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Nash Smoothing on the Test Bench: H_α–Essential Equilibria

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Abstract: We extend the analysis of van Damme (1987, Section 7.5) of the famous smoothing demand in Nash (1953) as an argument for the singular stability of the symmetric Nash bargaining solution among all Pareto efficient equilibria of the Nash demand game. Van Damme's analysis provides a clean mathematical framework where he substantiates Nash's conjecture by two fundamental theorems in which he proves that the Nash solution is among all Nash equilibria of the Nash demand game the only one that is H–essential. We show by generalizing this analysis that for any asymmetric Nash bargaining solution a similar stability property can be established that we call H_α–essentiality. A special case of our result for $\alpha = 1/2$ is $H_{1/2}$–essentiality that coincides with van Damme's H–essentiality. Our analysis deprives the symmetric Nash solution equilibrium of Nash's demand game of its exposed position and fortifies our conviction that, in contrast to the predominant view in the related literature, the only structural difference between the asymmetric Nash solutions and the symmetric one is that the latter one is symmetric.

While our proofs are mathematically straightforward given the analysis of van Damme (1987), our results change drastically the prevalent interpretation of Nash's smoothing of his demand game and dilute its conceptual importance.

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1 Introduction

In the abstract of Serrano in Homo Oeconomicus, *Sixty-Seven Years of the Nash Program: Time for Retirement?* Roberto Serrano writes: "*The program is thus turning sixty-seven years old, but I will argue it is not ready for retirement, as it is full of energy and one can still propose important directions to be explored.*"

The last two passages in the introduction of Nash (1953) beginning with the sentence: "*We give two independent derivations of our solution of the two-person cooperative game*" are the origin of what is now called *Nash program.* The solution referred to here is the *Nash bargaining* solution.

After a description and formalization of his *Negotiation Model*, Nash provided the object of our analysis by the following passage:

"*What we have is actually a two move game. Stages two and four do not involve any decisions by the players. The second move choices are made with full information about what was done in the first move. Therefore, the game of the second move alone may be considered separately (it is a game with a variable payoff function determined by the choices made at the first move) . . . The demand game defined by these payoff functions will generally have an infinite number of inequivalent equilibrium points. . . Thus the points do not lead us immediately to a solution of the game. But if we discriminate between them by studying their relative stabilities we can escape from this troublesome non–uniqueness.*

To do this we "smooth" the game to obtain a continuous payoff function and then study the limiting behavior of the equilibrium points of the smoothed game as the amount of smoothing approaches zero.

A certain general class of natural smoothing methods will be considered here. This class is broader than one might at first think, for many other methods that superficially seem different are actually equivalent."

While Nash's own analysis provides a clear picture of what he has in mind it is lacking a complete formal model and a precise analysis that would result in a mathematical theorem. Nevertheless, Luce and Raiffa write in Section 6.9 on the Nash demand game:

"*as Nash is well aware, there is in general a continuum of other inequivalent equilibrium pairs* [of payoffs].*The weak link in the argument is to single out this particular pair. Nash offers an ingenious and mathematically sound argument for doing so, but we fail to see why it is relevant. Nash then shows that this "solution" is the only necessary limit of the equilibrium points of smooth games.*"

Several authors have later, despite some critical remarks, disagreed with this statement regarding the relevance of Nash's smoothing and also regarding the mathematical soundness of Nash's treatment, and have attempted to remedy the deficiencies or to just give a concise presentation of Nash's analysis. Despite some small differences in their modellings they all confirm the stability.

Osborne and Rubinstein (1990) write in their Section 4.3.2: *The Perturbed Demand Game*: "*Given that the notion of Nash equilibrium puts so few restrictions on the nature of the outcome of a demand game, Nash considered a more discriminating notion of equilibrium, which is related to Selten's (1975) [trembling hand] perfect equilibrium*". They confirm Nash in their Proposition 4.3. Other formal treatments can be found in Binmore (1987, Section 4), van Damme (1987, Section

7.5), Peters (1992, Section 9.3), Roemer (1998, Section 2.2). See also Carlsson (1991), Kaneko (1981, Section 3) and Malueg (2010) for further comments on Nash's smoothing.

All these approaches follow Nash's idea to define a family of disturbed games where symmetry is already inherent. The families of perturbations of the simple demand game are defined with **the symmetric Nash product** as the fundamental ingredient. Consideration of Figure 1 in Nash (1953) illustrates very well that the perturbations are defined via the symmetric Nash product. From the picture one cannot see any argument why analogous figures centered at asymmetric Nash solution points should behave qualitatively differently!

