Schramm-Loewner Evolution Introduction

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Chapter 1
Introduction

In this introductory chapter, we look at iterations of conformal maps, random processes such as random walks and statistical physics and establish some connections.

1.1 Iteration of conformal maps

We assume that the reader has some familiarity with Complex Analysis.[1] Recall that a differentiable function $f : U \to \mathbb{C}$, where $U \subset \mathbb{C}$ is a set containing an open neighborhood of a point z_0, is *conformal* at z_0, if the map f preserves angles[2] at z_0. If this holds for all points of U, we call f a *conformal map*. Remember also that a function in a planar domain is conformal if and only if it is holomorphic and one-to-one.

Let $\mathbb{H}$ be the upper half-plane and let $f_k : \mathbb{H} \to \mathbb{H}$ be a sequence of conformal maps where $k \in \mathbb{Z}_{>0}$.[3] Define

$$f^{[\![1,n]\!]}(z) = f_1 \circ f_2 \circ \ldots \circ f_n(z).$$

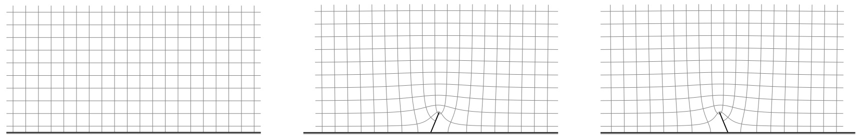
Suppose that each f_k maps $\mathbb{H}$ onto a set which is the complement (with respect to $\mathbb{H}$) of a bounded set K_k, whose boundary is a curve, and suppose that $|f_k(z)| \to \infty$ as $|z| \to \infty$. Then it turns out that f_k extends continuously to the closure $\overline{\mathbb{H}}$.

Suppose that the set K_k is a line segment $[\xi_k, \zeta_k]$, where $\xi_k \in \mathbb{R}$ is the base point and $\zeta_k \in \mathbb{H}$ is the tip point. By the continuity of f_k to the boundary, we can talk about the point $x_k \in \mathbb{R}$ which is mapped to the tip ζ_k by f_k. Define now $\hat{f}_k(z) =$

[1] The reader can use, for instance, Rudin's book [7] as a reference. Notice the *supplementary material* (appendices) of this book described in the preface, and also Chapter 3 below.

[2] In the sense that if P_1 and P_2 are smooth curves that form an angle θ at z_0, then also $f \circ P_1$ and $f \circ P_2$ form an angle θ at $f(z_0)$.

[3] Throughout this text we use the notations $\mathbb{Z}_{>0} = \{k \in \mathbb{Z} : k \geq 1\}$, $\mathbb{Z}_{\geq 0} = \{k \in \mathbb{Z} : k \geq 0\}$, $\mathbb{R}_{>0} = \{x \in \mathbb{R} : x > 0\}$, $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$ as well as $[\![j,k]\!]$ for the ordered set $j, j+1, j+2, \ldots, k-1, k$, where $j < k$ are integers.



(a) The grid lines in $\mathbb{H}$.

(b) The gridlines transformed by the mapping $z \mapsto \hat{f}_{\mathrm{R}}(z)$.

(c) The gridlines transformed by the mapping $z \mapsto \hat{f}_{\mathrm{L}}(z)$.

Fig. 1.1 Two elementary conformal transformations that are being iterated in the process illustrated in Figure 1.2. Grid lines can be used to illustrate the action of conformal maps.

$f_k(z+x_k) - \xi_k$. Then $\hat{f}_k$ is conformal and it maps $\mathbb{H}$ onto the complement of a line segment, whose base point is 0, it maps ∞ to ∞ and 0 to the tip of the line segment. It turns out (as we will later see) that it is useful to consider conformal maps that for large values of $|z|$ are close to identity, in the sense that they neither expand or shrink the grid as in Figure 1.1 far away from the origin.

