Three Moments in the Life of the Mathematical Diagram : Notes on the Syntax and Semantics Distinction

Kennedy, Juliette Cara

Cambridge University Press
2017

Kennedy , J C 2017 , Three Moments in the Life of the Mathematical Diagram : Notes on the Syntax and Semantics Distinction . in L de Freitas , N Sinclair & A Coles (eds) , What is a Mathematical Concept? . Cambridge University Press , Cambridge , pp. 55-75 . https://doi.org/10.1017/97813164711

http://hdl.handle.net/10138/312701
https://doi.org/10.1017/9781316471128

unspecified
acceptedVersion

Three Moments in the Life of the Mathematical Drawing: Notes on the Syntax/Semantics Distinction

Juliette Kennedy*

March 23, 2018

1 Introduction

The mathematician's life revolves around proving theorems, mainly, but at the same time the mathematician is a creator of visual objects, of *drawings*. Artifacts of the main action—the proof—these drawings are never saved, or published. And yet, it is often the case that the idea of the proof cannot be understood without the drawing, either in the form of a concrete token or in the form of a mental image. The proof, in such cases, seems to be inherently pictorial.

What is one to make of these drawings, whose transient presence is so important to the practice? In this essay we will suggest that, seen from what we will call the "single text" point of view, such drawings help us toward a different understanding of mathematical practice, one that assumes direct access to mathematical content; one that resists the bifurcation of that content into its (possible) semantic and syntactic modes.

*I would like to express my deepest thanks to Nathalie Sinclair for inviting me to contribute to this volume. For stimulating discussions and correspondence on the mathematical diagram I am very grateful to Nathalie, Andrew Arana, who took the time to share with me the thinking behind his important paper on the mathematical imagination [1], Arno Schubbach and Philip Welch. Finally, I would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme 'Mathematical, Foundational and Computational Aspects of the Higher Infinite', where work on this paper was undertaken.

1.1 Two Genres

The first thing to say is that the genre of the mathematical drawing seems to admit two categories. There are the Euclidean diagrams—*metonyms for epiphany*, or so they have been called,[1] created by the movement of the hand, but directly constitutive of the proof:

[ILLUSTRATION 1.]

And then there are the informal, illustrative drawings, as one might call them, the *incidental* drawings mathematicians create while working alone or in conversation with other mathematicians, or while giving lectures.

[ILLUSTRATIONS 2,3.]

Depending on what part of the scholarship one consults, the Euclidean diagram oscillates between being a proof and being a set of instructions for a proof—a bidirectional capability that is the source of extraordinary philosophical opportunities, as K. Manders has said[2]—and the diagram is accompanied by a text, a set of equations given in a sequence, directives to the reader to observe *congruences*.

How does this mixture of language and image produce knowledge, for us, and for the Greeks? The question is complicated by this peculiar ambiguity that surrounds Greek diagrammatic practice, the question whether the drawing is a proof or not. Even the Greek word *diagramma* has two meanings: diagram in the usual sense of an instructional image, and "proposition":

> This is a notorious fact about Greek practice: it is generally difficult to tell whether the authors speak about drawing a figure or about proving an assertion, and this is because the same words are used for both. And this again is because the diagram is the proof, it is the essence of the proof for the Greek, the metonym of the proof.[3]

There is a substantial literature on the Euclidean diagram. What we call *incidental* diagrams, on the other hand, have not been the object of critical

[1] Catton and Montelle, [5], p. 27. See below.

[2] [19], p. 68

[3] Reviel Netz, [22]. "Pappus, when commenting on earlier mathematical treatises, estimates the size of a treatise by the number of diagrams. He does not say only 'this work has 73 propositions': rather, he often says 'that work has 73 diagrams'. Of course, this shows first of all something about the Greek word *diagramma* which Pappus uses. It means much more than just "a diagram." It means something much closer to "a proposition" or "a proof."" See also Netz's [23].

attention. This lack of attention is odd, considering the fact that the production of visual imagery is such an essential part of mathematical practice.

What can we say about these incidental diagrams? Considered in the context of the discourse to which they are associated, they do not encode its logic so much as compress that discourse into an image, a visual *mathematical*.[4] Unlike the Euclidean diagram, which is radiant with absoluteness, the incidental diagram reminds us of all that is raw and unfinished, *unrigorous* and incomplete in mathematics. Taken as a single moment in the production of a piece of mathematical knowledge, they give us the mathematician in the grip of insight à l'état brut; they are the trail of the human serpent in the garden.[5]

This chapter contains a sample of such of drawings, Euclidean drawings and others, along with, here, a consideration of three moments in the life of the mathematical drawing: *Euclidean*, *architectural* and *incidental*, or respectively: the drawing as proof object, but heaven-sent, as it were, a token of the realm of pure ideas; the drawing in the light of the manual-practical agency of the architect; and finally the drawing as the carrier of lived sensibility.

2 First Moment: The Euclidean Diagram

Consider the case of the classical Greek mathematical diagram. From Catton and Montelle:

> ... the Greek mathematical diagram is a metonym for an epiphany, namely, an epiphany concerning manual practice, by which one rationally resolves how to prosecute the relevant agency, so that in consequence new concepts are formed.[6]

The classical Euclidean proof, under this view, involves a text, but it also involves the action of the hand under leadership by the text, an action in which, by a kind of "manual-practical leading," the student draws their way toward a

[4]The term "mathematical" is due to A. Arana, and is meant to be "inclusive of objects, concepts, structures, whatever we talk about when we talk about, think about, mathematics." Arana, personal communication.

