SUBORDINATION AND CONVOLUTION OF MULTIVALENT FUNCTIONS AND STARLIKENESS OF INTEGRAL TRANSFORMS

by

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SYMBOLS

Symbol	**Description**	**page**
$\mathcal{A}_p$	Class of all p-valent analytic functions f of the form $f(z)=z^p+\sum_{k=1+p}^{\infty}a_kz^k \quad (z\in\mathcal{U})$	16
$\mathcal{A}:=\mathcal{A}_1$	Class of analytic functions f of the form $f(z)=z+\sum_{k=2}^{\infty}a_kz^k \quad (z\in\mathcal{U})$	2
$(a)_n$	Pochhammer symbol	79
$\mathbb{C}$	Complex plane	1
$\mathcal{CCV}$	Class of close-to-convex functions in $\mathcal{A}$	6
$\mathcal{CCV}_\alpha$	Class of close-to-convex functions of order α in $\mathcal{A}$	6
$\mathcal{CCV}(\varphi,\psi)$	$\left\{f\in\mathcal{A}:\frac{f'(z)}{h'(z)}\prec\varphi(z),\quad h\in\mathcal{CV}(\psi)\right\}$	74
$\mathcal{CCV}(\alpha,\tau)$	$\left\{f\in\mathcal{A}:\mathrm{Re}\left(\frac{f'(z)}{h'(z)}\right)>\alpha,\quad h\in\mathcal{CV}(\tau)\right\}$	74
$\mathcal{CCV}_p^n(h)$	$\left\{f\in\mathcal{A}_p:\frac{1}{p}\frac{zf'(z)}{\phi_n(z)}\prec h(z),\ \phi\in\mathcal{ST}_p^n(h)\right\}$	56
$\mathcal{CCV}_{p,g}^n(h)$	$\left\{f\in\mathcal{A}_p:\frac{1}{p}\frac{z(g*f)'(z)}{(g*\phi)_n(z)}\prec h(z),\ \phi\in\mathcal{ST}_{p,g}^n(h)\right\}$	56
$\mathcal{CP}_p(\alpha,\lambda)$	$\left\{f\in\mathcal{A}_p:\mathrm{Re}\left(\frac{1}{p}\frac{(zf'(z))'}{p(1-\lambda)z^{p-1}+\lambda f'(z)}\right)+\alpha > \left\lvert\frac{(zf'(z))'}{p(1-\lambda)z^{p-1}+\lambda f'(z)}-\alpha\right\rvert,\quad z\in\mathcal{U}\right\}$	111
$\mathcal{CV}$	Class of convex functions in $\mathcal{A}$	5
$\mathcal{CV}(\alpha)$	Class of convex functions of order α in $\mathcal{A}$	5
$\mathcal{CV}_g(h)$	$\left\{f\in\mathcal{A}:1+\frac{z(f*g)''(z)}{(f*g)'(z)}\prec h(z)\right\}$	23
$\mathcal{CV}[A,B]$	$\{f\in\mathcal{A}:1+\frac{zf''(z)}{f'(z)}\prec\frac{1+Az}{1+Bz}\quad(-1\le B<A\le 1)\}$	12
$\mathcal{CV}(\varphi)$	$\left\{f\in\mathcal{A}:1+\frac{zf''(z)}{f'(z)}\prec\varphi(z)\right\}$	12

$\mathcal{CV}_p$	Class of convex functions in $\mathcal{A}_p$	16
$\mathcal{CV}_p(\beta)$	Class of convex functions of order β in $\mathcal{A}_p$	18
$\mathcal{CV}_p(\varphi)$	$\left\{f \in \mathcal{A}_p : \frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \phi(z)\right\}$	17
$\mathcal{CV}_p^n(h)$	$\left\{f \in \mathcal{A}_p : \frac{1}{p}\frac{(zf')'(z)}{f_n'(z)} \prec h(z)\right\}$	51
$\mathcal{CV}_{p,g}^n(h)$	$\left\{f \in \mathcal{A}_p : f * g \in \mathcal{CV}_p^n(h)\right\}$	52
$\mathcal{CVS}_p^n(h)$	$\left\{f \in \mathcal{A}_p : \frac{1}{p}\frac{2(zf')'(z)}{f_n'(z)+f_n'(-z)} \prec h(z),\ \frac{f_n'(z)+f_n'(-z)}{z^{p-1}} \neq 0 \text{ in } \mathcal{U}\right\}$	59
$\mathcal{CVS}_{p,g}^n(h)$	$\left\{f \in \mathcal{A}_p : f * g \in \mathcal{CVS}_p^n(h)\right\}$	59
$\mathcal{CVC}_p^n(h)$	$\left\{f \in \mathcal{A}_p : \frac{1}{p}\frac{2(zf')'(z)}{f_n'(z)+\overline{f_n'(\overline{z})}} \prec h(z),\ \frac{f_n'(z)+\overline{f_n'(\overline{z})}}{z^{p-1}} \neq 0 \text{ in } \mathcal{U}\right\}$	62
$\mathcal{CVC}_{p,g}^n(h)$	$\left\{f \in \mathcal{A}_p : f * g \in \mathcal{CVC}_p^n(h)\right\}$	62
$\mathcal{CVSC}_p^n(h)$	$\left\{f \in \mathcal{A}_p : \frac{1}{p}\frac{2(zf')'(z)}{f_n'(z)+\overline{f_n'(-\overline{z})}} \prec h(z),\ \frac{f_n'(z)+\overline{f_n'(-\overline{z})}}{z^{p-1}} \neq 0 \text{ in } \mathcal{U}\right\}$	66
$\mathcal{CVSC}_{p,g}^n(h)$	$\left\{f \in \mathcal{A}_p : f * g \in \mathcal{CVSC}_p^n(h)\right\}$	66
$\overline{co}(D)$	The closed convex hull of a set D	28
$f * g$	Convolution or Hadamard product of functions f and g	14
$\mathcal{H}(\mathcal{U})$	Class of analytic functions in $\mathcal{U}$	1
$\mathcal{H}[b,n]$	Class of analytic functions f in $\mathcal{U}$ of the form	1
	$f(z) = b + b_n z^n + b_{n+1} z^{n+1} + \cdots$	
$\mathcal{H}_0 := \mathcal{H}[0,1]$	Class of analytic functions f in $\mathcal{U}$ of the form	1
	$f(z) = b_1 z + b_2 z^2 + \cdots$	
$\mathcal{H} := \mathcal{H}[1,1]$	Class of analytic functions f in $\mathcal{U}$ of the form	1
	$f(z) = 1 + b_1 z + b_2 z^2 + \cdots$	
$\prec$	Subordinate to	11
k	Koebe function $k(z) = z/(1-z)^2$	2

$\mathbb{N}$	$\mathbb{N} := \{1, 2, \cdots\}$	15
$N_\delta(f)$	$\left\{z + \sum_{k=2}^{\infty} b_k z^k \;:\; \sum_{k=2}^{\infty} k\lvert a_k - b_k\rvert \leq \delta\right\}$	32
$N_\delta^p(f)$	$\left\{z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \;:\; \sum_{k=1}^{\infty} \frac{(p+k)}{p} \left\lvert a_{p+k} - b_{p+k}\right\rvert \leq \delta\right\}$	111
$\mathcal{P}(\beta)$	$\left\{f \in \mathcal{A} : \exists \phi \in \mathbb{R} \text{ with } \operatorname{Re} e^{i\phi}\left(f'(z) - \beta\right) > 0,\ z \in \mathcal{U}\right\}$	30
$\mathcal{P}_\alpha(\beta)$	$\Big\{f \in \mathcal{A} : \exists \phi \in \mathbb{R}$ with $\operatorname{Re} e^{i\phi}\left((1-\alpha)\frac{f(z)}{z} + \alpha f'(z) - \beta\right) > 0,\ z \in \mathcal{U}\Big\}$	31
$\mathcal{PST}$	Class of parabolic starlike functions in $\mathcal{A}$	8
$\mathcal{PST}(\alpha)$	Class of parabolic starlike functions of order α in $\mathcal{A}$	9
$\mathcal{PST}(\alpha, \beta)$	Class of parabolic β-starlike functions of order α in $\mathcal{A}$	9
$\mathcal{QCV}$	Class of quasi-convex functions in $\mathcal{A}$	6
$\mathbb{R}$	Set of all real numbers	2
$\mathcal{QCCV}_p^n(h)$	$\left\{f \in \mathcal{A}_p : \frac{1}{p}\frac{\left(zf'\right)'(z)}{\phi_n'(z)} \prec h(z),\ \phi \in \mathcal{CV}_p^n(h)\right\}$	56
$\mathcal{QCCV}_{p,g}^n(h)$	$\left\{f \in \mathcal{A}_p : \frac{1}{p}\frac{\left(z(g*f)'\right)'(z)}{(g*\phi)_n'(z)} \prec h(z),\ \phi \in \mathcal{CV}_{p,g}^n(h)\right\}$	56
Re	Real part of a complex number	5
$\mathcal{R}_\alpha$	Class of prestarlike functions of order α in $\mathcal{A}$	15
$\mathcal{R}_p(\alpha)$	Class of prestarlike functions of order α in $\mathcal{A}_p$	18
$\mathcal{R}(\beta)$	$\left\{f \in \mathcal{A} : \operatorname{Re}\left(f'(z) + zf''(z)\right) > \beta,\ z \in \mathcal{U}\right\}$	29
$\mathcal{R}_\gamma(\beta)$	$\{f \in \mathcal{A} : \exists \phi \in \mathbb{R}$ with $\operatorname{Re} e^{i\phi}\left(f'(z) + \gamma z f''(z) - \beta\right) > 0, \quad z \in \mathcal{U}\Big\}$	31
$\mathcal{S}$	Class of all normalized univalent functions f of the form $f(z) = z + a_2 z^2 + \cdots,\ z \in \mathcal{U}$	2
$\mathcal{SCCV}_\alpha$	Class of strongly close-to-convex functions of order α in $\mathcal{A}$	6