If we iterate maps of this form, for instance, $\hat{f}_1 \circ \hat{f}_2$, then the composition will be a map from $\mathbb{H}$ onto the complement of a piecewise smooth curve. The continuity of the curve at the points where the (images of) line segments meet, follows from the fact that 0 is the base point of $\hat{f}_2$ and 0 is mapped to the tip point of $\hat{f}_1$ by $\hat{f}_1$.

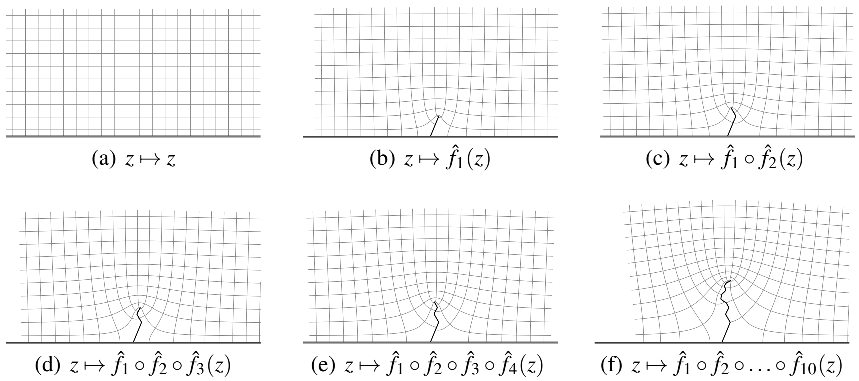
Figure 1.2 illustrates the iterates $\hat{f}^{[\![1,n]\!]}$. We have chosen two conformal maps (see Figure 1.1) that correspond to the line segments of the same length forming angles $\alpha\pi$ and $(1-\alpha)\pi$ with the positive real axis, and each f_k is one of the two maps.

The parameter n acts naturally as discrete time of the growth process. If we wish study a continuous time limit of the iterates $\hat{f}^{[\![1,n]\!]}$, we need to take large n and adjust the elementary conformal maps so that the sizes of the line segments are small, but the composed piecewise smooth curve reaches roughly to a constant height. This can be achieved by considering $\hat{\phi}_k(z) = n^{-a} \hat{f}_k(n^a z)$ where $a > 0$ is a suitable constant.

Let $F_t^{(n)}$ denote the iterate $\hat{\phi}^{[\![1,\lfloor nt \rfloor]\!]}$ for any $n \in \mathbb{Z}_{>0}$ and $t \in [0,1]$.[4] When n is large, the composed piecewise smooth curve corresponding to $\hat{\phi}^{[\![1,\lfloor nt \rfloor]\!]}$ increases by tiny steps as t is increased. It seems reasonable to expect that the limit $\lim_{n\to\infty} F_t^{(n)}$ exists and defines a continuous-time flow of the points of $\mathbb{H}$.[5] This is indeed the case at least when the sequence f_k are random, symmetrically distributed ($\hat{f}_{\mathrm{R}}$ and $\hat{f}_{\mathrm{L}}$ are equally likely) and independent. The continuous-time versions in the case of random, symmetric and independent sequences are the *Schramm–Loewner evolutions*.

[4] We use a common notion that $\lfloor x \rfloor$ is the largest integer smaller or equal to x.

[5] Such a limit is an example of *scaling limit*. Two typical features of a scaling limit are that there are scaling factor involved, such as n^{-a} and n^a above, which ensure that the limit exists, and that the limiting object will be described by continuous variables (another term is a continuum limit).



(a) $z \mapsto z$ (b) $z \mapsto \hat{f}_1(z)$ (c) $z \mapsto \hat{f}_1 \circ \hat{f}_2(z)$

(d) $z \mapsto \hat{f}_1 \circ \hat{f}_2 \circ \hat{f}_3(z)$ (e) $z \mapsto \hat{f}_1 \circ \hat{f}_2 \circ \hat{f}_3 \circ \hat{f}_4(z)$ (f) $z \mapsto \hat{f}_1 \circ \hat{f}_2 \circ \ldots \circ \hat{f}_{10}(z)$

Fig. 1.2 Consider conformal maps from the upper half-plane onto the complements of line segments. We can arrange so that ∞ is mapped to ∞ and that the base point of the line segment is 0 as well as the point which gets mapped to the tip of the segment. The figures here illustrate how iterations of such maps look like.

