Continuous Team Semantics*

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Abstract. We study logics with team semantics in computable metric spaces. We show how to define approximate versions of the usual independence/dependence atoms. For restricted classes of formulae, we show that we can assume w.l.o.g. that teams are closed sets. This then allows us to import techniques from computable analysis to study the complexity of formula satisfaction and model checking.

Keywords: team semantics · continuous logic · computable analysis · independence logic · dependence logic

1 Introduction

Team semantics is a semantical framework for logics of dependence and independence. Team semantics was originally invented by Hodges [11] and later systematically developed and made popular by Väänänen by the introduction of Dependence Logic [22]. In team semantics formulas are evaluated using sets of assignments (called teams) rather than single assignments as in first-order logic. Therefore, it is not surprising that the expressive power of many of the logics studied in team semantics exceeds the expressive power of first-order logic. The introduction of Independence Logic (FO($\perp$)) in [9] and inclusion ($\boldsymbol{x} \subseteq \boldsymbol{y}$) and exclusion atoms ($\boldsymbol{x} \mid \boldsymbol{y}$) (and the corresponding logics) [8] demonstrated the versatility of the framework and have led to several studies on the applications of team semantics in areas such as database theory, model theory, and quantum information theory (see, e.g. [10, 13, 6, 16, 12, 1]).

In this article we explore and apply team semantics in a metric context. The expressive power that team semantics makes available to us in the form of dependence and independence atoms comes at the prize that in the definitions, we have to quantify over the powerset of our structure. On the logical level, this

* This research was partially supported by the Royal Society International Exchange Grant 170051 “Continuous Team Semantics: On dependence and independence in a continuous world” and grant 308712 of the Academy of Finland.

lets the expressivity exceed first-order logic, and in many cases, reach existential second-order logic. From an algorithmic perspective, this involves an exponential blow-up in the relevant search space, and causes a number of hardness results.

In a metric context, we are facing having to deal with sets such as the powerset of the unit interval. This would seem to destroy the hope of any algorithmic approach, and very much opens the door to the specter of independence of ZFC. Fortunately, our first main results show that for certain well-behaved classes of sentences, it is safe to assume that all teams are closed. The hyperspace of closed subsets of a (compact separable) metric space is much better understood, and been made accessible to algorithmic approaches through the development of computable analysis. We make use of the opportunity, and show that model-checking and satisfaction are semidecidable for the aforementioned well-behaved formulae. This is the best possible result in full generality.

In the metric context, it is very natural to consider approximate versions of the usual dependence and independence atoms (see, e.g., [15, 23, 6] for related previous work on so-called metric functional dependencies and approximate dependencies in (non-metric) team semantics). As an added bonus, the approximate versions are compatible with our notions of well-behaved formulae, and thus greatly increase what we can express directly without leaving the realm of tameness. We conclude our investigation (for now) by considering the translatability between the approximate versions, and contrast these to the established translability results regarding the exact versions.

2 Definitions

We are working with a fixed structure, which here is a (compact separable[3]) metric space $(\mathbf{X}, d)$ together with certain predicates, i.e. subsets of $\mathbf{X}^n$ for $n \in \omega$. To simplify notation, we will usually not explicitly mention the structure, but simply take it for granted. We then proceed to define when $T \models \phi$ holds, where $T \subseteq \mathbf{X}^n$ is a *team*, and ϕ is a positive formula involving both basic predicates and certain special primitives. These definitions are completely standard, see [22, 8].

Variables are assumed to correspond to specific dimensions. We write $\pi_{\overline{x}}$ for the projection to the dimensions corresponding to the variables comprising the tuple $\overline{x}$; and π_{-x} for the projection to all dimensions except the one corresponding to the variable x. Note that the order n which the variables appear in $\overline{x}$ impacts the meaning of $\pi_{\overline{x}}$, e.g. $\pi_{xy}A$ and $\pi_{yx}A$ are related by $(a, b) \in \pi_{xy}A$ iff $(b, a) \in \pi_{yx}A$. We allow for the case of variables appearing multiple times in a tuple, this means the corresponding dimension will be duplicated in the result. By $\times$ we denote the usual cartesian product and, for R over $\overline{xz}$ and R' over $\overline{zy}$, the join $R \bowtie R'$ of R and R' is defined by

$$R \bowtie R' = \{\overline{xzy} \mid \overline{xz} \in R \text{ and } \overline{zy} \in R'\}.$$

[3] These requirements are used for the proofs, but are not strictly needed for our definitions to make sense.

$T \models P(x_1, \ldots, x_k)$ if $\forall (x_1, \ldots, x_k) \in T$ it holds that $P(x_1, \ldots, x_k)$.
$T \models \phi \wedge \psi$ if both $T \models \phi$ and $T \models \psi$.
$T \models \phi \vee \psi$ if there are T_1, T_2 such that $T_1 \models \phi$, $T_2 \models \psi$ and $T = T_1 \cup T_2$.
$T \models \forall x\ \phi$ if $(\mathbf{X} \times T) \models \phi$.
$T \models \exists x\ \phi$ if there is T' such that $\pi_{-x} T' = T$ and $T' \models \phi$.
$T \models =(\overline{x}, \overline{y})$ if for any $s, s' \in T$ it holds that if $s(\overline{x}) = s'(\overline{x})$ then $s(\overline{y}) = s'(\overline{y})$.
$T \models \overline{x} \perp \overline{y}$ if $(\pi_{\overline{x}} T) \times (\pi_{\overline{y}} T) = \pi_{\overline{xy}} T$.
$T \models \overline{x} \perp_{\overline{z}} \overline{y}$ if $(\pi_{\overline{xz}} T) \bowtie (\pi_{\overline{zy}} T) = \pi_{\overline{xzy}} T$.
$T \models \overline{x} \subseteq \overline{y}$ if $(\pi_{\overline{x}} T) \subseteq (\pi_{\overline{y}} T)$.

There is one case of a primitive where we will deviate from its usual definition. Usually, one would define

$T \models \overline{x} | \overline{y}$ if $\pi_{\overline{x}} T \cap \pi_{\overline{y}} T = \emptyset$. (**Classical definition**)

However, in a metric context *apartness* seems to be a far more natural notion than *disjointness* – and they obviously coincide in the traditional setting of finite models. We thus chose:

$T \models \overline{x} | \overline{y}$ if $d(\pi_{\overline{x}} T, \pi_{\overline{y}} T) > 0$ (**Our modified definition**)

We point out that in the semantics for $\exists$ and $\vee$, we are quantifying over teams. The precise scope of this quantification will vary in our investigation. We consider the case where the quantification ranges over the entire powerset of $\mathbf{X}$, the case where only closed teams are permitted, and briefly also the case where only open teams are permitted. These options are compared in Section 3.