It appears credible that the symmetric Nash solution is stable in the sense of Nash. So the formal proofs are not surprising. What surprised us is that nobody tried to check the stability of the other equilibria of the demand game. Only if they would turn out not to be stable under similar criteria the exposed singular role of the symmetric Nash equilibrium, accepted without exception in the literature, would be justified.

In this paper we are making the litmus test. After we had first looked for the most convincing and rigorous treatment, we decided for van Damme (1987) and Peters (1992). We choose van Damme (1987) which was earlier and had been followed and quoted by Peters (1992). Hence, we will follow, also close in our notation, the analysis in Section 7.5 of van Damme (1987). As he felt Nash's stability concept to be close to essentiality of equilibria as defined in Wu and Jiang (1962) he called it H–essentiality. We will define for any $\alpha \in (0,1)$ the concept of H_α–essentiality, that coincides for $\alpha = 1/2$ with van Damme's H–essentiality. We extend his Theorems 7.5.4 and 7.5.5 on $H_{1/2}$–essentiality of the symmetric Nash solution to H_α–essentiality of α–symmetric Nash solutions for any $\alpha \in (0,1)$. It is quite obvious that also the other quoted theorems in the literature can mutatis mutandis be extended in analogous ways.

2 Definitions

Our main object of analysis is the two-person cooperative bargaining game defined and analyzed in detail in Nash (1950) and Nash (1953), respectively.

The compact convex subset S of $\mathbb{R}^2$ represents the two-player feasible payoff vectors, the point $d \in S$ is the status quo point that describes the players' payoffs in the case of disagreement about any other feasible $x \in S$. Like van Damme (1987), we assume for convenience that S is d–comprehensive which means that for any $y \in S$, we have that if $x \in \mathbb{R}^2$ with $d \leq x \leq y$, then $x \in S$.

Moreover, we assume wlog that (S, d) is $(0,1)$–normalized, i.e., $d = 0$ and $\max\limits_{x \in S} x_1 = \max\limits_{x \in S} x_2 = 1$. So $(1,1)$ is the utopia point of S. As S is convex set, its efficient boundary ∂S is the graph of a continuous function $f : [0,1] \to [0,1]$ such that $(x_1, f(x_1)) \in \partial S$.

In this framework, we follow as far as possible van Damme (1987) in notation and terminology. We fix some arbitrary $\alpha \in (0,1)$, and consider wlog the case $\alpha \leq 1 - \alpha$. We sometimes denote the bargaining game $(S, 0)$ by S.

Let Γ be the Nash demand game associated with S and described by the following rules: the players state their demands x_1, x_2 simultaneously and independently, and if the demands are feasible, i.e., $(x_1, x_2) \in S$, then each player receives her demand, otherwise each player receives her disagreement outcome of 0. Formally speaking, $\Gamma = (\mathbb{R}^2_+, \mathbb{R}^2_+, R_1, R_2)$ is the demand game where for all i, $R_i(x_1, x_2) = x_i \chi_S(x_1, x_2)$ in which χ_S denotes the characteristic function of S with $\chi_S(x) = 1$ if $x \in S$, otherwise $\chi_S(x) = 0$.

Our class H_α of perturbations of the characteristic function χ_S is defined in

Definition 1.

Definition 1. *Let* $H_\alpha = \bigcup_{\epsilon>0} H_\alpha^\epsilon$ *be such that for all* $\epsilon > 0$, H_α^ϵ *is the set of functions satisfying*

$h_\alpha^\epsilon : \mathbb{R}_+^2 \longrightarrow (0,1]$ *is continous,* $h_\alpha^\epsilon(x) = 1$ *for all* $x \in S$ *and*

$\max\left\{h_\alpha^\epsilon(x), x_1^{2\alpha}x_2^{2(1-\alpha)}h_\alpha^\epsilon(x)\right\} < \epsilon$ *if* $\rho(x,S) > \epsilon$,
where ρ *denotes the Euclidean distance between* x *and* S.

Like in van Damme (1987), we collect functions which at points near to S still take values close to 1, but then, as Nash (1950) had called it, '*taper off very rapidly towards zero*', as x moves away from S.