1.2 On stochastic models and connection to statistical physics

1.2.1 Random walk and Brownian motion

We also assume some familiarity with Probability Theory.[6]

Recall that a *stochastic process* is a collection of random variables indexed by an ordered set which is interpreted as the *time* variable. Let's consider random walks on $\mathbb{Z}$ as an example. We will denote probability measures generally by P. Let X_k, $k \in \mathbb{Z}_{>0}$, be a sequence of random variables which take two possible values ± 1, i.e., $\mathsf{P}[X_k = -1] + \mathsf{P}[X_k = +1] = 1$. Assume that X_k, $k \in \mathbb{Z}_{>0}$, are independent[7] and fix some $x \in \mathbb{Z}$. The formula

$$S_t = x + \sum_{k=1}^{t} X_k$$

defines a stochastic process[8] $(S_t)_{t \in \mathbb{Z}_{\geq 0}}$. If the random variables X_k, $k \in \mathbb{Z}_{>0}$, have symmetric distribution, that is, $\mathsf{P}[X_k = -1] = \mathsf{P}[X_k = +1] = \frac{1}{2}$, then the process is called *symmetric simple random walk* on $\mathbb{Z}$.

Often we wish to derive a continuum limit of the simple random walk or other processes. Such a limit is a scaling limit in the same sense as in the previous section. For that purpose, we choose a constant $a > 0$ and consider the continuous-time process $(n^{-a} S_{\lfloor nt \rfloor})_{t \in \mathbb{R}_{\geq 0}}$. For suitably chosen constant a this process will converge

[6] The reader can use, for instance, Durrett's book [3] as a reference. Notice the *supplementary material* (appendices) of this book described in the preface, and also Chapter 2 below.

[7] Remember that for these given random variables, X_k, $k \in \mathbb{Z}_{>0}$, are independent if for any $n \in \mathbb{Z}_{>0}$ and for any $x_1, x_2, \ldots, x_n \in \{-1, +1\}$, $\mathsf{P}[X_k = x_k \text{ for all } k \in [\![1,n]\!]] = \prod_{k \in [\![1,n]\!]} \mathsf{P}[X_k = x_k]$.

[8] We use the notation $(X_t)_{t \in I}$ where usually $I = \mathbb{Z}_{\geq 0}$ or $I = \mathbb{R}_{\geq 0}$, to denote a stochastic process.

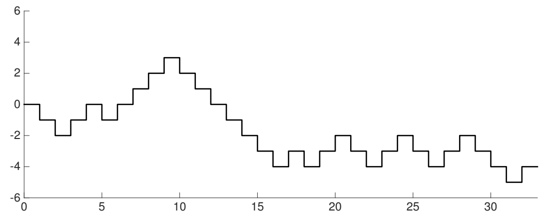


Fig. 1.3 Simple random walk

as $n \to \infty$ to a stochastic process $(B_t)_{t \in \mathbb{R}_{\geq 0}}$ called *Brownian motion*. From the *central limit theorem* (CLT) we know that $a = 1/2$ and that all the finite dimensional distributions (distributions of vectors of type $(B_{t_1}, B_{t_2}, \ldots, B_{t_k})$) are Gaussian.[9]

1.2.2 Ising model and other statistical physics models

The study of Schramm–Loewner evolutions is motivated by their applications to statistical physics. Those random curves appear in statistical physics under specific circumstances as *interfaces*, that is, domain walls separating parts of the system which differ in some microscopic property.

