The gap between school mathematics and university mathematics: prospective mathematics teachers' conceptions and mathematical thinking

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In Finland, both prospective and in-service mathematics teachers report a discontinuity between university-level mathematics and mathematics taught at comprehensive and secondary school. In this study, ten prospective mathematics teachers (PMTs) were interviewed to examine their conceptions of the nature of this gap as well as their mathematical thinking. The study's findings support research that has revealed difficulties experienced by PMTs in the secondary–tertiary transition and in connecting formal and informal components of mathematical thinking. Additionally, the study provides new insight into PMTs' conceptions of teacher knowledge, such as the relationship between knowledge of advanced mathematics and the knowledge needed in teaching situations. The findings offer guidelines for further studies that could help the development of mathematics teacher education.

Finnish mathematics teacher education includes a strong emphasis on advanced mathematics taught in mathematics departments. The underlying assumption of the tradition is that university-level mathematics enhances prospective mathematics teachers' (PMTs') knowledge of mathematics and therefore their teaching knowledge. In this paper, the term "university mathematics" refers to university-level studies in mathematics. In Finnish teacher education, such studies mainly focus on the basics of scientific mathematics such as analysis, linear algebra, logic and abstract algebra. The term "school mathematics" refers to mathematics studied at comprehensive and secondary school.

Although school mathematics and university mathematics mostly deal with related topics (such as calculus and analysis) and the same concepts (such as derivative), both prospective and in-service teachers report

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Hannula, J. (2018). The gap between school mathematics and university mathematics: prospective mathematics teachers' conceptions and mathematical thinking. *Nordic Studies in Mathematics Education*, 23 (1), X–Y.

a discontinuity between the two domains (Hähkiöniemi & Viholainen, 2004; Koponen, Asikainen, Viholainen & Hirvonen, 2016). According to research, this gap is evident in terms of the secondary–tertiary transition (see e.g. Education Committee of the EMS, 2013) as well as building mathematical knowledge for teaching based on university mathematics (Chin, 2013; Peled, 1999; Sirotic & Zazkis, 2007; Viholainen, 2008).

Several studies have revealed that the secondary–tertiary transition in mathematics is problematic for beginning undergraduates. The transition includes a change in mathematical content, sociomathematical norms and educational culture (Education Committee of the EMS, 2013), and therefore causes both cognitive and pedagogical shocks to beginning undergraduates (Clark & Lovric, 2009). Regarding cognitive aspects of the transition, a rigorous and axiomatic-deductive approach is emphasised at university, meaning that the transition includes a major change in mathematical thinking (Tall, 2008). Consequently, beginning undergraduates consider university mathematics to be more theoretical than school mathematics (Hähkiöniemi & Viholainen, 2004).

Regarding mathematical knowledge for teaching, studies indicate that mathematical content knowledge is a necessary basis for teacher's professional knowledge (Education Committee of the EMS, 2012). In Finnish teacher education, this knowledge is enhanced by studies in university mathematics. However, research has shown that PMTs have difficulties connecting formal aspects of university mathematics with the more informal reasoning emphasised at school (Chin, 2013; Peled, 1999; Sirotic & Zazkis, 2007; Viholainen, 2008).

As a gap between university and school mathematics is evident, the two can be described as different discourses (Sfard, 2014). Forming links between the two discourses is typically not an explicit part of Finnish mathematics teacher education (Yrjänäinen, 2011). In the Finnish context, bridging the gap between the discourses as well as understanding PMTs' beliefs and knowledge are thus crucial to the development of mathematics teacher education (cf. Tossavainen & Pehkonen, 2013). Similar research interests and developmental challenges are also addressed more generally in the Nordic and wider European contexts (e.g. Dreher, Lindmeier & Heinze, 2016; Jakobsen, Ribeiro & Mellone, 2014).

Understanding the gap between school mathematics and university mathematics from the point of view of PMTs requires an examination of PMTs' beliefs. According to Beswick (2012), a teacher's beliefs about the nature of mathematics as a discipline may differ from his or her beliefs about the nature of mathematics as a school subject. This may partly explain the findings of Koponen et al. (2016), which indicate that Finnish in-service teachers report university mathematics to lack a clear connection to the mathematics taught at school. A study by Even (2011), however, suggests that PMTs find university mathematics important 1) as a resource for teaching secondary school mathematics, 2) for improving understanding about what mathematics is, and 3) for reminding teachers about how learning mathematics feels. On the other hand, PMTs seem to understate the role of subject matter knowledge in teacher knowledge (Hoffkamp & Warmuth, 2015) and in-service teachers seem to emphasise the mathematical content at the level they are teaching, disregarding the broader mathematical context (Mosvold & Fauskanger, 2014). Additionally, according to Koponen (2017), Finnish PMTs find subject matter knowledge somewhat distinct from other areas of teacher knowledge, such as knowledge about students' misconceptions.