[5]We are referencing here William James, *Pragmatism*: "The trail of the human serpent is over everything."

[6][5], p. 27.

"needed practical epiphany"; draws their way, that is, toward an understanding of the proof:

> The neophyte learns only by engaging with manual actions under leadership by the text... in order truly to learn from Euclid, one needs one's hands, not only one's eyes and ones brain. This is at least to say that either manually or in the (practical) imagination, one must move about a diagram as one constructs it—yet also it must move one too, to a timeless insight. Otherwise one is not managing to do geometry with the Greeks.[7]

Catton and Montelle's [5] builds on K. Manders' ground-breaking 1995 [19], an analysis of the role of the diagram in Euclidean proofs that calls into question earlier, critical conceptions of Euclid's diagrammatic practice.[8] Manders' question, how can it be that the Euclidean drawing can serve as a carrier of inferential weight? is answered there in terms of what Manders calls the diagram's *exact* and *co-exact* properties, corresponding, roughly, to the diagram's incidental and invariant properties, respectively. When a diagram displays a triangle, for example, it displays a co-exact property; if a line segment occurring in the diagram has a particular magnitude, then that is an exact property of the diagram. Manders observes that, inferentially speaking, co-exact properties always trump exactness. Thus the Euclidean diagram encodes a kind of pictorial logic, and Manders' observation is that it is only by virtue of its co-exact properties that the diagram contributes to the proof.[9] As Manders puts it, "Euclidean demonstration has no alternative justificational resources."[10]

In addition to Catton and Montelle's [5], Manders' [19] has either given rise to, or coincided with, other developments. Shin has argued in her [31] that the Euclidean diagram draws on a syntax, a visual syntax that supports a special

---

[7] [5], ibid

[8] It has long been known that in Book I of Euclid's *Elements*, the proofs require more than what is stated in the Common Notions and Postulates. A statement of the view Manders nevertheless wishes to challenge is the following: "Euclid did not have an axiom system in mind, and did not develop geometry axiomatically, in Book I of the *Elements*." See Seidenberg, [30].

[9] This turns on the fact that the co-exact properties of a diagram are shared by all instances of it. Thus a single diagram can stand in for all configurations that lie in the range of the proof. For more about the so-called generality problem, see [31]. The question how a single diagram can support a general result, how a proof of a proposition relating to a specific diagram can be thought of as generalising to all possible diagrams of the same kind, was considered already by Aristotle.

[10] ibid, p. 67.

diagrammatic logic which can be studied in just the way that any other logic can be studied, i.e. for soundness and completeness and so forth. And others have formalised this logic in different ways, either *reconstructively*, as in Avigad, Dean and Mumma's [3] or in ways that do not necessarily defer to the original texts, i.e. seeking merely to extract a diagrammatic logic which is both correct and workable.[11]

This belief in the inferential power of the Euclidean diagram is new—or again new. Descartes instigated a revolution in mathematics by treating geometric concepts algebraically; and although this was an essential step in the development and rigorization of mathematics, this would suppress the Euclidean diagram, not only pedagogically but *ontologically*, that is to say in its very function within the proof. The judgement became entrenched in the practice subsequently, so that in 1882 Moritz Pasch would describe his axiomatisation of geometry as effecting the banishment not only of the "figure", but of sense altogether:

> In fact, if geometry is genuinely deductive, *the process of deducing must be in all respects independent of the sense of the geometrical concepts*, just as it must be independent of figures; only the relations set out between the geometrical concepts used in the propositions (respectively definitions) concerned ought to be taken into account.[12]

Pasch's view that true rigor does not allow for a drawing was adopted by David Hilbert in his 1894 work [16], raising the question whether the banishment of "sense" and the subsequent rupture between syntax and semantics in the twentieth century—a division of labor, so to speak, to which the modern logician is nowadays completely bound—did all this begin with the suppression of drawings? We will return to this point below.[13]

Pasch's remark needs to be understood against the background of an expanding algebraic discourse in mathematics—but this is just one way to move against the diagram. It is well known that Plato abjured thinking of geom-

[11] For an overview of the recent literature on diagrammatic reasoning in geometry and other areas, see Shin, Oliver and Mumma's [32].

[12] [25], 98. Emphasis ours. Translation from Schlimm [29]. See also Shin, Oliver and Mumma, [32].

[13] For an interesting later counterpoint to the "suppression of drawings point of view" see Hilbert and Cohn-Vossen's 1932 *Geometry and the Imagination*, [15]. See also Arana's discussion of [15] in his [1].

etry in practical terms, as a matter of "doing things";[14] as if the activity of the geometer were centered on the production of impermanent and temporal images—for what is more ephemeral than a drawing?—than with the timeless and acausal mathematicals of which we have true knowledge.