$\mathcal{SCV}_\alpha$	Class of strongly convex functions of order α in $\mathcal{A}$	6
$\mathcal{SP}_p(\alpha,\lambda)$	$\left\{f\in\mathcal{A}_p : \operatorname{Re}\left(\frac{1}{p}\frac{zf'(z)}{(1-\lambda)z^p+\lambda f(z)}\right)+\alpha > \left\lvert\frac{1}{p}\frac{zf'(z)}{(1-\lambda)z^p+\lambda f(z)}-\alpha\right\rvert,\quad z\in\mathcal{U}\right\}$	111
$\mathcal{SST}_\alpha$	Class of strongly starlike functions of order α in $\mathcal{A}$	6
$\mathcal{ST}$	Class of starlike functions in $\mathcal{A}$	5
$\mathcal{ST}[A,B]$	$\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec\frac{1+Az}{1+Bz}\quad(-1\le B<A\le 1)\}$	12
$\mathcal{ST}(\alpha)$	Class of starlike functions of order α in $\mathcal{A}$	5
$\mathcal{ST}(\varphi)$	$\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec\varphi(z)\right\}$	12
$\mathcal{ST}_p$	Class of starlike functions in $\mathcal{A}_p$	16
$\mathcal{ST}_p(\beta)$	Class of starlike functions of order β in $\mathcal{A}_p$	18
$\mathcal{ST}_p(\varphi)$	$\left\{f\in\mathcal{A}_p:\frac{1}{p}\frac{zf'(z)}{f(z)}\prec\phi(z)\right\}$	16
$\mathcal{ST}_s$	Class of starlike functions with respect to symmetric points in $\mathcal{A}$	7
$\mathcal{ST}_c$	Class of starlike functions with respect to conjugate points in $\mathcal{A}$	7
$\mathcal{ST}_{sc}$	Class of starlike functions with respect to symmetric conjugate points in $\mathcal{A}$	7
$\mathcal{ST}_g(h)$	$\left\{f\in\mathcal{A}:\frac{z(f*g)'(z)}{(f*g)(z)}\prec h(z)\right\}$	23
$\mathcal{ST}_s^n$	Class of starlike functions with respect to n-ply symmetric points in $\mathcal{A}$	24
$\mathcal{ST}_c^n$	Class of starlike functions with respect to n-ply conjugate points in $\mathcal{A}$	25

$\mathcal{ST}^n_{sc}$	Class of starlike functions with respect to n-ply symmetric conjugate points in $\mathcal{A}$	25
$\mathcal{ST}^n_p(h)$	$\left\{f \in \mathcal{A}_p : \frac{1}{p}\frac{zf'(z)}{f_n(z)} \prec h(z),\ \frac{f_n(z)}{z^p} \neq 0 \text{ in } \mathcal{U}\right\}$	51
$\mathcal{ST}^n_{p,g}(h)$	$\left\{f \in \mathcal{A}_p : f * g \in \mathcal{ST}^n_p(h)\right\}$	51
$\mathcal{STS}^n_p(h)$	$\left\{f \in \mathcal{A}_p : \frac{1}{p}\frac{2zf'(z)}{f_n(z)-f_n(-z)} \prec h(z),\ \frac{f_n(z)-f_n(-z)}{z^p} \neq 0 \text{ in } \mathcal{U}\right\}$	58
$\mathcal{STS}^n_{p,g}(h)$	$\left\{f \in \mathcal{A}_p : f * g \in \mathcal{STS}^n_p(h)\right\}$	58
$\mathcal{STC}^n_p(h)$	$\left\{f \in \mathcal{A}_p : \frac{1}{p}\frac{2zf'(z)}{f_n(z)+\overline{f_n(\overline{z})}} \prec h(z),\ \frac{f_n(z)+\overline{f_n(\overline{z})}}{z^p} \neq 0 \text{ in } \mathcal{U}\right\}$	62
$\mathcal{STC}^n_{p,g}(h)$	$\left\{f \in \mathcal{A}_p : f * g \in \mathcal{STC}^n_p(h)\right\}$	62
$\mathcal{STSC}^n_p(h)$	$\left\{f \in \mathcal{A}_p : \frac{1}{p}\frac{2zf'(z)}{f_n(z)-\overline{f_n(-\overline{z})}} \prec h(z),\ \frac{f_n(z)-\overline{f_n(-\overline{z})}}{z^p} \neq 0 \text{ in } \mathcal{U}\right\}$	65
$\mathcal{STSC}^n_{p,g}(h)$	$\left\{f \in \mathcal{A}_p : f * g \in \mathcal{STSC}^n_p(h)\right\}$	66
$\mathcal{U}$	Open unit disk $\{z \in \mathcal{C} : \lvert z\rvert < 1\}$	1
$\mathcal{U}_r$	Open disk $\{z \in \mathcal{C} : \lvert z\rvert < r\}$ of radius r	7
$\mathcal{UST}$	Class of uniformly starlike functions in $\mathcal{A}$	8
$\mathcal{UCV}$	Class of uniformly convex functions in $\mathcal{A}$	8
$\mathcal{UCV}(\alpha)$	Class of uniformly convex functions of order α in $\mathcal{A}$	9
$\mathcal{UCV}(\alpha,\beta)$	Class of uniformly β-convex functions of order α in $\mathcal{A}$	10
$\mathcal{V}^*$	The dual set of $\mathcal{V}$	27
$\mathcal{V}^{**}$	The second dual of $\mathcal{V}$	27
$\mathcal{W}_\beta(\alpha,\gamma)$	$\Big\{f \in \mathcal{A} : \exists\phi \in \mathbb{R}$ with $\operatorname{Re} e^{i\phi}\Big((1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z) - \beta\Big) > 0, z \in \mathcal{U}\Big\}$	31
$\Psi_n[\Omega, q]$	Class of admissible functions	19

SUBORDINASI DAN KONVOLUSI FUNGSI MULTIVALEN DAN PENJELMAAN KAMIRAN BAK–BINTANG

ABSTRAK

Tesis ini membincangkan fungsi analisis dan fungsi multivalen yang tertakrif pada cakera unit terbuka $\mathcal{U}$. Umumnya, fungsi-fungsi tersebut diandaikan ternormal, sama ada dalam bentuk

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

atau

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p},$$

dengan p integer positif tetap. Andaikan $\mathcal{A}$ sebagai kelas yang terdiri daripada fungsi-fungsi f dengan penormalan pertama, manakala $\mathcal{A}_p$ terdiri daripada fungsi-fungsi f dengan penormalan kedua. Tesis ini mengkaji lima masalah penyelidikan.

Pertama, andaikan $f^{(q)}$ sebagai terbitan peringkat ke-q bagi fungsi $f \in \mathcal{A}_p$. Dengan menggunakan teori subordinasi pembeza, syarat cukup diperoleh agar rantai pembeza berikut dipenuhi:

$$\frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}} \prec Q(z), \text{ atau } \quad \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 \prec Q(z).$$

Di sini, Q ialah fungsi superordinasi yang bersesuaian, $\lambda(p,q) = p!/(p-q)!$, dan $\prec$ menandai subordinasi. Sebagai hasil susulan penting, beberapa kriteria sifat univalen dan cembung diperoleh bagi kes $p = q = 1$.

Sifat bak-bintang terhadap titik n-lipat juga diitlakkan kepada kes fungsi mul-

tivalen. Hal ini melibatkan fungsi-fungsi $f \in \mathcal{A}_p$ yang memenuhi subordinasi

$$\frac{1}{p}\frac{zf'(z)}{\frac{1}{n}\sum_{k=0}^{n-1}\epsilon^{n-k}f(\epsilon^k z)} \prec h(z),$$

dengan h sebagai fungsi cembung ternormalkan yang mempunyai bahagian nyata positif serta $h(0) = 1$, n integer positif tetap, dan ϵ memenuhi $\epsilon^n = 1$, $\epsilon \neq 1$. Dengan cara yang serupa, kelas fungsi p-valen cembung, hampir-cembung dan kuasicembung terhadap titik n-lipat diperkenalkan, serta juga fungsi p-valen bak-bintang dan fungsi cembung terhadap titik simetri n-lipat, titik konjugat dan titik konjugat simetri. Sifat rangkuman kelas dan konvolusi bagi kelas-kelas tersebut dikaji.