2.1 Approximate dependence/independence atoms

We can use the metric available as part of our structure to define approximate versions of the dependence/independence atoms. As mentioned in the introduction, this has precedence in database theory, related to data cleaning. In many cases, there are two independent parameters describing how exactly we approximate the atoms. Typically, one parameter corresponds to relaxing the atom, the other to strengthening it. Depending on whether we chose strict or non-strict inequalities, one gets both open and closed versions of the atoms[4]. We denote the closed versions with $\overline{}$, the open versions are the non-decorated ones.

$T \models =^{\varepsilon}_{\delta} (\overline{x}, \overline{y})$ if for any $s, s' \in T$ it holds that if $d(s(\overline{x}), s'(\overline{x})) \leq \delta$ then $d(s(\overline{y}), s'(\overline{y})) < \varepsilon$.
$T \models \overline{=^{\varepsilon}_{\delta}} (\overline{x}, \overline{y})$ if for any $s, s' \in T$ it holds that if $d(s(\overline{x}), s'(\overline{x})) < \delta$ then $d(s(\overline{y}), s'(\overline{y})) \leq \varepsilon$.
$T \models \overline{x} \perp^{\delta,\varepsilon}_{\overline{z}} \overline{y}$ if for all $s, s' \in T$, if $d(s(\overline{z}), s'(\overline{z})) \leq \delta$ then there is $s'' \in T$ such that $d(s''(\overline{xz}), s(\overline{xz})) < \varepsilon$ and $d(s''(\overline{zy}), s'(\overline{zy})) < \varepsilon$.

[4] Of course, we could also mix the cases. However, part of the overall theme of this article is to control topological complexity, so this seems undesirable.

$T \models \overline{x}\overline{\perp^{\delta,\varepsilon}}_{\overline{z}}\overline{y}$ if for all $s, s' \in T$, if $d(s(\overline{z}), s'(\overline{z})) < \delta$ then there is $s'' \in T$ such that $d(s''(\overline{xz}), s(\overline{xz})) \leq \varepsilon$ and $d(s''(\overline{zy}), s'(\overline{zy})) \leq \varepsilon$.
$T \models \overline{x} \subseteq^{\varepsilon} \overline{y}$ if $(\pi_{\overline{x}}T) \subseteq B(\pi_{\overline{y}}T, \varepsilon)$.
$T \models \overline{x}\overline{\subseteq^{\varepsilon}}\overline{y}$ if $(\pi_{\overline{x}}T) \subseteq \overline{B}(\pi_{\overline{y}}T, \varepsilon)$.
$T \models \overline{x}|^{\varepsilon}\overline{y}$ if $d(\pi_{\overline{x}}T, \pi_{\overline{y}}T) > \varepsilon$.
$T \models \overline{x}\overline{|^{\varepsilon}}\overline{y}$ if $d(\pi_{\overline{x}}T, \pi_{\overline{y}}T) \geq \varepsilon$.

3 Restricting teams to closed sets

In this section, we show that for restricted formulae the semantics allowing arbitrary teams and the semantics allowing only closed teams coincide, and then give some examples how this breaks down for arbitrary formulae.

3.1 Closed formulae

Theorem 1. *For positive sentences involving closed basic predicates,* $\perp$, $\subseteq$, $\overline{=^{\varepsilon}_{\delta}}(\cdot,\cdot)$, $\overline{\perp^{\delta,\varepsilon}}_{\cdot}$, $\overline{\subseteq^{\varepsilon}}$ *and* $\overline{|^{\varepsilon}}$, *the usual team semantics and the teams-are-closed sets semantics agree.*

Proof. This is a special case of Corollary 1 below.

Lemma 1. *For an arbitrary team T, we find that*

1. $T \models P(\overline{x})$ *implies* $\overline{T} \models P(\overline{x})$, *where P is a basic closed predicate*
2. $T \models \overline{x} \perp \overline{y}$ *implies* $\overline{T} \models \overline{x} \perp \overline{y}$
3. $T \models \overline{x} \subseteq \overline{y}$ *implies* $\overline{T} \models \overline{x} \subseteq \overline{y}$
4. $T \models \overline{=^{\varepsilon}_{\delta}}(\overline{x}, \overline{y})$ *implies* $\overline{T} \models \overline{=^{\varepsilon}_{\delta}}(\overline{x}, \overline{y})$
5. $T \models \overline{x}\overline{\perp^{\delta,\varepsilon}}_{\overline{z}}\overline{y}$ *implies* $\overline{T} \models \overline{x}\overline{\perp^{\delta,\varepsilon}}_{\overline{z}}\overline{y}$
6. $T \models \overline{x}\overline{\subseteq^{\varepsilon}}\overline{y}$ *implies* $\overline{T} \models \overline{x}\overline{\subseteq^{\varepsilon}}\overline{y}$
7. $T \models \overline{x}\overline{|^{\varepsilon}}\overline{y}$ *implies* $\overline{T} \models \overline{x}\overline{|^{\varepsilon}}\overline{y}$

Proof. For 1-3, we only need that closure and projection commute. For 4,5 we note that since the premise of the implication in the definition has a strict inequality and the conclusion a non-strict one, taking the closure of the team has no impact. For 6,7 we are using the distance to the team, which is invariant under taking the closure.

Corollary 1. *Let ϕ be a positive formula involving basic closed predicates,* $\perp$, $\subseteq$, $\overline{=^{\varepsilon}_{\delta}}(\cdot,\cdot)$, $\overline{\perp^{\delta,\varepsilon}}_{\cdot}$, $\overline{\subseteq^{\varepsilon}}$ *and* $\overline{|^{\varepsilon}}$. *For an arbitrary team T, we find that $T \models \phi$ implies that $\overline{T} \models \phi$.*

Proof. Induction over the structure of ϕ. Lemma 1 provides the base case. The only non-trivial steps are $\exists$ and $\vee$, where we use that projection commutes with closure for the former, and that $T = T_1 \cup T_2$ implies $\overline{T} = \overline{T}_1 \cup \overline{T}_2$ for the latter.

3.2 Open formulae

Theorem 2. *For positive sentences involving basic open predicates and $|$, the following all agree:*

1. *the semantics allowing arbitrary sets as teams*
2. *the semantics demanding teams to be open sets*
3. *the semantics demanding teams to be closed sets*

Proof. By Lemma 2, truth in (1) implies truth in (2). By Lemma 3, truth in (2) implies truth in (3). That truth in (3) implies truth in (1) is trivial.

Lemma 2. *Let ϕ be a positive formula involving basic open predicates and $|$, let $T \models \phi$ and let $\overline{x} \in T$. Then there is some $\varepsilon > 0$ such that $T \cup B(\overline{x}, \varepsilon) \models \phi$, with only open teams being used as witnesses.*

Proof. We proceed by induction over the structure of ϕ. Universal and existential quantifier are trivial. For conjunctions, we use that the relevant formulae are downwards-closed, and take the minimum ε from both sides. For disjunctions, we note that from $\overline{x} \in T = T_1 \cup T_2$ we find $\overline{x} \in T_1$ or $\overline{x} \in T_2$, proceed to use the induction hypothesis on the relevant side, and then use that $T \cup B(\overline{x}, \varepsilon) = (T_i \cup B(\overline{x}, \varepsilon)) \cup T_{3-i}$.

