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On a class of periodic Dirichlet series with functional equation

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Abstract. The structure of the extended Selberg class of degree one was completely revealed by Kaczorowski and Perelli [10]. In this short paper, we give a new characterization of the functions with periodic coefficients in that class by giving a simple relation that the coefficients have to satisfy.
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1. Introduction and statement of results

1.1. The extended Selberg class

The extended Selberg class $\mathcal{S}^\sharp$ of functions introduced in [10] is the class of Dirichlet series $f(s)$ converging absolutely for $\mathrm{Re} s > 1$, possessing a meromorphic continuation to the whole complex plane with the only possible pole at $s = 1$ and satisfying a functional equation of the form

$$\Phi_f(s) = \omega_f \overline{\Phi_f(1-\overline{s})}, \tag{1}$$

where

$$\Phi_f(s) = f(s) Q_f^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j), \tag{2}$$

with $Q_f > 0$, $r \geq 0$, $\lambda_j > 0$, $|\omega_f| = 1$, $\mathrm{Re}\mu_j \geq 0$, $j = 1, \dots, r$. The numbers $\lambda_1, \dots, \lambda_r$ in the functional equation are not unique; however, the numbers

$$d_f = 2\sum_{j=1}^{r} \lambda_j \quad \text{and} \quad q_f = 2\pi Q_f^2/\beta_f, \quad \text{where} \quad \beta_f = \prod_{j=1}^{r} \lambda_j^{-2\lambda_j}$$

are invariants (i.e. they depend only upon $f \in \mathcal{S}^\sharp$), called the *degree* and the *conductor* of f, respectively, see, e.g. [14]. We denote by $\mathcal{S}_1^\sharp$ the subclass of the extended Selberg class consisting of degree one elements of $\mathcal{S}^\sharp$.

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1.2. Dirichlet series with periodic coefficients

Let $A = \{a(n)\}_{n=1}^{\infty}$ be a periodic sequence of complex numbers with period $q > 1$ (meaning that q is the smallest positive integer such that $a(n+q) = a(n)$ for every $n \geq 1$) and let

$$f(s) = f(A, s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \tag{3}$$

be the associated Dirichlet series, absolutely convergent for $\mathrm{Re}s > 1$.

In the sequel, we denote by A a $q-$periodic sequence with integral period $q > 1$. We also identify A with the mapping $A : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$ and say that A is an even or odd $q-$periodic sequence if the associated mapping is even/odd.

The series $f(A, s)$ can be written as a linear combination of Hurwitz zeta functions (see (8) below), thus it possesses an analytic continuation to the whole complex plane except for a possible pole at $s = 1$, when $a(1) + \ldots + a(q) \neq 0$ (see e.g. [7], [17]). Therefore, $f(A, s)$ given by (3) belongs to $\mathcal{S}^\sharp$ if and only if it satisfies the functional equation (1).

If the Dirichlet series $f(A, s)$ associated to A belongs to $\mathcal{S}^\sharp$, then necessarily $d_f = 1$, see [10]. Moreover, it is proved in [10] that $f(A, s) \in \mathcal{S}_1^\sharp$ if and only if $q_f \in \mathbb{N}$ and $f(A, s)$ can be written as a linear combination

$$f(A, s) = \sum_{\chi} P(s, \chi) L(s, \chi) \tag{4}$$

of Dirichlet $L-$functions $L(s, \chi)$ taken over the set of even/odd primitive Dirichlet characters χ modulo q with coefficients equal to Dirichlet polynomials $P(s, \chi)$ associated to character χ and satisfying certain conditions. (This representation is unique, according to results of [9].)

1.3. Statement of results

The representation of $q-$periodic Dirichlet series (3) as a linear combination (4) is rather complicated, especially for numerical implementations and experiments, in the sense that it is rather difficult to determine for a given even/odd $q-$periodic sequence A whether series (3) is in $\mathcal{S}_1^\sharp$. Moreover, there are many numerical methods related to converse theorems (a la Hamburger), where numerical computations produce first N coefficients in the Dirichlet series satisfying a certain functional equation with gamma factors, see e.g. [2], [12] and [13]. A simple, computationally effective criterion in terms of coefficients of the periodic series (3) which will give a necessary and sufficient condition for the series with given coefficients to belong to the extended Selberg class can be very useful in applications of such methods, see also [6].

The main purpose of this note is to give a new, simple characterization of functions $f(A, s) \in \mathcal{S}_1^\sharp$ in terms of relations satisfied by the "defining coefficients" $a(n)$ in the Dirichlet series representation (3). Our main theorem is the following.