We define the disturbed game $\Gamma(h_\alpha^\epsilon) = (\mathbb{R}_+^2, \mathbb{R}_+^2, R_{\alpha_1}^\epsilon, R_{\alpha_2}^\epsilon)$ with $R_{\alpha_1}^\epsilon(x) = x_1(h_\alpha^\epsilon(x))^{1/2\alpha}$ and $R_{\alpha_2}^\epsilon(x) = x_2(h_\alpha^\epsilon(x))^{1/2(1-\alpha)}$. Notice that for $\alpha = 1/2$ this amounts to $R_{(1/2)_i}^\epsilon(x) = x_i h_{1/2}^\epsilon(x) = R_i^h(x)$ as in van Damme's framework.

Definition 2. *An equilibrium* x *of* Γ *is called* H_α*-essential if for any* $(h_\alpha^\epsilon)_{\epsilon\to 0^+}$ *with* $h_\alpha^\epsilon \in H^\epsilon$, *there exists* $(x_\alpha^\epsilon)_{\epsilon\to 0^+}$ *such that* $x_\alpha^\epsilon \to x$ *as* $\epsilon \to 0$ *and* x_α^ϵ *is an equilibrium of* $\Gamma(h_\alpha^\epsilon)$.

Let x_α^* be the maximizer of $x_1^\alpha x_2^{1-\alpha}$ on S, i.e., the α–symmetric Nash solution of the game (S,d) for the given α. Our first theorem generalizes Theorem 7.5.4 in van Damme (1987).

Theorem 1. x_α^* *is an* H_α*-essential equilibrium of* Γ.

Proof. Note that $(x_{\alpha_1}^*)^\alpha (x_{\alpha_2}^*)^{1-\alpha} > 0$ and that for $h_\alpha^\epsilon \in H_\alpha^\epsilon$ when $\epsilon \in (0, (x_{\alpha_1}^*)^{2\alpha}(x_{\alpha_2}^*)^{2(1-\alpha)})$, there exists a point x^ϵ where by continuity of the function $x_1^{2\alpha}x_2^{2(1-\alpha)}h_\alpha^\epsilon$ reaches its maxiumum. By the definition of H_α^ϵ, we get $\rho(x^\epsilon, S) \le \epsilon$.

The maximality of x^ϵ implies for all $x_1 \in \mathbb{R}_+$, $x_1^{2\alpha}(x_2^\epsilon)^{2(1-\alpha)}h_\alpha^\epsilon(x_1, x_2^\epsilon) \le (x_1^\epsilon)^{2\alpha}(x_2^\epsilon)^{2(1-\alpha)}h_\alpha^\epsilon(x_1^\epsilon, x_2^\epsilon)$. By cancelling $(x_2^\epsilon)^{2(1-\alpha)} > 0$, we get $x_1^{2\alpha}h_\alpha^\epsilon(x_1, x_2^\epsilon) \le$

$(x_1^\epsilon)^{2\alpha} h_\alpha^\epsilon(x_1^\epsilon, x_2^\epsilon)$, hence $R_{\alpha_1}^\epsilon(x_1, x_2^\epsilon) \leq R_{\alpha_1}^\epsilon(x^\epsilon)$ for all $x_1 \in \mathbb{R}_+$. Similarly, one derives $R_{\alpha_2}^\epsilon(x_1^\epsilon, x_2) \leq R_{\alpha_2}^\epsilon(x^\epsilon)$ for all $x_2 \in \mathbb{R}_+$. This establishes that x^ϵ is an equilibrium of $\Gamma(h_\alpha^\epsilon)$.

Now, consider the (generalized) sequence $(x^\epsilon)_\epsilon$. We get $(x_{\alpha_1}^*)^{2\alpha}(x_{\alpha_2}^*)^{2(1-\alpha)} \leq \lim\limits_{\epsilon\to 0}(x_1^\epsilon)^{2\alpha}(x_2^\epsilon)^{2(1-\alpha)}$, since $(x_{\alpha_1}^*)^{2\alpha}(x_{\alpha_2}^*)^{2(1-\alpha)} = (x_{\alpha_1}^*)^{2\alpha}(x_{\alpha_2}^*)^{2(1-\alpha)} h_\alpha^\epsilon(x_\alpha^*) \leq (x_1^\epsilon)^{2\alpha}(x_2^\epsilon)^{2(1-\alpha)} h_\alpha^\epsilon(x^\epsilon) \leq (x_1^\epsilon)^{2\alpha}(x_2^\epsilon)^{2(1-\alpha)}$. Moreover, $\lim\limits_{\epsilon\to 0}(x_1^\epsilon)^{2\alpha}(x_2^\epsilon)^{2(1-\alpha)} \leq (x_{\alpha_1}^*)^{2\alpha}(x_{\alpha_2}^*)^{2(1-\alpha)}$ since $\lim\limits_{\epsilon\to 0} x^\epsilon \in \partial S$. Thus, $\lim\limits_{\epsilon\to 0} x^\epsilon = x_\alpha^*$.