It is interesting that Pasch's view did not suppress diagrammatic practice outside of geometry. New diagrammatic systems were introduced at the time by logicians, most notably Frege and Pierce, who are credited (independently) with introducing, in the form of these systems, modern quantificational logic. Analytic philosophers have been mostly concerned with first order logic as it is conceived apart from any diagrammatic presentation of it—until recently, when Manders' analysis of the Euclidean diagram stoked interest in *logical* diagrammatic practice. Thus Frege's logical diagrams have been considered by Danielle Macbeth in her 2014 [18], who asks in that work (in analogy to the question about Euclidean diagrams), how do Frege's logical diagrams serve as vehicles for the acquisition of *logical* knowledge? The answer, according to Macbeth, has to do with those features of Frege's logical practice which involve reasoning *in* the diagram and not *about* the diagram, a "distinction between describing or reporting a chain of reasoning in some natural language and *displaying or embodying* a chain of reasoning..."[15] As for Peirce's logical practice, many have read into Peirce's diagrammatic logic, and the "flow of experience" which supports it, a similar, ontological reliance on the diagram—a *logical* practice in which the diagram is, embodies or displays the proof.[16]

We are concerned here with the mathematical drawing, Euclidean and incidental drawings, and the question, how can it be that the act of drawing an image, either manually or in the practical imagination,[17] can deliver mathe-

[14] As A. Arana writes in his [1], Plato's views "were at odds with Greek diagrammatic practice" (cf. O'Meara). Nevertheless Plato's influence was sufficient to drive into the discourse surrounding the diagram, and more widely into the discourse surrounding mathematical practice altogether, a conception of mathematical concepts as unembodied, timeless and acausal, and mathematics itself as descriptive, a set of truths about an objectual domain. As Arana describes Plato's view, "if geometry is to produce knowledge rather than mere opinion it must shun apprehension by construction with its apparent temporality."

[15] ibid, p. 73, emphasis ours.

[16] It is beyond the scope of this paper to treat the diagrammatic practice either of Peirce, whose diagrammatic practice has sparked a large literature, or of Frege.

[17] This is the distinction in what Giaquinto calls "visual thinking," i.e. "thinking with external visual representations (e.g., diagrams, symbol arrays, kinematic computer images) and thinking with internal visual imagery; often the two are used in combination, as when we are required to visually imagine a certain spatial transformation of an object represented by a diagram on paper or on screen." See [14].

matical understanding, when this should be a matter of pure analysis?[18] If the diagram's *rational moment*, the "needed epiphany," in Catton and Montelle's terminology, is the outcome of reason in its "manual-practical" mode, how is this rational moment accomplished? How is it that pure analysis, on its own, can't get us there?

Catton and Montelle's radical answer is this: Insofar as reason is restricted to its analytic mode, or, for Catton and Montelle, its *logical* mode, reason must be in a special sense *undone*; space must be cleared for the operation of reason in its synthetic mode:

> Whether the diagram's rational moment chiefly concerns inference - that is to say, whether the diagram chiefly assists rational progression from some already articulate thoughts to some other - we question... If we seek to understand the nature of the reasoning in Greek geometry, it is good neither to equate reasoning with logic, nor even to equate it with inference of a diagram-assisted kind from some articulate thoughts to some other.
>
> To understand Greek geometry in its own intellectual form ... is to consider the synthetic function of reason - as deeper-lying and still more important within Greek geometry than the analytic function of reason.[19]

Catton and Montelle's wish to sever reason from analysis, or even, perhaps, logic, creates difficulties everywhere. But their question whether a notion of inference can be developed that harmonises with (what they have called) manual/practical agency—the primary capability, as they argue in their [5], that enables the construction of the Euclidean diagram—is an interesting one. Of course traditional theories of inference have always asked whether (and how) reason can function not only purely analytically but in a way that makes essential use of other, synthetic, or synthetic-constructive, modes of reasoning. The line of thought goes back to Proclus, whose commentaries on Euclid were widely read, also in the modern era, who invoked the *imagination* in order to

[18]For the purposes of this paper here and subsequently we understand the pure analytic functioning of reason in a very strict sense, as purely syntactic and void of content. An alternative view would not see the pure analytic functioning of reason as void of content in our sense.

[19][5], p. 27.

explain constructive modes of reasoning in geometry—the imagination in the two-dimensional sense, that is, thought of as a faculty which presents or represents the diagram to the mind, generating projections "as if on a screen" so that the understanding might act upon them.[20]

The idea that the imagination enables synthetic reasoning wove itself into the veil of philosophy from then on. The idea would be greatly elaborated by Kant, who devised a complex theory of synthetic modes of judgement in mathematics, built around the analytic-synthetic distinction, and the further parsing of these two categories of judgement into the *a priori* and the *a posteriori*.[21] Kant's move inaugurated the *semantic tradition*, so-called, in which synthesis came eventually to be thought of as having to do with *meanings*; and the synthetic functioning of reason came to be thought of as a faculty of intuition, in which such meanings could be grasped.[22]

The subsequent, *post*-Kantian development is what concerns us here, in particular Bolzano's separation of contentual or *world-involving inference* from mere logical consequence in his monumental 1837 work *Wissenschaftslehre*—a distinction that has survived and more than that is utterly central to logic nowadays, the distinction between semantic and syntactic consequence.

The syntax/semantic distinction does not come without critique, Quine's 1951 [26] (questioning the analytic-synthetic distinction altogether), being perhaps the most well known. Of interest to us here is the idea that such a critique might be motivated by considerations of diagram use in mathematics. This is the idea that the diagram-assisted proof functions as a *single text*, an *ensemble intégré*; and thus the rational moment enabled by the diagram is prima facie indecomposable in Bolzano's sense, i.e. the inference cannot be split into its syntactic and its semantic modes.