Theorem 1. *Let a Dirichlet series $f(s)$ be defined by* (3) *and not identically zero. We have $f \in \mathcal{S}^\sharp$ with periodic coefficients and conductor q, for some positive integer q, if and only if $f(s) = f(A, s)$ is associated to an even or odd $q-$periodic sequence*

$A = \{a(n)\}_{n=1}^{\infty}$ of coefficients in (3) *such that the following system of equations holds:*

$$\overline{a(n)}(\mathcal{F}_q A)(m) = (\mathcal{F}_q A)(n)\overline{a(m)}, \text{ for all } 1 \leq m \leq q, \tag{5}$$

where $n \geq 1$ is the smallest integer such that $a(n) \neq 0$ and

$$(\mathcal{F}_q A)(m) := \frac{1}{\sqrt{q}} \sum_{j=1}^{q} a(j) \exp\left(\frac{2\pi jm}{q} i\right) \tag{6}$$

is the finite Fourier transform of the function A on $\mathbb{Z}/q\mathbb{Z}$.

In other words, the subset of $\mathcal{S}_1^\sharp$ consisting of periodic functions of conductor q coincides with the set of Dirichlet series $f(A, s)$ associated to $q-$periodic even/odd defining sequence A which satisfies system (5). Actually, as we will see below, it is sufficient that system (5) is satisfied for indices $m \in \{\delta, \dots, \lfloor \frac{q}{2} \rfloor\}$, where $\delta = 0$ when A is even, and $\delta = 1$ when A is odd.

Let $\widetilde{\mathcal{S}}^\sharp$ denote the subclass of $\mathcal{S}^\sharp$ consisting of Dirichlet series with periodic coefficients. Theorem 1 is quite useful in deeper understanding of the structure of the vector space (over $\mathbb{R}$) consisting of functions from $\widetilde{\mathcal{S}}^\sharp$ with a fixed conductor q_f and the invariant ω_f^* given by (19). For example, we will show in Section 4 that $\widetilde{\mathcal{S}}^\sharp$ is invariant under the mapping $A \mapsto \mathcal{F}_q\overline{A}$, where $\overline{A} = \{\overline{a(n)}\}_{n=1}^{\infty}$. Moreover, the main result is useful in the construction of various examples of Dirichlet series from $\widetilde{\mathcal{S}}^\sharp$ whose coefficients satisfy certain additional conditions (e.g. the series with first n coefficients that are alternating or equal to zero).

The appearance of the finite Fourier transform of the sequence of defining coefficients in the characterization of the space $\widetilde{\mathcal{S}}^\sharp$ seems natural, especially from the point of view of modular forms, in the sense that system (5) can be viewed as some type of modularity condition. Namely, in the modular forms setting it is natural to associate an $L-$function to a modular form by taking the coefficients $a(n)$ in the Fourier expansion of a form to be "defining coefficients" in the representation (3). The invariance of $\widetilde{\mathcal{S}}^\sharp$ under the mapping $A \mapsto \mathcal{F}_q\overline{A}$ shows that the analogous construction of functions from $\widetilde{\mathcal{S}}^\sharp$ is possible, with modular forms replaced by even/odd functions $A : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$ satisfying system (5). In other words, $\widetilde{\mathcal{S}}^\sharp$ can be viewed as the set of $L-$functions associated to even/odd functions $A : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$ satisfying system (5) by taking the defining coefficients $a(n)$ in the series representation (3) to be the nth Fourier coefficient of A.

1.4. Related results

Our main theorem gives a necessary and sufficient condition for a $q-$periodic Dirichlet series (3) to satisfy the functional equation (1) with $d_f = 1$.

Similar results, giving a necessary and sufficient condition that a certain Dirichlet series satisfies a functional equation of a given type, are proved in e.g. [4], [17]. In other words, starting from a functional equation with gamma factors satisfied by a Dirichlet series (of some type), it is possible to deduce certain arithmetical identities for coefficients of Dirichlet series satisfying this equation; for a detailed overview of methods and results, see [11] and references therein.

We find it important to notice that our equation (5) can not be deduced from arithmetic identities derived in [4], [17], [18] and [11] by specializing to the case of degree one functions in the Selberg class.

2. Finite Fourier transform

The finite Fourier transform $\mathcal{F}_q A$ of the function $A : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$ is a mapping $\mathcal{F}_q : \mathbb{Z}/q\mathbb{Z} \to \mathbb{Z}/q\mathbb{Z}$ defined by (6), see e.g. [1, p. 850]. It is a unitary linear operator on $L^2(\mathbb{Z}/q\mathbb{Z})$, which is the Hermitian space of complex valued functions g on $\mathbb{Z}/q\mathbb{Z}$ with the norm defined as $||g||^2 := \sum_{j=1}^{q} g(j)\overline{g(j)}$. We start with the following lemma.