Although some of the recent studies discussed above have addressed beliefs about teacher knowledge, still relatively little is known about prospective secondary school teachers' conceptions of the relationship between school mathematics and university mathematics. The aim of this study, therefore, is to contribute to existing knowledge by 1) investigating prospective secondary school mathematics teachers' conceptions of university mathematics in relation to school mathematics and teacher knowledge, and 2) examining their mathematical reasoning.

Theoretical background

The theoretical background of the study consists of conceptualisations of teacher knowledge and mathematical thinking. These conceptualisations are discussed in detail in the following two subsections.

Teacher knowledge

Current research on teacher knowledge has typically been built upon Shulman's (1987) distinction between content knowledge (subject matter knowledge), pedagogical knowledge and pedagogical content knowledge (see Ball, Thames & Phelps, 2008; Carrillo, Climent, Contreras & Muñoz-Catalán, 2013; Hoover, Mosvold, Ball & Lai, 2016). In Shulman's distinction, content knowledge means general knowledge of the subject (e.g. mathematics). Pedagogical knowledge is defined as general pedagogical knowledge that is not specific to the subject, whereas the special amalgam of content and pedagogy is defined as pedagogical content knowledge (Shulman, 1987).

Since Shulman's distinction was made, many researchers have further developed conceptualisations of teacher knowledge. A large amount of the contemporary research (see Hoover et al., 2016) on teacher knowledge has been based on the *Mathematical knowledge for teaching* (MKT) model (Ball et al., 2008). In this model, subject matter knowledge (SMK) is divided into common content knowledge (CCK), specialised content knowledge (SCK) and horizon content knowledge (HCK). Similarly, pedagogical content knowledge (PCK) is divided into knowledge of content and students (KCS), knowledge of content and teaching (KCT) and knowledge of content and curriculum (KCC) (figure 1).

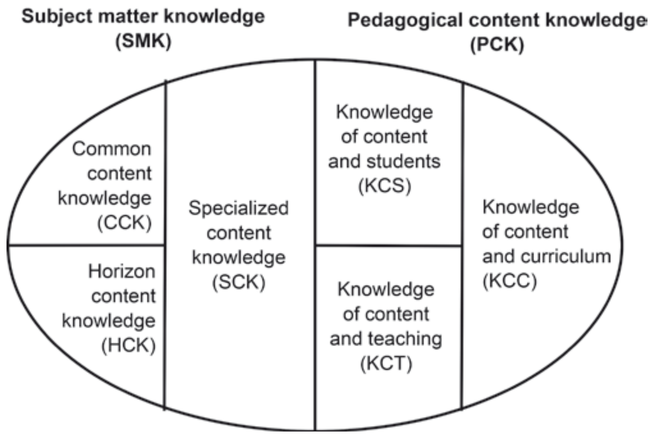

Figure 1. *The Mathematical knowledge for teaching model (Ball et al., 2008)*

CCK is the area of mathematical knowledge that is not specifically for teachers; it is needed in other professions as well (e.g. engineering). SCK, on the other hand, is the part of content knowledge that is specifically for teachers (e.g. modifying tasks). HCK is defined as "awareness how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p. 403). KCS includes such knowledge of students as their typical misconceptions and questions, whereas KCT includes knowledge of the teaching process such as how the teacher can sequence lessons or respond to students' questions. KCC is described as knowledge of how mathematical content is set in the curriculum.

Researchers using the MKT model in the empirical studies have faced difficulties in drawing a line between the components of SMK (e.g. Carrillo et al., 2013; Figueiras, Ribeiro, Carrillo, Fernández & Deulofeu, 2011). These "boundary problems" have also been acknowledged by Ball et al. (2008). In addition, as Hurrell (2013) states, the domains are not unconnected, rather they all interact with each other. HCK is stated to be the most problematic category of MKT (Ball et al., 2008; Fernánez & Figueiras, 2014; Jakobsen, Thames & Ribeiro, 2013). Jakobsen and colleagues

(2013, p. 3128) provide a refined definition of HCK as "an orientation to and familiarity with the discipline (or disciplines) that contribute to the teaching of the school subject at hand, providing teachers with a sense for how the content being taught is situated in and connected to the broader disciplinary territory". In this sense, HCK also includes "explicit knowledge on ways of and tools for knowing in the discipline that enables teachers to understand and make judgements of students' statements and reasoning" (Jakobsen et al., 2013, p. 3128).

Some authors (e.g. Dreher et al., 2016) state that the MKT model does not fully take into account the content knowledge needed for teaching secondary school mathematics, and postulate the concept of school-related content knowledge (SRCK). SRCK consists of "knowledge about the curricular structure and its legitimation as well as knowledge about the interrelations between school mathematics and academic mathematics in top-down and in bottom-up direction" (Dreher et al., 2016, p. 223). In the present study, SRCK is located within HCK, as the knowledge described in the definition of SRCK is included in the refined definition of HCK by Jakobsen et al. (2013).