We take a moment to explain what is meant by this idea. The following example appears in Avigad (et al) [3]:

> Let L be a line. Let a and b be points on L, and let c be between a and b. Let d be between a and c, and let e be between c and b. Is

[20] The comparison of the imagination with a screen is due to Proclus. See Arana, [1].

[21] Kant defined synthesis as "the act of putting different representations together, and grasping what is manifold in them in one cognition" (A77/B103); a process that "gathers the elements for cognition, and unites them to form a certain content" (A78/B103).

[22] Our view of the emergence of the semantic tradition is in agreement with that of A. Coffa in his landmark work *The Semantic Tradition from Kant to Carnap: To the Vienna Station*, [6].

d necessarily between a and e?[23]

As the authors remark:

> ...it is hard to make sense of the question without drawing a diagram or picturing the situation in your mind's eye; but doing so should easily convince you that the answer is "yes." With the diagram in place, there is nothing more that needs to be said. *The inference is immediate, whether or not we are able to cite the axioms governing the betweenness predicate that would be used to justify the assertion in an axiomatic proof system.*[24]

If we let P be the proposition of the above proof, that d is between a and e, then what Avigad et al are noting here is that the inference to P, the principal (and indeed only) inference of the diagram, is delivered as a matter of pure, *immediate*, visual perception.[25]

This is a simple example. But if one considers the various forms of inference at play in more complex Euclidean diagrams, then it is possible that the principal inference associated with these diagrams consists of an aggregate of inferences of the form, inference to P. The diagram-assisted proof is thus, in our terminology, enabled by a collection of *single texts*.[26]

Are other propositions in mathematics assimilated as *single texts*? Possible candidates might include the *Pigeon-hole Principle*, which states that if f is a function from n to k, with $k < n$, then for some $m_1, m_2 \leq n$, $f(m_1) = f(m_2)$.[27] Another example might include the well-ordering of sets of natural numbers.

[23] p. 705

[24] [3], ibid. Emphasis ours.

[25] One might consult neurobiology on the question whether inferences of the form, inference to P are indeed immediate as opposed to sequential, for reasons having to do with, for example, the rate of neuronal firing. It is possible that neurobiological models implementing facial recognition, for example—a much more complicated capability perceptually than than inferences we consider here—cannot be constructed on the basis of a model in which the subject recognizes different features sequentially, because the rate of neuronal firing is too slow. The subject executes various processes rather in parallel.

[26] As to parsing the inference to P into its syntactic and semantic modes, of course one could view the proof this way; what we are claiming is that analysing the inference this way does not preserve the inference in its original form. On this point it is interesting to consider de Freitas and Viana's [9], in which meanings are attached to Venn diagrams, but in a departure from other approaches, these meanings are not propositions, but sets.

[27] Or more colloquially: given ten apples and nine lunch boxes, such that all the apples must be placed in all the lunch boxes, one lunch box must receive (at least) two apples.

When does the aggregate: text plus visual mathematical, fail to cohere as a single text? A theorem of Bolzano (1817) states that if a continuous function assumes a negative value at a real number a, and a positive value at a real number b, then it must have a zero in the interval $[a, b]$. The theorem follows immediately from the relevant diagram—and yet it does not.[28] Establishing the theorem requires least upper bound principle (in fact the two statements are equivalent). Thus the relevant inference is *not* enabled by a single act of perception, *despite every appearance that it does.*

Resolving the difficulty requires a nuanced view of Bolzano's motivation for proving the theorem in the first place: the search for explanation in the form of objective grounds, along with a conception of the mathematical discipline of analysis, that seeks to ground its theorems strictly outside of geometry. As Bolzano wrote:

> There is certainly no question concerning the *correctness*, nor indeed the *obviousness*, of this geometrical proposition. But it is clear that it is an intolerable offence against *correct method* to derive truths of *pure*, or general mathematics (i.e. arithmetic, algebra and analysis) from considerations that belong to a merely *applied*, or special part, namely *geometry*. . . For in fact, if one considers that the proofs of the science should not merely be *confirmations*, but rather *justifications*, i.e. presentations of the objective ground for the truth concerned, then it is self-evident that the strictly scientific proof, or the objective reason, of a truth which holds equally for *all* quantities, whether in space or not, cannot possibly lie in a truth which holds merely for quantities that are in *space*.[29]

Bolzano goes on to say, that construed as a geometric truth, the theorem consists of a number of component concepts, or "simple truths. . . reasons for other truths and never themselves consequences." A theorem, then, has its basis in these simple truths, and thus "must be proved by a derivation from these other truths."

To summarise Bolzano's view: construed as a geometrical truth, the theorem follows immediately from the relevant diagram. Construed as a general

[28] See for example Courant and Robbins: "Here for the first time it was recognised that many apparently obvious statements concerning continuous functions can and must be proved if they are to be used in full generality." [7], p. 312.