Lemma 1. *Let $q \geq 2$ be an integer, and let $A = \{a(n)\}_{n=1}^{\infty}$ define an even or odd function on $\mathbb{Z}/q\mathbb{Z}$, not identically zero. Let n denote the smallest positive integer such that $a(n) \neq 0$, and put $\delta = 0$ if A is even, and $\delta = 1$ if A is odd. If the sequence $A = \{a(n)\}_{n=1}^{\infty}$ satisfies the system of equations*

$$\overline{a(n)}(\mathcal{F}_q A)(m) = (\mathcal{F}_q A)(n)\overline{a(m)}, \text{ for all } \delta \leq m \leq \left\lfloor \frac{q}{2} \right\rfloor, \tag{7}$$

where in the case $\delta = 0$ we define $a(0) = a(q)$, then there exists a complex constant $\omega_0(A)$ of modulus one, such that $(\mathcal{F}_q A)(n) = \omega_0(A)\overline{a(n)}$ for all $n = 1, \ldots, q$.

Proof. First, we prove that the equation $\overline{a(n)}(\mathcal{F}_q A)(m) = (\mathcal{F}_q A)(n)\overline{a(m)}$ also holds for all $\lfloor \frac{q}{2} \rfloor < m \leq q$ as well. For a fixed $\delta \leq l \leq \lfloor \frac{q}{2} \rfloor$ we have

$$\begin{aligned}(\mathcal{F}_q A)(q-l) &= \frac{1}{\sqrt{q}} \sum_{j=1}^{q} a(j) \exp\left(\frac{-2\pi j l}{q} i\right) \\ &= \frac{1}{\sqrt{q}} \sum_{k=0}^{q-1} a(q-k) \exp\left(\frac{2\pi k l}{q} i\right) = (-1)^{\delta} (\mathcal{F}_q A)(l),\end{aligned}$$

where the last equality was obtained by applying the equation $a(q-k) = (-1)^{\delta} a(k)$, for $\delta \leq k \leq \lfloor \frac{q}{2} \rfloor$. The same equation, together with (7), yields $\overline{a(n)}(\mathcal{F}_q A)(q-m) = (\mathcal{F}_q A)(n)\overline{a(q-m)}$, for all $\delta \leq m \leq \lfloor \frac{q}{2} \rfloor$, meaning that (5) holds.

Now, the statement follows from equation (5) together with the fact that the finite Fourier transform is a unitary linear operator on $L^2(\mathbb{Z}/q\mathbb{Z})$ and hence $\omega_0(A) = (\mathcal{F}_q A)(n)/\overline{a(n)}$ is a complex number of modulus one. $\square$

Remark 1. *Let us mention here that throughout this paper, with a slight abuse of notation, we will identify the periodic sequence $\{a(n)\}_{n=1}^{\infty}$ of period $q > 1$ with the function $A : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$ and write $A = \{a(n)\}_{n=1}^{\infty}$. Moreover, for the sake of notational simplicity, in the sequel, we also introduce the zeroth element of the sequence A by $a(0) = a(q)$ using periodicity. In case when $A = \{a(n)\}_{n=1}^{\infty}$ is odd, we also make note that $a(q) = a(0) = a(q-q) = -a(q)$, which implies that $a(q) = 0$.*

In the case when the sequence A consists of real or purely imaginary numbers, we have the following corollary.

Corollary 1. *Let $q \geq 2$ be an integer, and let A be an even or odd function on $\mathbb{Z}/q\mathbb{Z}$, whose values are either all real or all purely imaginary numbers, not identically zero. Let n denote the smallest positive integer such that $a(n) \neq 0$. If the sequence $A = \{a(n)\}_{n=1}^{\infty}$ satisfies the system of equations* (7)*, then there exists a complex constant $\omega_0(A) \in \{-1, 1, i, -i\}$ such that $(\mathcal{F}_q A)(l) = \omega_0(A)a(l)$ for all $l = 1, \ldots, q$.*

Proof. The statement of Lemma 1 yields that $\omega_0(A)$ is an eigenvalue of the operator $\mathcal{F}_q$ associated to an eigenfunction A. According to [1, p. 850], the only possible eigenvalues of $\mathcal{F}_q$ are 1, -1, i and $-i$. The proof is complete. □

3. Properties of Dirichlet series with periodic coefficients

Our next theorem proves that an even/odd Dirichlet series with $q-$periodic coefficients belongs to the extended Selberg class and has conductor q if and only if its coefficients satisfy the system of equations (7). Namely, we have the following theorem.

Theorem 2. *Let $q \geq 2$ be an integer, and assume that $f(A, s)$ is a Dirichlet series associated to the even or odd $q-$periodic sequence $A = \{a(m)\}_{m=1}^{\infty}$, not identically equal to zero. The function $f(A, s)$ belongs to the extended Selberg class and has conductor q if and only if the values $a(m)$ satisfy the system of equations* (7)*, where $n \geq 1$ is the smallest integer such that $a(n) \neq 0$.*

Proof. First, we prove that if system (7) is satisfied, then $f(A, s)$ belongs to the extended Selberg class.