Sifat mengawetkan rangkuman bagi pengoperasian kamiran juga diperluaskan. Dua pengoperasian kamiran $F : \mathcal{A}^n \times \overline{\mathcal{U}}^2 \to \mathcal{A}$ dan $G : \mathcal{A}^n \times \overline{\mathcal{U}}^2 \to \mathcal{A}$ dibincangkan, dengan

$$F(z) = F_{f_1,\cdots,f_n;z_1,z_2}(z) = \int_0^z \prod_{j=1}^{n}\left(\frac{f_j(z_2\zeta) - f_j(z_1\zeta)}{(z_2 - z_1)\zeta}\right)^{\alpha_j} d\zeta \quad (z_1, z_2 \in \overline{\mathcal{U}}),$$

$$G(z) = G_{f_1,\cdots,f_n;z_1,z_2}(z) = z\prod_{j=1}^{n}\left(\frac{f_j(z_2 z) - f_j(z_1 z)}{(z_2 - z_1)z}\right)^{\alpha_j} \quad (z_1, z_2 \in \overline{\mathcal{U}}).$$

Pengoperasian tersebut merupakan pengitlakan hasil kajian-kajian terdahulu. Sifat mengawetkan bak-bintang, cembung, dan hampir-cembung dikaji, bukan sahaja bagi fungsi f_j yang terletak di dalam kelas-kelas tertentu, tetapi juga bagi fungsi f_j yang terletak di dalam kelas fungsi bak-bintang ala Ma-Minda dan cembung ala Ma-Minda.

Satu penjelmaan kamiran menarik yang mendapat perhatian pelbagai kajian dewasa ini ialah $V_\lambda : \mathcal{A} \to \mathcal{A}$ dengan

$$V_\lambda(f)(z) := \int_0^1 \lambda(t) \frac{f(tz)}{t} dt.$$

Di sini λ merupakan fungsi nyata tak negatif terkamirkan pada $[0, 1]$ yang memenuhi syarat $\int_0^1 \lambda(t)dt = 1$. Penjelmaan tersebut mempunyai penggunaan signifikan dalam teori fungsi geometri. Tesis ini mengkaji sifat bak-bintang penjelmaan V_λ pada kelas

$$\mathcal{W}_\beta(\alpha, \gamma) := \Big\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \text{ with } \operatorname{Re} e^{i\phi}\Big((1 - \alpha + 2\gamma)\frac{f(z)}{z} + (\alpha - 2\gamma)f'(z) + \gamma z f''(z) - \beta\Big) > 0, z \in \mathcal{U} \Big\}$$

dengan menggunakan Prinsip Dual. Sebagai hasil susulan penting, nilai terbaik $\beta < 1$ diperoleh yang mempastikan fungsi-fungsi $f \in \mathcal{A}$ yang memenuhi syarat

$$\operatorname{Re}\Big(f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z)\Big) > \beta$$

in $\mathcal{U}$ adalah semestinya bak-bintang pada $\mathcal{U}$. Contoh-contoh penting turut dibangunkan sepadan dengan pilihan tertentu fungsi teraku λ.

Tesis ini diakhiri dengan memperkenalkan dua subkelas multivalen pada $\mathcal{A}_p$. Kelas-kelas tersebut terdiri daripada fungsi bak-bintang parabola teritlak peringkat α jenis λ, ditandai $\mathcal{SP}_p(\alpha, \lambda)$, dan kelas fungsi cembung parabola peringkat α jenis λ, ditandai $\mathcal{CP}_p(\alpha, \lambda)$. Kedua-dua kelas tersebut ditunjukkan tertutup terhadap konvolusi dengan fungsi prabak-bintang. Turut diperoleh adalah kriteria baru bagi fungsi-fungsi untuk terletak di dalam kelas $\mathcal{SP}_p(\alpha, \lambda)$. Jiranan- δ bagi fungsi-fungsi di dalam kelas-kelas tersebut juga dicirikan.

SUBORDINATION AND CONVOLUTION OF MULTIVALENT FUNCTIONS AND STARLIKENESS OF INTEGRAL TRANSFORMS

ABSTRACT

This thesis deals with analytic functions as well as multivalent functions defined on the unit disk $\mathcal{U}$. In most cases, these functions are assumed to be normalized, either of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

or

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p},$$

p a fixed positive integer. Let $\mathcal{A}$ be the class of functions f with the first normalization, while $\mathcal{A}_p$ consists of functions f with the latter normalization. Five research problems are discussed in this work.

First, let $f^{(q)}$ denote the q-th derivative of a function $f \in \mathcal{A}_p$. Using the theory of differential subordination, sufficient conditions are obtained for the following differential chain to hold:

$$\frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}} \prec Q(z), \text{ or } \quad \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 \prec Q(z).$$

Here Q is an appropriate superordinate function, $\lambda(p;q) = p!/(p-q)!$, and $\prec$ denotes subordination. As important consequences, several criteria for univalence and convexity are obtained for the case $p = q = 1$.

The notion of starlikeness with respect to $n-$ply points is also generalized to

the case of multivalent functions. These are functions $f \in \mathcal{A}_p$ satisfying

$$\frac{1}{p} \frac{zf'(z)}{\frac{1}{n} \sum_{k=0}^{n-1} \epsilon^{n-k} f(\epsilon^k z)} \prec h(z),$$

where h is a normalized convex function with positive real part satisfying $h(0) = 1$, n a fixed positive integer, and ϵ satisfies $\epsilon^n = 1,\ \epsilon \neq 1$. Similar classes of p-valent functions to be convex, close-to-convex and quasi-convex functions with respect to n-ply points, as well as p-valent starlike and convex functions with respect to n-ply symmetric points, conjugate points and symmetric conjugate points respectively are introduced. Inclusion and convolution properties of these classes are investigated.

Membership preservation properties of integral operators are also extended. Two integral operators $F : \mathcal{A}^n \times \overline{\mathcal{U}}^2 \to \mathcal{A}$ and $G : \mathcal{A}^n \times \overline{\mathcal{U}}^2 \to \mathcal{A}$ are considered, where

$$F(z) = F_{f_1,\cdots,f_n;z_1,z_2}(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{f_j(z_2\zeta) - f_j(z_1\zeta)}{(z_2 - z_1)\zeta} \right)^{\alpha_j} d\zeta \quad (z_1, z_2 \in \overline{\mathcal{U}}),$$

$$G(z) = G_{f_1,\cdots,f_n;z_1,z_2}(z) = z \prod_{j=1}^{n} \left(\frac{f_j(z_2 z) - f_j(z_1 z)}{(z_2 - z_1)z} \right)^{\alpha_j} \quad (z_1, z_2 \in \overline{\mathcal{U}}).$$

These operators are generalization of earlier works. Preservation properties of starlikeness, convexity, and close-to-convexity are investigated, not only for functions f_j belonging to those respective classes, but also for functions f_j in the classes of Ma-Minda type starlike and convex functions.

An interesting integral transform that has attracted many recent works is the transform $V_\lambda : \mathcal{A} \to \mathcal{A}$ given by

$$V_\lambda(f)(z) := \int_0^1 \lambda(t) \frac{f(tz)}{t} dt,$$

where λ is an integrable non-negative real-valued function on $[0,1]$ satisfying $\int_0^1 \lambda(t)dt = 1$. This transform has significant applications in geometric function theory. This thesis investigates starlikeness of the transform V_λ over the class

$$\mathcal{W}_\beta(\alpha,\gamma) := \Bigg\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \text{ with } \operatorname{Re} e^{i\phi}\Big((1-\alpha+2\gamma)\frac{f(z)}{z} \\ + (\alpha-2\gamma)f'(z) + \gamma z f''(z) - \beta\Big) > 0,\ z \in \mathcal{U} \Bigg\}$$

using the Duality Principle. As a significant consequence, the best value of $\beta < 1$ is obtained that ensures functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re}\left(f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z)\right) > \beta$$

in $\mathcal{U}$ are necessarily starlike. Important examples are also determined for specific choices of the admissible function λ.

Finally, two multivalent subclasses of $\mathcal{A}_p$ are introduced. These classes consist of generalized parabolic starlike functions of order α and type λ, denoted by $\mathcal{SP}_p(\alpha,\lambda)$, and parabolic convex functions of order α and type λ, denoted by $\mathcal{CP}_p(\alpha,\lambda)$. It is shown that these two classes are closed under convolution with prestarlike functions. Additionally, a new criterion for functions to belong to the class $\mathcal{SP}_p(\alpha,\lambda)$ is derived. We also describe the δ-neighborhood of functions belonging to these classes.

CHAPTER 1

INTRODUCTION

1.1 Univalent Functions

Let $\mathbb{C}$ be the complex plane. A function f is *analytic* at z_0 in a domain D if it is differentiable in some neighborhood of z_0, and it is *analytic on a domain* D if it is analytic at all points in D. A function f which is analytic on a domain D is said to be *univalent* there if it is a one-to-one mapping on D, and f is *locally univalent* at $z_0 \in D$ if it is univalent in some neighborhood of z_0. It is evident that f is locally univalent at z_0 provided $f'(z_0) \neq 0$. The Riemann Mapping theorem is an important theorem in geometric function theory. It states that every simply connected domain which is not the whole complex plane can be mapped conformally onto the unit disk $\mathcal{U}$ =$\{z \in \mathbb{C} : |z| < 1\}$.