Although many alternative conceptualisations of teacher knowledge have also been suggested (e.g. Carrillo et al., 2013; Rowland, 2009), in this study the MKT model was adopted due to its established position. Studies have shown that the knowledge components of the MKT model are important for effective teaching (e.g. Baumert et al., 2010; Hill, Rowan & Ball, 2005). Furthermore, the model has been utilised in prior research in the Finnish context, and according to Koponen et al. (2016), Finnish in-service teachers consider current teacher education to be mainly sufficient in terms of CCK but somewhat insufficient in terms of SCK.

Mathematical thinking

Research on mathematical thinking includes several approaches that may focus on different aspects of the subject such as the pedagogical, cultural or cognitive (Sternberg, 1996). In this paper, the term "mathematical thinking" refers to the cognitive aspects of mathematical thinking. These aspects are closely related to teacher knowledge, as SMK includes knowledge of mathematical concepts, processes and representations. In this study, the theoretical constructs of *concept image* and *three worlds of mathematics* are used to examine PMTs' mathematical thinking.

The distinction between concept image and concept definition (Tall & Vinner, 1981) has been fundamental in understanding the relationship between a learner's thinking and mathematical theory (Bingolbali & Monaghan, 2008). Tall and Vinner (1981) define concept image as the total cognitive structure that is associated with a mathematical concept. Concept image may include mental pictures, symbolic processes and axioms, for example. Concept image is subjective by nature whereas the formal concept definition is the definition accepted by the mathematical community. As the concept image of a student continuously changes, the term *evoked concept image* is used to refer to a student's concept image at a certain time and in a certain situation. These evoked concept images depend on the kind of content students have been exposed to during their studies (Bingolbali & Monaghan, 2008).

Concept images may include different forms of information or knowledge. In the three worlds of mathematics framework (Tall, 2004), mathematical thinking is divided into 1) the conceptual-embodied world of mathematics, 2) proceptual-symbolic world of mathematics, and 3) axiomatic-formal world of mathematics.

The conceptual-embodied world includes embodied thinking about mathematical concepts and processes such as pictures and physical objects. As an illustration, the embodiment of the calculation $1/4 \cdot 1/2$ can be presented as a picture: one fourth of a half makes one eighth (figure 2).

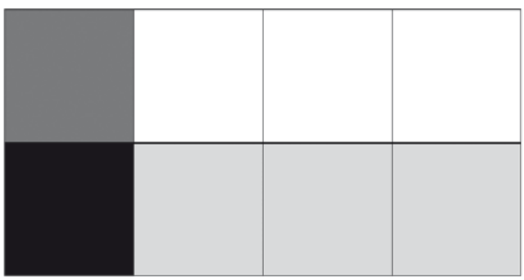

Figure 2. *Calculation* $1/4 \cdot 1/2 = 1/8$ *in the conceptual-embodied world*

On the other hand, calculations such as $1/4 \cdot 1/2 = (1 \cdot 1)/(4 \cdot 2) = 1/8$ can be represented and learned in a symbolic manner. In the framework of the three worlds of mathematics, this kind of knowledge belongs to the proceptual-symbolic world. The word "proceptual" refers to an "amalgam of process and concept in which process and product is represented by the same symbolism" (Gray & Tall, 1991, p. 73). Additionally, the multiplication of fractions is based on formal mathematical theory where the definition of multiplication of rational numbers is $a/b \cdot c/d = ac/bd$. This kind of knowledge of mathematical theory belongs to the axiomatic-formal world of mathematics.

As described above, concept images may consist of mental pictures, symbols and axioms. Mental pictures can be seen as part of the conceptual-embodied world (later the first world), symbolic processes as part of the proceptual-symbolic world (later the second world) and axioms as part of the axiomatic-formal world of mathematics (later the third world). Although all these worlds are apparent in both school and university mathematics, the transition from school mathematics to university mathematics includes a change in emphasis from the first and second world to the third world (Tall, 2004; Tall, 2008).

Viholainen (2008) classifies mathematical reasoning into f*ormal reasoning* based on axioms, definitions and proven theorems, and *informal reasoning* based on visual or physical interpretations of mathematical concepts. Viholainen (2008) showed that making connections between formal and informal reasoning regarding the differentiability and continuity of functions can be especially difficult for PMTs. Similarly, PMTs may approach the sine function only in terms of triangles, or alternatively in terms of the series $x - x^3/3! + x^5/5! - x^7/7! + \ldots$ without making coherent links between the two approaches (Chin, 2013). Sirotic and Zazkis (2007) have also shown inconsistencies between PMTs' formal and intuitive knowledge, such as defining irrational numbers and fitting them into a number line. In this paper, the terms ”formal” and ”informal” are used with reference to the three worlds of mathematics: informal components of mathematical thinking are those associated with the first and second world, with formal components being those that can be classified into the third world.

In this study, the framework of the three worlds of mathematics is utilised, as it provides an overall view of the learner's mathematical thinking, whereas many other frameworks are more domain-specific (Chin, 2013). The framework also specifies the SMK, which includes abilities in graphing and symbolic procedures as well as knowledge of axiomatic-formal mathematics. In particular, HCK includes awareness of how the topics of school mathematics are situated in and connected to the broader disciplinary territory. That is, HCK includes the ability to connect the formal and informal components of mathematical thinking.