□

We describe the distance of a point $x \in \mathbb{R}^2$ outside S to S by the Minkowski functional or *gauge* of S, denoted γ_S which is defined as $\gamma_S : \mathbb{R}_+^2 \setminus S \to \mathbb{R}_+$ where $\gamma_S(x) = inf\{t > 0 \mid x/t \in S\}$. As S is fixed in our analysis, we skip S and denote this gauge γ. Since the set S is convex and closed with $0 \in S$, the function γ is continuous [cf. Aliprantis and Border (1994, Theorem 4.37)]. In order to make the analysis simpler, we assume that the function f defined above is differentiable. Then ∂S becomes smooth and γ differentiable on $\mathbb{R}_+^2 \setminus S$ due to its positive linear homogeneity.

As for any $x \in \partial S$, we have $f(x_1/\gamma(x)) = x_2/\gamma(x)$ and we get by partial differentiation with respect to x_1 and x_2

$$f'(x_1/\gamma(x))(\gamma(x) - x_1\gamma_1(x)) = -x_2\gamma_1(x) \tag{1}$$

$$f'(x_1/\gamma(x))x_1\gamma_2(x) = x_2\gamma_2(x) - \gamma(x), \tag{2}$$

where $\gamma_i(x) = \dfrac{\partial\gamma(x)}{\partial x_i}$ for $i = 1, 2$.

By Eq. 1 and Eq. 2, one can derive in few steps

$$\gamma(x) = x_1\gamma_1(x) + x_2\gamma_2(x).$$

Define

$$h^\epsilon_\alpha(x) = \begin{cases} 1 & \text{if} \quad x \in S, \\ e^{-(\gamma(x)-1)^2/\epsilon} & \textit{otherwise.} \end{cases}$$

Analogously to van Damme (1987), one can show that $h^\epsilon_\alpha \in H^\epsilon_\alpha$ for all $\epsilon > 0$.

Theorem 2. *For $\epsilon > 0$, $\Gamma(h^\epsilon_\alpha)$ has a unique equilibrium x^ϵ, and $x^\epsilon \to x^*_\alpha$ as $\epsilon \to 0$.*

Proof. Clearly, an interior point of S cannot be an equilibrium point.

$$\begin{aligned} \partial R^\epsilon_{\alpha_1}(x)/\partial x_1 &= (h^\epsilon_\alpha(x))^{1/2\alpha} + x_1(1/2\alpha)(h^\epsilon_\alpha(x))^{-1+1/2\alpha}\partial h^\epsilon_\alpha(x)/\partial x_1 \\ &= (h^\epsilon_\alpha(x))^{1/2\alpha}[1 - x_1\gamma_1(x)(\gamma(x)-1)/\alpha\epsilon], \end{aligned}$$

$$\begin{aligned} \partial R^\epsilon_{\alpha_2}(x)/\partial x_2 &= (h^\epsilon_\alpha(x))^{1/2(1-\alpha)} + x_2(1/2(1-\alpha))(h^\epsilon_\alpha(x))^{-1+1/2(1-\alpha)}\partial h^\epsilon_\alpha(x)/\partial x_2 \\ &= (h^\epsilon_\alpha(x))^{1/2(1-\alpha)}[1 - x_2\gamma_2(x)(\gamma(x)-1)/(1-\alpha)\epsilon]. \end{aligned}$$

If $x \in \partial S$, $\partial R^\alpha_i(x)/\partial x_i = 1$ for all $i = 1, 2$. So, an equilibrium point is not an element of S. At the equilibrium point x^*, $\partial R^\alpha_i(x^*)/\partial x_i = 0$ for all $i = 1, 2$. This gives us

$$\epsilon = x_1\gamma_1(x)(\gamma(x)-1)/\alpha = x_2\gamma_2(x)(\gamma(x)-1)/(1-\alpha). \tag{3}$$

By combining this with $\gamma(x) = x_1\gamma_1(x) + x_2\gamma_2(x)$, we get

$$\alpha(1-\alpha)\gamma(x) = (1-\alpha)x_1\gamma_1(x) = \alpha x_2\gamma_2(x). \tag{4}$$

Substituting Eq. 4 into Eq. 3 gives $\gamma(x)(\gamma(x)-1) = \epsilon$. Hence, $\gamma(x) = (1 + \sqrt{1+4\epsilon})/2$. Observe that $\gamma(x) \to 1$ as $\epsilon \to 0$.