Theorem 1.1 (Riemann Mapping Theorem) [40, p. 11] *Let D be a simply connected domain which is a proper subset of the complex plane. Let ζ be a given point in D. Then there is a unique univalent analytic function f which maps D onto the unit disk $\mathcal{U}$ satisfying $f(\zeta) = 0$ and $f'(\zeta) > 0$.*

Let $\mathcal{H}(\mathcal{U})$ be the class of analytic functions in $\mathcal{U}$ and $\mathcal{H}[b, n]$ be the subclass of $\mathcal{H}(\mathcal{U})$ consisting of functions of the form

$$g(z) = b + b_n z^n + b_{n+1} z^{n+1} + \cdots . \tag{1.1}$$

Denote by $\mathcal{H}_0 \equiv \mathcal{H}[0, 1]$ and $\mathcal{H} \equiv \mathcal{H}[1, 1]$. If $g \in \mathcal{H}[b_0, 1]$ is univalent in $\mathcal{U}$, then $g(z) - b_0$ is again univalent in $\mathcal{U}$ as the addition of a constant only translates the image. Since g is univalent in $\mathcal{U}$, then $b_1 = g'(0) \neq 0$, and hence $f(z) = (g(z) - b_0)/b_1$ is also univalent in $\mathcal{U}$. Conversely, if f is univalent in $\mathcal{U}$, then so is

g. Putting $b_n/b_1 = a_n,\ n = 1, 2, 3\cdots$ in (1.1) gives the normalized form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots .$$

In the treatment of univalent analytic functions in $\mathcal{U}$, it is sufficient to consider the class $\mathcal{A}$ consisting of all functions f analytic in $\mathcal{U}$ normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. A function f in $\mathcal{A}$ has a Taylor series of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathcal{U}).$$

The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. The function k in the class $\mathcal{S}$ given by

$$k(z) = \frac{z}{(1-z)^2} = \frac{1}{4}\left(\left(\frac{1+z}{1-z}\right)^2 - 1\right) = \sum_{n=1}^{\infty} n z^n \quad (z \in \mathcal{U}) \tag{1.2}$$

is called the Koebe function. It maps $\mathcal{U}$ onto the complex plane except for a slit along the half-line $(-\infty, -1/4]$. The Koebe function and its rotations $e^{-i\beta}k(e^{i\beta}z),\ \beta \in \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers), play a very important role in the study of $\mathcal{S}$. They often are the extremal functions for various problems in $\mathcal{S}$. In 1916, Bieberbach [20] proved the following theorem for functions in $\mathcal{S}$.

Theorem 1.2 (Bieberbach's Theorem) [40, p. 30] *If $f \in \mathcal{S}$, then $|a_2| \le 2$, with equality if and only if f is a rotation of the Koebe function k.*

In the same paper, Bieberbach conjectured that, for $f \in \mathcal{S}$, $|a_n| \le n$ is generally valid. For the cases $n = 3$, and $n = 4$, the conjecture was proved respectively by Löwner [69], and Garabedian and Schiffer [50]. Much later in 1985, de Branges [22] proved the Bieberbach's conjecture for all coefficients with the help of the hypergeometric functions. Bieberbach's theorem has important implications in the theory of univalent functions. These include the famous covering theorem which states that if $f \in \mathcal{S}$, then the image of $\mathcal{U}$ under f must cover the open disk centered at the origin of radius $1/4$.

Theorem 1.3 (Koebe One-Quarter Theorem) [40, p. 31] *The range of every function $f \in \mathcal{S}$ contains the disk $\{w : |w| < 1/4\}$.*

The Koebe function shows that the radius one-quarter is sharp. Another important consequence of the Bieberbach's theorem is the Distortion Theorem which provides sharp upper and lower bounds for $|f'(z)|$.

Theorem 1.4 (Distortion Theorem) [40, p. 32] *For each $f \in \mathcal{S}$,*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3} \quad (|z| = r < 1).$$

The Distortion Theorem can be applied to obtain sharp upper and lower bounds for $|f(z)|$, known as the Growth Theorem.

Theorem 1.5 (Growth Theorem) [40, p. 33] *For each $f \in \mathcal{S}$,*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2} \quad (|z| = r < 1).$$

Again the Koebe function demonstrates sharpness of both theorems above.

Another implication of the Bieberbach's theorem is the Rotation Theorem which provides sharp upper bound for $|\arg f'(z)|$.

Theorem 1.6 (Rotation Theorem) [40, p. 99] *For each $f \in \mathcal{S}$,*

$$|\arg f'(z)| \leq \begin{cases} 4sin^{-1}r \quad (r \leq \dfrac{1}{\sqrt{2}}), \\ \\ \pi + \log \frac{r^2}{1-r^2} \quad (r \geq \frac{1}{\sqrt{2}}), \end{cases}$$

where $r = |z| < 1$. The bound is sharp for each $z \in \mathcal{U}$.

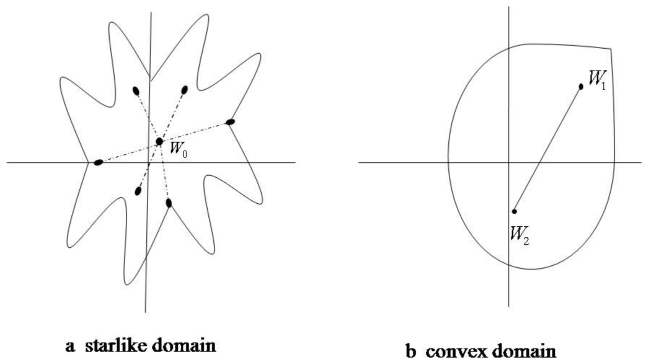

Figure 1.1: Starlike and convex domains

The very long gap between the Bieberbach's conjecture [20] of 1916 and its proof in 1985 by de Branges [22] motivated researchers to work in several directions. One of these directions was to prove the Bieberbach's conjecture $|a_n| \leq n$ for subclasses of $\mathcal{S}$ defined by geometric conditions. Among these classes are the classes of starlike functions, convex functions, close-to-convex functions, and quasi-convex functions. A set $D \subset \mathbb{C}$ is called *starlike* with respect to $w_0 \in D$ if the line segment joining w_0 to every other point $w \in D$ lies in the interior of D (see Figure 1.1a). The set D is called *convex* if for every pair of points w_1 and w_2 in D, the line segment joining w_1 and w_2 lies in the interior of D (see Figure 1.1b). A function $f \in \mathcal{A}$ is said to be a starlike function if $f(\mathcal{U})$ is a starlike domain with respect to 0, and $f \in \mathcal{A}$ is a convex function if $f(\mathcal{U})$ is a convex domain. Analytically, these geometric properties are respectively equivalent to the conditions [40, 51, 52, 55, 93]

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad \text{and} \quad \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,$$

where $\mathrm{Re}(w)$ is the real part of the complex number w. The Koebe function k in (1.2) is an example of a starlike function. The function

$$f(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n$$

which maps $\mathcal{U}$ onto the half-plane $\{w : \mathrm{Re}\, w > -1/2\}$ is convex. The subclasses of $\mathcal{A}$ consisting of starlike and convex functions are denoted respectively by $\mathcal{ST}$ and $\mathcal{CV}$. An important relationship between convex and starlike functions was first observed by Alexander [5] in 1915.

Theorem 1.7 (Alexander's Theorem) [40, p. 43] *Let $f \in \mathcal{A}$. Then $f \in \mathcal{CV}$ if and only if $zf' \in \mathcal{ST}$.*

Robertson [105] in 1936 introduced the concepts of starlike and convex functions of order α, $0 \leq \alpha < 1$. A function $f \in \mathcal{A}$ is said to be *starlike or convex of order* α if it satisfies the condition

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \alpha \quad \text{or} \quad \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \alpha \quad (0 \leq \alpha < 1).$$

These classes will be denoted respectively by $\mathcal{ST}(\alpha)$ and $\mathcal{CV}(\alpha)$. Evidently $\mathcal{ST}(0) = \mathcal{ST}$ and $\mathcal{CV}(0) = \mathcal{CV}$.

For $0 < \alpha \leq 1$, Brannan and Kirwan [23] introduced the classes of strongly starlike and strongly convex functions of order α. A function $f \in \mathcal{A}$ is said to be *strongly starlike of order* α if it satisfies

$$\left|\arg \frac{zf'(z)}{f(z)}\right| \leq \frac{\alpha\pi}{2} \quad (z \in \mathcal{U},\ 0 < \alpha \leq 1),$$

and is *strongly convex of order* α if

$$\left|\arg \left(1 + \frac{zf''(z)}{f'(z)}\right)\right| \leq \frac{\alpha\pi}{2} \quad (z \in \mathcal{U},\ 0 < \alpha \leq 1).$$

The subclasses of $\mathcal{A}$ consisting of strongly starlike and strongly convex functions of order α are denoted respectively by $\mathcal{SST}_{\alpha}$ and $\mathcal{SCV}_{\alpha}$. Since the condition $\operatorname{Re} w(z) > 0$ is equivalent to $|\arg w(z)| < \pi/2$, it follows that $\mathcal{SST}_1 \equiv \mathcal{ST}$ and $\mathcal{SCV}_1 \equiv \mathcal{CV}$.

In 1952, Kaplan [61] introduced the class of close-to-convex functions. A function $f \in \mathcal{A}$ is said to be *close-to-convex* if there is a function $g \in \mathcal{CV}$ such that $\operatorname{Re}\big(f'(z)/g'(z)\big) > 0$ for all $z \in \mathcal{U}$, or equivalently, if there is a function $g \in \mathcal{ST}$ such that $\operatorname{Re}\big(zf'(z)/g(z)\big) > 0$ for all $z \in \mathcal{U}$. The class of all close-to-convex functions in $\mathcal{A}$ is denoted by $\mathcal{CCV}$. A function $f \in \mathcal{A}$ is said to be *close-to-convex of order* α, $0 \leq \alpha < 1$, if there is a function $g \in \mathcal{CV}$ such that $\operatorname{Re}\big(f'(z)/g'(z)\big) > \alpha$ for all $z \in \mathcal{U}$. This class is denoted by $\mathcal{CCV}_{\alpha}$.