[29] [28], p. 160

mathematical truth, the theorem requires proof.[30]

Of interest here, our starting point, was Manders's work on diagrammatic reasoning, and on the heels of Manders' work Macbeth's extension of it to logical diagrams. What is missing? From Manders:

> The conception of the role of diagrams in traditional geometrical thought given here should be understood as incomplete, probably on philosophically central points. While I go beyond inferential reconstruction in traditional logical terms' the roles to which we in the twentieth century are sensitive and know to articulate are broadly but constrainedly 'inferential'. It is plain that we are missing a lot.[31]

The mind/body binary that Plato drove so deeply into epistemology, the idea of the body as a presence that corrupts and confuses, and in every other possible way works against rational thought; an idea which nowadays expresses itself in the thought that the body is, at best, epistemologically irrelevant, is still with us. Philosophical discourse on the imagination draws heavily on visual metaphors, as we saw, the idea of the imagination as a faculty enabling a special kind of two dimensional representation, not in the sense of an embodied capability but representation in the sense of projection onto a screen.[32]

---

[30]The single text point of view is not new. Foucault reads the language/world binary prior to the sixteenth century as no binary at all, but as a single text. As Curtis Franks has written in the opening of his "Logical completeness, form and content: an archaeology", [11] quoting Foucault's [10]:

> The signifying function of words in the sixteenth century, according to Foucault, depends not on our acquaintance with them, or with their use, "but with the very language of things..." [A]ll the thinkers of the Renaissance ... were "meticulously contemplating a nature which was, from top to bottom, written," for they inhabited "an unbroken tissue of words and signs, of accounts and characters, of discourse and forms." ([10], p. 59, 36, 40.)
>
> In the seventeenth century, we are told, things are different. Language is believed to be arbitrary, its relationship to the world contingent on the details of its fallible design and conventional use. "As a result," Foucault urges, "the entire episteme of Western culture found its fundamental arrangements modified. And, in particular, the empirical domain which sixteenth century man saw as a complex of kinships, resemblances, and affinities, and in which language and things were endlessly interwoven—this whole vast field was to take on a new configuration." (ibid, p. 54.)
>
> We are instead in doubt about what before could not meaningfully be questioned: whether our systems of signs adequately fit the world. [11], p. 78

[31][19], pp. 81-82

[32]Though the active imagination appears already in Proclus, in that "the understanding

But we now know that the synthetic, or active imagination—if we still want to call it that—is more complex. Thus it seems plausible that the imagination is in some way essentially embodied, and that the production of internal imagery exceeds the two dimensional, being experienced, again, bodily.

De Freitas and Sinclair construe thinking as bodily experience *outright*:

> According to phenomenological currents within this [empiricist JK] tradition, thinking and reasoning, and any other related cognitive constructs, are always external or located in the flesh; "Thinking is not a process that takes place 'behind' or 'underneath' bodily activity, but is the bodily activity itself."[33]

And they too concentrate on manual agency:

> In studying a student's tactile and multi-modal engagement with a cube, Roth shows how the movement of the hands erupts or emerges without intention or governing concept. These haptic encounters are somehow more originary than language, somehow detached or free from the "knowing" that is bound to signification. It is in the hand that the memory of prior encounters with cubes is immanent. Roth suggests that there is a more originary pre- verbal "I can" that coordinates this encounter with the cube, and that the world begins to emerge through touch and the coordination of movements of eyes and hands. He privileges the movement of the hand itself, its "auto-affection", as an embodied activity that is prior to all verbal framing.[34]

This leads us to our second moment in the life of the mathematical drawing: the drawing in the light of the manual-practical agency of the architect.

draws upon the screen, and that is an activity: an activity of mind, but an activity... modeled on the physical activity being represented. ... older views of imagination [did not take JK] it for a merely receptive facility." Arana, personal communication.

[33] [8], citing [21].

[34] [8], citing [27].

3 Second Moment: "What is the hand?"

[ILLUSTRATION: ARCH.]

In his monograph *The Thinking Hand*[35]—an ode to manual agency—the architect Juhani Pallasmaa gives us, indeed celebrates, the hand. "A prodigious precision instrument that seems to have its own understanding, will and desires,"[36] more than any other part of the body, it is the hand that serves as a bodily interface between the architectural task and the imagination of the architect; it is the hand—*the thinking hand*, as Pallasmaa calls it—whose probing action drives the imagination of the architect toward the achievement of the task, the task at hand; it is the hand, in the end, that is the locus of the architect's potency, and for all of us, the locus of our human potency.

The architect's task, Pallasmaa tells us, is "something to be lived, rather than understood"; and it is achieved not rationally-conceptually, but *bodily*, from "unconceptualized and lived existential knowledge."[37] Architectural ideas are thus, for Pallasmaa, biological ideas, and architectural pedagogy is a transfer of a kind of skilled sensation "from the muscles of the teacher to the muscles of the student" through a kind of manual mimesis.

The hand is thus, for Pallasmaa, the periphery of the self, the place where subjectivity pours outwards into the world. But it is also the place where the world forces itself inward, forces itself into subjectivity through the hand, inflecting and modulating that subjectivity through manual action.