Plugging Eq. 4 into Eq. 1 gives us

$$f'(x_1/\gamma(x)) = -\alpha(1/\gamma(x))x_2/(1-\alpha)(1/\gamma(x))x_1,$$

by which we know that the point $x/\gamma(x) \in \partial S$ satisfies $f'(x_1) = -\alpha x_2/(1-\alpha)x_1$.

Now, consider x^*_α. It maximizes $x_1^\alpha x_2^{1-\alpha} = x_1^\alpha f(x_1)^{1-\alpha}$ on S. By FOC, we easily get $f'(x^*_{\alpha_1}) = -\alpha x^*_{\alpha_2}/(1-\alpha)x^*_{\alpha_1}$. This shows that $\gamma^{-1}(x^\epsilon)x^\epsilon$ is the α–symmetric Nash solution of (S, d). Hence, $x^\epsilon = \gamma(x^\epsilon)x^*_\alpha$ is the unique equilibrium of $\Gamma(h^\epsilon_\alpha)$.

□

Theorem 1 and Theorem 2 established what we wanted to show: Every α–symmetric Nash solution represents the unique H_α–essential equilibrium of the Nash demand game. The (symmetric) Nash solution is not "more stable" than any α–symmetric Nash solution.

3 Concluding Remarks

We have deprived in this article the symmetric Nash bargaining solution of its exposed role as the unique H–essential ("stable") equilibrium of Nash's demand game. In fact all equilibria of this game are H_α–essential for exactly one $\alpha \in (0, 1)$, but none is H_α–essential for more than one $\alpha \in (0, 1)$.

Despite the long history of belief in a special structural position of the symmetric Nash solution, none of the other approaches to the Nash solutions give reason to this view.

Anbar and Kalai (1978), Binmore et al. (1986), Trockel (1996, 2000), Duman and Trockel (2016) present different non-cooperative support results for all α–symmetric Nash solutions, with the symmetric one just as a special case, namely $\alpha = 1/2$. Any use of symmetry as a postulate like in the Nash axioms needs some justification like equity, justice. For a Pareto based justification see Trockel (2008).

The similarity of $H_{1/2}$–essentiality with the trembling hand perfectness of Selten stressed in the literature like for instance in our above quotation from Osborne and Rubinstein (1990) or in Kaneko (1981, p. 312) turned out now articulately to be inadequate. While all efficient equilibria of Nash's demand game share the same types of stability (namely H_α–essentiality), this is not the case with a trembling hand equilibrium that is inherently structurally different from non–perfect equilibria. This fact is nicely demonstrated by the example of Chain Store Game (see Selten (1978)), where only one of the two Nash equilibria is subgame perfect. Although in Milgrom and Roberts (1982) and Kreps and Wilson (1982) in certain perturbations of the chain store game both pendants of Selten's equilibria are sequential only the (pendant of) the subgame equilibrium in Selten's game is even trembling hand perfect. [‡]

Our results document a different direct analogy to Walrasian equilibria and their efficiency. The well–known pendant of the *First Welfare Theorem* asserts that every Nash equilibrium (except the utopia point of the underlying bargaining problem) of the Nash demand game, hence in particular the $H_{1/2}$–equilibrium, is Pareto efficient.

In this article we have proven an analogue of the *Second Welfare Theorem*: Every Pareto efficient payoff vector of the Nash demand game results from an H_α-essential Nash equilibrium for some $\alpha \in (0, 1)$.

Analogous results, where prices rather than the alphas act as parameters, had been proven in Trockel (1996) for (tatonnement–stable) Walrasian equilibria rather than H_α-essential Nash equilibria.

[‡] In fact, in order to get sequentiality of both equilibria one does not need any perturbation of the data of Selten's game (see Trockel (1986), Duman (2020)).

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