The base cases for the induction are the basic open predicates and $|$, in both cases the claim follows from the definition.

Lemma 3. *Let ϕ be a positive formula involving basic open predicates and $|$, let $T \models \phi$ for open T, let $n \in \mathbb{N}$. Define $T_n = \{\overline{x} \in T \mid d(\overline{x}, T^C) \geq 2^{-n}\}$. Then $T_n \models \phi$, with only closed teams used as witnesses.*

Proof. Straight-forward from downwards-closure.

Downwards-closure of $|$ makes the proofs of Lemmas 2, 3 very simple, but this proof does not extent to the open approximate atoms that are not downwards-closed. We conjecture that downwards-closure is not actually needed for the statements to hold, but leave this for future work at this stage.

3.3 Counterexamples

We proceed to give some counterexamples showing that if we allow using both open and closed basic predicates in a formula, then the semantics for arbitrary teams and the semantics where we allow only closed teams differ. Using closed teams only lets us express some topological properties of the carrier metric space, and reveals some similarities to constructive mathematics.

Example 1. The formula

$$\forall x \forall y \ (x = y) \vee (x \neq y)$$

is a tautology for arbitrary teams, but expresses that the space is discrete for closed teams. Note that this is in line with how the formula works in constructive mathematics.

Example 2. The formula

$$\exists x \forall y\ (x = y) \vee (x \neq y)$$

is a tautology for arbitrary teams, but expresses that the space contains some isolated point for closed teams. Note that this is in line with how the formula works in constructive mathematics.

Example 3. The formula

$$\forall x \exists y \exists z\ \ ((x = y \vee x = z) \wedge y|z)$$

holds over $\mathbf{X} = [0,1]$ if arbitrary teams are allowed, but not if teams have to be closed sets. The reason is that the y and the z values have to be disjoint non-empty sets covering $\mathbf{X}$. For closed teams, this formula expresses that the space is disconnected.

Example 4. The formula

$$\forall x \exists y\ (x \neq y) \wedge = (x, y)$$

holds over every model with at least two elements, if arbitrary teams are allowed. If teams are restricted to closed sets, it asserts the negation of the fixed-point-property for $\mathbf{X}$.

4 Background on computability on metric spaces and for closed sets

We wish to study the algorithmic properties of questions such as satisfiability and model checking for our continuous team semantics. The algorithmic aspects of logics with team semantics have been studied extensively over finite structures (see the survey [7]). This requires notions of effectivity and computability for separable metric spaces, for hyperspaces of closed subsets of metric spaces, and finally for the entire collection of compact separable metric spaces. The field of computable analysis provides all of these notions. The standard reference is [24], but we follow [18]. Another short introduction to the area is [3].

As we lack the space for a rigorous development of the area, we will restrict our undertaking to a cursory description of the needed notions and special cases. The foundational concept in computable analysis is a *represented space*, which is just a set X together with a partial surjection $\delta :\subseteq \mathbb{N}^\mathbb{N} \to X$. Here δ tells us how the elements of interest are coded. We can then lift the usual notion of computation on $\mathbb{N}^\mathbb{N}$ to any represented space by letting our machine model act on names for elements.

A class of represented spaces of particular interest for us are the *computable metric spaces*: We take a separable metric space with a designated dense sequence $(a_n)_{n \in \mathbb{N}}$ such that $s < d(a_n, a_m) < t$ is recursively enumerable in $n, m \in \mathbb{N}$, $s, t \in \mathbb{Q}$. Then a point x is coded by giving a sequence p of indices such that

$d(a_{p(n)}, x) < 2^{-n}$. A computable metric space $\mathbf{X}$ is *computably compact*, if the set of finite sequences $(n_0, r_0), \ldots, (n_\ell, r_\ell)$ such that $\mathbf{X} \subseteq \bigcup_{i \leq \ell} B(a_{n_i}, 2^{-r_i})$ is recursively enumerable. Computable metric spaces always have two further properties we will use; they are computably Hausdorff and computably overt.

We use several hyperspace constructions, i.e. constructions of certain spaces of subsets of a given represented space. We have the space $\mathcal{O}(\mathbf{X})$ of *open subsets*, the space $\mathcal{A}(\mathbf{X})$ of closed subsets and the space $\mathcal{V}(\mathbf{X})$ of compact subsets.

The open subsets are characterized by $x \in U$ being semidecidable (recognizable) in $x \in \mathbf{X}$ and $U \in \mathcal{O}(\mathbf{X})$. For a computable metric space $\mathbf{X}$, this can concretely be achieved by coding $U \in \mathcal{O}(\mathbf{X})$ as $\langle p, q \rangle$ with $U = \bigcup_{\{n \mid p(n) \neq 0\}} B(a_{p(n)+1}, 2^{-q(n)})$. The closed subsets are the formal complements of the open sets, i.e. the codes for $A \in \mathcal{A}(\mathbf{X})$ are just the codes for $(X \setminus A) \in \mathcal{O}(\mathbf{X})$.

The overt subsets $\mathcal{V}(\mathbf{X})$ are assumed to be closed extensionally, but have different codes and subsequently very different associated computable operations from $\mathcal{A}(\mathbf{X})$. The overt subsets are characterized by $U \cap A \neq \emptyset$ being semidecidable (recognizable) in $U \in \mathcal{O}(\mathbf{X})$ and $A \in \mathcal{V}(\mathbf{X})$. In a computable metric space they can be coded as a list of all basic open balls intersecting them.

Since $\mathcal{A}(\mathbf{X})$ and $\mathcal{V}(\mathbf{X})$ pointwise have the same elements, we can also construct $\mathcal{A} \wedge \mathcal{V}(\mathbf{X})$, where a set is coded by the combination of its $\mathcal{A}(\mathbf{X})$-code and its $\mathcal{V}(\mathbf{X})$-code. Over a computably compact computable metric space $\mathbf{X}$, the space $\mathcal{A} \wedge \mathcal{V}(\mathbf{X})$ corresponds to the space of closed subsets equipped with the Hausdorff metric. In this case, the space is characterized by making $d : \mathbf{X} \times (\mathcal{A} \wedge \mathcal{V}(\mathbf{X})) \to \mathbb{R}$ computable.

4.1 Overtness and compactness

The relevance of the notions of compactness and overtness for spaces in general, and for our purposes in particular, is exhibit by the following lemmas tying it in to the preservation of the complexity of formula under quantification.