We start with the representation of $f(A, s)$ in terms of the linear combination of Hurwitz zeta functions (see e.g. [17] and [18]):

$$f(A, s) = \frac{1}{q^s} \sum_{j=1}^{q} a(j)\zeta\left(s, \frac{j}{q}\right), \tag{8}$$

showing that axioms (i) and (ii) of the extended Selberg class are satisfied by $f(A, s)$. Now, we claim that the function

$$\Lambda_f(s) := \left(\sqrt{\frac{q}{\pi}}\right)^s \Gamma\left(\frac{s+\delta}{2}\right) f(A, s), \tag{9}$$

where we put $\delta = 0$ if A is even, and $\delta = 1$ if A is odd, satisfies the functional equation

$$\Lambda_f(s) = \omega\overline{\Lambda_f(1 - \overline{s})}, \tag{10}$$

with the constant

$$\omega_f = \omega(A) = \frac{1}{i^\delta} \cdot \frac{(\mathcal{F}_q A)(n)}{\overline{a(n)}} \tag{11}$$

of absolute value one (by Lemma 1).

The representation (8) together with the functional equation for the Hurwitz zeta function yields

$$f(A,s)=\frac{2\Gamma(1-s)}{q^s(2\pi q)^{1-s}}\sum_{j=1}^{q}a(j)\sum_{k=1}^{q}\sin\left(\frac{\pi s}{2}+\frac{2k\pi j}{q}\right)\zeta\left(1-s,\frac{k}{q}\right). \tag{12}$$

Using the relation $(\mathcal{F}_qA)(q-k)=(-1)^{\delta}(\mathcal{F}_qA)(k)$, we obtain

$$\begin{aligned} f(A,s) = {} & \frac{2\Gamma(1-s)}{q^{s-1/2}(2\pi q)^{1-s}}\sum_{k=1}^{q}\zeta\left(1-s,\frac{k}{q}\right)(\mathcal{F}_qA)(k)\\ & \frac{1}{2i}\left(\exp(i\pi s/2)-(-1)^{\delta}\exp(-i\pi s/2)\right), \end{aligned}$$

or, equivalently,

$$f(A,s)=\frac{2\Gamma(1-s)}{q^{s-1/2}(2\pi q)^{1-s}}\frac{1}{i^{\delta}}\sin\left(\frac{\pi(s+\delta)}{2}\right)\sum_{k=1}^{q}(\mathcal{F}_qA)(k)\zeta\left(1-s,\frac{k}{q}\right). \tag{13}$$

Therefore, equation (7) (or, equivalently (5)) yields

$$f(A,s)=\frac{2\Gamma(1-s)}{q^{s-1/2}(2\pi)^{1-s}}\frac{(\mathcal{F}_qA)(n)}{\overline{a(n)}}\frac{1}{i^{\delta}}\sin\left(\frac{\pi(s+\delta)}{2}\right)\overline{f(A,1-\overline{s})}. \tag{14}$$

The functional relation $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$ together with the doubling formula for the gamma function gives

$$\frac{2\Gamma(1-s)}{q^{s-1/2}(2\pi)^{1-s}}\sin\left(\frac{\pi(s+\delta)}{2}\right)=q^{-s+1/2}\pi^{s-1/2}\frac{\Gamma\left(\frac{1-s}{2}+\frac{\delta}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{\delta}{2}\right)}. \tag{15}$$

Substituting this into equation (14), we obtain

$$f(A,s)=\frac{1}{i^{\delta}}\cdot\frac{(\mathcal{F}_qA)(n)}{\overline{a(n)}}\cdot q^{-s+1/2}\pi^{s-1/2}\frac{\Gamma\left(\frac{1-s}{2}+\frac{\delta}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{\delta}{2}\right)}\overline{f(A,1-\overline{s})},$$

which proves the first part of our theorem. (From the functional equation, it is obvious that the conductor of f is exactly q.)

Let us now prove that if $f(A,s)\in\mathcal{S}^{\sharp}$ is not identically zero with conductor q, then equation (5) holds.

Functional equation (1) for the completed function (2) implies that

$$f(A,s)=cQ_f^{1-2s}g(s)\cdot\overline{f(A,1-\overline{s})}, \tag{16}$$

where c is a complex constant of modulus one, $Q_f>0$ and

$$g(s)=\frac{\prod_{j=1}^{r}\overline{\Gamma(\lambda_j(1-\overline{s})+\mu_j)}}{\prod_{j=1}^{r}\Gamma(\lambda_j s+\mu_j)}.$$

On the other hand, equation (13) is obtained from the definition of the function f, using only the properties of the finite Fourier transform. Therefore, for $\mathrm{Re} s < 0$, using the definition of the Hurwitz zeta function, the $q-$periodicity of the finite Fourier transform and (15), we may write (13) as

$$f(A,s) = \frac{1}{i^\delta} q^{-s+1/2} \pi^{s-1/2} \frac{\Gamma\left(\frac{1-s}{2}+\frac{\delta}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{\delta}{2}\right)} \sum_{m=1}^{\infty} \frac{(\mathcal{F}_q A)(m)}{m^{1-s}}.$$