Research questions

This study aims to explain the gap between school mathematics and university mathematics from the point of view of PMTs who are no longer in the secondary–tertiary transition phase or in working life yet. This gap can be reflected on in terms of the PMTs' beliefs using the MKT model as well as in terms of their mathematical thinking using the framework of the three worlds of mathematics. The following research questions were consequently formed:

1. What kind of conceptions of university mathematics do prospective mathematics teachers have in relation to school mathematics and teacher knowledge?

2. What kind of evoked concept images do prospective mathematics teachers produce of the mathematical concepts that are discussed at both the school and university level?

As the research questions strive to gain a description of PMTs' beliefs and concept images, a qualitative research approach was adopted and semi-structured interviews (RQ1) as well as written tasks (RQ2) were used in the data gathering.

Method

Context

In Finland, all qualified teachers must earn a master's degree (300 ECTS credits), typically completed in 5 to 7 years. Finnish teacher education has two distinct teacher education programmes: pedagogy-orientated class teacher education and subject-orientated subject teacher education. Class teachers teach several subjects (including mathematics) in the first six years of comprehensive school (with students aged 7 to 13 years). Subject teachers typically teach one or two subjects in the last three years of comprehensive school (with students aged 13 to 16 years) or upper secondary school (with students aged 16 to 19 years).[1]

The context of this study is mathematics subject teacher education. A mathematics teaching degree consists of subject studies in mathematics (at least 150 ECTS credits), subject studies in a minor subject (such as physics, at least 60 ECTS credits) and educational studies (60 ECTS credits). The participants were PMTs who had completed the 3-ECTS credit mathematics course "University mathematics from the teachers' perspective" (later UMTP) prior to the interview. The course was held by the author. The course included calculus, number systems, vectors and logic from a teacher knowledge perspective. That is, the content was discussed from both the SMK and PCK perspectives. The course was specially designed to form links between school mathematics and university mathematics, which is not typically an explicit part of Finnish teacher education. As the aim of the course was to strengthen the MKT of the PMTs, it was tentatively assumed that the participants in this study might have even richer evoked concept images than other PMTs and that they could discuss teacher knowledge at least to the same extent as PMTs in general.

Participants

The participants' ($n = 10$) background information was asked about at the beginning of each interview. All participants were PMTs, of whom eight had mathematics as their major subject and two had mathematics as a minor subject[2]. The participants were at different stages of their studies: two at the beginning stage (second year), three at the middle stage (third to fourth year), and five at the final stage (fifth year or more). Most of the participants had completed studies in education. A summary of the background of the participants is given in table 1.

Table 1. *The participants of the study*

Participant	Stage of studies	ECTS credits	Studies in education
1	second year	around 150	no
2	second year	around 120	no
3	fifth year	around 300	yes
4	more than sixth year	around 250	yes
5	more than sixth year	around 300	yes
6	fourth year	around 230	yes
7	third year	around 180	no
8	more than sixth year	around 350	yes
9	fourth year	around 250	yes
10	more than sixth year	around 290	yes

All participants had some experience working as a teacher in addition to their studies. All described their teaching experiences as positive and felt rather competent in terms of teaching.

Data collection

All students of the UMTP course given in the autumn of 2014 (27 students), 2015 (28 students) and 2016 (33 students) were invited for an individual interview after the course. Four students volunteered in the spring of 2015 (participants 1–4), one in the spring of 2016 (participant 5) and five in the spring of 2017 (participants 6–10). The interview sessions consisted of two parts: a semi-structured interview and a written part that was filled out after the semi-structured interview.

The semi-structured interview was conducted to answer RQ1. A semi-structured format was applied, as it allows the participant to build his or her own narrative through open-ended questions and a flexible structure (Galletta & Cross, 2013). That is, the semi-structured interview is an effective method for understanding participants' experiences and conceptualisations. The interview focused on four themes: 1) mathematics learning and teaching *in the school context*, 2) mathematics, learning and teaching *in the university context*, 3) teacher knowledge, and 4) the participant's view of mathematics (table 2). One pilot interview was carried out before the actual interviews to refine the theme list and possible interview questions. As the discussion on the fourth theme did not help in answering the research questions, the focus of analysis is on the first three themes. The average length of the interviews was 41 minutes, with the minimum and maximum lengths being 29 and 61 minutes. The transcripts of the interviews were written verbatim but without speech intonations and breaks.