Lemma 4. *The following are equivalent for a represented space* $\mathbf{X}$*:*

1. $\mathbf{X}$ *is computably overt.*
2. $\exists : \mathcal{O}(\mathbf{X} \times \mathcal{Y}) \to \mathcal{O}(\mathbf{Y})$ *is computable for all represented spaces* $\mathbf{Y}$ *(respectively some space containing a computable point)*
3. $\forall : \mathcal{A}(\mathbf{X} \times \mathcal{Y}) \to \mathcal{A}(\mathbf{Y})$ *is computable for all represented spaces* $\mathbf{Y}$ *(respectively some space containing a computable point)*

Proof. The equivalence of 1 and 2 is a special case of [18, Proposition 40]. The equivalence of 2 and 3 is by duality.

Lemma 5. *The following are equivalent for a represented space* $\mathbf{X}$*:*

1. $\mathbf{X}$ *is computably compact.*
2. $\forall : \mathcal{O}(\mathbf{X} \times \mathcal{Y}) \to \mathcal{O}(\mathbf{Y})$ *is computable for all represented spaces* $\mathbf{Y}$ *(respectively some space containing a computable point)*
3. $\exists : \mathcal{A}(\mathbf{X} \times \mathcal{Y}) \to \mathcal{A}(\mathbf{Y})$ *is computable for all represented spaces* $\mathbf{Y}$ *(respectively some space containing a computable point)*

Proof. The equivalence of 1 and 2 is a special case of [18, Proposition 42]. The equivalence of 2 and 3 is by duality.

4.2 Computable operations on the closed and overt sets

We proceed to recall or establish the basic properties of the space $\mathcal{A} \wedge \mathcal{V}(\mathbf{X})$ which we shall use in the following.

Theorem 3 (Park, Park, Park, Seon and Ziegler [17]). *For a computably compact computable metric space* $\mathbf{X}$*, the space* $\mathcal{A} \wedge \mathcal{V}(\mathbf{X})$ *is a computably compact computable metric space again.*

Corollary 2. *For a computably compact computable metric space* $\mathbf{X}$*, the space* $\mathcal{A} \wedge \mathcal{V}(\mathbf{X})$ *is computably overt.*

Corollary 3. *For a computably compact computable metric space* $\mathbf{X}$*, the space* $\mathcal{A} \wedge \mathcal{V}(\mathbf{X})$ *is computably compact.*

Lemma 6. *The following maps are computable for computably compact* $\mathbf{Y}$*, and countably-based* $\mathbf{X}$*,*$\mathbf{Y}$*:*

1. $\pi_x : \mathcal{A} \wedge \mathcal{V}(\mathbf{X} \times \mathbf{Y}) \to \mathcal{A} \wedge \mathcal{V}(\mathbf{X})$
2. $\times : \mathcal{A} \wedge \mathcal{V}(\mathbf{X}) \times \mathcal{A} \wedge \mathcal{V}(\mathbf{Y}) \to \mathcal{A} \wedge \mathcal{V}(\mathbf{X} \times \mathbf{Y})$

Proof. 1. We can show separately that $\pi_x : \mathcal{A}(\mathbf{X} \times \mathbf{Y}) \to \mathcal{A}(\mathbf{X})$ and $\pi_x : \mathcal{V}(\mathbf{X} \times \mathbf{Y}) \to \mathcal{V}(\mathbf{X})$ are computable. The former is [18, Proposition 8 (8)] (using computable compactness of $\mathbf{Y}$), the latter is [18, Proposition 21 (6)].
2. Again, this can be shown separately for $\mathcal{A}$ and $\mathcal{V}$. The former is [18, Proposition 6 (8)]. For the latter, we use the fact that $\mathbf{X}$,$\mathbf{Y}$ being countably-based implies that $\mathcal{O}(\mathbf{X} \times \mathbf{Y})$ effectively is the product topology, i.e. that there is a computable multi-valued operation $\mathrm{Decompose} : \mathcal{O}(\mathbf{X} \times \mathbf{Y}) \rightrightarrows \mathcal{C}(\mathbb{N}, \mathcal{O}(\mathbf{X}) \times \mathcal{O}(\mathbf{Y}))$ such that $(U_i, V_i)_{i \in \mathbb{N}} \in \mathrm{Decompose}(O)$ iff $O = \bigcup_{n \in \mathbb{N}} U_n \times V_n$. Now if $(U_i, V_i)_{i \in \mathbb{N}} \in \mathrm{Decompose}(O)$ we find that O intersects $A \times B \subseteq \mathbf{X} \times \mathbf{Y}$ iff $\exists n \in \mathbb{N}\ U_n \cap A \neq \emptyset \wedge V_n \cap B \neq \emptyset$. By currying, this is all we need.

Lemma 7. *In a computably compact computable metric space* $\mathbf{X}$*, the map* $(A, \varepsilon) \mapsto \overline{B}(A, \varepsilon) : \mathcal{A}(\mathbf{X}) \times \mathbb{R}_+ \to \mathcal{A}(\mathbf{X})$ *is computable.*

Proof. We have $y \notin \overline{B}(A, \varepsilon)$ iff $\overline{B}(y, \varepsilon) \cap A = \emptyset$. The latter is an open property by computable compactness.

Lemma 8. *The following are closed predicates on* $\mathcal{A} \wedge \mathcal{V}(\mathbf{X}) \times \mathcal{A} \wedge \mathcal{V}(\mathbf{X})$*:*

1. $\subseteq$
2. $=$

Proof. Note that $A \subseteq B$ iff $A \cap B^C = \emptyset$, and that by definition of $\mathcal{A}$ and $\mathcal{V}$, $A \cap B^C \neq \emptyset$ is already an open predicate on $\mathcal{V}(\mathbf{X}) \times \mathcal{A}(\mathbf{X})$. For 2., just observe that $A = B$ iff $A \subseteq B \wedge B \subseteq A$.

Lemma 9. $\cup : \mathcal{A} \wedge \mathcal{V}(\mathbf{X}) \times \mathcal{A} \wedge \mathcal{V}(\mathbf{X}) \to \mathcal{A} \wedge \mathcal{V}(\mathbf{X})$ *is computable.*

Proof. This is the combination of [18, Proposition 6(3)] and [18, Proposition 21(3)].

4.3 Compact metric structures

We shall now discuss the connection of the theory of computable metric subspace and the induced hyperspaces to the notion of a structure as used to interpret logical formulas. First, we note that we have the represented space Pol of Polish spaces and the represented space KPol of compact separable metric spaces. A similar hyperspace of countably-based spaces was introduced and studied in [20]. In the space Pol, we code a separable space $\mathbf{X}$ by presupposing $\mathbb{N}$ as a dense set, and then providing all distances $d_{\mathbf{X}}(n, m)$. This uniquely determines a Polish space by considering the completion. The space KPol is not merely the subspace of Pol restricted to compact spaces, but here we additionally code all finite covers $B(n_0, 2^{-k_0}) \cup \ldots \cup B(n_\ell, 2^{-k_\ell})$ into the name of the space. We point out that all arguments given above[5] regarding the properties of computably compact computable metric spaces hold uniformly in a compact metric space given as an element of KPol.