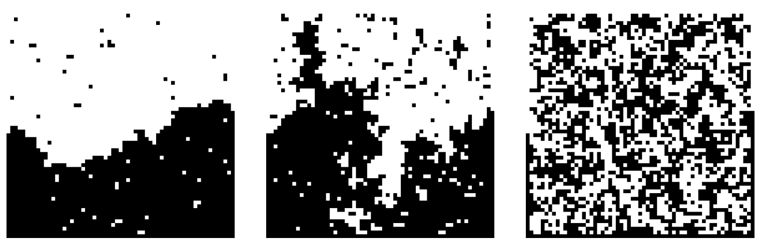


Fig. 1.4 Ising model with Dobrushin boundary conditions for $T < T_c$, $T = T_c$ and $T > T_c$. Here the black pixels are vertices with $\sigma = +1$ and the white pixels are vertices with $\sigma = -1$. An *interface* is a broken line separating white and black regions.

A typical example of a lattice model of statistical physics (i.e., a simplified model defined on a lattice such as $\mathbb{Z}^d$) is the *Ising model*, which models ferromagnetic material. Each site v is occupied by an elementary magnet, *spin*, which takes values $\sigma_v \in \{\pm 1\}$. The Ising model is defined by an energy functional

$$H(\underline{\sigma}) = -\sum \sigma_v \sigma_w.$$

[9] Remember that the result that a sum of independent and identical centered random variables scaled by $n^{-1/2}$ converges to a Gaussian random variable in distribution, is called the central limit theorem.

Here $\underline{\sigma} = (\sigma_v)_{v \in V}$ is the spin configuration of the system and V is a finite subset of the square lattice $\mathbb{Z}^2$ (we focus here on two-dimensional model). The sum in H is over neighboring pairs of sites. The more there are pairs of aligned spins, the more this functional favors the configuration (that is, the configuration has smaller energy) — this can be seen as the source of the ferromagnetic phenomenon.

In the Ising model, we take the configuration $\underline{\sigma} \in \{\pm 1\}^V$ to be random. Its law is given by the Boltzmann distribution corresponding to the energy functional H, i.e., the probability of observing $\underline{\sigma}$ is proportional to $\exp(-\beta H(\underline{\sigma}))$. Here $\beta = 1/T$, the inverse temperature, is a parameter.

The behavior of the systems depends drastically on the temperature T, as the reader can see from Figure 1.4. In the figure we use so called Dobrushin boundary conditions, where we force the spins on the two complementary boundary arcs to be constant -1 on one of them and $+1$ on the other. The *interface* which is the broken line separating the large $+1$-cluster and the large -1-cluster, can be studied when these boundary conditions are used.

The *scaling limit* of the interface is obtained by fixing a shape, say, a square and the Dobrushin boundary conditions on its boundary and then by approximating that shape by finite subsets of a lattice with a lattice mesh parameter. The scaling limit is the limit as the lattice mesh tends to zero.

The phase transition of the model can be explained in terms of interface in the following way. There is a critical temperature T_c such that for $T < T_c$ for large systems looked far away (i.e. in the scaling limit) the interface is close to the minimal energy line with fluctuations of order $\sqrt{N}$, where N is the side length of the box. As T approaches T_c the fluctuations grow and at T_c they are of the size of the system. Therefore $T = T_c$ is the smallest value of the parameter where we expect a non-trivial scaling limit for the interface. The fact that the scaling limit at $T < T_c$ is non-random is a result of [6]. For $T > T_c$, when looked far away, the spins behave more or less independently of the of each other and the interface looks like the interface of $T = \infty$, for which value the spin configuration is truly totally disordered.

1.2.3 Conformal invariance of the scaling limits

Schramm–Loewner evolutions give an efficient tool for verifying conformal invariance in the context of random curves of statistical physics and their scaling limits.