Table 2. *The interview outline*

Theme	Example questions
1 Mathematics, learning and teaching in school context	– What was mathematics like as a school subject? – What kind of content did you learn at school? – What was the teaching and learning like?
2 Mathematics, learning and teaching in university context	– What is mathematics like as a university subject? – How have you experienced university-level mathematics at different stages of your studies? – What kind of content have you learned? – What has the teaching and learning been like? – Have you seen any connections between university mathematics and school mathematics? If so, what are they?
3 Teacher knowledge	– What kind of knowledge does a mathematics teacher need? – What does subject matter knowledge include? How important is it for a mathematics teacher? Why is it important? – What does pedagogical knowledge include? How important is it for a mathematics teacher? Why is it important?
4 View of mathematics (*not included in the analysis*)	– What is mathematics? – What kind of discipline is it? – What kind of subject is it?

The written part was done to answer RQ2. In the written part, the participants answered four mathematical questions. In each question, the participants were first asked to give a definition and/or description of a concept. Additionally, they were asked to answer a question regarding a truth value of a related theorem or definition, and explain their thinking (figure 3).

1. What is a derivative of a function?
2. Function f (from real numbers to real numbers) is differentiable everywhere and its derivative is positive everywhere. Is f monotonically increasing everywhere? Why / why not?

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1. What is a vector?
2. Let **a** and **b** be vectors. Is it true that $3(\mathbf{a} + \mathbf{b}) = 3\mathbf{a} + 3\mathbf{b}$? Why / why not?

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1. What does congruent line segments mean? How can one define congruent line segments?
2. Does the following hold: If the line segments a and b are congruent and the line segments c and d are congruent then line segments $a + c$ and $b + d$ are congruent. Why / why not?

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1. What is a rational number? What is a real number?
2. Let a, b, c and d be rational numbers. Does the following hold: $\frac{a}{b} : \frac{c}{d} = \frac{ad}{bc}$? Why / why not?

Figure 3. *The questions of the written part of the interview (translated from Finnish by the author)*

The concepts in question (derivative, vector, congruence, rational and real numbers) were selected because they are all discussed in both school- and university-level courses and thus can be approached through informal as well as formal thinking. All participants had been exposed to these concepts at upper secondary school. Additionally, all had been exposed to derivative and vectors in previous university courses as well as in the UMTP course. The latter also included a formal as well as informal approach to number systems. Only participants 3, 4, 5, 6 and 10, however, had studied geometry at the university level. The participants were advised to answer the question and explain their thinking through any approach they preferred and, if possible, using various approaches.

Data analysis

The analysis of the interviews and written answers was based on the content analysis method (Elo & Kyngäs, 2008). A combination of deductive and inductive content analysis was used to analyse the semi-structured interviews. First, a rough analysis matrix was formed using two themes based on prior research literature. Categories and sub-categories were then formed inductively (table 3). This process was based on the deductive-inductive path presented by Elo and Kyngäs (2008, p. 110).

The written answers were analysed utilising the framework of the three worlds of mathematics as a basis for a straightforward deductive content analysis. That is, the participants' definitions and explanations

Table 3. *Example of the deductive-inductive content analysis process*

Transcript	Theme (formed according to research literature)	Category (formed inductively)	Sub-category (formed inductively)
9: " [...] no matter how good you would be pedagogically but if you can't grasp mathematics. [...] if you don't know the substance you don't necessarily find the illustrations."	University mathematics as a basis for teacher knowledge	University mathematics as a basis for SCK	University mathematics as a basis for illustration and representations

were coded as 1, 2 or 3 (referring to the three worlds of mathematics) depending on the approach used. Many explanations were coded, for instance, as both 1 and 2 because both worlds were used or combined in the explanation (figure 4). After the coding, the most prominent aspects of the data were analysed in more detail.

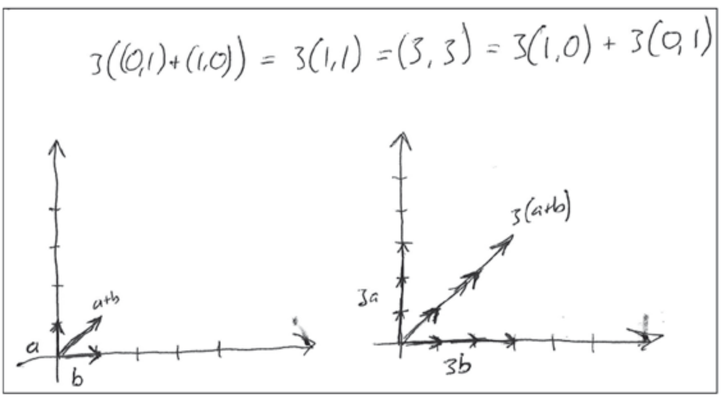

Figure 4. *The explanation of participant 7 combining the first and second world of mathematics*

Findings

Conceptions of university mathematics

The participants' conceptions are examined within two broad themes derived from prior research literature. The first theme, "Secondary–tertiary transition", includes participants' conceptions of the gap between school mathematics and university mathematics from the secondary–tertiary transition point of view. The second theme, ”University mathematics as a basis for teacher knowledge”, includes participants' conceptions of the role of university mathematics in teachers' professional knowledge. The categories distinguished within these themes are presented in table 4.