Pallasmaa begins his book with this passage from Rilke, and his metaphor of the *delta*:

> There are hands that walk, hands that sleep and hands that wake; criminal hands weighted with the past, and hands that are tired and want nothing more, hands that lie down in a corner like sick animals who know no one can help them. But then hands are a complicated organism, a delta from which life from the most distant sources flows together, surging into the great current of action. Hands have stories; they even have their own culture and their own particular beauty. We grant them the right to have their own development, their own

[35] [24]
[36] ibid, p. 24
[37] ibid, p. 15

wishes, feelings, moods and occupations...[38]

For Pallasmaa, then, the essence of manual agency is that it is, like a delta, *bi-directional*. Manual acts exercise so-called *illocutionary force*:[39] so the hand blesses, the hand refuses, the hand expresses friendship and love, the hand demeans and abases—as Pallasmaa tells us, there is a vast theatre of manual gestures, such that when they are performed they find resolution in domains beyond the hand, that is, beyond the gesture. But at the same time the hand modulates the world's inward flow, the world's impact on the self, in a "softening of the existential boundary, the fusion of the world and the self," especially in artistic experience.[40]

And what of reason? Pallasmaa makes an observation that is reminiscent of that made by Catton and Montelle in connection with the acquisition of geometric knowledge, when he notes that during creative acts of architecture, a similar suppression takes place, of reason in its analytic mode; the suppression of, in his terminology, "rational-conceptual reason." In the drawings of Alvar Aalto for example, Pallasmaa notes the "seminal role [in them] of the absent-minded hand and its seemingly unconscious and aimless play" in some of Alvar Aalto's architectural sketches—subconscious sketches of mountain landscapes, and of "many suns."[41]

To the question "what *is* the hand?," Pallasmaa tells us:

> Everyday use of the word as well as classical *surface anatomy* tells us that it is the organ that extends from the wrist to the fingertips. From the point of view of *biomechanical anatomy*, the hand would be seen as an integral part of the entire arm. But the arm also functions in a dynamic coordination with the muscles of the neck, back, and even the legs, and in fact with the rest of the body... When I raise my hand for an oath or a greeting, or give my fingerprints as evidence of my identity, the hand stands for my entire persona. Altogether,

[38] cited in [24], p. 29

[39] Austin's theory of speech acts develops the idea that an utterance can have so-called illocutionary force, as in the utterance "I declare thee man and wife," which, when it is said, effectuates a legal marriage. Put simply, an illocutionary act is the act performed in making the utterance, as opposed to the act of making the utterance. See Austin, *How to Do Things with Words*, [2].

[40] Pallasmaa, op cit, p. 19

[41] Pallasmaa, op cit, p. 74

> we are bound to admit that the hand is everywhere in our body, as well as in all our actions and our thoughts, and thus the hand is fundamentally beyond definability.

Thus the mind-hand barrier, following Pallasmaa's thought here, is porous, or at the end of the day, no barrier at all—the hand is everywhere in our body and in our thoughts.

We noted the admonition of Catton and Montelle, that if one is to "do geometry with the Greeks" one must "move about" the Euclidean diagram as one constructs it. Pallasmaa takes us from "moving about" to full habitation:

> While drawing, a mature designer and architect is not focused on the lines of the drawing, as he is envisioning the object itself, and in his mind holding the object in his hand or occupying the space being designed. During the design process, the architect *occupies the very structure that the lines represent.* As a consequence of the mental transfer from the actuality of the drawing or the model to the material reality of the project, the images with which the designer advances are not mere visual renderings; they constitute a fully haptic and multi-sensory reality of the imagination. The architect moves about freely in the imagined structure, however large and complex it may be, as if walking in a building and touching all its surfaces and sensing their materiality and texture. This is an intimacy that is surely difficult, if not impossible, to simulate through computer-aided means of modelling and simulation.[42]

Pallasmaa's architect is equipped with a fully embodied, *complete* capability, one that fuses into a *single* entity hand, imagination, diagram and task. He lays out a potent confrontation between the architect and the architectural task, in which the architect transports himself into the drawing, and beyond that into the structure the drawing represents, as if the architectural task were like a dream in the mind of the architect, but one that is fully veridical.

Unlike the epistemology of mathematics, architectural practice has not otherized the body in its critical discourse. Pallasmaa's architect is then, not

[42] p 59, emphasis added

metaphysician but *homo faber*: the creator of a public world that shapes and structures every aspect of human existence. As such, Pallasmaa's architect is fully grounded, a maker of his own fortune: *homo faber suae quisque fortunae*—an inhabitant, in other words, of drawings.

4 Third Moment: The Incidental Drawing

[ILLUS.]

The genre of the mathematical drawing admits, as we noted, a second form, what we have called the *incidental* drawing. These drawings are not proofs, though that claim has been made for them on occasion; and they are almost always discarded—in fact it seems wrong to preserve them, as if to circumscribe an epiphany, an intimate, rational moment, with a souvenir. However we do show them in this volume, "pivotal sources of mathematical meaning" as they are.[43] Treated as artworks—in the present environment thinking of these drawings as artworks is unproblematic—there is a hiddenness about them, a sense of inwardness; in fact much of their power as drawings seems to come from their hiddenness. As works of art they give us *the beauty of the awkward line*. They are meant to depict mathematical gestalts in as simple and unaffected a way as possible, but, aesthetically speaking at least, their exact properties, as Manders calls them, also seem to matter.

Perhaps one can think of such drawings as a border-crossing device carrying lived sensibility—the arc of a projectile, the flow of a river, the bodily experience of space and time—into the mathematician's exact discourse. In that sense what these drawings help us to see is that mathematical practice is essentially *bidirectional*; that the mathematician's existential condition, as it were, is one of ambivalence: the mathematician looks forward, toward exactness, rigor, algebraization, and formalisation in an exact, formal language; but the mathematician also looks backward, toward raw meaning and lived sensibility.