Based on physical arguments, the scaling limit is expected to be scale invariant. In fact, under some hypothesis such as partial rotation invariance of the Hamiltonian ($\pi/4$-rotation invariance of the Ising model on $\mathbb{Z}^2$) and short range of the interactions, it is expected that the scaling limit is even conformally invariant. Conformal invariance could be described to be *local* rotation, scale and translation invariance. Here "local" refers to the fact that the factor that we use in e.g. scale invariance can vary over the domain. Consult, for instance, the introduction of [5] for an introduction to the physical theories of phase transitions. The conformal invariance property of the Ising model should be understood concretely in the following way.

If we start from any two shapes (simply connected domains) and approximate both with sequences of discrete domains then the laws of the interfaces are equal in the scaling limit, in the sense that they are conformal images of each other.

This property is related to the *conformal Markov property* of iterates of conformal maps. Namely consider the conditional law of $\hat{f}^{[\![1,n+m]\!]}$ given that we know $\hat{f}_k$, $k \in [\![1,n]\!]$. That conditional law is just the law of $\hat{f}^{[\![m+1,n+m]\!]}$ transformed by the (known) conformal map $\hat{f}^{[\![1,n]\!]}$. This is an evidence of a connection between statistical physics and the iterates of conformal maps. We call the argument *Schramm's principle*, see [10], the original article by Schramm [8] or Section 5.1.1 below.

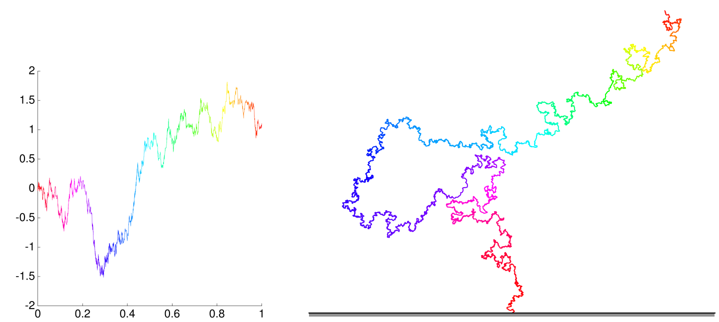


Fig. 1.5 Realizations of a 1D Brownian motion (left) and the corresponding SLE(3) (right) driven by the Brownian motion. SLEs are random curves which are fractal, in the sense, that they contain statistically similar details repeating on different length scales.

1.3 An example: percolation model and Cardy's formula

In this section we will present an example with some details that highlight the main topics of this text and the example is one of the main application of the theory of Schramm–Loewner evolution. The full argument is presented later in the text.

Consider the triangular lattice which is formed by the centers of the regular hexagonal tiling of the plane. Take a finite, simply connected[10] subgraph of the triangular lattice. We call the centers of the hexagons sites.

In the *site percolation model*, each site carries a random variable which takes value *open* or *closed*.[11] In any pictures, we color the corresponding hexagon green

[10] Simply connectedness means that the domain consisting of the hexagons is a simply connected domain (i.e. with no holes) — in other words, if we have a closed path of hexagons in the domain, it cannot disconnect any point in the complement of the domain from infinity.

[11] From the modelling perspective, the open sites represent channels through which a substance, say, water can flow. Therefore if we inject water into the sites of a set A_1, the water will flow to all the sites connected by a path of open sites to A_1. In particular we are interested in connection events that for fixed A_1 and A_2 there exists a connected path from A_1 to A_2 that stays in a set B.

if the site is open and red if the site is closed. The decision, whether a site is open or closed, is made randomly, independently and from the same distribution at each site. This leaves only one parameter in the model, which is the quantity $p := \mathsf{P}[\text{the site } x \text{ is open}] \in [0,1]$ which is independent of x.