Table 4. *The categorisation of the data*

Theme	Category	Sub-categories
Secondary–tertiary transition	Change in way of thinking	proofs, formal approach, problem-solving process
	Change in content	continuity/discontinuity of content, relationship between mathematics and everyday life, bridging courses
	Change in affect	shock, difficulties in learning, fresh start
	Institutional change	personal responsibility
University mathematics as a basis for teacher knowledge	University mathematics, school mathematics and HCK	HCK as being ”one step ahead”, HCK as a relationship between mathematical concepts, the SRCK aspect of HCK, HCK within university-level content
	University mathematics as a basis for SCK	basis for illustration and representations
	University mathematics as a basis for PCK	basis for explaining mathematics, basis for approaching the content from different angles
	University mathematics as a basis for practical capability	self-confidence, teachers' credibility

Secondary–tertiary transition

The participants discussed the transition to university mathematics in terms of the change in the way of thinking, change in content and change in self-confidence. One participant also referred to institutional change. Six participants explained the change in the way of thinking in terms of formal or proof-based thinking.

8: The change in thinking is so big and, and. Then you have to write, like, formal mathematics.

Additionally, participant 9 highlighted the change in problem-solving processes: at school the emphasis is on procedural calculations, whereas at university one must use different kinds of evaluation techniques that lead to formulating formal proofs.

9: [...] the way we solved the tasks. [...] if we have calculated some limit, for example. At upper secondary school, [...] you take the highest powers of *n* as a common factor in the nominator and denominator and that's it. But here [at university] we first evaluate [the expression], either upwards or downwards, depending on the task.

The majority of participants (8 of 10) saw the content of school mathematics and university mathematics as at least somewhat mutually distinct. The most typical experience of this was, during the first analysis or linear algebra courses, finding it difficult to understand how the content was related to school mathematics content. Two of the participants even stated that although the mathematical concept had the same name at university and at school, it felt like an entirely different concept.

6: [...] for example the limit thing. It is discussed at upper secondary school but I felt that it is a different entity here [at university] than what it was at school and they didn't have anything in common.

Two participants also specified this distinction in terms of real life.

4: It [university mathematics] was so distant from the real world.

Participant 9, in contrast, saw university mathematics as a logical continuation from school mathematics.

9: [...] the content goes further and much deeper [...]. But it is just logical [...] there are no matrices at school but it is natural and logical that new information is provided.

Three participants emphasised the importance of bridging courses such as "Revision of secondary school mathematics" and "Introduction to university mathematics". Participant 8, for instance, felt the former was important as it helps in gaining a complete picture of school mathematics. Regarding the latter, participant 7 saw the course as important because during it he learned proof techniques and set theory that helped him to understand the content of further courses in university mathematics.

Most of the participants (7 of 10) reported difficulties in learning the first courses of university mathematics because the content appeared so different compared to school mathematics. This also affected their view of themselves as learners of mathematics. Participant 6, for instance, found it shocking that at school she managed mathematics well but at university she was no longer "a good student". Two participants, however, saw the possibility of a fresh start, meaning that they thought university mathematics can be learned without a solid knowledge of school mathematics.

7: [...] if there was a shock it was, like, a positive one. [...] everybody said that ”welcome from secondary school, forget everything that they have said there”. [...] And I was like, ”don't worry, I have already mainly forgotten”. It was really a nice start.

Participant 10 highlighted the institutional change: at school, one is more responsible to one's teacher (meaning that learning is more guided), whereas at university one is supposed to work more upon one's own initiative.

University mathematics as a basis for teacher knowledge

The participants discussed university mathematics as a basis for teacher knowledge in terms of SMK, PCK as well as practical capability. Three participants found SMK to be important, in terms of being one step ahead of the students, and thus spoke of HCK on a general level.

3: [...] the teacher should be one step ahead or should have a broader knowledge base than what the teaching itself needs.

In addition, five participants highlighted the aspect of knowing the disciplinary territory that school mathematics is based on (the SRCK aspect of HCK).

7: Indeed, no-one has said [at secondary school] that you have a set and two things you can do to them [the elements of the set]. Instead, if you are solving equations, you move and change the sign. [...] But no, [...] you [...] use the additive inverse or multiplicative inverse.

Four participants emphasised the importance of understanding the relationships between the concepts and processes deeply enough (structural aspect of HCK). One of them, however, stated that the concepts discussed in university mathematics are easier to connect to each other than to concepts discussed in school mathematics. Two participants (6 and 8) saw university mathematics as a basis for explaining the school content and knowing different ways to approach it.

8: Well, of course you have to master the content you teach but it is really important to be able to explain it as clearly as possible and in different ways and with examples.

Participant 9 also suggested that university mathematics helps in finding representations and answering students' questions, and thus saw it in relation to SCK and KCT.

9: [...] no matter how good you would be pedagogically but if you can't grasp mathematics. [...] if you don't know the substance you don't necessarily find the illustrations. And you can't adequately answer the students' "why" questions.

In addition to these implications for teacher knowledge, participants 4 and 6 discussed the importance of university mathematics for practical capability: knowledge of university mathematics gives the teacher credibility and self-confidence as a subject teacher.