Together with the functional equation (16), this yields for $\mathrm{Re} s < 0$,

$$\sum_{m=1}^{\infty} \frac{(\mathcal{F}_q A)(m)}{m^{1-s}} = i^\delta c q^{s-1/2} \pi^{-s+1/2} \frac{\Gamma\left(\frac{\delta}{2}+\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}+\frac{\delta}{2}\right)} Q_f^{1-2s} g(s) \overline{f(A, 1-\overline{s})}.$$

We put $\Xi_m := \sqrt{q}(\mathcal{F}_q A)(m)$ and let k be the smallest positive integer such that $\Xi_k \neq 0$. Recalling that n is the smallest positive integer such that $a(n) \neq 0$, we can write the above equation as

$$\sum_{m=k}^{\infty} \frac{\Xi_m}{(m/k)^{1-s}} = G(s) \sum_{m=n}^{\infty} \frac{\overline{a(m)}}{(m/n)^{1-s}}, \tag{17}$$

where we put

$$G(s) = i^\delta c \left(\frac{k}{n}\right)^{1-s} Q_f \sqrt{\pi} \left(\frac{q}{\pi Q_f^2}\right)^s \frac{\Gamma\left(\frac{\delta}{2}+\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}+\frac{\delta}{2}\right)} \prod_{j=1}^{r} \frac{\Gamma(\lambda_j(1-s)+\overline{\mu_j})}{\Gamma(\lambda_j s + \mu_j)}.$$

Then, letting $\mathrm{Re} s \to -\infty$ in (17) we get that

$$G(s) \to \frac{\Xi_k}{a(n)}, \text{ as } \mathrm{Re} s \to -\infty. \tag{18}$$

We claim that $k = n$ and that $G(s) = \frac{\Xi_n}{a(n)}$ for all $s \in \mathbb{C}$, which, together with (17), would imply that (7) holds true. In order to prove our claim, we first apply the Stirling formula for the asymptotic behavior of $\log \Gamma(s)$ to deduce that, as $|s| \to \infty$, we have (for $\delta \in \{0, 1\}$)

$$\begin{aligned}
\log G(s) = {} & s \log s \, (1 - d_f) \\
& + s \left[d_f - 1 + \log\left(\frac{q\beta_f}{2\pi Q_f^2}\right) + \frac{i\pi}{2}(1-d_f) - \log(k/n) \right] \\
& + \log s \left[\frac{1}{2}(d_f - 1) - 2i \sum_{j=1}^{r} \mathrm{Im}\mu_j \right] + C + O\left(\frac{1}{|s|}\right),
\end{aligned}$$

where the constant C depends on δ, k, n, c q, Q_f, λ_j, μ_j, $j = 1, ..., r$.

Note that from the expression above it easily follows that $f(A, s) \in \mathcal{S}^\sharp$ implies that the coefficient multiplying $s \log s$ needs to be equal to zero, i.e. $d_F = 1$, as mentioned in the introduction.

Since $\log G(s) \to \log\left(\frac{\Xi_k}{a(n)}\right)$, as $\mathrm{Re}s \to -\infty$, inserting $s = -\sigma + i\epsilon$, for some small, positive ϵ into (19), we deduce that the factors multiplying $s\log s$, s and $\log s$ must be equal to zero. Since the conductor of f is q, the term multiplying s in (19) becomes $-\log(k/n)$, which implies that $k = n$ and, moreover, that $G(s)$ is bounded as $|s| \to \infty$.

Therefore, in order to prove that $G(s) = \frac{\Xi_n}{a(n)}$, it is now left to prove that $G(s)$ is an entire function. Coefficients Ξ_m of the Dirichlet series on the left-hand side of (17) are bounded and $q-$periodic. Therefore, for $\mathrm{Re}(-s) = \sigma$ large enough, we have

$$\left|\sum_{m=n}^{\infty} \frac{\Xi_m}{(m/n)^{1-s}}\right| \geq |\Xi_n| - \sum_{m=n+1}^{\infty} \frac{|\Xi_m|}{(m/n)^{1+\sigma}} > 0.$$

Since n^{s-1} is a non-vanishing function, this shows that the series $\sum_{m=1}^{\infty} \frac{\Xi_m}{(m/n)^{1-s}}$ is non-vanishing for $\mathrm{Re}s$ small enough. Analogously, we deduce that the series $\overline{f(A, 1-\overline{s})}$ is non-vanishing for $\mathrm{Re}s$ small enough; hence, there exists a non-negative integer m_0 such that both Dirichlet series appearing in (17) are non-vanishing for $\mathrm{Re}s < -m_0$. For that reason, both $G(s)$ and $G(s)^{-1}$ are non-vanishing and holomorphic for $\mathrm{Re}s < -m_0$.