Consider first the crossing probability for a fixed shape with varying size. More specifically, take rhombi $R_N = \{x\mathbf{e}_1 + y\mathbf{e}_2 : x,y \in [\![1,N]\!]\}$ where $\mathbf{e}_1 = 1$ and $\mathbf{e}_2 = \exp(\mathrm{i}\pi/3)$ are two vectors in the plane that generate the triangular lattice. Denote by $f(p,N)$ the probability of a left-to-right crossing of R_N. Clearly f is monotone in p.[12] As illustrated in Figure 1.6, as N tends to infinity the crossing probability tends to a sharp step function. More accurately $\lim_{N\to\infty} f(p,N)$ equals to 0, $\frac{1}{2}$ and 1, when $p < \frac{1}{2}$, $p = \frac{1}{2}$ and $p > \frac{1}{2}$, respectively. We would arrive to a similar conclusion if we had taken a rhombus with a different aspect ratio. The only difference is that the limit of the crossing probability at $p = \frac{1}{2}$ is not necessarily $\frac{1}{2}$, but it can take some other value in $(0,1)$. The parameter $p = \frac{1}{2}$ is *critical* in the sense that outside criticality limits of crossing probabilities are trivial, either 0 or 1. In fact, the limit when $p = \frac{1}{2}$ depends non-trivially on the aspect ratio of the rhombus.

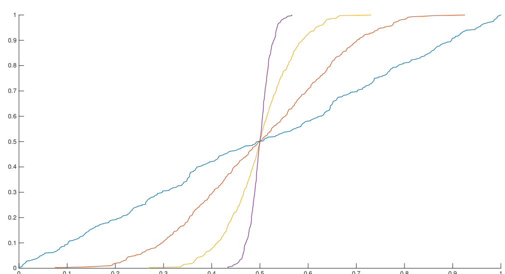


Fig. 1.6 The crossing probabilities of a left-to-right crossing in a rhombus R_N of side length N. The crossing probability is estimated using a computer simulation and plotted as a function of p for different values of N ($N = 1$ blue, $N = 4$ orange, $N = 16$ yellow, $N = 64$ purple). Different values of p are coupled using standard approach that uses uniform random variables. The sample size is 200 for each value of N.

1.3.1 Cardy's formula from SLE(6)

We will describe here how to derive a formula for the crossing probability using a *conformal invariance hypothesis*. Consider for simplicity the crossing probability in a rectangle $[0,aL] \times [0,a]$, where $a > 0$ and $L > 0$, for an open crossing from $\{0\} \times [0,a]$ to $\{aL\} \times [0,a]$. Map the rectangle conformally onto the upper half-plane $\mathbb{H}$ such that $(0,0) \mapsto U_0, (aL,0) \mapsto V_0, (0,a) \mapsto W_0, (aL,a) \mapsto \infty$. The exact form of the mapping doesn't play a role here.

[12] The reader should stop to think this for a moment, though.

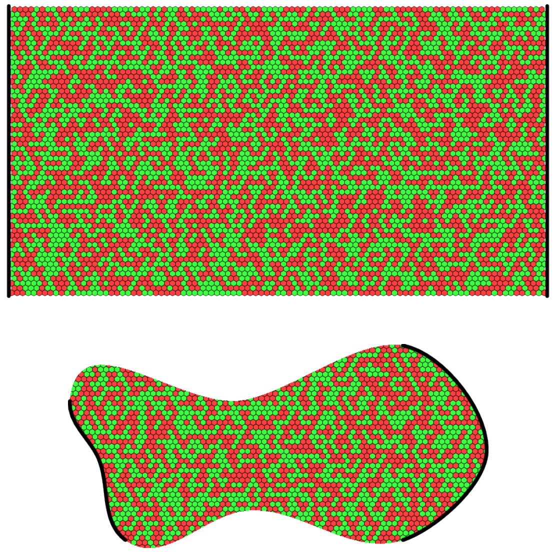


Fig. 1.7 Percolation on two different shapes. Cardy's formula tells that the probability of an open crossing of the quadrilaterals depends only on the conformal modulus in the scaling limit, at criticality, and gives an explicit expression for it.

Next introduce a new layer of hexagons around the rectangle, as in Figure 1.8. Assign boundary conditions such that the hexagons on $([0,aL]\times\{0\})\cup(\{aL\}\times[0,a])$ are closed and on $([0,aL]\times\{a\})\cup(\{0\}\times[0,a])$ open. Then there will be an interface separating the closed cluster and the open cluster that touch the boundary. In Figure 1.8, this is the blue path.