To introduce the notion of a structure, we first need the notion of a signature. A *signature* consists of function and relation symbols, each with some finite arity. Once a signature is fixed, we define a structure to be an underlying set A, together with a function $f_i : A^{n_i} \to A$ for each function symbol of arity n_i, and a subset $R_i \subseteq A^{n_i}$ for each relation symbol of arity n_i. Note that, contrary to convention, we do not make equality available for free. Instead, we can only use equality in our formula if it is provided as a relation by the signature/structure. We lift this to compact metric spaces as follows:

Definition 1. *A compact metric closed (respectively open) structure (over a given signature) consists of a compact separable metric space* $\mathbf{X}$ *as carrier, a continuous function* $f_i : \mathbf{X}^{n_i} \to \mathbf{X}$ *for each function symbol of arity* n_i *in the signature, and a closed (respectively open) subset* $R_i \subseteq \mathbf{X}^{n_i}$ *for each relation symbol of arity* n_i *in the signature.*

We write ACMS *(respectively* OCMS*) for the represented space of compact metric closed (respectively open) structures, where the carrier is given as* $\mathbf{X} \in \mathrm{KPol}$*, the functions as* $f_i \in \mathcal{C}(\mathbf{X}^{n_i}, \mathbf{X})$ *and the relations as* $R_i \in \mathcal{A}(\mathbf{X}^{n_i})$ *(respectively as* $R_i \in \mathcal{O}(\mathbf{X}^{n_i})$*).*

5 Topological complexity

We proceed to study the topological complexity of the atoms, of formula satisfaction and of model checking. Since we have a topology on the space of closed and overt sets we are using for teams, and on the spaces of structures, these are all well-defined notions.

[5] In fact, example of statements that are computable for each computable metric space, yet are not computable uniformly in the metric space are very rare in the literature. See [19, Proposition 14] for such a rare example.

5.1 Topological complexity of dependence atoms

Proposition 1. *The following are closed predicates in the team:*

1. $T \models P(\overline{x})$, *where P is a basic closed predicate*
2. $T \models \overline{x} \perp \overline{y}$
3. $T \models \overline{x} \subseteq \overline{y}$
4. $T \models \overline{=^{\varepsilon}_{\delta}}(\overline{x}, \overline{y})$
5. $T \models \overline{x}\overline{\perp^{\delta,\varepsilon}}_{\overline{z}}\overline{y}$
6. $T \models \overline{x}\overline{\underline{\subseteq}^{\varepsilon}}\overline{y}$
7. $T \models \overline{x}|^{\varepsilon}\overline{y}$

Proof. By Lemmas 6, 8, 1-3 follow immediately from the definitions. For 4-5, we have a universal quantification over the team, and then a closed property, which makes for a closed property by Lemma 5. For 6, note this is obtained by combining Lemmas 6, 7 and 8. Finally, 7. just follows from the continuity of the Hausdorff distance on $\mathcal{A} \wedge \mathcal{V}(\mathbf{X})$.

Proposition 2. *The following are open predicates in the team:*

1. $T \models P(\overline{x})$, *where P is a basic open predicate*
2. $T \models \overline{x}|\overline{y}$
3. $T \models =^{\varepsilon}_{\delta} (\overline{x}, \overline{y})$
4. $T \models \overline{x} \perp^{\delta,\varepsilon}_{\overline{z}} \overline{y}$
5. $T \models \overline{x} \subseteq^{\varepsilon} \overline{y}$
6. $T \models \overline{x}|^{\varepsilon}\overline{y}$

Proof. For 1., note that this means $\pi_{\overline{x}}T \subseteq P$. By Lemma 6, we can compute $\pi_{\overline{x}}T$ as a closed set, which we also have as a compact set due to the fact that we are working in a compact space. By definition of compact sets, this makes the predicate open. Claim 2. follows immediately from the definition by Lemmas 6, 8. Items 3., 4., 5. and 6. are analogous to their closed counterparts in Proposition 1.

Proposition 3. *The following are Π^0_2-complete predicates in the team:*

1. $T \models =(\overline{x}, y)$
2. $T \models \overline{x} \perp_{\overline{z}} \overline{y}$

Proof. We obtain a lower bound for $T \models =(\overline{x}, y)$ from Lemma 10, and an upper bound for $T \models \overline{x} \perp_{\overline{z}} \overline{y}$ by noting that $\forall \varepsilon \exists \exists \delta T \models \overline{x}\delta \perp^{\delta,\varepsilon}_{\overline{z}} \overline{y}$ is equivalent to $T \models \overline{x} \perp_{\overline{z}} \overline{y}$. These are then linked by noting that $=(\overline{x}, y) \equiv y \perp_{\overline{x}} y$.

Lemma 10. *$T \models =(x, y)$ is Π^0_2-hard over $\{0,1\}^{\mathbb{N}}$.*

Proof. Given some $p \in \{0,1\}^{\mathbb{N}}$, we compute some $T_p \in \mathcal{A} \wedge \mathcal{V}(\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}})$ such that $T_p \models =(x, y)$ iff p contains infinitely many 1s. We point out that a set $A \in \mathcal{A} \wedge \mathcal{V}(\{0,1\}^{\mathbb{N}})$ can be represented as a sequence $(W_k)_{k \in \mathbb{N}}$ where

$W_k \subseteq \{0,1\}^{2^k}$ satisfy that $\forall w \in W_k \exists u \in \{0,1\}^{2^k}$ $wu \in W_{k+1}$ and $q \in A \Leftrightarrow \forall k$ $q_{\leq 2^k} \in W_k$.

We define our sequence inductively, and take into account the standard bijection $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}}$. We start with $W_1 = \{00, 01, 10, 11\}$. Whenever $p(k) = 1$, then we let $W_{k+1} = \{\langle wu, uu\rangle \mid \langle w, u\rangle \in W_k\}$. Whenever $p(k) = 0$, then $W_{k+1} = \{wu \mid w \in W_k \wedge u \in \{0,1\}^{2^k}\}$.

To argue that this construction works as intended, let us first consider the case where p has only finitely many 1s. Let K be sufficiently large that $p(j) = 0$ for all $j \geq K$. Pick some $\langle w, u\rangle \in W_K$. Now the construction ensures that $w\{0,1\}^{\mathbb{N}} \times u\{0,1\}^{\mathbb{N}} \subseteq T_p$, hence $T_p \not\models =(x, y)$.