Since $G(s)\overline{G(1-\overline{s})} = c\overline{c}q$, both $G(s)$ and $G(s)^{-1}$ are non-vanishing and holomorphic for $\mathrm{Re}s > 1 + m_0$ as well.

The function $\prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)^{-1}$ has zeros at points s such that $\lambda_j s + \mu_j = -\ell$, $j = 1, \ldots, r$, where ℓ runs through the set of all non-negative integers. Since $\lambda_j > 0$ and μ_j is fixed, for all ℓ large enough, the zeros $s = \frac{-\ell-\mu_j}{\lambda_j}$ lie in the half-plane $\mathrm{Re}s < -m_0$ and, therefore must cancel with the poles of $\Gamma\left(\frac{s}{2} + \frac{\delta}{2}\right)$ (as those are the only possible poles in the half-plane $\mathrm{Re}s < -m_0$).

The function $\Gamma\left(\frac{s}{2} + \frac{\delta}{2}\right)$ has a pole when $-\frac{s}{2} - \frac{\delta}{2}$ is a non-negative integer. This implies that $\frac{\ell+\mu_j}{\lambda_j}$ must be an even ($\delta = 0$)/odd ($\delta = 1$) non-negative integer, for all ℓ large enough. Since f is actually of degree one, according to [10, Remark on p. 211] $\lambda_j = \frac{1}{2m_j}$, where m_j is a positive integer. Since $\frac{\ell+\mu_j}{\lambda_j}$, ℓ and $\frac{1}{\lambda_j}$ are non-negative integers, we conclude that μ_j must be a real rational number. Since $\Re\mu_j \geq 0$, the numbers μ_j must be non-negative rational numbers. Moreover, we easily deduce that $2m_j\mu_j$ is an even (case $\delta = 0$) or odd (case $\delta = 1$) non-negative integer; hence $\mu_j = \alpha_j/2m_j$ for some even/odd non-negative integer α_j for $\delta = 0/\delta = 1$. This proves that the set of all zeros of $(\Gamma(\lambda_j s + \mu_j))^{-1}$ is a subset of the set of poles of $\Gamma\left(\frac{s}{2} + \frac{\delta}{2}\right)$.

Reasoning analogously, we conclude that the set of poles of $\Gamma\left(\frac{s}{2} + \frac{\delta}{2}\right)$ is a subset of zeros of $\left(\Gamma\left(\lambda_j s + \mu_j\right)\right)^{-1}$. Therefore, the functions $\prod_{j=1}^{r} \Gamma\left(\lambda_j s + \mu_j\right)^{-1}$ and $\left(\Gamma\left(\frac{s}{2} + \frac{\delta}{2}\right)\right)^{-1}$ are entire functions of order one whose sets of zeros coincide, so there must exist a polynomial $P_1(s)$ of degree at most one such that

$$\prod_{j=1}^{r} \Gamma\left(\lambda_j s + \mu_j\right)^{-1} = \exp\left(P_1(s)\right)\Gamma\left(s/2 + \delta/2\right)^{-1}.$$

Substituting $1-s$ in the place of s, and recalling that $\overline{\mu_j}=\mu_j$, we obtain

$$\prod_{j=1}^{r}\Gamma\left(\lambda_j(1-s)+\overline{\mu_j}\right)^{-1}=\exp\left(P_1(1-s)\right)\Gamma\left((1-s)/2+\delta/2\right)^{-1}.$$

Hence,

$$G(s)=cq^sQ_f^{1-2s}\pi^{1/2-s}\exp\left(P_1(s)-P_1(1-s)\right),$$

which is entire. By Liouville's theorem, a function which is holomorphic and bounded, must be constant. This completes the proof. □

4. Proof of the main theorem and its corollaries

4.1. Proof of the main result

Proof. Assume first that the periodic Dirichlet series f belongs to $\mathcal{S}^\sharp$ with conductor q, for some positive integer q. Then, $f\in\mathcal{S}_1^\sharp$ and according to [10, Theorem 2], with $\theta=0$, we may write $f(s)=\sum_{l=1}^{\infty}a(l)\cdot l^{-s}$, where the coefficients $a(l)$ are $q-$periodic and the sequence $A=\{a(l)\}_{l=1}^{\infty}$ is either even or odd (since $a(l)$ are given as linear combinations of even/odd characters modulo q). Application of Theorem 2 yields that the coefficients $a(l)$ satisfy system (5), which proves the first part of the theorem.