Here too the incidental drawing helps us to devise better ways of considering the traditional categories of truth and proof, syntax and semantics. We referred earlier to the distinction between meaning and barren logical form, and the idea that the syntax/semantics distinction emerged in the late nineteenth century on the heels of a long discourse in mathematics centred on diagrammatic inference,

[43] REF Nathalie

a discourse that culminated with Pasch's suppression of the Euclidean diagram in 1882.[44]

Specialized to inference the syntax/semantics distinction comes to us through Bolzano, among others, who distinguished contentual, world-involving inference from mere logical or tautological inference, as we saw,[45] though the notion would be fully articulated only later, by Tarski in his 1933 [33], and completely articulated by Tarski and Vaught in their 1958 [34]. Specialised to geometry the distinction comes down to us through Pasch, "the father of rigour in geometry,"[46] who demanded of a rigorous geometrical discourse, that it does away with meanings: "During the deduction it is useful and legitimate, but in no way necessary, to think of the meanings of the terms; in fact, if it is necessary to do so, the inadequacy of the proof is made manifest."[47]

The model-theoretic conception of language, as Danielle Macbeth calls it, the fruit of the tradition that begins with the suppression of meaning in the nineteenth century, partitions mathematical discourse, or as Macbeth would have it, burdens mathematical discourse, with a dual structure: there is deductive inference, which is compressed into a formal, uninterpreted calculus, a mechanical manipulation of empty signs according to syntactic rules; and then there are domains of interpretation, set-theoretical objects that interpret the deductive formalism, and more than that, licence talk of truth.

As a self-account for the mathematical logician it has turned out to be very useful. But a century of foundations has taught us that splitting meaning off from language at the same time fails to capture the sense of the practice itself, which is contentual and meaningful. Perhaps the problem of developing contentual inference—putting the fly back into the fly-bottle, as it were[48]—is part of the reason why recent philosophers of mathematics have taken a turn towards the practice.

[44]Pasch's move took place not in isolation but in the context of the development of rigor in science altogether. Daston and Galison [17] trace objectivist imperatives in science overall to the 1830s, when, compelled by what they claim was a growing fear of subjectivity in science, "mechanical, or non-interventionist objectivity" began to suppress contentual discourse, and by extension, theories that would involve, through an injudicious and uncareful use of words, the *self*.

[45]see [4]

[46]Freudenthal, [12], p. 237)

[47]Pasch 1882; quoted in Nagel [20], p. 237)

[48]paraphrasing Wittgenstein in Philosophical Investigations: "a philosophical problem has the form: 'I don't know my way about.'" (PI 123), and hence the aim of philosophy is "to show the fly the way out of the fly-bottle" (PI 309)

The message of the incidental drawing underscores this move toward the practice. The careless use of words, and for that matter, diagrams; the lack of attention to their unintended meanings, can—and has (for the concept of continuity, famously)—introduced instability into mathematics. This makes urgent the formulation of a dual structure, one consisting of a precise inference expressed in a formal logical signature, in which mathematical reasoning is identified with an uninterpreted logical calculus; and secondly the concept of a domain of interpretation, in which meaning is handled by severing it from inference, relegating it to the idea of what holds of a concept in such a domain.

The mathematician (who is not a logician) has never internalised this dual structure, despite a century of progress in the foundations of mathematics. Unlike the logician, the mathematician grounds herself in natural language—*and in the production of images.*

5 Conclusion

We began this essay with Euclidean diagrammatic practice, and the philosophical opportunities raised by it. One suggestion was to resolve the problem through the articulation of the notion of the imagination, seen as a manifold enabling the operation of synthetic reasoning. The suggestion did not gain traction, in the light of the rigorization of mathematics in the nineteenth century, which would lead mathematicians away from the diagram, and the idea of *sense*, or as we would say nowadays, from mathematics conceived purely semantically—not from the point of view of the practice, but from the point of view of its foundations.

This in turn left unsolved the problem of manual practical agency, the problem of articulating a notion of inference harmonizing with it. We looked for a resolution in the single text point of view, enabled by the immediate assimilation of a visual mathematical. We then looked to architectural practice, the architect's power, the capability he has, to occupy the structure that the drawing represents.

For the mathematician too, this is the essence of her capability: the power to occupy a structure, to inhabit the very structure that the drawing represents. As a power of thinking—the power to project oneself into a drawing, or beyond that, into a structure—it is fallible. Mistakes are made; special cases, instances of assumptions that were not made evident by the diagram, turn out to be counterexamples. Like Euclidean drawings, whose wild progeny they are, the incidental drawing reminds us that a stream of lived sensibility underlies mathematical practice, a substrate of meaning that pulls the mathematician toward the direct visual assimilation of content—explaining, perhaps, why fallibility occurs so rarely in mathematics.

We leave this as a suggestion, as we leave to the embodiment theorists, the question of the *embodied mathematical*. Embodied cognition has been investigated amply in the general case, and is now beginning to be investigated in the context of mathematical cognition. What a welcome development! that the discourse of the Timaeus, the Platonic conception of mathematics as a field of pure concepts that leaves the "meat" of the body behind, is beginning to loosen its grip on us.[49]

References

[1] Andrew Arana. Imagination in mathematics. In Amy Kind, editor, *The Routledge Handbook of Philosophy of Imagination*. Routledge, Oxford, 2016.