Conversely, assume for the sake of a contradiction that p contains infinitely many 1s, yet $T_p \not\models =(x, y)$. Pick witnesses a, b_1, b_2 for the latter, i.e. satisfying that $(a, b_1) \in T_p$ and $(a, b_2) \in T_p$, yet $b_1 \neq b_2$. Pick K such that $p(K) = 1$ and $(b_1)_{\leq 2^K} \neq (b_2)_{\leq 2^K}$. We must have that $\langle a_{\leq 2^{K+1}}, (b_1)_{\leq 2^{K+1}}\rangle \in W_{K+1}$ and $\langle a_{\leq 2^{K+1}}, (b_2)_{\leq 2^{K+1}}\rangle \in W_{K+1}$, but these cannot both be of form $\langle wu, uu\rangle$, contradiction.

5.2 Complexity of formula satisfaction

Theorem 4. *Let ϕ be a positive formula involving closed basic predicates, $\perp$, $\subseteq$, $\overline{=^{\varepsilon}_{\delta}}(\cdot,\cdot)$, $\overline{\perp^{\delta,\varepsilon}}\cdot$, $\overline{\subseteq^{\varepsilon}}$ and $\overline{|^{\varepsilon}}$. Then $T \models \phi$ defines a closed predicate in the team (uniformly in ϕ).*

Proof. By induction on the structure of ϕ. The base cases are provided by Proposition 1.

Dealing with $\wedge$ is trivial.

Let $\phi = \phi_1 \vee \phi_2$. By induction hypothesis, $T_i \models \phi_i$ is a closed predicate in T_i. Then $T = T_1 \cup T_2 \wedge T_1 \models \phi_1 \wedge T_2 \models \phi_2$ is a closed predicate in (T, T_1, T_2), since $\cup$ is computable on $\mathcal{A} \wedge \mathcal{V}(\mathbf{X})$ by Lemma 9 and $\mathcal{A} \wedge \mathcal{V}(\mathbf{X})$ is Hausdorff by Lemma 8. By Corollary 3, Lemma 5 applies and lets us conclude that quantifying existentially over T_1 and T_2 still leaves us with a closed predicate.

For $\phi = \forall x\ \psi$, we note that $T \mapsto \mathbf{X} \times T$ is computable, and that $T \models \phi$ is preimage of the closed predicate $T' \models \psi$ under that map.

For $\phi = \exists x\ \psi$, we invoke Lemma 5 by means of Corollary 3.

Since we have not yet established whether the semantics for arbitrary teams and closed teams still agree when also the approximate open predicates are permitted, for now we study the complexity of satisfaction only in the case where satisfaction is unambiguous:

Theorem 5. *Let ϕ be a positive formula involving basic open predicates and $|$. Then $T \models \phi$ defines a open predicate in the team (uniformly in ϕ). As a consequence, if $T \models \phi$, then we can effectively find some $n \in \mathbb{N}$ such that any T' with $d(T, T')$ satisfies $T' \models \phi$.*

Proof. By induction on the structure of ϕ. The base cases are provided by Proposition 2.

Dealing with $\wedge$ is trivial.

Let $\phi = \phi_1 \vee \phi_2$. By induction hypothesis, $T_i \models \phi_i$ is an open predicate in T_i. Given T_0, T_1 with $T_i \models \phi_i$, we can find some $n \in \mathbb{N}$ such that if $d(T_i, T_i')$, then $T_i' \models \phi_i$. The map $T_i \mapsto OT_i := \{x \in \mathbf{X}^k \mid d(x, T_i) < 2^{-n}\} :\subseteq \mathcal{A} \wedge \mathcal{V}(\mathbf{X}^k) \to \mathcal{O}(\mathbf{X}^k)$ is computable. We can extend this to a computable total map (i.e. define it also for $T_i \not\models \phi$ by setting $OT_i = \emptyset$ in that case. Now $T_1 \models \phi_1 \wedge T_2 \models \phi_2 \wedge T \subseteq OT_1 \cup OT_2$ is an open predicate in (T, T_1, T_2). Clearly, whenever the T_i are suitable witnesses for $T \models \phi$, this predicate is satisfied. Conversely, if the predicate is satisfied, consider $T_i' = \{x \in T \mid d(x, T_i) \leq d(x, T_{2-i})\}$ and note that $d(T_i, T_i') < 2^{-n}$, hence $T_i' \models \phi$. Thus, our modified predicate is equivalent to the existence of witnesses for $T \models \phi$. Corollary 2 lets us invoke Lemma 4 to remove the existential quantifier over T_i.

For $\phi = \forall x\ \psi$, we note that $T \mapsto \mathbf{X} \times T$ is computable, and that $T \models \phi$ is preimage of the open predicate $T' \models \psi$ under that map.

Let $\phi = \exists x\ \psi$. Similar to the argument above, the map $T' \mapsto OT' : \mathcal{A} \wedge \mathcal{V}(\mathbf{X}^{k+1}) \to \mathcal{O}(\mathbf{X}^k)$ is computable; mapping T' with $T' \models \psi$ to $OT' = \{y \in \mathbf{X}^k \mid d(x, \pi_{-x}T') < 2^{-n}\}$, where n is chosen such that if $d(T', T'') < 2^{-n}$, then $T'' \models \psi$; and mapping $T' \not\models \psi$ to $OT' = \emptyset$. Now $T' \models \psi \wedge T \subseteq OT'$ is an open predicate in (T, T'), and as above, equivalent to the existence of a witness for $T \models \phi$. Corollary 2 lets us invoke Lemma 4 to remove the existential quantifier over T'.

5.3 The complexity of model checking

As the proofs of Theorem 4 and Theorem 5 are fully uniform, we can obtain a classification of the model checking problem. We shall write $\mathcal{L}_+(\perp, \subseteq, \overline{=^{\varepsilon}_{\delta}}(\cdot,\cdot), \overline{\perp^{\delta,\varepsilon}}\cdot, \overline{\subseteq^{\varepsilon}}, \overline{|^{\varepsilon}})$ for the set of positive sentences involving basic predicates, $\perp$, $\subseteq$, $\overline{=^{\varepsilon}_{\delta}}(\cdot,\cdot)$, $\overline{\perp^{\delta,\varepsilon}}\cdot$, $\overline{\subseteq^{\varepsilon}}$ and $\overline{|^{\varepsilon}}$. Likewise, we write $\mathcal{L}_+(|)$ for the set of positive sentences involving basic predicates and $|$. These sets are coded in the obvious way, including the real-valued parameters. Note that formulae from $\mathcal{L}_+(\perp, \subseteq, \overline{=^{\varepsilon}_{\delta}}(\cdot,\cdot), \overline{\perp^{\delta,\varepsilon}}\cdot, \overline{\subseteq^{\varepsilon}}, \overline{|^{\varepsilon}})$ can use all potential choices for ε and δ, but that no quantification over these parameters is available. We recall our convention that equality is not automatically available, but would need to be provided by interpreting some binary relation symbol accordingly. We then find:

Corollary 4. *It is semidecidable whether a formula* $\mathcal{L}_+(\perp, \subseteq, \overline{=^{\varepsilon}_{\delta}}(\cdot,\cdot), \overline{\perp^{\delta,\varepsilon}}\cdot, \overline{\subseteq^{\varepsilon}}, \overline{|^{\varepsilon}})$ *does* **not hold** *in a structure* $\mathfrak{S} \in$ ACMS*.*

Proof. From Theorem 4. Note that a sentence ϕ is satisfied in a structure with carrier $\mathbf{X}$ iff $\{1\} \models \phi$ for the trivial non-empty team $\{1\} \subseteq \mathbf{X}^0$.