Reade [104] introduced the class of strongly close-to-convex functions of order α, $0 < \alpha \leq 1$. A function $f \in \mathcal{A}$ is said to be *strongly close-to-convex of order* α if there is function $\phi \in \mathcal{CV}$ satisfying

$$\left|\arg \frac{f'(z)}{\phi'(z)}\right| \leq \frac{\alpha\pi}{2} \quad (z \in \mathcal{U},\ 0 < \alpha \leq 1).$$

The subclass of $\mathcal{A}$ consisting of strongly close-to-convex functions of order α is denoted by $\mathcal{SCCV}_{\alpha}$. When $\alpha = 1$, $\mathcal{SCCV}_1 \equiv \mathcal{CCV}$.

In 1980, Noor and Thomas [80] introduced the class of quasi-convex functions. A function $f \in \mathcal{A}$ is said to be *quasi-convex* if there is a function $g \in \mathcal{CV}$ such that $\operatorname{Re}\big((zf'(z))'/g'(z)\big) > 0$ for all $z \in \mathcal{U}$. The class of all quasi-convex functions in $\mathcal{A}$ is denoted by $\mathcal{QCV}$.

A function $f \in \mathcal{A}$ is said to be *starlike with respect to symmetric points in* $\mathcal{U}$ if for every r less than and sufficiently close to one and every ζ on $|z| = r$, the angular velocity of $f(z)$ about the point $f(-\zeta)$ is positive at $z = \zeta$ as z traverses the circle $|z| = r$ in the positive direction, that is,

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z) - f(-\zeta)}\right) > 0 \quad (z = \zeta,\ |\zeta| = r).$$

This class was introduced and studied in 1959 by Sakaguchi [115]. Let the class of these functions be denoted by $\mathcal{ST}_s$. An equivalent description of this class is given by the following theorem.

Theorem 1.8 [115] *Let $f \in \mathcal{A}$. Then $f \in \mathcal{ST}_s$ if and only if*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0 \quad (z \in \mathcal{U}).$$

Further investigations into the class of starlike functions with respect to symmetric points can be found in [35, 79, 85, 117, 128, 130–132, 135]. El-Ashwah and Thomas [41] introduced and studied the classes consisting of starlike functions with respect to conjugate points, and starlike functions with respect to symmetric conjugate points defined respectively by the conditions

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z) + \overline{f(\overline{z})}}\right) > 0, \quad \operatorname{Re}\left(\frac{zf'(z)}{f(z) - \overline{f(-\overline{z})}}\right) > 0.$$

Let the classes of these functions be denoted respectively by $\mathcal{ST}_c$ and $\mathcal{ST}_{sc}$.

Ford [44] observed that convex or starlike functions inherit hereditary properties. In other words, if $f \in \mathcal{S}$ is starlike or convex, then $f(\mathcal{U}_r)$ is also a starlike or a convex domain, where $\mathcal{U}_r = \{z : |z| < r\}$.

Theorem 1.9 (Ford's Theorem) [52, p. 114] *Let f be in $\mathcal{S}$. If $f(\mathcal{U})$ is a convex domain, then for each positive $r < 1$, $f(\mathcal{U}_r)$ is also a convex domain. If $f(\mathcal{U})$ is starlike with respect to the origin, then for each positive $r < 1$, $f(\mathcal{U}_r)$ is also starlike with respect to the origin.*

It follows from the above theorem that convex (starlike) functions map circles centered at the origin in the unit disk onto convex (starlike) area. However this geometric property does not hold in general for circles whose centers are not at the origin. This motivated Goodman [53, 54] to introduce the classes $\mathcal{UCV}$ and $\mathcal{UST}$ of uniformly convex and uniformly starlike functions. An analytic function $f \in \mathcal{S}$ is uniformly convex (uniformly starlike) if f maps every circular arc γ contained in $\mathcal{U}$ with center ζ also in $\mathcal{U}$ onto a convex (starlike with respect to $f(\zeta)$) arc. Analytic descriptions of these classes are given by the following theorem.

Theorem 1.10 [53, 54] *Let $f \in \mathcal{A}$. Then*

(a) $f \in \mathcal{UCV}$ if and only if

$$\operatorname{Re}\left(1 + (z-\zeta)\frac{f''(z)}{f'(z)}\right) \geq 0 \quad (\,(z,\zeta) \in \mathcal{U} \times \mathcal{U})\,.$$

(b) $f \in \mathcal{UST}$ if and only if

$$\operatorname{Re}\frac{f(z)-f(\zeta)}{(z-\zeta)f'(z)} \geq 0 \quad (\,(z,\zeta) \in \mathcal{U} \times \mathcal{U})\,.$$

Rønning [106] (also, see [70]) gave a more applicable one variable analytic characterization for $\mathcal{UCV}$. A normalized analytic function $f \in \mathcal{A}$ belongs to $\mathcal{UCV}$ if and only if it satisfies

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \mathcal{U}).$$

Goodman [54] gave examples that demonstrated the Alexander's relation (Theorem 1.7) does not hold between the classes $\mathcal{UCV}$ and $\mathcal{UST}$. Later Rønning [107] introduced the class of parabolic starlike functions $\mathcal{PST}$ consisting of functions

$F = zf'$ such that $f \in \mathcal{UCV}$. It is evident that $f \in \mathcal{PST}$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right| \quad (z \in \mathcal{U}).$$

Let

$$\Omega = \{w : \operatorname{Re} w > |w-1|\} = \left\{w : (\operatorname{Im} w)^2 < 2\operatorname{Re} w - 1\right\}.$$

Clearly, Ω is a parabolic region bounded by $y^2 = 2x - 1$. The function $f \in \mathcal{UCV}$ if and only if $\left(1 + zf''/f'\right) \in \Omega$. Similarly, $f \in \mathcal{PST}$ if and only if $zf'/f \in \Omega$. For this reason, a function $f \in \mathcal{PST}$ is called a parabolic starlike function. A survey of these functions can be found in [108]. In [106, 109], Rønning generalized the classes $\mathcal{UCV}$ and $\mathcal{PST}$ by introducing a parameter α in the following way: a function $f \in \mathcal{A}$ is in $\mathcal{PST}(\alpha)$ if it satisfies the analytic characterization

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)} - \alpha\right) > \left|\frac{zf'(z)}{f(z)} - 1\right| \quad (\alpha \in \mathbb{R},\ z \in \mathcal{U}),$$

and $f \in \mathcal{UCV}(\alpha)$, the class of uniformly convex functions of order α, if it satisfies

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right) > \left|\frac{zf''(z)}{f'(z)}\right| \quad (\alpha \in \mathbb{R},\ z \in \mathcal{U}).$$

He also introduced the more general classes $\mathcal{PST}(\alpha, \beta)$ consisting of parabolic β-starlike functions of order α that satisfies the condition

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)} - \alpha\right) > \beta\left|\frac{zf'(z)}{f(z)} - 1\right| \quad (-1 < \alpha \leq 1,\ \beta \geq 0,\ z \in \mathcal{U}). \tag{1.3}$$

Analogously, the class $\mathcal{UCV}(\alpha,\beta)$ consists of uniformly β-convex functions of order α satisfying the condition

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}-\alpha\right)>\beta\left|\frac{zf''(z)}{f'(z)}\right| \quad (-1<\alpha\leq 1,\ \beta\geq 0,\ z\in\mathcal{U}). \tag{1.4}$$

Indeed, it follows from (1.3) and (1.4) that $f\in\mathcal{UCV}(\alpha,\beta)$ if and only if $zf'\in\mathcal{PST}(\alpha,\beta)$. The geometric representation of the relation (1.3) is that the class $\mathcal{PST}(\alpha,\beta)$ consists of functions f for which the function $\left(zf'/f\right)$ takes values in the parabolic region Ω, where

$$\Omega=\{w:\operatorname{Re}w-\alpha>\beta|w-1|\}=\left\{w:\operatorname{Im}w<\frac{1}{\beta}\sqrt{(\operatorname{Re}w-\alpha)^2-\beta^2\left(\operatorname{Re}w-1\right)^2}\right\}.$$

Clearly, $\mathcal{PST}(\alpha,1)=\mathcal{PST}(\alpha)$ and $\mathcal{UCV}(\alpha,1)=\mathcal{UCV}(\alpha)$.

The transform

$$\int_0^z\frac{f(t)}{t}dt\equiv\int_0^1\frac{f(tz)}{t}dt$$

introduced by Alexander [5] is known as Alexander transform of f. Using Alexander's Theorem (Theorem 1.7), it is clear that $f\in\mathcal{ST}$ if and only if the Alexander transform of f is in $\mathcal{CV}$. Libera [67] and Livingston [68] investigated the transform

$$2\int_0^1 f(tz)dt,$$

and Bernardi [17] later considered the transform

$$(c+1)\int_0^1 t^{c-1}f(tz)dt, \quad (c>-1) \tag{1.5}$$

which generalizes the Libera and Livingston transform. For that reason, the transform (1.5) is called the generalized Bernardi-Libera-Livingston integral transform. It is well-known [17] that the classes of starlike, convex and close-to-convex functions are closed under the Bernardi-Libera-Livingston transform for all $c > -1$.