Now, assume that $f(s)=f(A,s)$ is an even or odd $q-$periodic Dirichlet series such that the system of equations (5) holds. Then Theorem 2 yields that $f\in\mathcal{S}^\sharp$. Moreover, the completed function $\Lambda_f(s)$ defined by (9), satisfies equation (10) and hence f is of degree 1 with conductor q. The proof is complete. □

4.2. Corollaries of the main result

In addition to the invariants d_f and q_f, it can be shown that the expressions

$$\omega_f^*=\omega_f e^{-i\pi(\eta_f+1)/2}\left(\frac{Q_f^2}{\beta_f}\right)^{i\theta_f}\prod_{j=1}^{r}\lambda_j^{-2i\mathrm{Im}\mu_j} \tag{19}$$

and $\xi_f:=2\sum_{j=1}^{r}(\mu_j-1/2)=\eta_f+i\theta_f$ are also invariants. The following corollary gives a simple representation of the invariant ω_f^* for $f\in\widetilde{\mathcal{S}}^\sharp$.

Corollary 2. *For $f(s)=f(A,s)\in\widetilde{\mathcal{S}}^\sharp$ the invariant ω_f^* can be written as*

$$\omega_f^*=(-1)^{\mathrm{Re}\xi_f+1}\frac{\mathcal{F}_qA(n)}{\overline{a(n)}},$$

where n is the smallest positive integer such that $a(n)\neq 0$.

Proof. When $f(s)=f(A,s)\in\widetilde{\mathcal{S}}^\sharp$, then $\omega_f^*=(i)^{-\delta}\omega(A)$, where $\delta=\mathrm{Re}\xi_f+1=0$ if A is even and $\delta=\mathrm{Re}\xi_f+1=1$ if A is odd. This, together with equation (11), completes the proof. □

A straightforward application of the properties of the finite Fourier transform yields the following corollary of Theorem 1.

Corollary 3. *For a positive integer q and a complex number ω^* of modulus 1, let $\widetilde{V}^\sharp(q,\delta,\omega^*) = \widetilde{\mathcal{S}}^\sharp(q,\delta,\omega^*) \cup \{0\}$ denote the vector space over $\mathbb{R}$ identified with the set of functions $f(A,s) \in \widetilde{\mathcal{S}}^\sharp$ of conductor q and invariant ω^* given by (19), associated to an even ($\delta = 0$) or odd ($\delta = 1$) $q-$periodic sequence $A = \{a(n)\}_{n=1}^\infty$ of coefficients in (3) and put $\overline{A} = \{\overline{a(n)}\}_{n=1}^\infty$. Then, the following statements hold true:*

i) The mapping $A \mapsto \overline{A}$, is an isomorphism between the spaces $\widetilde{V}^\sharp(q,\delta,\omega^*)$ and $\widetilde{V}^\sharp(q,\delta,(-1)^\delta\overline{\omega^*})$.

ii) The mapping $A \mapsto \mathcal{F}_q\overline{A}$ is an isomorphism between the spaces $\widetilde{V}^\sharp(q,\delta,\omega^*)$ and $\widetilde{V}^\sharp(q,\delta,\overline{\omega^*})$.

iii) The mapping $A \mapsto \mathcal{F}_q A$ is an isomorphism between the spaces $\widetilde{V}^\sharp(q,\delta,\omega^*)$ and $\widetilde{V}^\sharp(q,\delta,(-1)^\delta\omega^*)$.

Proof.

i) Conjugation does not change the parity of the function, hence by conjugating system (5) and the equation $\overline{(\mathcal{F}_q A(m))} = (-1)^\delta(\mathcal{F}_q\overline{A})(m)$ we deduce that equation (5) is fulfilled by $\overline{A}$. This also implies that $\omega(\overline{A}) = \overline{\omega(A)}$, which together with $\omega_f^* = (i)^{-\delta}\omega(A)$, yields the statement.

ii) The Fourier inversion formula yields that $a(n) = \mathcal{F}_q(\mathcal{F}_q A)(-n)$. Moreover, $\mathcal{F}_q A(-n) = \overline{\mathcal{F}_q\overline{A}(n)}$, hence we have for $m,n \in \{1,...,q\}$:

$$\overline{\mathcal{F}_q\overline{A}(n)} = \mathcal{F}_q A(-n) = (-1)^\delta\mathcal{F}_q A(n); \quad \mathcal{F}_q(\mathcal{F}_q\overline{A})(m) = \overline{a(-m)} = (-1)^\delta\overline{a(m)},$$

hence

$$\overline{\mathcal{F}_q\overline{A}(n)}\mathcal{F}_q(\mathcal{F}_q\overline{A})(m) = \mathcal{F}_q A(n)\overline{a(m)}$$

and the equation is symmetric with respect to m and n. For this reason, system (5) holds if and only if

$$\overline{\mathcal{F}_q\overline{A}(n)}\mathcal{F}_q(\mathcal{F}_q\overline{A})(m) = \mathcal{F}_q(\mathcal{F}_q\overline{A})(n)\overline{\mathcal{F}_q\overline{A}(m)}, \quad \text{for all } 1 \le m \le q.$$

