R}$ is non-Archimedean: $n \prec \tau^{-1}$ for any $n \in \mathbb{N}$.

B.2.2 Topology in $\mathcal{R}$, convergence and Cauchy-completness.

Most definitions and statements in this subsection are from [4] and [28].
Definition B.2.10 (Order topology)**.** *We call a subset M of $\mathcal{R}$ open if for any $\alpha_0 \in M$ there exists an $\epsilon \succ 0, \epsilon \in \mathcal{R}$ such that an ϵ-ball*

$$O(\alpha_0, \epsilon) = \{\alpha \in M \mid |\alpha - \alpha_0| \preceq \epsilon\}$$

is a subset of M.

All sets $O(\alpha_0, \epsilon)$ form a basis of topology.
Theorem B.2.11. *With the above topology, $\mathcal{R}$ becomes nonconnected topological space. It is Hausdorff. There is no countable base. The topology induced to $\mathbb{R}$ is the discrete topology. The topology is not locally compact.*
Definition B.2.12. *We call the sequence $\{\alpha_n\}_{i=n}^{\infty}$ in $\mathcal{R}$* strongly convergent *to the limit $\alpha \in \mathcal{R}$ if it converges to α with respect to an order topology, i.e. for every $\epsilon \succ 0, \epsilon \in \mathcal{R}$ there exists $N_0 \in \mathbb{N}$ such that $|\alpha_n - \alpha| \preceq \epsilon$ for any $n > N_0$.*

Theorem B.2.13. *Let $\alpha \in \mathcal{R}$. Then α is uniquely characterized by an ascending (finite or infinite) sequence $\{q_i\}$ of support points and corresponding sequence $\{\alpha(q_i)\}$ of function values, and the sequence $\alpha_n = \sum_{i=1}^{n} \alpha(q_i)\tau^{q_i}$ converges strongly to the limit α. Hence we can write*

$$\alpha = \sum_{i=1}^{\infty} \alpha(q_i)\tau^{q_i}. \tag{B.2.7}$$

Theorem B.2.14 (Cauchy-completness of $\mathcal{R}$)**.** *$\{\alpha_n\}$ is a Cauchy sequence in $\mathcal{R}$ (i. e. for any positive $\epsilon \in \mathcal{R}$ exists $N_0 \in \mathbb{N}$ such that $|\alpha_l - \alpha_m| \preceq \epsilon$ for all $l, m \geq N_0$), if and only if $\{\alpha_n\}$ converges strongly.*

Appendix C

Some Known Physical Laws and Statements

In this section we recall some basic physical concepts and laws from the theory of electrical networks. See, e.g. [9], [12], [14], [15], [16], [20], [27] and [31] for the references.

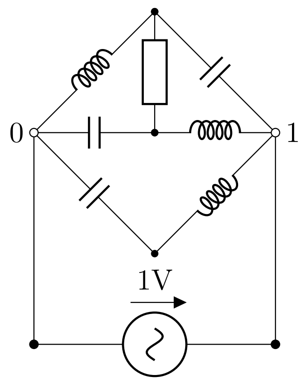


Figure C.1: Physical electrical network

In an electrical network three types of passive elements (impedances) can occur:

1. resistor, the voltage drop on which by Ohm's law is

$$V_R(t) = RI_R(t), \tag{C.0.1}$$

where $I_R(t)$ is the current in this segment of the circuit at time t and $R \in (0, +\infty)$ is the resistance;

2. inductor, the voltage drop on which by Faraday's law is

$$V_L(t) = LI'_L(t), \tag{C.0.2}$$

where $I_L(t)$ is the current in this segment of the circuit at time t and $L \in (0, +\infty)$ is the inductance;

3. capacitor, the voltage drop on which is

$$V_C(t) = \frac{Q(t)}{C}, \tag{C.0.3}$$

where $Q(t)$ is the charge of the capacitor at time t (also, $Q'(t) = I_C(t)$, where $I_C(t)$ is the current in this segment of the circuit at time t) and $C \in (0, +\infty)$ is the capacity.

Statement C.0.1 (Kirchhoff's law)**.** *At any node in an electrical circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of that node:*

$$\sum_k I_k(t) = 0. \tag{C.0.4}$$

For any electrical network, when external periodic voltage of frequency ω is applied, the time-varying voltage $V(t)$ can be written as

$$V(t) = \widehat{V} e^{i\omega t},$$

where $\widehat{V}$ is a complex number that is independent of t, ω is the frequency of an alternating current. The current $I(t)$ can be represented in the similar way

$$I(t) = \widehat{I} e^{i\omega t}$$

(see e.g. [15, p.22-1]).

Statement C.0.2 (Kirchhoff's complex law)**.** *At any node in an electrical circuit, the sum of complex currents flowing into that node is equal to the sum of complex currents flowing out of that node:*

$$\sum_k \widehat{I}_k = 0. \tag{C.0.5}$$

For the complex voltage $\widehat{V}$ on a segment of an electrical circuit the following is true:

$$\widehat{V} = \widehat{I} Z, \tag{C.0.6}$$

where $\widehat{I}$ is a complex current in the segment and Z is its impedance. For the passive elements the impedances are as follows:

- $R \in (0, \infty)$ for the resistor,
- $Li\omega$, $L \in (0, \infty)$ for the inductor,
- $\dfrac{1}{Ci\omega}$, $C \in (0, \infty)$ for the capacitor,

where ω is the frequency of an alternating current, i is the imaginary unit (see e.g. [14, p.23-6]).

The equation (C.0.6) is a complex generalization of the Ohm's law.

Statement C.0.3 (Series law)**.** *[14, p. 25-9] The impedance of two passive elements, connected in series, is equal to the sum of their impedances.*

Figure C.2: Physical series law

Statement C.0.4. *[Conservation of complex power] In any AC network, operating in a steady state, the sum of complex powers delivered by sources is equal to the sum of complex powers absorbed by passive elements. Here the complex power can be calculated by the following formula:*

$$S = \widehat{V}\overline{I}, \tag{C.0.7}$$

where S is a complex power, $\widehat{V}$ is a complex voltage, $\overline{I}$ is a complex conjugate of a complex current. Then $\operatorname{Re} S$ *is equal to the real power and* $\operatorname{Im} S$ *is equal to the reactive power.*

Statement C.0.5 (Superposition theorem)**.** *[20, p.75 and p.329] In any network containing more than one source, the current in, or the potential difference (voltage) across, any branch can be found by considering each source separately and adding their effects: omitted sources of electromotive force are replaced by impedances equal to their internal impedances.*

List of Figures

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