An analytic function f is *subordinate* to g in $\mathcal{U}$, written $f \prec g$, or $f(z) \prec g(z) \quad (z \in \mathcal{U})$, if there exists a function w analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. In particular, if the function g is univalent in $\mathcal{U}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Recall that a function $f \in \mathcal{A}$ belongs to the class of starlike functions $\mathcal{ST}$, convex functions $\mathcal{CV}$, or close-to-convex functions $\mathcal{CCV}$ if it satisfies respectively the analytic condition

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad \text{and} \quad \operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right) > 0, \quad g(z) \in \mathcal{CV}.$$

A function in any one of these classes is characterized by either of the quantities $zf'(z)/f(z)$, $1 + zf''(z)/f'(z)$ or $f'(z)/g'(z)$ lying in a given region in the right half plane; the region is convex and symmetric with respect to the real axis [71]. Since $p(z) = (1 + z)/(1 - z)$ is a normalized analytic function mapping $\mathcal{U}$ onto $\{w : \operatorname{Re} w > 0\}$, in terms of subordination, the above conditions are respectively equivalent to

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \qquad 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z} \quad \text{and} \quad \frac{f'(z)}{g'(z)} \prec \frac{1+z}{1-z}.$$

Ma and Minda [71] gave a unified presentation of various subclasses of starlike and convex functions by replacing the superordinate function $p(z) = (1+z)/(1-z)$ by a more general analytic function φ with positive real part and normalized by the conditions $\varphi(0) = 1$ and $\varphi'(0) > 0$. Further it is assumed that φ maps the unit disk $\mathcal{U}$ onto a region starlike with respect to 1 that is symmetric with respect to the real axis. They introduced the following general classes that enveloped several well-known classes as special cases:

$$\mathcal{CV}(\varphi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\},$$

and

$$\mathcal{ST}(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}.$$

When

$$\varphi(z) = \varphi_\alpha(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \quad (0 \leq \alpha < 1),$$

the classes $\mathcal{CV}(\varphi_\alpha)$ and $\mathcal{ST}(\varphi_\alpha)$ reduce to the familiar classes $\mathcal{CV}(\alpha)$ and $\mathcal{ST}(\alpha)$ of univalent convex and starlike functions of order α.

When

$$\varphi(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B \leq A \leq 1),$$

the classes $\mathcal{CV}(\varphi)$ and $\mathcal{ST}(\varphi)$ reduce respectively to the class $\mathcal{CV}[A, B]$ of *Janowski convex functions* and the class $\mathcal{ST}[A, B]$ of *Janowski starlike functions* [60, 90]. Thus

$$\mathcal{CV}[A, B] =: \mathcal{CV}\left(\frac{1 + Az}{1 + Bz}\right) \text{ and } \mathcal{ST}[A, B] =: \mathcal{ST}\left(\frac{1 + Az}{1 + Bz}\right).$$

When

$$\varphi(z) = 1 + \frac{2}{\pi^2}\left(\log \frac{1 + \sqrt{z}}{1 - \sqrt{z}}\right)^2,$$

the classes $\mathcal{CV}(\varphi)$ and $\mathcal{ST}(\varphi)$ reduce to the familiar classes of uniformly convex functions $\mathcal{UCV}$ and its associated class $\mathcal{PST}$.

Define the functions $h_\varphi \in \mathcal{ST}(\varphi)$ and $k_\varphi \in \mathcal{CV}(\varphi)$ respectively by

$$\frac{zh'_\varphi(z)}{h_\varphi(z)} = \varphi(z) \quad (z \in \mathcal{U},\ h_\varphi \in \mathcal{A})$$

$$1 + \frac{zk''_\varphi(z)}{k'_\varphi(z)} = \varphi(z) \quad (z \in \mathcal{U},\ k_\varphi \in \mathcal{A}).$$

In [71], Ma and Minda showed that the functions h_φ and k_φ turned out to be extremal for certain functionals in $\mathcal{ST}(\varphi)$ and $\mathcal{CV}(\varphi)$. In addition, they derived distortion, growth, covering and rotation theorems for the classes $\mathcal{ST}(\varphi)$ and $\mathcal{CV}(\varphi)$ and obtained sharp order of growth for coefficients of these classes.

Theorem 1.11 (Distortion Theorem for $\mathcal{CV}(\varphi)$) [71, Corollary 1] *For each $f \in \mathcal{CV}(\varphi)$,*

$$k'_\varphi(-r) \leq |f'(z)| \leq k'_\varphi(r) \quad (|z| = r < 1).$$

Equality holds for some $z \neq 0$ if and only if f is a rotation of k_φ.

Theorem 1.12 (Growth Theorem for $\mathcal{CV}(\varphi)$) [71, Corollary 2] *For each $f \in \mathcal{CV}(\varphi)$,*

$$-k_\varphi(-r) \leq |f(z)| \leq k_\varphi(r) \quad (|z| = r < 1).$$

Equality holds for some $z \neq 0$ if and only if f is a rotation of k_φ.

Theorem 1.13 (Covering Theorem for $\mathcal{CV}(\varphi)$) [71, Corollary 3] *Suppose $f \in \mathcal{CV}(\varphi)$. Then either f is a rotation of k_φ or $f(\mathcal{U}) \supseteq \{w : |w| \leq -k_\varphi(-1)\}$. Here $-k_\varphi(-1)$ is understood to be the limit of $-k_\varphi(-r)$ as r tends to 1.*

Theorem 1.14 (Rotation Theorem for $\mathcal{CV}(\varphi)$) [71, Corollary 4] *For each $f \in \mathcal{CV}(\varphi)$,*

$$|\arg f'(z)| \leq \max_{|z|=r} \arg\left(k'_\varphi(z)\right) \quad (|z| = r < 1).$$

Equality holds for some $z \neq 0$ if and only if f is a rotation of k_φ.

Next, we state the corresponding results for the class $\mathcal{ST}(\varphi)$. These results follows from the correspondence between $\mathcal{ST}(\varphi)$ and $\mathcal{CV}(\varphi)$.

Theorem 1.15 (Distortion Theorem for $\mathcal{ST}(\varphi)$) [71, Theorem 2] *If $f \in \mathcal{ST}(\varphi)$ with $\min_{|z|=r} |\varphi(z)| = |\varphi(-r)|$ and $\max_{|z|=r} |\varphi(z)| = |\varphi(r)|$, then*

$$h'_{\varphi}(-r) \leq |f'(z)| \leq h'_{\varphi}(r) \quad (|z| = r < 1).$$

Equality holds for some $z \neq 0$ if and only if f is a rotation of h_{φ}.

Theorem 1.16 (Growth Theorem for $\mathcal{ST}(\varphi)$) [71, Corollary 1'] *If $f \in \mathcal{ST}(\varphi)$, then*

$$-h_{\varphi}(-r) \leq |f(z)| \leq h_{\varphi}(r) \quad (|z| = r < 1).$$

Equality holds for some $z \neq 0$ if and only if f is a rotation of h_{φ}.

Theorem 1.17 (Covering Theorem for $\mathcal{ST}(\varphi)$) [71, Corollary 2'] *Suppose $f \in \mathcal{ST}(\varphi)$. Then either f is a rotation of h_{φ} or $f(\mathcal{U}) \supseteq \{w : |w| \leq -h_{\varphi}(-1)\}$. Here $-h_{\varphi}(-1)$ is the limit of $-h_{\varphi}(-r)$ as r tends to 1.*

Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be analytic in $|z| < R_1$, and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ be analytic in $|z| < R_2$. The *convolution*, or *Hadamard product*, of f and g is the function $h = f * g$ given by the power series

$$h(z) = (f * g)(z) = \sum_{n=1}^{\infty} a_n b_n z^n. \tag{1.6}$$

This power series is convergent in $|z| < R_1 R_2$. The term "convolution" arose from the following equivalent representation

$$(f * g)(z) = \frac{1}{2\pi i} \int_{|\zeta|=\rho} f\left(\frac{z}{\zeta}\right) g(\zeta) \frac{d\zeta}{\zeta} \quad \left(\frac{|z|}{R_1} < \rho < R_2\right).$$

The geometric series
$$l(z) = \sum_{n=1}^{\infty} z^n = \frac{z}{1-z},$$
acts as the identity element under convolution [40, pp. 246-247] for the class $\mathcal{A}$.

The functions f and zf' can be represented in terms of convolution as

$$f(z) = f * \frac{z}{1-z} \quad \text{and} \quad zf'(z) = f * \frac{z}{(1-z)^2}.$$

Using Alexander's theorem (Theorem 1.7), a function $f \in \mathcal{A}$ is convex if and only if $f * \big(z/(1-z)^2\big)$ is starlike. So the classes $\mathcal{ST}$ and $\mathcal{CV}$ can be unified by considering $\mathcal{S}_g = \{f \in \mathcal{A} : f * g \in \mathcal{ST}\}$ for an appropriate g. For $g(z) = z/(1-z)$, $\mathcal{S}_g = \mathcal{ST}$, while for $g(z) = z/(1-z)^2$, $\mathcal{S}_g = \mathcal{CV}$.