Corollary 5. *It is semidecidable whether a formula* $\phi \in \mathcal{L}_+(|)$ **holds** *in a structure* $\mathfrak{S} \in$ OCMS*.*

Proof. From Theorem 5. Note that a sentence ϕ is satisfied in a structure with carrier $\mathbf{X}$ iff $\{1\} \models \phi$ for the trivial non-empty team $\{1\} \subseteq \mathbf{X}^0$.

A priori, having even decidability may seem desirable. This, however, is completely out of the question:

Proposition 4. *It is undecidable whether $\forall x\ R(x)$ holds in a structure $\mathfrak{S} \in$ AMCS or $\mathfrak{S} \in$ OMCS, even if we restrict to the case where the carrier space is the one-point space* $\mathbf{1}$.

Proof. In the restricted case, the question becomes whether R is interpreted as the universal predicate or as the empty predicate. This is undecidable for $R \in \mathcal{O}(\mathbf{1})$ or $R \in \mathcal{A}(\mathbf{1})$.

6 Translations between approximate atoms

In classical dependence logic the expressive power of logics appended with various combinations of dependence/independence atoms has been studied. The comparisons rely on translations of the atoms. In an approximate setting we don't get exact translations, but can 'sandwich' atoms between parameterised variants of a formula using some other atoms.

In the translations we use as underlying logic first order logic without equality, and replace equality by either open or closed metric predicates.

We give three 'translations' between dependency atoms. We only show the open versions here, but the closed counterparts are proved similarly.

Proposition 5. *1. If $\varepsilon > \delta \geq 0$, then $=^{\varepsilon}_{\delta} (\overline{x}, \overline{y}) \quad \Rightarrow \quad \overline{y} \perp^{\delta,\varepsilon}_{\overline{x}} \overline{y}$.*
2. For any $\delta \geq 0, \varepsilon > 0$, $\overline{y} \perp^{\delta,\varepsilon/2}_{\overline{x}} \overline{y} \quad \Rightarrow \quad =^{\varepsilon}_{\delta} (\overline{x}, \overline{y})$.

Proof. For the first, assume $T \models =^{\varepsilon}_{\delta} (\overline{x}, \overline{y})$ and let $s, s' \in T$ be such that $d(s(\overline{x}), s'(\overline{x})) \leq \delta$. Then $d(s(\overline{y}), s'(\overline{y})) < \varepsilon$, so s satisfies the independence witness requirement $d(s(\overline{xy}), s(\overline{xy})) < \varepsilon$ and $d(s(\overline{xy}), s'(\overline{xy})) < \varepsilon$.

For the second claim, assume $T \models \overline{y} \perp^{\delta,\varepsilon/2}_{\overline{x}} \overline{y}$ and let $s, s' \in T$ be such that $d(s(\overline{x}), s'(\overline{x})) \leq \delta$. Then there is $s'' \in T$ such that $d(s''(\overline{xy}), s(\overline{xy})) < \varepsilon$ and $d(s''(\overline{xy}), s'(\overline{xy})) < \varepsilon$, and the claim follows by the triangle inequality.

The next proposition is a metric modification of Galliani's proof from [8]. The remarkable thing is, that the Boolean encoding he uses can be made to work in this metric setting.

Proposition 6. *Assuming all models considered have diameter at least D,*

$$\begin{aligned}\overline{x} \subseteq^{\varepsilon} \overline{y} \quad \Rightarrow \quad & \forall v_1 \forall v_2 \forall \overline{z}(\\ & (d(\overline{z}, \overline{x}) > \delta/2 \wedge d(\overline{z}, \overline{y}) > \varepsilon) \vee \\ & (d(v_1, v_2) < d_2 + \delta \wedge d(v_1, v_2) > d_1 - \delta) \vee \\ & (d(v_1, v_2) > d_2 \wedge d(\overline{z}, \overline{y}) > \varepsilon) \vee \\ & ((d(v_1, v_2) < d_1 \vee d(\overline{z}, \overline{y}) < \varepsilon + \delta/2) \wedge \overline{z} \perp^{\delta} v_1 v_2))\end{aligned}$$

for any $d_1 < d_2 < D$ and $0 < \delta < d_1, D - d_2$.

Proof. Assume $T \models \overline{x} \subseteq^{\varepsilon} \overline{y}$. Let $T' = T[M/v_1][M/v_2][M/\overline{z}]$.[6] Let

$$\begin{aligned}
T_1 &= \{s \in T' : d(s(\overline{z}), s(\overline{x})) > \delta/2 \,\&\, d(s(\overline{z}), s(\overline{y}) > \varepsilon\},\\
T_2 &= \{s \in T' : d(s(v_1), s(v_2)) < d_2 + \delta \,\&\, d(v_1, v_2) > d_1 - \delta\},\\
T_3 &= \{s \in T' : d(s(v_1), s(v_2)) > d_2 \,\&\, d(s(\overline{z}), s(\overline{y})) > \varepsilon\},\\
T_4 &= T' \backslash (T_1 \cup T_2 \cup T_3).
\end{aligned}$$

So we need to show that anything not in $T_1 \cup T_2 \cup T_3$ satisfies the fourth disjunct. Now, if $s \in T_4$ is such that $d(s(v_1), s(v_2)) \geq d_1 > d_1 - \delta$, then (as it is not in T_2) $d(s(v_1), s(v_2)) \geq d_2 + \delta > d_2$. Thus (since $s \notin T_3$) $d(s(\overline{z}), s(\overline{y})) \leq \varepsilon < \varepsilon + \delta/2$. So the first conjunct is satisfied.