This, together with the statement of Theorem 1, shows that $f(A,s) \in \widetilde{\mathcal{S}}^\sharp$ if and only if $f(\mathcal{F}_q\overline{A},s) \in \widetilde{\mathcal{S}}^\sharp$. From the proof of Theorem 2, it is obvious that the mapping $A \mapsto \mathcal{F}_q\overline{A}$ fixes the conductor q and, moreover, for $f(A,s) \in \widetilde{\mathcal{S}}^\sharp$ the constant ω_f in the functional equation axiom (1) is

$$\omega_f = \omega(A) = i^{-\delta}\frac{(\mathcal{F}_q A)(n)}{\overline{a(n)}} = i^{-\delta}\frac{\overline{(\mathcal{F}_q\overline{A})(n)}}{\mathcal{F}_q(\mathcal{F}_q\overline{A})(n)} = (-1)^\delta\omega(\mathcal{F}_q\overline{A})^{-1}.$$

Noticing that $\omega_f^* = (-i)^\delta\omega(A)$ completes the proof.

iii) Follows directly from i) and ii).

□

5. Examples

The characterization of a $q-$periodic Dirichlet series from the extended Selberg class in terms of the relations satisfied by the coefficients in their Dirichlet series, derived in this paper, can be easily applied in the construction of a $q-$periodic Dirichlet series with certain properties. It is particularly useful in situations when one needs to find out whether a given even/odd periodic sequence generates a function from $\mathcal{S}^\sharp$, in which case one needs to check if system (7) holds.

The main result is also useful in situations when one needs to construct a Dirichlet series with periodic coefficients which have some given values at specific arguments, see e.g. [3], where Davenport-Heilbronn type functions have been constructed or [15]. In the following examples we will demonstrate the construction of such functions having first k coefficients equal to zero, for arbitrary k.

Example 1. *Let $q = 12$, $\omega_f^* = -1$ and define*

$$b(n) = \begin{cases} 0, & n \equiv 1 \pmod{12} \\ 0, & n \equiv 2 \pmod{12} \\ 1, & n \equiv 3 \pmod{12} \\ \xi, & n \equiv 4 \pmod{12} \\ \eta, & n \equiv 5 \pmod{12} \\ 0, & n \equiv 6 \pmod{12} \\ -\eta, & n \equiv 7 \pmod{12} \\ -\xi, & n \equiv 8 \pmod{12} \\ -1, & n \equiv 9 \pmod{12} \\ 0, & n \equiv 10 \pmod{12} \\ 0, & n \equiv 11 \pmod{12} \\ 0, & n \equiv 12 \pmod{12}. \end{cases}$$

The system of equations (7) *for $q = 12$ and the odd sequence $b(n)$ reduces to*

$$\begin{aligned} \eta + \sqrt{3}\xi + 2 &= 0 \\ \eta + \xi &= 0 \\ 2(\eta - 1)\bar{\xi} + \sqrt{3}(\eta - \xi) &= 0 \\ 2(\eta - 1)\bar{\eta} + \sqrt{3}\xi &= \eta + 2. \end{aligned}$$

The solution $\xi = -1 - \sqrt{3}$ and $\eta = 1 + \sqrt{3}$ of the given system produces an odd 12-periodic Dirichlet series belonging to the extended Selberg class with the first two coefficients equal to zero.

A construction of an even 12-periodic Dirichlet series belonging to the extended Selberg class with the first two coefficients equal to zero can be done analogously.

Example 2. *Let* $q = 8$, $\omega_f^* = 1$ *and define*

$$c(n) = \begin{cases} 0, & n \equiv 1 \pmod 8 \\ 0 & n \equiv 2 \pmod 8 \\ 1, & n \equiv 3 \pmod 8 \\ \xi, & n \equiv 4 \pmod 8 \\ 1, & n \equiv 5 \pmod 8 \\ 0, & n \equiv 6 \pmod 8 \\ 0, & n \equiv 7 \pmod 8 \\ \eta, & n \equiv 8 \pmod 8. \end{cases}$$

A system of equations (7) *consists of five equations and one of them is trivially satisfied, while the remaining ones are:*

$$\begin{aligned} \xi + \sqrt{2} &= \eta \\ \eta + \xi &= 0 \\ \left(\eta - \xi + \sqrt{2}\right) \bar{\xi} &= \eta + \xi - 2 \\ \left(\eta - \xi + \sqrt{2}\right) \bar{\eta} &= \eta + \xi + 2. \end{aligned}$$

The solution $\xi = -1/\sqrt{2}$ *and* $\eta = 1/\sqrt{2}$ *of the given system produces an even 8-periodic Dirichlet series belonging to the extended Selberg class with the first two coefficients equal to zero.*

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