An important subclass of $\mathcal{A}$ defined by using convolution is the class of prestarlike functions introduced by Ruscheweyh [111]. For $\alpha < 1$, the class $\mathcal{R}_\alpha$ of prestarlike functions of order α is defined by

$$\mathcal{R}_\alpha := \left\{ f \in \mathcal{A} : f * \frac{z}{(1-z)^{2-2\alpha}} \in \mathcal{ST}(\alpha) \right\},$$

while $\mathcal{R}_1$ consists of $f \in \mathcal{A}$ satisfying $\operatorname{Re} f(z)/z > 1/2$. Prestarlike functions have a number of interesting geometric properties. For instance, $\mathcal{R}_0$ is the class of univalent convex functions $\mathcal{CV}$, and $\mathcal{R}_{1/2}$ is the class of univalent starlike functions $\mathcal{ST}(1/2)$ of order $1/2$.

1.2 Multivalent Functions

A function f is *p-valent* (or *multivalent* of order p) if for each w_0 (w_0 may be infinity), the equation $f(z) = w_0$ has at most p roots in $\mathcal{U}$, where the roots are counted with their multiplicities, and for some w_1 the equation $f(z) = w_1$ has exactly p roots in $\mathcal{U}$ [52]. For a fixed $p \in \mathbb{N} := \{1, 2, \cdots\}$, let $\mathcal{A}_p$ denote the class of all analytic functions f of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \tag{1.7}$$

that are p-valent in the open unit disk $\mathcal{U}$, and for $p = 1$, let $\mathcal{A}_1 := \mathcal{A}$.

The *convolution*, or *Hadamard product*, of two p-valent functions

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad \text{and} \quad g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}$$

is defined as

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p}.$$

A p-valent function $f \in \mathcal{A}_p$ is *starlike* if it satisfies the condition

$$\frac{1}{p} \mathrm{Re}\, \frac{zf'(z)}{f(z)} > 0 \quad (f(z)/z \neq 0, \; z \in \mathcal{U}),$$

and is *convex* if it satisfies the condition

$$\frac{1}{p} \mathrm{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0 \quad (f'(z) \neq 0, \; z \in \mathcal{U}).$$

The subclasses of $\mathcal{A}_p$ consisting of starlike and convex functions are denoted respectively by $\mathcal{ST}_p$ and $\mathcal{CV}_p$. More generally, let φ be an analytic function with positive real part in $\mathcal{U}$, $\varphi(0) = 1$, $\varphi'(0) > 0$, and φ maps the unit disk $\mathcal{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The classes $\mathcal{ST}_p(\varphi)$ and $\mathcal{CV}_p(\varphi)$ consist respectively of p-valent functions f *starlike* with respect to φ and p-valent functions f *convex* with respect to φ in $\mathcal{U}$ given by

$$\mathcal{ST}_p(\varphi) := \left\{ f \in \mathcal{A}_p : \frac{1}{p} \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\},$$

and

$$\mathcal{CV}_p(\varphi) := \left\{ f \in \mathcal{A}_p : \frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \right\}.$$

These classes were introduced and investigated by Ali et al. in [8]. The functions $h_{\varphi,p}$ and $k_{\varphi,p}$ defined respectively by

$$\frac{1}{p}\frac{zh'_{\varphi,p}(z)}{h_{\varphi,p}(z)} = \varphi(z) \quad (z \in \mathcal{U}, h_{\varphi,p} \in \mathcal{A}_p),$$
$$\frac{1}{p}\left(1 + \frac{zk''_{\varphi,p}(z)}{k'_{\varphi,p}(z)}\right) = \varphi(z) \quad (z \in \mathcal{U}, k_{\varphi,p} \in \mathcal{A}_p),$$

are important examples of functions in $\mathcal{ST}_p(\varphi)$ and $\mathcal{CV}_p(\varphi)$. A result analogues to Alexander' theorem (Theorem 1.7) was obtained by Ali et al. in [8].

Theorem 1.18 [8, Theorem 2.1] *The function f belongs to $\mathcal{CV}_p(\varphi)$ if and only if $(1/p)zf' \in \mathcal{ST}_p(\varphi)$.*

When $p = 1$ these classes reduced to the $\mathcal{ST}(\varphi)$ and $\mathcal{CV}(\varphi)$ classes introduced by Ma and Minda [71].

When

$$\varphi(z) = \frac{1+z}{1-z},$$

the classes $\mathcal{ST}_p(\varphi)$ and $\mathcal{CV}_p(\varphi)$ reduce to the familiar classes of p-valent starlike and convex functions $\mathcal{ST}_p$ and $\mathcal{CV}_p$. In addition if $p = 1$ these classes are respectively the classes of univalent starlike and convex functions $\mathcal{ST}$ and $\mathcal{CV}$.

When

$$\varphi(z) = \varphi_\beta(z) = \frac{1 + (1-2\beta)z}{1-z} \quad (0 \le \beta < 1)$$

the classes $\mathcal{ST}_p(\varphi_\beta)$ and $\mathcal{CV}_p(\varphi_\beta)$ reduce to the familiar classes of p-valent starlike and convex functions of order β introduced by Patil and Thakare in [88]:

$$\mathcal{ST}_p(\beta) := \left\{ f \in \mathcal{A}_p : \frac{1}{p}\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \right\},$$
$$\mathcal{CV}_p(\beta) := \left\{ f \in \mathcal{A}_p : \frac{1}{p}\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta \right\}.$$

For $p \in \mathbb{N}$ and $\alpha < 1$, Kumar and Reddy [12] defined the class $\mathcal{R}_p(\alpha)$ of p-valent prestarlike functions of order α by

$$\mathcal{R}_p(\alpha) = \left\{ f \in \mathcal{A}_p : f(z) * \frac{z^p}{(1-z)^{2p(1-\alpha)}} \in \mathcal{ST}_p(\alpha) \right\}.$$

They obtained necessary and sufficient coefficient conditions for a function f to be in the class $\mathcal{R}_p(\alpha)$. Evidently, this class reduces to the class of prestarlike functions $\mathcal{R}(\alpha)$ introduced by Ruscheweyh [111] for $p = 1$.

1.3 Differential Subordination

In the theory of complex-valued functions there are many differential conditions which shape the characteristics of a function. A simple example is the Noshiro-Warschawski Theorem [40, Theorem 2.16, p.47]: If f is analytic in a convex domain D, then

$$\mathrm{Re}\,\left(f'(z)\right) > 0 \Rightarrow f \text{ is univalent in } D.$$

This theorem and many known similar differential implications dealt with real-valued inequalities that involved the real part, imaginary part or modulus of a complex expression. In 1981 Miller and Mocanu [74] replaced the differential inequality, a real valued concept, with its complex analogue of differential subordination.

Let $\psi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ be an analytic function and let h be univalent in the unit disk $\mathcal{U}$. If p is analytic in $\mathcal{U}$ and satisfies the second-order differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{1.8}$$

then p is called a *solution* of the differential subordination. A univalent function q is called a *dominant* if $p(z) \prec q(z)$ for all p satisfying (1.8). A dominant q_1 satisfying $q_1(z) \prec q(z)$ for all dominants q of (1.8) is said to be the *best dominant* of (1.8). The best dominant is unique up to a rotation of $\mathcal{U}$. If $p \in \mathcal{H}[a, n]$, then p is called an (a, n)-*solution*, q an (a, n)-*dominant*, and q_1 the *best* (a, n)-*dominant*. Let $\Omega \subset \mathbb{C}$ and let (1.8) be replaced by

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega \quad (z \in \mathcal{U}), \tag{1.9}$$

where Ω is a simply connected domain containing $h(\mathcal{U})$. Even though this is a "differential inclusion" and $\psi(p(z), zp'(z), z^2p''(z); z)$ may not be analytic in $\mathcal{U}$, the condition in (1.9) will also be referred to as a *second-order differential subordination*, and the same definition for solution, dominant and best dominant as given above can be extended to this generalization. The monograph [75] by Miller and Mocanu provides a detailed account on the theory of differential subordination.

Denote by $\mathcal{Q}$ the set of all functions q that are analytic and injective on $\overline{\mathcal{U}} \backslash E(q)$, where

$$E(q) = \{\zeta \in \partial\mathcal{U} : \lim_{z \to \zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathcal{U} \setminus E(q)$.

Definition 1.1 [75, Definition 2.3a, p. 27] *Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ satisfying the admissibility condition*

$$\psi(r, s, t; z) \notin \Omega \tag{1.10}$$

whenever $r = q(\zeta),\ s = k\zeta q'(\zeta),$ *and*

$$\mathrm{Re}\left(\frac{t}{s}+1\right) \geq k\mathrm{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right),$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ *and* $k \geq n$. *Additionally,* $\Psi_1[\Omega, q]$ *will be written as* $\Psi[\Omega, q]$.

If $\psi : \mathbb{C}^2 \times \mathcal{U} \to \mathbb{C}$, *then the admissibility condition (1.10) reduces to*

$$\psi(q(\zeta), k\zeta q'(\zeta); z) \notin \Omega,$$

$z \in \mathcal{U}$, $\zeta \in \partial\mathcal{U} \setminus E(q)$ *and* $k \geq n$.

If $\psi : \mathbb{C} \times \mathcal{U} \to \mathbb{C}$, *then the admissibility condition (1.10) becomes*

$$\psi(q(\zeta); z) \notin \Omega,$$

$z \in \mathcal{U}$, *and* $\zeta \in \partial\mathcal{U} \setminus E(q)$.

A foundation result in the theory of first and second order differential subordination is the following theorem:

Theorem 1.19 [75, Theorem 2.3b, p.28] *Let* $\psi \in \Psi_n[\Omega, q]$ *with* $q(0) = a$. *If* $p \in \mathcal{H}[a, n]$ *satisfies*

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega, \tag{1.11}$$

then $p \prec q$.