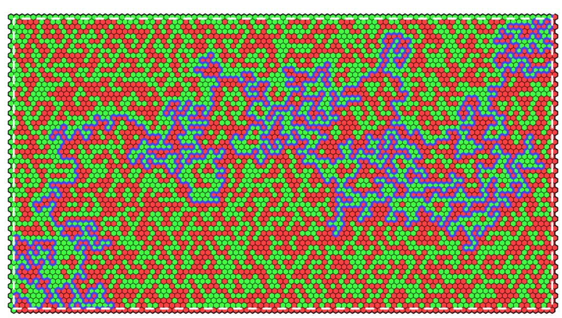


Fig. 1.8 After introducing the extra layer of hexagons for boundary conditions, there will be interface that separates the red and blue clusters that touch the boundary.

We can read the crossing event from the interface. Namely, the left-to-right crossing exists if and only if the interface hits $\{aL\} \times [0,a]$ before $[0,aL] \times \{a\}$.

Let's next consider the probability conditionally on the initial segment of the interface. Suppose that the interface is $\gamma(t)$, $t \in [0,T]$. The conditional probability of an open crossing given the initial segment $\gamma(s)$, $s \in [0,t]$, is a crossing probability but now in the complement of $\gamma[0,t]$ in the rectangle from the union of $\{0\} \times [0,a]$ and the left-hand side of $\gamma[0,t]$ to $\{aL\} \times [0,a]$. It is natural to transform that domain also onto the upper half-plane and take the points $\gamma(t),(aL,0),(0,a),(aL,a)$ to U_t, V_t, W_t, ∞, respectively.

We make an assumption that the scaling limit of the interface is conformally invariant and more specifically, we make a guess that the scaling limit is a process called SLE(6). Under further assumptions it holds that

$$U_t = \sqrt{6} B_t, \qquad \dot{V}_t = \frac{2}{V_t - U_t}, \qquad \dot{W}_t = \frac{2}{W_t - U_t}$$

where $\dot{X}_t = \partial_t X_t$. The first equality is the fact that the process is SLE(6) and the two others represent the Loewner flow of the marked points.

Set $Z_t = (U_t - W_t)/(V_t - W_t)$, which is a quantity called cross-ratio. That is equivalent of mapping $\mathbb{H}$ with marked points U_t, V_t, ∞, W_t onto $\mathbb{H}$ with marked points $z, 1, \infty, 0$. We further map the latter domain using a conformal map of the form $\phi(z) = C \int^z w^{-2/3}(1-w)^{-2/3} \mathrm{d}w$ onto a equilateral triangle PQR. Suppose that $\phi(W) = P$, $\phi(V) = Q$ and ϕ. Then $\zeta_t := \phi(Z_t) \in PQ$.

Based on stochastic calculus we can verify that the process ζ_t is a time change of a Brownian motion on PQ and thus the crossing probability, which can be reformulated as the probability that the process ζ_t hits Q before P, can be calculated. After an argument from stochastics (time-changed Brownian motions are conserved on average) and some algebra we end up to famous *Cardy's formula*

$$\lim_{N \to \infty} f(p_c, N) = \frac{\phi(z) - \phi(0)}{\phi(1) - \phi(0)} = \frac{3\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} z^{\frac{1}{3}} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}, \frac{4}{3}; z\right)$$

where $p_c = \frac{1}{2}$, Γ is the gamma function and ${}_2F_1$ is the hypergeometric function. The original articles on Cardy's formula are [2, 4, 9, 11].

1.4 On reading this book

The next two chapters review background material on Stochastic Calculus and Complex Analysis. The reader familiar with those topics may choose to jump directly to the main chapters, Chapters 4–6. Those chapters build on the prior chapters and are easiest read in the order of presentation. Appendices with additional material are provided in separate documents (see the preface) and cited occasionally here.

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