[2] J.L. Austin. *How to do Things with Words: The William James Lectures delivered at Harvard University in 1955, 1962.*

[3] Jeremy Avigad, Edward Dean, and John Mumma. A Formal System for Euclid's *elements*. *Rev. Symb. Log.*, 2(4):700–768, 2009.

[4] Bernard Bolzano. *Bernard Bolzano—Gesamtausgabe. Reihe I. Schriften. Band 13. Teil 3.* Friedrich Frommann Verlag Günther Holzboog GmbH & Co., Stuttgart, 1992. Wissenschaftslehre. §§349–391. [Theory of science. Sections 349–391], Edited and with an introduction by Jan Berg.

[5] P. Catton and C. Montelle. To diagram, to demonstrate: To do, to see, and to judge in greek geometry. *Philosophia Mathematica*, 20(1):25–57, 2012.

[6] Alberto Coffa. *The Semantic Tradition from Kant to Carnap: to the Vienna Station.*

[49] Galison and Jones, [13], p. 12

[7] Richard Courant and Herbert Robbins. *What Is Mathematics?* Oxford University Press, New York, 1941.

[8] Elizabeth de Freitas and Nathalie Sinclair. Diagram, Gesture, Agency: Theorising Embodiment in the Mathematics Classroom. *Educational Studies in Mathematics*, 2011.

[9] R. de Freitas and P. Viana. Set venn diagrams applied to inclusions and non-inclusions. *Journal of Logic, Language and Information*, 24:457–485, 2015.

[10] M. Foucault. *The Order of Things: An archaeology of the Human Sciences.* Random House, New York, 1970. translated from Le Mots et Choses.

[11] Curtis Franks. Logical completeness, form, and content: an archaeology. In J. Kennedy, editor, *Interpreting Gödel.* Cambridge University Press, Cambridge, (to appear).

[12] H. Freudenthal. The Main Trends in the Foundations of Geometry in the 19th Century. In *Logic, Mathodology and Philosophy of Science.* Stanford University Press, 1962.

[13] P. Galison and C. Jnes. Picturing science, producing art. In Peter Galison and Caroline Jones, editors, *Picturing Science, Producing Art.* Routledge, Oxford, 1998.

[14] Marcus Giaquinto. The epistemology of visual thinking in mathematics. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy.* Winter 2015 edition, 2015.

[15] D. Hilbert and S. Cohn-Vossen. *Geometry and the Imagination.* Chelsea Publishing Company, New York, N. Y., 1952. Translated by P. Neményi.

[16] David Hilbert. Die grundlagen der geometrie. In M. Hallett and U. Majer, editors, *David Hilbert's Lectures on the Foundations of Geometry (18911902).* Springer, 2004.

[17] Peter Galison Lorraine Daston. The image of objectivity. *Representations*, (40):81–128, 1992.

[18] Danielle Macbeth. *Realizing Reason.* Oxford University Press, Oxford, 2014.

[19] Kenneth Manders. The euclidean diagram (1995). In *The Philosophy of Mathematical Practice*, pages 80–134. Oxford, Oxford, 2008.

[20] Ernest Nagel. The formation of modern conceptions of formal logic in the development of geometry. *Osiris*, 7:142–224, 1939.

[21] R. Nemirovsky and F. Ferrara. Mathematical Imagination and Embodied cognition. *Educational Studies in Mathematics*, 70:159–174, 2009.

[22] Reviel Netz. Greek mathematical diagrams: Their use and their meaning. *For the Learning of Mathematics*, 18(3):33–40, 1998.

[23] Reviel Netz. *The shaping of deduction in Greek mathematics*, volume 51 of *Ideas in Context.* Cambridge University Press, Cambridge, 1999. A study in cognitive history.

[24] Juhani Pallasmaa. *The Thinking Hand.* John Wiley & Sons Inc., 2009.

[25] M. Pasch. Vorlesungen über neuere Geometrie. Leipzig. Teubner (1882)., 1882.

[26] W. V. Quine. Two Dogmas of Empiricism. In *The philosophy of language*, pages 39–52. Oxford Univ. Press, New York, 1996.

[27] W. Roth. Incarnation: Radicalizing the Embodiment of Mathematics. *For the Learning of Mathematics*, 30(2):8–17, 2010.

[28] S. B. Russ. A translation of Bolzano's Paper on the Intermediate Value Theorem. *Historia Math.*, 7(2):156–185, 1980.

[29] Dirk Schlimm. Paschs philosophy of mathematics. *The Review of Symbolic Logic*, 3:93–118, 3 2010.

[30] A. Seidenberg. Did Euclid's *elements, book i,* Develop Geometry Axiomatically? *Arch. History Exact Sci.*, 14(4):263–295, 1975.

[31] Sun-Joo Shin. *The Logical Status of Diagrams.* Cambridge University Press, Cambridge, 1994.

[32] Sun-Joo Shin, Oliver Lemon, and John Mumma. Diagrams. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Winter 2014 edition, 2014.

[33] Alfred Tarski. *Logic, Semantics, Metamathematics*. Hackett Publishing Co., Indianapolis, IN, second edition, 1983. Papers from 1923 to 1938, Translated by J. H. Woodger, Edited and with an introduction by John Corcoran.

[34] Alfred Tarski and Robert L. Vaught. Arithmetical Extensions of Relational Systems. *Compositio Math*, 13:81–102, 1958.