It is evident that the dominant of a differential subordination of the form (1.11) can be obtained by checking that the function ψ is an admissible function. This requires that the function ψ satisfies (1.10). Considering the special case when $\Omega = h(\mathcal{U})$ is a simply connected domain, and h is a conformal mapping of $\mathcal{U}$ onto

Ω, the following second-order differential subordination result is an immediate consequence of Theorem 1.19. The set $\Psi_n[h(\mathcal{U}), q]$ is written as $\Psi_n[h, q]$.

Theorem 1.20 [75, Theorem 2.3c, p.30] *Let* $\psi \in \Psi_n[h, q]$ *with* $q(0) = a$. *If* $p \in \mathcal{H}[a, n]$, $\psi(p(z), zp'(z), z^2p''(z); z)$ *is analytic in* $\mathcal{U}$, *and*

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{1.12}$$

then $p \prec q$.

The next theorem yields best dominant of the differential subordination (1.12)

Theorem 1.21 [75, Theorem 2.3f, p.32] *Let* h *be univalent in* $\mathcal{U}$ *and* $\psi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$. *Suppose that the differential equation*

$$\psi(q(z), nzq'(z), n(n-1)zq'(z) + n^2z^2q''(z); z) = h(z)$$

has a solution q, *with* $q(0) = a$, *and one of the following conditions is satisfied:*

(i) $q \in \mathcal{Q}$ *and* $\psi \in \Psi_n[h, q]$,

(ii) q *is univalent in* $\mathcal{U}$ *and* $\psi \in \Psi_n[h, q_\rho]$ *for some* $\rho \in (0, 1)$, *or*

(iii) q *is univalent in* $\mathcal{U}$ *and there exists* $\rho_0 \in (0, 1)$ *such that* $\psi \in \Psi_n[h_\rho, q_\rho]$ *for all* $\rho \in (\rho_0, 1)$.

If $p \in \mathcal{H}[a, n]$, $\psi(p(z), zp'(z), z^2p''(z); z)$ *is analytic in* $\mathcal{U}$, *and* p *satisfies*

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z),$$

then $p \prec q$, *and* q *is the best* (a, n)*-dominant.*

When dealing with first-order differential subordination, the following theorem is useful.

Theorem 1.22 [75, Theorem 3.4h, p.132] *Let q be univalent in $\mathcal{U}$ and let θ and ϕ be analytic in a domain D containing $q(\mathcal{U})$, with $\phi(w) \neq 0$ when $w \in q(\mathcal{U})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that either*

(i) *h is convex, or*

(ii) *Q is starlike.*

In addition, assume that

(iii) $\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \left(\frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)} \right) > 0.$

If p is analytic in $\mathcal{U}$, with $p(0) = q(0)$, $p(\mathcal{U}) \subset D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z),$$

then $p \prec q$, and q is the best dominant.

Let $f \in \mathcal{A}_p$ be given by (1.7). Upon differentiating both sides of f q-times with respect to z, the following differential operator is obtained:

$$f^{(q)}(z) = \lambda(p;q)z^{p-q} + \sum_{k=1}^{\infty} \lambda(k+p;q)a_{k+p}z^{k+p-q},$$

where

$$\lambda(p;q) := \frac{p!}{(p-q)!} \quad (p \geq q; p \in \mathbb{N}; q \in \mathbb{N} \cup \{0\}).$$

Several researchers have investigated higher-order derivatives of multivalent functions, see for example [10, 11, 37, 56–58, 81, 89, 120, 141]. Recently, by use of the well-known Jack's Lemma [59, 75], Irmak and Cho [57] obtained interesting results for certain classes of functions defined by higher-order derivatives. We shall continue this investigation in Chapter 2.

1.4 Functions with Respect to n-ply Points

As defined on p. 14, the convolution of two functions f and g with power series

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^n \text{ and } g(z) = z + \sum_{n=1}^{\infty} b_n z^n$$

convergent in $\mathcal{U}$ is defined by

$$(f * g)(z) := z + \sum_{n=1}^{\infty} a_n b_n z^n \quad (z \in \mathcal{U}).$$

Pólya and Schoenberg in 1958 [91] posed two important conjectures:

1. If f and $g \in \mathcal{CV}$, then $f * g \in \mathcal{CV}$.

2. If $f \in \mathcal{CV}$ and $g \in \mathcal{ST}$, then $f * g \in \mathcal{ST}$.

Using Alexander's theorem, (Theorem 1.7), it is clear that any one of these conjectures implies the other. These conjunctures were later proved by Ruscheweyh and Sheil-Small [114].

Let $h : \mathcal{U} \to \mathbb{C}$ be a convex function with positive real part in $\mathcal{U}$, $h(0) = 1$, and g be a given fixed function in $\mathcal{A}$. Shanmugam [116] introduced the classes $\mathcal{ST}_g(h)$ and $\mathcal{CV}_g(h)$ consisting of functions f satisfying

$$\frac{z(f * g)'(z)}{(f * g)(z)} \prec h(z) \text{ and } 1 + \frac{z(f * g)''(z)}{(f * g)'(z)} \prec h(z).$$

Note that for $g(z) = z/(1-z)$, the class $\mathcal{ST}_g(h) \equiv \mathcal{ST}(h)$ and the class $\mathcal{CV}_g(h) \equiv \mathcal{CV}(h)$. He introduced these classes [116] and other related classes, and investigated inclusion and convolution properties by using the convex hull method [113, 114] and the method of differential subordination [75]. Ali *et al.* [8] investigated the subclasses of $p-$valent starlike and convex functions, and obtained several subordination and convolution properties, as well as sharp distortion, growth and rotation estimates. These works were recently extended by Supramaniam *et al.* [127]. Similar problems but for the class of meromorphic functions were also recently investigated by Mohd *et al.* [78].

For a fixed positive integer n, $\epsilon^n = 1$, $\epsilon \neq 1$ and $f \in \mathcal{A}$, define the function with $n-$ply points $f_n \in \mathcal{A}$ by

$$f_n(z) := \frac{1}{n} \sum_{k=0}^{n-1} \epsilon^{n-k} f(\epsilon^k z). \tag{1.13}$$

It is clear that $f_1(z) = f(z)$ and $f_2(z) = (f(z) - f(-z))/2$. A function $f \in \mathcal{A}$ is *starlike with respect to n-symmetric points* if it satisfies

$$\operatorname{Re} \frac{zf'(z)}{f_n(z)} > 0. \tag{1.14}$$

Denote the class of these functions by $\mathcal{ST}_s^n$. For $n = 2$, the class $\mathcal{ST}_s^n$ reduces to the class $\mathcal{ST}_s$ consisting of the starlike functions with respect to symmetric points in $\mathcal{U}$ introduced by Sakaguchi [115]. If k is an integer, then the following identities follow directly from (1.13) :

$$\begin{aligned} f_n(\epsilon^k z) &= \epsilon^k f_n(z), \\ f_n'(\epsilon^k z) &= f_n'(z) = \frac{1}{n} \sum_{m=0}^{n-1} f'(\epsilon^m z), \\ \epsilon^k f_n''(\epsilon^k z) &= f_n''(z) = \frac{1}{n} \sum_{m=0}^{n-1} \epsilon^m f''(\epsilon^m z). \end{aligned}$$

More generally, the condition (1.14) can be generalized to the subordination

$$\frac{zf'(z)}{f_n(z)} \prec h(z),$$

where h is a given convex function, with $h(0) = 1$ and $\mathrm{Re}\,(h) > 0$. El-Ashwah and Thomas [41] introduced the classes $\mathcal{ST}_c$ and $\mathcal{ST}_{sc}$ consisting of the starlike functions with respect to conjugate points in $\mathcal{U}$ and the starlike functions with respect to symmetric conjugate points in $\mathcal{U}$ respectively. In 2004, Ravichandran [101] introduced the classes of starlike, convex and close-to-convex functions with respect to $n-$ply symmetric points, conjugate points and symmetric conjugate points, and obtained several convolution properties. Other investigations into the classes defined by using conjugate and symmetric conjugate points can be found in [4,38,62,133,134,136,137,139]. These classes of functions will be treated further in Chapter 3.

1.5 Integral Operators

The study of integral operators is an important problem in the field of Geometric Function Theory. In [21], Biernacki falsely claimed that $\int_0^z (f(\zeta)/\zeta)\, d\zeta$ is univalent whenever f is univalent. Moved by this, Causey [34] considered a related problem of finding conditions on $\delta \in \mathbb{C}$ such that the integral operator $F_\delta : \mathcal{A} \to \mathcal{A}$ given by

$$(F_\delta f)(z) = \int_0^z \left(\frac{f(\zeta)}{\zeta}\right)^\delta d\zeta$$

is univalent whenever f is univalent. It is known [64] that $F_\delta \in \mathcal{S}$ when $|\delta| < 1/4$. The case $\delta = 1$ was earlier considered by Alexander [5] and he proved that F_1 is in $\mathcal{CV}$ whenever f is in $\mathcal{ST}$. In [73], Merkes obtained various extension of inclusion results for certain subclasses of $\mathcal{S}$. He showed that

$$F_\delta(\mathcal{ST}) \subset \mathcal{S} \text{ whenever } |\delta| \leq 1/2. \tag{1.15}$$

There is no larger disk $|\delta| \leq R,\ R > 1/2$, such that the inclusion (1.15) holds.