Next consider $s, s' \in T_4$. If $d(s(\overline{z}), s(\overline{y})) \leq \varepsilon$, then $s'' = s[s'(v_1v_2)/v_1v_2] \in T_4$ and it witnesses the independence atom with respect to s and s'. If, on the other hand, $d(s(\overline{z}), s(\overline{y})) > \varepsilon$, then (by $s \notin T_1 \cup T_2 \cup T_3$) $d(s(\overline{z}), s(\overline{x})) \leq \delta/2$ and $d(s(v_1), s(v_2)) \leq d_2$, and thus $d(s(v_1), s(v_2)) < d_1$. Since T, and thus also T' satisfies $\overline{x} \subseteq^{\varepsilon} \overline{y}$, there is $s^+ \in T'$ such that $d(s^+(\overline{y}), s(\overline{x})) < \varepsilon$, so $d(s^+(\overline{y}), s(\overline{z})) < \varepsilon + \delta/2$. Let $s'' = s^+[s'(v_1v_2)/v_1v_2][s(\overline{x})/\overline{z}]$. Then $d(s''(\overline{z}), s(\overline{z})) = d(s(\overline{x}), s(\overline{z})) \leq \delta/2 < \delta$ and $s''(v_1v_2) = s'(v_1v_2)$ so we are done if we can show $s'' \in T_4$. But by the values for v_1v_2, $s'' \notin T_2$, and $d(s''(\overline{z}), s''(\overline{y})) = d(s(\overline{x}), s^+(\overline{y})) < \varepsilon$ so $s'' \notin T_1 \cup T_3$.

Proposition 7.

$$\begin{aligned}
\overline{x} \subseteq^{\varepsilon+\delta} \overline{y} \quad \Leftarrow \quad & \forall v_1 \forall v_2 \forall \overline{z}(\\
& (d(\overline{z}, \overline{x}) > \delta/2 \wedge d(\overline{z}, \overline{y}) > \varepsilon) \vee\\
& (d(v_1, v_2) < d_2 + \delta \wedge d(v_1, v_2) > d_1 - \delta) \vee\\
& (d(v_1, v_2) > d_2 \wedge d(\overline{z}, \overline{y}) > \varepsilon) \vee\\
& ((d(v_1, v_2) < d_1 \vee d(\overline{z}, \overline{y}) < \varepsilon + \delta/2) \wedge \overline{z} \perp^m v_1v_2))
\end{aligned}$$

for any $d_1 < d_2 < D$ *and* $0 < \delta < d_1, D - d_2$ *and with* $m = \min\{\frac{d_2 - d_1}{2}, \frac{\delta}{2}\}$.

Proof. Assume T satisfies the formula on the right hand side and let $s \in T$ be arbitrary. Let $T' = T[M/v_1][M/v_2][M/\overline{z}]$ and choose $s' \in T'$ such that $s'(\overline{z}) = s'(\overline{x}) = s(\overline{x})$ and $d(s'(v_1), s(v_2)) < d_1$ (exists by universal quantification). Further choose $s^+ \in T'$ such that $d(s^+(\overline{z}), s^+(\overline{y})) \leq \varepsilon$ and $d(s^+(v_1), s^+(v_2)) \geq d_2 + \delta$. Then neither of s' and s^+ satisfies any of the first three disjuncts of the right hand side formula, so both must be in the part of T' satisfying the fourth. Thus there is s'' in this part satisfying $d(s''(\overline{z}), s'(\overline{z})) < m$, $d(s''(v_1v_2), s^+(v_1v_2)) < m$. So $d(s''(v_1), s''(v_2)) > d_2 - 2m > d_1$, and we must have $d(s''(\overline{z}), s''(\overline{y})) \leq \varepsilon + \delta/2$. So $d(s''(\overline{y}), s(\overline{x})) = d(s''(\overline{y}), s'(\overline{z})) < \varepsilon + \delta/2 + m \leq \varepsilon + \delta$.

Note that in the following translation we only have a translation for the dependence atom $=^{\varepsilon}_{\varepsilon} (\cdot)$ and not the general form $=^{\varepsilon}_{\delta} (\cdot)$. We show the proof for

[6] $T[M/x] = \{s(a/x) : s \in T, a \in M\}$ denotes the team one gets by adding every possible value for x to each assignment of T.

the open forms, but the closed ones go through practically verbatim. Here the closed version of the dependence atom may feel a bit more natural, as the open form talks about a contraction.

Proposition 8. *1.* $=^{\varepsilon}_{\varepsilon}(\bar{x}, y) \quad \Rightarrow \quad \forall z(d(z,y) < 2\varepsilon \vee \bar{x}z|^{\varepsilon}\bar{x}y)$.
2. $=^{\varepsilon}_{\varepsilon}(\bar{x}, y) \quad \Leftarrow \quad \forall z(d(z,y) < \varepsilon \vee \bar{x}z|^{\varepsilon}\bar{x}y)$.

Proof. For the first direction assume $T \models =^{\varepsilon}_{\varepsilon}(\bar{x}, y)$ and let $T' = T[M/z]$. Let $Y_1 = \{s \in T' : d(s(z), s(y)) < 2\varepsilon\}$ and $Y_2 = T' \setminus Y_1$. Now if $s, s' \in Y_2$ we cannot have $d(s(\bar{x}z), s'(\bar{x}y)) \leq \varepsilon$, as then we would have $d(s(y), s'(y)) < \varepsilon$ and thus $d(s(z), s(y)) \leq d(s(z), s'(y)) + d(s'(y), s(y)) < 2\varepsilon$, a contradiction. Thus Y_2 satisfies the second disjunct.

For the other direction, assume T satisfies the right hand side, and $s, s' \in T$ are such that $d(s(\bar{x}), s'(\bar{x})) \leq \varepsilon$. If $d(s(y), s'(y)) \geq \varepsilon$, then $s^+ := s[s'(y)/z]$ and $s^{'+} := s'[s(y)/z]$ cannot satisfy the first disjunct on the right hand side, so we must have $d(s^+(\bar{x}z), s^{'+}(\bar{x}y)) > \varepsilon$,a contradiction.

7 Outlook

We have introduced team semantics for logics with both exact and approximate dependence/independence atoms in the setting of metric spaces. We have shown that for compact carrier spaces, requiring teams to be closed sets leads to very nice behaviour, provided that our formulae contain only atoms of a certain type. While not all atoms are permitted in a single formula for this, we have approximate version of all atoms available. For formulae using the *open* variants of the approximate atoms, some questions remain open for now, but we believe that this will be straight-forward to settle.

There are some potential connections to other areas of logic we wish to explore in the future. On the one hand, as observed in Subsection 3.3 requiring teams to be closed adds a flavour of constructive math to the resulting statements. Concretely, there could be some relationship between satisfaction of formula interpreted via team semantics with closed teams, and provability in systems such as $\mathrm{RCA}_0 + \mathrm{WKL}$ from reverse math [21] or BISH + WKL from intuitionistic reverse math (e.g. [14, 5]).

On the other hand, the translations in Section 6 do not really give us a true comparison of logics, as they don't contain a measure of accuracy of the translations. A remedy seems to be considering many-valued logics, e.g., continuous first order logic from [2], that have a built-in grading of the strength of implications. Such a logic opens up a plethora of questions of the right choice of semantics, as it allows both for new connectives and enables new ways of aggregating truth values over a team.

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