4: [...] I think that a teacher is convincing if she masters her own subject [...]

Overall, the participants discussed university mathematics in terms of the structural and SRCK aspects of HCK. Nevertheless, although three participants (6, 8 and 9) discussed some of the implications of university mathematics for in-action teacher knowledge, other participants who highlighted the importance of a broad knowledge base and structural knowledge did not mention the in-action aspects of HCK, such as the effects of advanced mathematical knowledge on understanding students' reasoning.

Mathematical thinking and evoked concept images

Derivative

All participants explained the visual (first world) interpretation of the derivative. Four also gave the definition as the limit of the difference quotient. The most prominent aspect, however, was that although all participants knew or believed that if the derivative is positive the function must be increasing, most struggled with the explanation. Five participants gave a first world explanation of the fact and three assumed it to be true but could not verify it. Two of the participants reasoned erroneously, deducing the fact from the continuity or the existence of zero of the function.

Vector

Participants gave various definitions of a vector: participants 4, 5 and 10 referred to formal definitions as an element of a vector space or a set of congruent directed line segments. Other participants referred more informally to symbols such as (x, y) (three participants) and/or visual interpretations such as arrows (six participants). Only participants 1 and 10 tried to connect the claim $3\mathbf{a} + 3\mathbf{b} = 3(\mathbf{a} + \mathbf{b})$ to the formal theory. Participant 1 stated that if the structure is not a ring then the distribution law does not hold (figure 5). However, if the structure is a vector space then the law holds. That is, she could not correctly place vector spaces in formal theory. Participant 10, interestingly, claimed that "the sum of the vectors and scalar multiplication should satisfy the linear condition". That is, he linked the distribution property of vector space to linear mappings: scalar multiplication as mapping should have the property of linear mapping ($L(u+v) = L(u) + L(v)$). Six of the participants gave visual explanations (first world) to the claim and three calculated an example (second world) to show the correctness of the claim. Participant 10, on the other hand, gave only formal reasons for the claim.

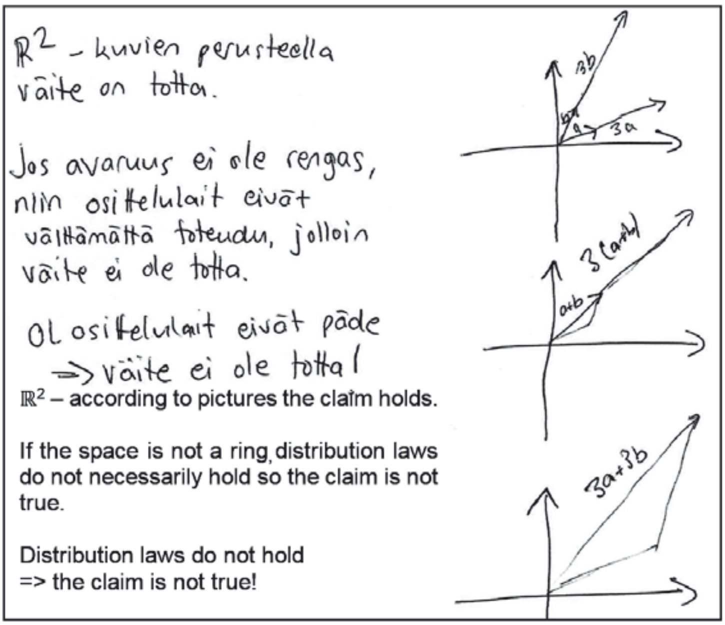

Figure 5. *Explanation of Participant 1 for distributive property of vectors*

Congruence

Participants 3 and 4 referred to a physical interpretation of the congruence as line segments that have the same length. Five participants stated that congruent line segments should have the same length and also be parallel. Additionally, two participants gave no answer to the question. Participant 10 gave a (for the most part correct) formal definition using equivalence relations. Nobody referred to a definition that would use geometric mappings. The question about the addition of line segments was mainly explained with pictures (first world) and the two participants who knew the correct physical interpretation were also able to give a coherent answer to the question.

Number systems

All participants gave the definition of rational numbers as a quotient of two integers. Real numbers were typically defined as ”rational numbers + irrational numbers” (8 participants) and/or ”the whole number line” (3 participants). Only participant 10 defined real numbers formally as the limit of a convergent rational number sequence [3]. Nobody referred to field axioms or completeness. Only participant 9 illustrated the subset structure of number systems. However, four participants could explain the division of rational numbers by referring to a formal (third world) definition of division as multiplying by multiplicative inverse. On the other hand, six participants only referred to calculation rules (second world): the division of fractions can be calculated as a multiplication.

Summary

Overall, the first and second world of mathematics were dominant in participants' evoked concept images. That is, the descriptions, definitions and explanations were more frequently based on physical and symbolic interpretations emphasised in school mathematics than axiomatic-formal theory emphasised in university mathematics. For instance, vectors and their distributive property were typically discussed in terms of arrows and symbolic calculations. Connections between these informal aspects were also formed. Connecting formal and informal components of mathematical thinking, however, seemed unnatural and difficult for the participants. Participants 1 and 4 used some axiomatic-formal definitions of the concepts but failed to use them in explaining such claims as $3\mathbf{a} + 3\mathbf{b} = 3(\mathbf{a} + \mathbf{b})$. Furthermore, participant 10 used several axiomatic-formal definitions but did not connect them to the first or second world of mathematics.

Discussion and conclusion

Regarding RQ1, the present study showed that the gap between school mathematics and university mathematics in the secondary–tertiary transition phase reported in prior research (Education Committee of the EMS, 2013) is also evident among Finnish PMTs. In addition to typical cognitive and pedagogical shocks experienced in the transition phase (Clark & Lovric, 2009), the findings revealed difficulties that may arise in terms of forming HCK in the early phase of studies. For instance, beginning PMTs may see concepts such as limit as completely different entities at school and at university. Additionally, concerning RQ1, research has suggested that pre-service and in-service teachers may disregard the importance of SMK and particularly HCK (Hoffkamp & Warmuth, 2015; Mosvold & Fauskanger, 2014). The participants in this study typically considered university mathematics and structural aspects of HCK important, but rarely specified this importance in terms of in-action teacher knowledge. As presented by Jakobsen et al. (2013), HCK enables the use of advanced mathematical knowledge (such as knowledge of indirect proof techniques) in teaching situations (such as making sense of students' reasoning). Further studies are needed to examine more closely PMTs' conceptions of HCK in relation to the teaching process.

As for RQ2, the study showed the participants mainly producing concept images based on the first and second world of mathematics as the axiomatic-formal theory was rarely utilised in the definitions, descriptions and explanations. Furthermore, the evoked third world concept images were not coherently linked to the other worlds, meaning that if a third world definition was given it was either in contradiction to an informal explanation (participant 1) or formal thinking was used exclusively (participant 10). Thus, the study supports and extends the findings of earlier studies that have described PMTs' difficulties in connecting the formal and informal sides of mathematics (Chin, 2013; Sirotic & Zazkis, 2007; Viholainen, 2008). These findings suggest possible problems regarding forming solid teacher knowledge. For instance, apart from visual illustrations, not one participant could explain why the positivity of the derivative implicates the increase of the function. This kind of knowledge, however, is crucial for a teacher when answering students' "why" questions (Ball et al., 2008).

The issues concerning RQ1 and RQ2 appear to be interrelated. The majority of the participants reported that they had difficulty making connections between university and school mathematics during their studies. On the other hand, coherently connecting the formal and informal components of mathematical thinking was lacking in their written answers. This seems to indicate that PMT's conceptions of the relationship between university and school mathematics and PMT's mathematical thinking go hand in hand.

The participants in this study comprised a limited, non-random sample of PMTs and therefore their conceptions and thinking cannot be generalised to a larger population. Nevertheless, the study shows that PMTs in different stages of their studies who have taken a course specifically aimed to develop teacher knowledge may 1) report difficulties in connecting university and school mathematics, as well as 2) lack the ability to connect formal and informal aspects of mathematical thinking. As the formal aspects are emphasised in university courses, a possible reason for such results could be that university mathematics courses do not have the desired effects on PMTs' concept images. It may also be, as suggested by Koponen et al. (2016), that the formal aspects are not sufficiently connected to the informal aspects during the courses. Although this study raised possible issues regarding the gap between university mathematics and school mathematics, further in-depth studies of PMTs' concept images of different topics as well as their conceptions of HCK should be conducted to gain a more precise picture of these multifaceted questions.

Generally, the challenges regarding secondary–tertiary transition have been well acknowledged and a growing amount of research-based development has been carried out in this area (Education Committee of the EMS, 2012; Oikkonen, 2009). Nevertheless, in Finnish teacher education at least, less effort has been put into developing university mathematics studies from the perspective of PMTs (Koponen et al., 2016; Tossavainen & Pehkonen, 2013). Such development from an SMK perspective seems important, as it is a necessary but not a sufficient condition for developing PCK (Baumert et al., 2010). This study together with prior research (Koponen et al., 2016; Koponen, 2017) seems to indicate that one prominent concern of current teacher education is sufficient support for the development of PMTs' HCK.

Acknowledgements

This research was supported by the Jenny and Antti Wihuri Foundation. I would like to express my gratitude to my colleagues Juulia Lahdenperä and Juuso Nieminen as well as the anonymous reviewers for their wise comments that helped me to improve the manuscript.

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Notes

1 For more detailed description of Finnish teacher education see e.g. (Niemi & Jakku-Sihvonen, 2011).

2 The major subjects of the two mathematics minors is not specified in this paper to ensure the anonymity of the participants.

3 This definition is not, strictly speaking, correct, but it can be assumed that the participant is thinking of one possible correct construction of real numbers from Cauchy sequences.

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